Robust Global Control of Unknown Nonlinear Systems

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Abstract: The problem of controlling nonlinear systems with unknown model is considered. A testing and design process that leads to the construction of robust state-feedback controllers for such systems is described. The framework presented in the note is based on the use of bang-bang controllers, and it applies to a wide range of nonlinear systems.

Keywords: unknown nonlinear systems, nonlinear feedback control, robust nonlinear control, State feedback, asymptotic tracking.

1. INTRODUCTION

This note describes a process that derives robust state-feedback controllers for nonlinear systems with unknown description. As opposed to linear systems, where techniques for controlling systems with unknown description are well established (Astrom and Wittenmark (2008)), it seems that no such techniques are available for broad families of nonlinear systems.

The design framework presented in this note yields robust state-feedback controllers for nonlinear systems whose model is unknown. It is valid for large families of nonlinear systems. It does not attempt to estimate parameters of the unknown systems that need to be controlled. In fact, the systems under consideration are nonlinear systems that may not be characterizable by finite sets of parameters. Rather than focus on system parameters, we concentrate on the use of signals that are inherently finitary: bang-bang signals.

Bang-bang signals switch between the upper and the lower input amplitude bounds of the controlled system. Thus, bang-bang signals admit only a finite number of values at each instant of time. This fact allows us to develop a finitary process for testing an unknown nonlinear system to derive information that is sufficient for the design of bang-bang state feedback controllers that fulfill desired control objectives. In view of (Hammer (2021, 2025)), bang-bang controllers, or closely related pseudo bang-bang controllers, can approximate the performance of any controller. Consequently, restriction to bang-bang controllers does not limit performance options.

The design framework of this note can be incorporated into artificial intelligence/machine learning algorithms to automatically build robust controllers for unknown nonlinear systems.

The control configuration is shown in Figure 1, where the state-feedback controller φ controls the system Σ; the latter's state is $x(t)$, and its input is $u(t)$. Structural limitations impose a constraint of $K > 0$ on the input

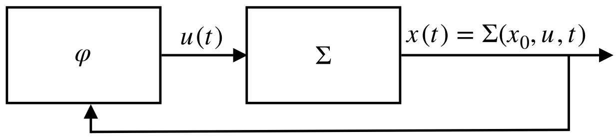

Fig. 1. The control configuration

amplitude and a constraint of $A > 0$ on the state norm of Σ.

The objective is to derive state-feedback controllers φ that asymptotically stabilize Σ. As the description of Σ is not known, testing of an available sample Σ_0 of Σ is performed to derive data that facilitates the design of φ. The tested system Σ_0 may differ from the system Σ placed in the configuration of Figure 1. We allow Σ to be any member of a family $\mathcal{F}_\gamma(\Sigma_0)$ of systems that differ by no more than γ from Σ_0.

As asymptotic stabilization with bang-bang controllers may involve infinite switchings near the zero state, we exchange to a linear controller near the origin of state space. To that end, we assume that Σ can be approximated by a stabilizable linear system Λ within a small neighborhood $\mathcal{N}$ of the zero state. Then, φ operates as a bang-bang controller outside $\mathcal{N}$ and as a linear controller inside $\mathcal{N}$. This note concentrates on bang-bang controllers that guide Σ to $\mathcal{N}$; derivation of linear controllers is well known. Denote by $\rho(a)$ a ball of radius a centered at zero. Let $\chi > 0$ be such that

$$\rho(2\chi) \subseteq \mathcal{N}. \tag{1}$$

Problem 1. Faced with a family $\mathcal{F}_\gamma(\Sigma_0)$ of unknown systems closely related to an unknown system Σ_0 available for testing, develop a testing protocol for Σ_0 that leads to the construction of robust bang-bang state-feedback controllers φ that guide all members of $\mathcal{F}_\gamma(\Sigma_0)$ to $\rho(2\chi)$. $\square$

This note continues the extensive literature of adaptive control that was seeded more than half a century ago by the works of Kalman (1958); Belman (1961); Mishkin and Braun (1961), and others. To mention a few recent contributions in this area, we list the paper by Ren and

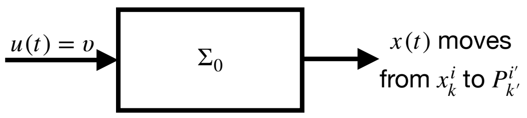

Fig. 2. The testing process

Yang (2019) on adaptive control of a certain class of input-affine cyber-physical systems subject to unknown nonlinearities and false data injection; the paper by Liu et al. (2020) on integral barrier Lyapunov functions that investigates adaptive control of a certain class of switched input-affine nonlinear systems; and the paper by Ortega et al. (2020) on identification of systems with a linear regression model for their unknown parameters. More generally, this note contributes to the general theory of nonlinear control (e.g., Nijmeijer and van der Schaft (2016) and the references cited there).

Regarding organization, Section 2 provides an overview of the framework developed in this paper. Section 3 describes the class of nonlinear systems considered; Section 4 discusses the impact of uncertainty; Sections 5 and 6 build the technical foundation; Section 7 designs robust bang-bang state-feedback controllers; Section 8 analyzes an example; and Section 9 is a brief summary.

2. DESIGN OUTLINE

The framework of this paper consists of two stages: *(i)* testing of a sample of the controlled system; and *(ii)* using the test results to derive a controller. The utilization of bang-bang controllers substantially simplifies the task since, for a system with m inputs, a bang-bang controller φ generates at each instant of time a constant signal from the family of 2^m members

$$\mathcal{C}(m) = \left\{ \upsilon = (\upsilon^1, \upsilon^2, \ldots, \upsilon^m)^\top : |\upsilon^i| = K, i = 1, \ldots, m \right\}.$$

For example, for a single input system, $\mathcal{C}(1) = \{K, -K\}$.

To derive the controller φ, we monitor the response of the tested system Σ_0 to constant inputs from $\mathcal{C}(m)$, using a grid Γ of initial states. The grid Γ consists of spherical shells of thickness $\Delta > 0$ (Section 5); it is produced by spheres $\sigma(r_i)$ of radii r_i centered at zero, where Δ is selected so that $p := (A - \chi)/\Delta$ is an integer; then,

$$r_i = \chi + i\Delta, i = 0, 1, \ldots, p. \tag{2}$$

Each sphere $\sigma(r_i)$ is partitioned into segments $\{P_k^i\}$, $k = 1, 2, \ldots, q(i)$, that are determined by a real number $\delta > 0$: each segment P_k^i is included in a ball of radius δ centered at a *designated state* $x_k^i \in \rho(r_i)$. The number $q(i)$ of $\{P_k^i\}$ members is often not overly large.

In testing, the response of Σ_0 to constant inputs $\upsilon \in \mathcal{C}(m)$ is monitored from each designated state x_k^i to record the first member $P_{k'}^{i'}$ reached by Σ_0, as shown in Figure 2.

This information is recorded in a directed graph $G(\Delta)$, where each vertex represents one member P_k^i; directed edges point from the vertex P_k^i to the vertex $P_{k'}^{i'}$. On each edge, we mark (υ, τ), where $\upsilon \in \mathcal{C}(m)$ is the input used and τ is the time it took to reach $P_{k'}^{i'}$ from P_k^i (Section 7). Next, methods of graph theory (e.g., Bollobás (1998)) are deployed to find directed paths $\Pi = \{P_k^i, P_{k_1}^{i_1}, P_{k_2}^{i_2}, \ldots, P_1^0\}$ from each vertex P_k^i to the vertex P_1^0 representing the target $\mathcal{N}$. The pairs (υ, τ) on the path's edges engender a bang-bang input signal, forming a robust bang-bang state-feedback controller φ that guides every system $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ to $\mathcal{N}$ (Section 7).

3. BASICS

3.1 Notation

Denote by R the real numbers and by R^+ the non-negative real numbers. The absolute value of a number r is $|r|$. For a vector $x = (x_1, x_2, \ldots, x_n)^\top \in R^n$, denote $|x| := \max\{|x_1|, |x_2|, \ldots, |x_n|\}$ and $|x|_2 = (x^\top x)^{1/2}$. Given a function $u : R^+ \to R^m$, denote $|u|_\infty := \sup_{t \geq 0} |u(t)|$. For a real number $M > 0$, denote by $[-M, M]^n$ the set of all $x \in R^n$ with $|x| \leq M$. The ball of radius $r > 0$ centered at $x = 0$ is $\rho(r) := \{x \in R^n : |x|_2 \leq r\}$; the ball centered at y is $\rho(r, y) := \{x \in R^n : |x - y|_2 \leq r\}$. For a set $S \subseteq R^n$,

$$\rho(r, S) := \bigcup_{y \in S} \rho(r, y).$$

The sphere of radius r around the origin is $\sigma(r) := \{x \in R^n : |x|_2 = r\}$, and $\sigma(r, y) := \{x \in R^n : |x - y|_2 = r\}$ is the sphere of radius r centered at y.

3.2 The controlled system

The system Σ of Figure 1 is time-invariant:

$$\Sigma : \ \dot{x}(t) = f(x(t), u(t)), t \geq 0, x(0) = x_0, \tag{3}$$

where the *recursion function* $f : R^n \times R^m \to R^n$ is unknown, but continuously differentiable; $x(t) \in R^n$ is the state; and $u(t) \in R^m$ is the input. The amplitude constraints are $|x(t)|_2 \leq A$ and $|u(t)| \leq K$ for all $t \geq 0$. The class of input signals is then

$$U(K) := \{u : R^+ \to R^m : |u|_\infty \leq K \text{ and } u \text{ is measurable}\}.$$

Initial states x_0 are taken from the ball $\rho(\eta)$, where $\eta \in (0, A)$ is specified. We also assume that $f(0, 0) = 0$.

To incorporate model uncertainty, set

$$\begin{aligned} f(x, u) &= f_0(x, u) + f_\gamma(x, u); \\ f_0(0, 0) &= 0, f_\gamma(0, 0) = 0, \end{aligned} \tag{4}$$

where f_0 is the recursion function of the tested system Σ_0, and f_γ represents modeling uncertainty. The functions f_0 and f_γ are both unknown but continuously differentiable. The tested system Σ_0 is given by $\dot{x}(t) = f_0(x(t), u(t)), t \geq 0, x(0) = x_0$.

To avoid 'freezing' of Σ outside the origin, we require

Assumption 2. For every state $x \in \rho(A)$, except possibly $x = 0$, there is an input $\upsilon \in \mathcal{C}(m)$ for which $f_0(x, \upsilon) \neq 0$. □

3.3 Effects of uncertainty

Considering that the functions f_0 and f_γ are continuously differentiable over the compact domain $\rho(A) \times [-K, K]^m$, it follows that there are constants $B, \gamma > 0$ such that

$$\begin{aligned} |f_0(x, u) - f_0(x', u')| &\leq B(|x - x'| + |u - u'|), \\ |f_\gamma(x, u) - f_\gamma(x', u')| &\leq \gamma(|x - x'| + |u - u'|) \end{aligned} \tag{5}$$

for all $(x, u), (x', u') \in \rho(A) \times [-K, K]^m$, where γ is the *uncertainty parameter*. The family of all systems represented by (3) and satisfying (4) and (5) is denoted by $\mathcal{F}_\gamma(\Sigma_0)$. We seek robust bang-bang state-feedback controllers φ that properly control all members of $\mathcal{F}_\gamma(\Sigma_0)$.

4. PRELIMINARIES

4.1 Bang-bang steps

A bang-bang step is a time during which a bang-bang controller provides a constant input to the controlled system Σ. It is possible that consecutive bang-bang steps provide the same input, so the number of switchings may be lower than the number of bang-bang steps. We impose the following.

Assumption 3. A bang-bang controller φ drives the controlled system Σ from initial states in $\rho(\eta)$ to $\mathcal{N}$ within a time of $\Theta > 0$, with no more than d bang-bang steps of duration not exceeding $T > 0$. □

The values of Θ, d, and T can be changed later, if needed.

During a bang-bang step, Σ_0 is an autonomous system given by the equation $\dot{x}(t) = f_0(x(t), \upsilon)$, where $\upsilon \in \mathcal{C}(m)$ is a constant input. For a set $S \subseteq \rho(A)$ of states at $t = 0$, the states Σ_0 can reach at a time $t \geq 0$ with the input υ is called the *reachable set* $\mathcal{R}_0(t, \upsilon, S)$. The *flow function* $F_0(t, \upsilon) : S \to \mathcal{R}_0(t, \upsilon, S)$ is known to be a homeomorphism (e.g., Hirsch et al. (2012)).

The set of all states that Σ_0 can reach with a constant input $\upsilon \in \mathcal{C}(m)$ during a time interval $[0, \tau]$, after starting from states in S at $t = 0$, is

$$\Phi_0(\tau, \upsilon, S) = \bigcup_{t\in[0,\tau]} \mathcal{R}_0(t, \upsilon, S). \tag{6}$$

Allowing all constant inputs $\upsilon \in \mathcal{C}(m)$, the system Σ_0 can reach the states

$$\Phi_0(\tau, S) = \bigcup_{\upsilon\in\mathcal{C}(m)} \Phi_0(\tau, \upsilon, S).$$

To reach a particular state in $\Phi_0(\tau, S)$, the system Σ_0 must start from an appropriate initial state in S, and receive an appropriate constant input $\upsilon \in \mathcal{C}(m)$ for an appropriate time $t \leq \tau$.

4.2 The impact of uncertainty

The next statement, which follows from (3), (4), (5), and Assumption 3, clarifies the impact of the uncertainty parameter γ (see Hammer (2025) for proof).

Proposition 4. There is a real constant $c > 0$ such that $|\Sigma(x_0, u, t) - \Sigma_0(x_0, u, t)| \leq \gamma c$ for all systems $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$, all states $x_0 \in \rho(\eta)$, all input signals $u \in U(K)$, and all times $t \in [0, \Theta]$. □

5. THE SPACING INTERVAL

The interval Δ of (2) is called the *spacing interval*. It creates a family $\mathcal{S}$ of spheres that help determine the switching points of the bang-bang controller φ. Basically, Δ is determined by requiring that a constant input $\upsilon \in \mathcal{C}(m)$ move the state of Σ_0 by at least Δ in one bang-bang step. We need the following notation to show that Δ exists.

A neighborhood of radius $\varepsilon > 0$ on a sphere $\sigma(r)$ around a state $x \in \sigma(r)$ is $s(r, \varepsilon, x) := \sigma(r) \cap \rho(\varepsilon, x)$; its complement

$$s^c(r, \varepsilon, x) := \sigma(r) \setminus s(r, \varepsilon, x) \tag{7}$$

consists of states on $\sigma(r)$ no closer than ε to x. Denote

$$\begin{aligned} &\Psi_0(r, \alpha, \Delta, \upsilon, x) := \\ &\Phi_0(T, \upsilon, x) \cap \{\sigma(r - \Delta) \cup \sigma(r + \Delta) \cup s^c(r, \Delta/\alpha, x)\} \end{aligned} \tag{8}$$

Then, following holds.

Lemma 5. There is a $\Delta_0 > 0$ such that, for every $r \in [\chi, A]$ and $x \in \sigma(r)$, there is a constant input $\upsilon \in \mathcal{C}(m)$ satisfying

$$\Psi_0(r, \alpha, \Delta_0, \upsilon, x) \neq \varnothing. \tag{9}$$

Proof. (outline) By contradiction, assume that there are sequences $\{x_k\}_{k=1}^{\infty} \subseteq \sigma(r)$ and $\{\delta_k\}_{k=1}^{\infty}$, where $\lim_{k\to\infty} \delta_k = 0$, such that $\Psi_0(r, \alpha, \delta_k, \upsilon, x_k) = \varnothing$ for all $\upsilon \in \mathcal{C}(m)$ and all $k = 1, 2, \ldots$ By compactness, $\{x_k\}_{k=1}^{\infty}$ has a limit point $x' \in \sigma(r)$. Then, we get $\Psi_0(r, \alpha, \delta', \upsilon, x') = \varnothing$ for all $\delta' \geq 0$ and all $\upsilon \in \mathcal{C}(m)$. This entails that $f_0(x', \upsilon) = 0$ for all $\upsilon \in \mathcal{C}(m)$, contradicting Assumption 2. □

By continuity of the flow function, any $\Delta \leq \Delta_0$ also satisfies Lemma 5. A value of Δ can be found experimentally as follows (the selection of α is discussed in Section 5).

Procedure 6. (Finding Δ_0). Fix an $\alpha > 0$.

Step 1: Select $\Delta_0 > 0$; build a grid Γ in $\rho(A)$, using radial spacing of Δ_0 and tangential spacing of Δ_0/α.

Step 2: Test the response of Σ_0 with constant inputs $\upsilon \in \mathcal{C}(m)$ for a duration not exceeding T, using states of Γ as initial states. If, for every state of Γ, there is an input $\upsilon \in \mathcal{C}(m)$ for which (9) holds, then terminate the procedure and record Δ_0.

Step 3: Else, repeat from Step 1, using $\Delta_0/2$ for Δ_0. □

By Lemma 5, Procedure 6 terminates.

6. THE PARTITION

6.1 Mode of operation

We select a real number $\delta > 0$ so that states not further than δ apart have 'similar' behavior over short period of time; selection of δ is discussed below. Using δ, we partition each sphere $\sigma(r_i)$ of (2) into a finite number of subsets $\{P^i_k\}$. Each P^i_k is determined by a designated state x^i_k and δ, so that

$$P^i_k \subseteq \rho(\delta, x^i_k).$$

To select δ, we need to describe the mode of operation of the bang-bang controller φ we aim to design.

Refer to Figure 3. Assume that the initial state x_0 of Σ_0 is within the subset $P^{i_0}_{k_0}$, The designated state of $P^{i_0}_{k_0}$ is $x^{i_0}_{k_0}$. Let $\upsilon_1 \in \mathcal{C}(m)$ be a constant input that drives Σ_0 from $x^{i_0}_{k_0}$ to a state $x_1 \in P^{i_1}_{k_1}$ in a time τ_1. Now, apply υ_1 to Σ_0 from x_0, and let x' be the state reached at the time τ_1. The resulting deviation is $D_1 := |x' - x_1|$. This is the first bang-bang step deviation.

For the second bang-bang step, use a constant input $\upsilon_2 \in \mathcal{C}(m)$ that drives Σ_0 from the designated state $x^{i_1}_{k_1}$ of $P^{i_1}_{k_1}$ to reach a state $x_2 \in P^{i_2}_{k_2}$ in τ_2 time. Now, apply the constant input υ_2 to Σ_0 starting from x' for the same time

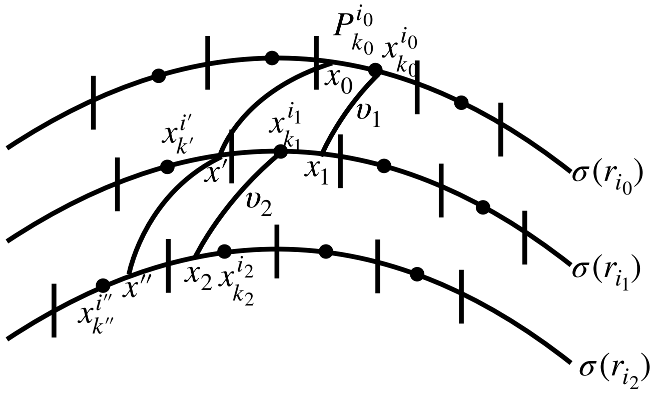

Fig. 3. Principle of operation

τ_2; let x'' be the state reached. The resulting deviation is $D_2 := |x'' - x_2|$.

In general, we have two progressions:

(i) A progression among subsets $\{P^i_k\}$, where, at step $j \in \{0, 1, \dots\}$, the state of Σ_0 moves from the designated state $x^{i_j}_{k_j}$ of $P^{i_j}_{k_j}$ to a subset $P^{i_{j+1}}_{k_{j+1}}$, while driven by a constant input $\upsilon_{j+1} \in \mathcal{C}(m)$ and reaching $P^{i_{j+1}}_{k_{j+1}}$ after τ_{j+1} time. This is a potentially discontinuous path, since, after reaching a state x in $P^{i_j}_{k_j}$, the next step starts at $x^{i_j}_{k_j}$, which may be a state different from x.

Assume that a state $x^* \in \sigma(\chi)$ is reached after q such steps, where $q \le d$ by Assumption 3. Denote $P^0_1 := \sigma(\chi)$. We obtain a sequence

$$\Pi_0 = \{P^{i_0}_{k_0}, P^{i_1}_{k_1}, \dots, P^{i_q}_{k_q}\} \tag{10}$$

of subsets through which the system traveled, where $P^{i_q}_{k_q} = P^0_1$.

(ii) A continuous path progression, starting Σ_0 from the initial state x_0, which is in $P^{i_0}_{k_0}$, and using the bang-bang input sequence

$$u_{x_0}(t) := \begin{cases} \upsilon_1 & t \in [0, \tau_1), \\ \upsilon_2 & t \in [\tau_1, \tau_1 + \tau_2), \\ \cdots & \\ \upsilon_q & t \in [\sum_{j=1}^{q-1} \tau_j, \sum_{j=1}^{q} \tau_j]. \end{cases}$$

This induces the bang-bang state-feedback controller

$$\varphi(x_0) := u_{x_0} \quad (q \text{ bang-bang steps}). \tag{11}$$

Let x^*_0 be the state Σ_0 reached at the end of the input u_{x_0}, starting from x_0.

The largest deviation between progressions *(i)* and *(ii)* is

$$D := \sup_{x_0 \in \rho(\eta)} |x^*_0 - x^*|. \tag{12}$$

The deviation D depends, among other factors, on the distance between the initial state x_0 and the designated state $x^{i_0}_{k_0}$ of the initial subset $P^{i_0}_{k_0}$ to which x_0 belongs.

Denote by ε the largest distance between x_0 and the designated state $x^{i_0}_{k_0}$ of $P^{i_0}_{k_0}$. We define the *deviation ratio* μ given by

$$\mu := \max\{D/\varepsilon, 1\}. \tag{13}$$

Now, the last member of the sequence Π_0 of (10) is $P^{i_q}_{k_q} = P^0_1 = \sigma(\chi)$. If the deviation satisfies $D \le \chi/2$, then the controller φ of (11) guides Σ_0 to the ball $\rho(3\chi/2)$ from every initial state in $\rho(\eta)$. As $\rho(3\chi/2) \subset \rho(2\chi) \subseteq \mathcal{N}$ (see (1)), the controller φ guides Σ_0 to $\mathcal{N}$. Given the deviation ratio μ and the requirement $D \le \chi/2$, we obtain the maximal distance ε as

$$\varepsilon = \chi/(2\mu). \tag{14}$$

An estimate of μ is derived later in this section.

6.1.1 The spacing interval Δ

Consider an initial condition x_0 that is between adjacent spheres $\sigma(r_i)$, $\sigma(r_{i+1})$. A slight reflection shows that the largest radial distance between x_0 and a designated state occurs when x_0 is in the middle between the two spheres, namely, at a radial distance of $\Delta/2$ from each. Tangentially, the largest distance is δ. Thus, the overall distance between x_0 and the closest designated state is $(\delta^2 + (\Delta/2)^2)^{1/2}$. As ε is the maximal allowed distance, we need $\delta^2 + (\Delta/2)^2 \le \varepsilon^2$. Using $\Delta/2 = \delta$ for simplicity, we get $\delta = \varepsilon/\sqrt{2}$. Substituting into (14) yields

$$\delta = \chi/(2^{3/2}\mu). \tag{15}$$

Having selected $\Delta/2 = \delta$, and using Δ_0 of Procedure 6 with $\alpha = 1/2$, the spacing interval is

$$\Delta = \min\{2\delta, \Delta_0\}. \tag{16}$$

6.2 The deviation ratio

It is shown in Hammer (2025) that the system Σ_0 has a finite deviation ratio μ. To find μ experimentally, we can use

Procedure 7. (Finding a deviation ratio μ).
Let Δ_0 be given by Procedure 6. In $\rho(A)$, construct a grid Γ with interval Δ_0. Denote by $c_1, c_2, \dots, c_g$ the cells of the grid, and by $\upsilon_1, \upsilon_2, \dots, \upsilon_{2^m}$ the members of $\mathcal{C}(m)$.

Step 1: Set $\varepsilon = \Delta_0/2$.
Step 2: Set $s = 1$.
Step 3 Let x, x' be states, where x is the center of cell c_s and $|x' - x| = \varepsilon$.
Step 4: Set $\ell = 1$.
Step 5: At the time t at which $\Sigma_0(x,\ \upsilon_\ell,\ t)$ enters another cell, record the distance $a(s, \ell) := |\Sigma_0(x, \upsilon_\ell, t) - \Sigma_0(x', \upsilon_\ell, t)|$. If $\Sigma_0(x, \upsilon_\ell, t)$ does not reach a different cell within the time T, then set $a(s, \ell) := 0$ and proceed to Step 6.
Step 6: If $\ell < 2^m$, replace ℓ by $\ell + 1$; return to Step 5.
Step 7: If $\ell = 2^m$ and $s < g$, replace s by $s + 1$; return to Step 3.
Step 8: If $\ell = 2^m$ and $s = g$, set $\beta = \max\{a(\kappa, \kappa') : \kappa \in \{1, 2, \dots, g\}, \kappa' \in \{1, 2, \dots, 2^m\}\}$; end procedure. □

When ε is not too large, Procedure 7 provides an estimate of the maximal deviation over one bang-bang step. Recalling the bound d of Assumption 3, it can be shown that the deviation of d steps satisfies $D \le \varepsilon(\beta^d + \sum_{j=0}^{d-1} \beta^j/\sqrt{2})$ (see Hammer (2025) for details). Consequently,

$$\mu = \max\{\beta^d + \frac{1}{\sqrt{2}} \sum_{j=0}^{d-1} \beta^j, 1\}. \tag{17}$$

6.3 The partition $P = \{P_k^i\}$

Construction 8. (The partition P). Use μ of (17), δ of (15), and Δ of (16).

Step 1: Build the spheres $\sigma(r_i)$, $r_i = \chi + i\Delta$, $i = 1, 2, \ldots, p$. For $i = 0$, set $P^0 = \{P_1^0\}$, where $P_1^0 := \sigma(\chi)$, $q(0) := 1$, and $x_1^0 = 0$.
Step 2: Set $i := 1$.
Step 3: On $\sigma(r_i)$, build a partition P^i of disjoint subsets $\{P_k^i\}$ with designated states $\{x_k^i\}$, so that $P_k^i \subseteq \rho(\delta, x_k^i) \cap \sigma(r_i)$. Let $q(i)$ be the number of members of P^i.
Step 4: For $i < p$, replace i by $i + 1$; return to Step 3.
End: If $i = p$, set $P := \{P^0, P^1, \ldots, P^p\}$. □

7. ROBUST BANG-BANG CONTROLLERS

A bang-bang state-feedback controller φ is built in two steps, qualitatively described as follows.

(i) The partition $P = \{P_k^i\}$ of Construction 8 is used to build a directed graph $G(\Delta)$ with vertices named after the members of $\{P_k^i\}$ and directed edges showing the propagation of the state of Σ_0 from designated states $\{x_k^i\}$ to neighboring members of $\{P_k^i\}$, when driven by constant inputs from $\mathcal{C}(m)$.
(ii) The directed graph $G(\Delta)$ is analyzed to find directed paths from each vertex to the vertex $P_1^0 = \sigma(\chi)$. These paths induce the bang-bang state-feedback controller φ of (11).

Construction 9. (of the directed graph $G(\Delta)$). Let $\upsilon_1, \upsilon_2, \ldots, \upsilon_{2^m}$ be the members of $\mathcal{C}(m)$, and let $P = \{P_k^i\}$ be the partition of Procedure 8 with the designated states $\{x_k^i\}$, $i = 0, 1, 2, \ldots, p$, $k = 1, 2, \ldots, q(i)$.

Step 1: Build a graph with vertices P_k^i, $i \in \{0, 1, 2, \ldots, p\}$, $k \in \{1, 2, \ldots, q(i)\}$ and no edges.
Step 2: Set $i = 1$.
Step 3: Set $k = 1$.
Step 4: Set $j = 1$.
Step 5: Apply to Σ_0 the constant input υ_j from the initial state x_k^i for no longer than T. If the state of Σ_0 does not enter a member different from P_k^i, go to Step 7.
Step 6: Let $P_{k'}^{i'} \neq P_k^i$ be the first member of P met by the state of Σ_0. Mark a directed edge from the vertex P_k^i to the vertex $P_{k'}^{i'}$; on this edge, mark the pair (υ_j, τ), where τ is the duration of the path from x_k^i to $P_{k'}^{i'}$.
Step 7: If $j < 2^m$, replace j by $j + 1$, and go to Step 5.
If $j = 2^m$ and $k < q(i)$, replace k by $k + 1$, and go to Step 4.
If $j = 2^m$ and $k = q(i)$ and $i < p$, replace i by $i + 1$, and go to Step 3.
End: If $j = 2^m$ and $k = q(i)$ and $i = p$, the construction of $G(\Delta)$ ends. □

Recall that initial states are restricted to the ball $\rho(\eta)$. Denote by $\lambda > 0$ the smallest integer for which

$$\lambda\Delta \geq \eta. \tag{18}$$

Procedure 10. (Analysis of $G(\Delta)$). Let $\lambda > 0$ be given by (18), and let $G(\Delta)$ be the directed graph of Construction 9.

Step 0: Use methods of graph theory (e.g., Bollobás (1998)) to mark directed paths in $G(\Delta)$ from every vertex P_k^i to P_1^0, $i \in \{1, 2, \ldots, \lambda\}$, $k \in \{1, 2, \ldots, q(i)\}$, if such paths exist.
Step 1: If there is a directed path from every vertex P_k^i to P_1^0, $i \in \{1, 2, \ldots, \lambda\}$, $k \in \{1, 2, \ldots, q(i)\}$, terminate the procedure.
Step 2: Otherwise, under the conditions of Theorem 12 below, there is no controller that satisfies the requirements of Problem 1. Terminate the procedure. □

When Procedure 10 ends in Step 1, there are bang-bang state-feedback controllers φ that take Σ_0 to $\rho(3\chi/2)$, as follows.

Construction 11. (Building a bang-bang controller φ). Assume that Procedure 10 ends in Step 1. Let x_0 be the initial state of Σ_0, let $x_{k_0}^{i_0}$ be a designated state for which $x_0 \in \rho(\varepsilon, x_{k_0}^{i_0})$, where ε is given by (14); and let Π_0 be a directed path in $G(\Delta)$ from $P_{k_0}^{i_0}$ to P_1^0. Then, $\varphi(x_0)$ is given by (11). □

Theorem 12. The following are true for $i \in \{1, 2, \ldots, \lambda\}$ and $k \in \{1, 2, \ldots, q(i)\}$.

(i) If the directed graph $G(\Delta)$ includes a directed path from every vertex P_k^i to the vertex P_1^0, then the bang-bang state-feedback controller of Construction 11 takes the tested system Σ_0 from every initial state in $\rho(\lambda\Delta)$ to $\rho(3\chi/2)$.
(ii) If, for every $\Delta > 0$, there are i, k for which the vertex P_k^i of $G(\Delta)$ has no directed path to any vertex of a partition P^{i^*}, where $\rho(r_{i^*}) \supseteq \rho(\xi, \mathcal{N})$ for some $\xi > 0$ and $i > i^*$, then there is no bang-bang state-feedback controller that guides Σ_0 from every initial state in $\rho(\lambda\Delta)$ to $\mathcal{N}$. □

Recent work (Hammer (2024, 2025)) has shown that bang-bang, or closely related pseudo bang-bang, state-feedback controllers can approximate the performance of any controller for most practical nonlinear systems. Thus, when Theorem 12*(ii)* holds, there are no controllers that achieve the objective of Problem 1.

Proof. (of Theorem 12) Referring to (12) and (13), Part *(i)* of the theorem follows from the inequality $D \leq \chi/2$. Part *(ii)* of the theorem is proved in Hammer (2025). □

The controller φ of Theorem 12*(i)* takes the tested system Σ_0 into $\rho(3\chi/2)$. As we have $\rho(2\chi) \subseteq \mathcal{N}$ by (1), there is further $\chi/2$ flexibility to reach the target set $\mathcal{N}$. This allows φ to accommodate the family $\mathcal{F}_\gamma(\Sigma_0)$, as follows.

Corollary 13. Let c be as given in Proposition 4. If $\gamma \leq \chi/(2c)$, then the bang-bang state-feedback controller φ of Construction 11 takes every system $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ to $\mathcal{N}$. □

8. EXAMPLE

The unknown controlled system is related to the Michaelis-Menten equation (Cao (2011)):

$$\Sigma : \begin{aligned} \dot{x}^1(t) &= \frac{a(x^2(t)+2)x^1(t)}{b+x^2(t)} - u(t), \\ \dot{x}^2(t) &= -\frac{dx^2(t)(x^1(t)+2)}{5+x^2(t)}; \end{aligned} \tag{19}$$

here, the state is $x(t) = (x^1(t), x^2(t))^\top$; the input is $u(t)$; the bounds are $K = 2$, $A = 2$, and $T = 3$; initial states

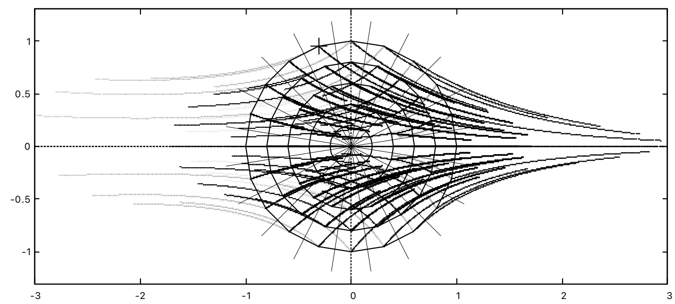

Fig. 4. The response

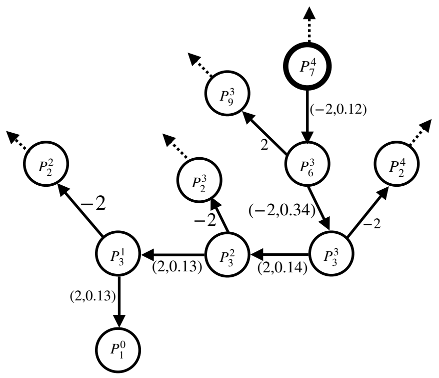

Fig. 5. Part of $G(0.2)$ that starts at P_7^4.

are in $\rho(1)$; and $\chi = 0.2$. The parameters $a \in [2.9, 3.1]$, $b \in [4.9, 5.1]$, and $d \in [4.9, 5.1]$ are constants. The user does not know the system's model (19). For the tested system Σ_0, take $a = 3.0$, $b = 5.0$, and $d = 5.0$.

Initial testing shows that we can use $\Delta = 0.2$ and that the system can be guided to $\rho(0.2)$ from initial states in $\rho(1)$ without exiting $\rho(1)$. Consequently, for the balls $\rho(r_i)$, we have $\mathrm{r}_i = 0.2 + i0.2$, $i = 1, 2, 3, 4$. For the partition of the spheres $\sigma(r_i)$ (circles in this case), it is simpler here to use angular sections $[k\pi/10 - \pi/20, k\pi/10 + \pi/20]$, $k = 0, 1, \ldots, 19$. The designated states x_k^i are on the circles $\sigma(r_i)$ at the angles $\phi_k = k\pi/10$, $k \in \{0, 1, \ldots, 19\}$.

In Construction 9, apply the constant inputs $\upsilon = 2$ and $\upsilon = -2$ at each designated state x_k^i for a time not exceeding 3. The results appear in Figure 4, where light lines show the response to $\upsilon = 2$ and darker lines show the response to $\upsilon = -2$. From Figure 4 we can build the directed graph $G(\Delta)$. Figure 5 shows the part of $G(\Delta)$ for the initial state x_7^4. From Figure 5 we obtain the bang-bang state-feedback controller $\varphi(x)$ shown below.

$$\varphi(x) := \begin{cases} \text{start with } -2 & \text{at } x = (\cos(3\pi/5), \sin(3\pi/5)); \\ \text{hold } -2 & \text{for a time of } 0.46; \\ \text{switch to } 2 & \text{at time } 0.46; \\ \text{hold } 2 & \text{until reaching } P_1^0. \end{cases}$$

This controller is for initial states x located in the vicinity of x_7^4, i.e., within the ball $\rho(0.1, x_7^4)$. The response induced by $\varphi(x)$ from the initial state x_7^4 is shown in Figure 6.

9. CONCLUSION

This note introduces a framework for the design of robust bang-bang state-feedback controllers for systems with un-

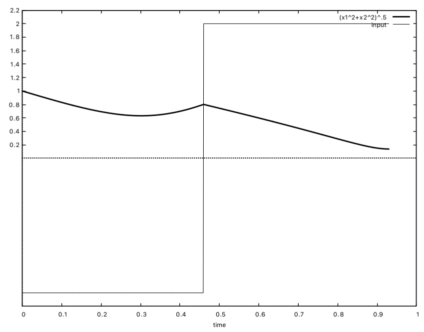

Fig. 6. Closed-loop system performance

known model. The design is based on data collected by testing a sample of the controlled system. The results of the testing lead to a a directed graph $G(\Delta)$; directed paths in this graph determine robust bang-bang controllers.

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