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Sampled-Data Systems: Maximal Sampling Period

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***Abstract*— This note presents a methodology for the design and the implementation of robust sampled-data systems with maximal sampling periods. The methodology applies to nonlinear input-affine systems. It is shown that optimal outcomes can be approximated by bang-bang controllers that are easy to design and implement.**

I. Introduction

Periodic sampling underlies the operation of sampled-data systems, one of the most common classes of systems in modern control engineering. In this note, we develop robust controllers that permit the maximal sampling period consistent with a specified operating error bound. Longer sampling periods bring many advantages, allowing more complex inter-sample computations in real time, reducing energy expenditures, and lowering data loads in feedback communication links ([1], [2], [3]).

As shown in Figure 1, the controlled system Σ has state $x(t)$ and input signal $u(t)$; the latter is generated by the controller C. Due to structural limitations of Σ, the amplitude of $u(t)$ cannot exceed a specified bound K. The feedback signal to C is provided by a periodic sampler with period T.

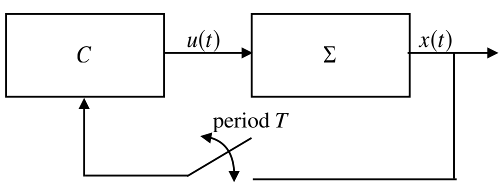

Fig. 1: The control configuration

Operating errors may increase during the inter-sample time due to the lack of feedback. Generally, the magnitude of these errors will grow as the sampling period T increases. We concentrate on the existence and the design of optimal robust controllers that use the maximal sampling period T, while keeping the magnitude of operating errors below a specified bound of ℓ. We examine the existence of such controllers in Section III, where we show that they exist for a broad family of nonlinear input-affine systems; the main requirement for their existence is a certain controllability property the controlled system must possess.

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Specifically, our control objective is to keep the state $x(t)$ of the controlled system in the vicinity of a target state x_{target}. Assuming that the state coordinates were appropriately shifted, we can take the target state as the origin $x = 0$. The operating error is then the deviation of $x(t)$ from the zero state. This deviation is expressed by the squared inner product $|x(t)|_2^2 := x^\top(t)x(t)$. The requirement is to keep $|x(t)|_2^2 \leq \ell$ for all times t, where $\ell > 0$ is a specified *operating error bound*. Denoting $\rho(\ell) := \{x \in R^n : |x|_2^2 \leq \ell\}$, the requirement is to keep $x(t) \in \rho(\ell)$ for all times t. Note that this requirement also implies a certain form of stabilization of the controlled system Σ.

Referring to Figure 1, the input signal of the controlled system Σ is created by the controller C. Generally speaking, the implementation of an optimal controller C may be a difficult task, since optimal input functions needed to operate Σ are often elaborate vector-valued functions of time. To make the process of designing and implementing optimal controllers easier, we use bang-bang controllers, namely, controllers that create bang-bang signals as input to the controlled system Σ. We show in Section IV that bang-bang controllers can approximate optimal performance as closely as desired. As bang-bang signals are determined by lists of scalars – their switching times, the complexity of designing and building bang-bang controllers is much lower than that of general controllers. The following two questions summarize our discussion so far.

Problem 1.1. *(i)* Do there exist robust controllers C that use the maximal sampling period T, while keeping operating error below a specified bound ℓ?
(ii) Is there a simple way to design and build such controllers, when they exist? □

This note draws on many earlier studies, including [4]–[27], the references cited in these studies, and many others. The note is organized as follows. Section II introduces the mathematical framework, while Section III shows that Problem 1.1*(i)* has an affirmative response: such optimal robust controllers exist under broad conditions. Section IV shows that optimal performance can be approximated by bang-bang controllers. An example is presented in Section V, together with a discussion of numerical methods that can be used to solve such optimization problems. A comparison to the widely used sample-and-hold method is also included in Section V. Final observations are made in Section VI.

II. Formal statement of objectives

We use the compactified set R of real numbers (obtained by adding $\pm\infty$ to the real numbers); R^n is then the compactified set of $n-$dimensional real vectors. Denote by R^+ the non-negative real numbers. The absolute value of a real number r is $|r|$; the $L^\infty-$norm of a matrix $V=(V_{ij})$ is $|V|:=\max_{i,j}|V_{ij}|$. Finally, for a matrix-valued function of time $W: R^+ \to R^{n\times m}: t \mapsto W(t)$, we denote by $|W|_\infty := \sup_{t\geq 0}|W(t)|$ the $L^\infty-$norm. We refer to the L^∞-norm as the *amplitude*.

A. The class of systems

The system Σ of Figure 1 is described by the equation

$$\Sigma: \quad \dot{x}(t) = a(x(t)) + b(x(t))u(t), \ x(0) = x_0; \qquad (1)$$

it has the state $x(t)\in R^n$, the input signal $u(t)\in R^m$, and the initial state x_0. The functions $a: R^n \to R^n$ and $b: R^n \to R^{n\times m}$ are continuous and satisfy the Lipschitz conditions

$$|a(x)-a(y)| \leq \alpha^+|x-y| \text{ and } |b(x)-b(y)| \leq \alpha^+|x-y|$$

for all $x,y \in R^n$. Here, $\alpha^+ > 0$ is a specified constant. Needless to say, Σ is a nonlinear input-affine time-invariant system.

To represent uncertainties affecting Σ, the functions a and b of (1) are interpreted as a sum of specified functions a_0, b_0 – the nominal functions; and unspecified functions a_γ, b_γ that represent uncertainties:

$$a(x) = a_0(x) + a_\gamma(x), \ b(x) = b_0(x) + b_\gamma(x). \qquad (2)$$

The functions $a_0, a_\gamma : R^n \to R^n$ and $b_0, b_\gamma : R^n \to R^{n\times m}$ are continuous and satisfy the Lipschitz conditions

$$\begin{aligned} &|a_0(x)-a_0(y)| \leq \alpha|x-y|, && |a_\gamma(x)-a_\gamma(y)| \leq \gamma|x-y|,\\ &|b_0(x)-b_0(y)| \leq \alpha|x-y|, && |b_\gamma(x)-b_\gamma(y)| \leq \gamma|x-y|, \quad (3)\\ &a_0(0)=0, \ |b_0(0)| \leq \alpha; && |a_\gamma(0)| \leq \gamma, \ |b_\gamma(0)| \leq \gamma, \end{aligned}$$

for all $x,y \in R^n$. The constants $\alpha, \gamma > 0$ are provided as part of the specifications, and $\alpha^+ = \alpha+\gamma$. The number γ is the *uncertainty parameter*; it represents uncertainties and is usually small. The nominal system is

$$\Sigma_0: \quad \dot{x}(t) = a_0(x(t)) + b_0(x(t))u(t), \ x(0) = x_0.$$

B. Basics

Let $L_2^{\omega,m}$ be the Hilbert space of Lebesgue measurable functions $f,g: R^+ \to R^m$ with the inner product $\langle f,g\rangle := \int_0^\infty e^{-\omega t} f^\top(s)g(s)ds$, where $\omega > 0$. This inner product assures that all bounded functions have a bounded inner product ([17,18]).

The system Σ of Figure 1 accepts only input signals of amplitude bounded by $K>0$; therefore, all input signals of Σ must belong to the set

$$U(K) = \{u \in L_2^{\omega,m} : |u|_\infty \leq K\}. \qquad (4)$$

Notation 2.1. $\mathcal{F}_\gamma(\Sigma_0)$ is the family of systems (1), (2), (3), all with the same initial state x_0 and the same input signal $u \in U(K)$. The state of a system Σ is $\Sigma(x_0,u,t) := x(t)$. □

As the family of systems $\mathcal{F}_\gamma(\Sigma_0)$ represents uncertainty about the controlled system, it is not known which member of $\mathcal{F}_\gamma(\Sigma_0)$ is the active system Σ in the loop. Therefore, all members of $\mathcal{F}_\gamma(\Sigma_0)$ share the same initial state x_0 and input signal u. The initial state x_0 is known, since it is derived from feedback information.

C. A periodic framework

In Figure 1, the controller C receives periodic samples $x(kT)$ of the state $x(t)$ of Σ, $k=\ldots,-1,0,1,2,\ldots$ We call $x_0 = x(0)$ the *initial state*; it is the initial state of the interval $[0,T]$. Generally, x_{kT} is the initial state of the interval $[kT,(k+1)T]$ and the terminal state of the previous interval $[(k-1)T,kT]$.

To simplify, we take advantage of the time invariance of the controlled system Σ and the controller C. As a result of this time invariance, the behavior of the closed-loop system over any time interval $[kT,(k+1)T]$ is the same as its behavior over the time interval $[0,T]$. In other words, we can limit our attention to the time interval $[0,T]$. In this interpretation, the initial state can be any one of the states x_{kT}, $k=\ldots-1,0,1,2,\ldots$; each one of these states is known and provided by the feedback, as the loop closes momentarily at all integer multiples of the sampling period T.

To circumvent the burden of characterizing all states $\{x_{kT}\}_{k>-\infty}$, we choose a real number $\sigma\in[0,\ell]$, and devise the controller C to drive every member $\Sigma\in\mathcal{F}_\gamma(\Sigma_0)$ from all states $x_{kT}\in\rho(\sigma)$ to states $x_{(k+1)T}\in\rho(\sigma)$, while keeping $x(t)\in\rho(\ell)$ for all $t\in[kT,(k+1)T]$, $k=\ldots-1,0,1,2,\ldots$ Equivalently, by time invariance, it suffices to design C so as to driven every system $\Sigma\in\mathcal{F}_\gamma(\Sigma_0)$ from every initial state $x_0\in\rho(\sigma)$ to a state $x_T\in\rho(\sigma)$, while keeping $x(t)\in\rho(\ell)$ for all $t\in[0,T]$. We can summarize this discussion as follows.

Conclusion 2.2. In Figure 1, let $\Sigma\in\mathcal{F}_\gamma(\Sigma_0)$ be the controlled system, let C be a time invariant controller, let $T>0$ be the sampling period, and let $\ell>0$ and $\sigma\in[0,\ell]$ be real numbers. Then, *(i)* and *(ii)* are equivalent:
(i) C drives every system $\Sigma\in\mathcal{F}_\gamma(\Sigma_0)$ from all initial states $x_0\in\rho(\sigma)$ to states $x_T\in\rho(\sigma)$, keeping $x(t)\in\rho(\ell)$ for all $t\in[0,T]$.
(ii) C drives every system $\Sigma\in\mathcal{F}_\gamma(\Sigma_0)$ from all states $x_{kT}\in\rho(\sigma)$ to states $x_{(k+1)T}\in\rho(\sigma)$, keeping $x(t)\in\rho(\ell)$ for all $t\in[kT,(k+1)T]$, $k=\ldots-1,0,1,2,\ldots$ □

As a result, our objective can be restated as follows.

Problem 2.3. *(a)* Find the longest time $T(\sigma)>0$ such that, for every initial state $x_0\in\rho(\sigma)$, there is an input signal $u_{x_0}\in U(K)$ that drives every system $\Sigma\in\mathcal{F}_\gamma(\Sigma_0)$ so as to satisfy the following conditions: *(i)* $\Sigma(x_0,u_{x_0},T(\sigma))\in\rho(\sigma)$, and *(ii)* $\Sigma(x_0,u_{x_0},t)\in\rho(\ell)$ for all $t\in[0,T(\sigma)]$.
(b) Find the sample radius σ^* that yields longest time $T(\sigma^*)$. □

D. The mathematical framework

1) Inputs: By Conclusion 2.2*(i)*, we must keep $x(t)\in\rho(\ell)$ for all $t\in[0,T]$. The class of input signals that attain that is

$$U_\ell(x_0,K,\ell,\Sigma,T) := \left\{u\in U(K) : \sup_{t\in[0,T]}|\Sigma(x_0,u,t)|_2^2 \leq \ell\right\}.$$

As it is not known which member $\Sigma\in\mathcal{F}_\gamma(\Sigma_0)$ is active, we can use only input signals that belong to $U_\ell(x_0,K,\ell,\Sigma,T)$ for all $\Sigma\in\mathcal{F}_\gamma(\Sigma_0)$. This yields the intersection

$$U_\ell(x_0,K,\ell,\gamma,T) := \cap_{\Sigma\in\mathcal{F}_\gamma(\Sigma_0)} U_\ell(x_0,K,\ell,\Sigma,T). \qquad (5)$$

Conclusion 2.2 also requires to take Σ to $\rho(\sigma)$ at the time T. The appropriate set of input signals is

$$U_\sigma(x_0,K,\sigma,\Sigma,T) := \left\{u \in U(K) : |\Sigma(x_0,u,T)|_2^2 \le \sigma\right\}.$$

The set of input signals that satisfy this requirement for all $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ is the intersection

$$U_\sigma(x_0,K,\sigma,\gamma,T) := \cap_{\Sigma\in\mathcal{F}_\gamma(\Sigma_0)} U_\sigma(x_0,K,\sigma,\Sigma,T). \tag{6}$$

Now, Conclusion 2.2 requires both (5) and (6) to hold, so we get the set

$$U'(x_0) := U_\ell(x_0,K,\ell,\gamma,T) \cap U_\sigma(x_0,K,\sigma,\gamma,T). \tag{7}$$

Finally, (7) must be in effect for all initial states $x_0 \in \rho(\sigma)$; excluding cases for which (7) is not valid for all initial states $x_0 \in \rho(\sigma)$ leads to the set

$$U(x_0,K,\ell,\sigma,\gamma,T) = \begin{cases} U'(x_0) & \text{if } U'(x_0) \neq \varnothing \text{ for all } x_0 \in \rho(\sigma) \\ \varnothing & \text{otherwise.} \end{cases}$$

Note that the relation $\sigma \le \ell$ implies that $U(x_0,K,\ell,\sigma,\gamma,0) = U(K)$ for all $x_0 \in \rho(\sigma)$; needless to say, $T = 0$ is not a sampling period.

2) Maximal sampling periods: Assume Σ starts from an initial state $x_0 \in \rho(\sigma)$ and is driven by a particular input signal $u \in U(K)$. Then, the maximal time Σ can dwell in $\rho(\ell)$ and return to $\rho(\sigma)$ at the end is

$$T(x_0,\ell,\sigma,\Sigma,u) = \sup_{t\ge 0}\{u \in U(x_0,K,\ell,\sigma,\gamma,t)\}. \tag{8}$$

Considering that Σ can be any member of $\mathcal{F}_\gamma(\Sigma_0)$ – it is unknown which specific member it is, we must use the same input signal u for all $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$. With the input signal u, the longest time all members $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ can dwell in $\rho(\ell)$ and return to $\rho(\sigma)$ at the end is

$$T(x_0,\ell,\sigma,\gamma,u) = \inf_{\Sigma\in\mathcal{F}_\gamma(\Sigma_0)} T(x_0,\ell,\sigma,\Sigma,u). \tag{9}$$

The longest such time achievable by any input signal is then

$$T(x_0,\ell,\gamma,\sigma) = \sup_{u\in U(K)} T(x_0,\ell,\gamma,\sigma,u). \tag{10}$$

Next, as x_0 can be any member of $\rho(\sigma)$, the longest sampling period for a sample radius σ is

$$T(\ell,\gamma,\sigma) = \inf_{x_0\in\rho(\sigma)} T(x_0,\ell,\gamma,\sigma). \tag{11}$$

Section III shows that optimal input signals $u^*_{x_0}(\sigma) \in U(K)$ satisfying $T(x_0,\ell,\gamma,\sigma,u^*_{x_0}(\sigma)) = T(\ell,\gamma,\sigma)$ do exist.

Lastly, we seek the longest sampling period for any σ:

$$T^*(\ell,\gamma) := \sup_{0\le\sigma\le\ell} T(\ell,\gamma,\sigma). \tag{12}$$

If there is an optimal sample radius σ^*, it satisfies $T(\ell,\gamma,\sigma^*) = T^*(\ell,\gamma)$. Section III shows that optimal sample radii exist. Then, $T^*(\ell,\gamma)$ is the longest possible sampling period. Section IV shows that, without significantly degrading performance, optimal input signals $u^*_{x_0}(\sigma^*)$ can be replaced by bang-bang signals to simplify implementation. Our objectives are then as follows.

Problem 2.4. *(i)* Obtain conditions that guarantee the existence of optimal sample radii σ^*, namely, sample radii yielding a maximal sampling period.
(ii) Find conditions under which there is, for every initial state $x_0 \in \rho(\sigma^*)$, an optimal input signal $u^*(x_0,\ell,\gamma)$ that satisfies $T(x_0,\ell,\gamma,\sigma^*,u^*(x_0,\ell,\gamma)) = T^*(\ell,\gamma)$.
(iii) Find an input signal $u^\pm$ that approximates optimal performance and is easy to calculate and implement. □

E. Controllability requirements

To fulfill the requirements of Problem 2.3, we need to drive the controlled system Σ from an initial state $x_0 \in \rho(\sigma)$ back to a state in $\rho(\sigma)$, without exiting $\rho(\ell)$ along the way; this must be accomplished while using only input signals bounded by K. This objective leads us to the following controllability property (see also [26]–[28]).

Definition 2.5. Let $K,\ell,\sigma > 0$ be real numbers with $\sigma \in [0,\ell]$. A member $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ is $(K,\ell,\sigma)-$*controllable* if the following is valid for every initial state $x_0 \in \rho(\sigma)$: there are real numbers $t' < \infty$ and $\sigma' \in [0,\sigma)$ for which there is an input signal $u_{x_0} \in U(K)$ such that $\Sigma(x_0,u_{x_0},t') \in \rho(\sigma')$ and $\Sigma(x_0,u_{x_0},t) \in \rho(\ell)$ for all $t \in [0,t']$. The entire family $\mathcal{F}_\gamma(\Sigma_0)$ is $(K,\ell,\sigma)-$*controllable* if all its members are $(K,\ell,\sigma)-$controllable with the same σ'. □

In fact, $(K,\ell,\sigma)-$controllability is almost a necessary condition for solving Problem 2.3. The only added requirement is the contraction $\sigma' < \sigma$; Problem 2.3 requires only $\sigma' = \sigma$. This contraction property helps us deal with uncertainties about the exact description of the model of Σ, uncertainties represented by the family $\mathcal{F}_\gamma(\Sigma_0)$.

Checking $(K,\ell,\sigma)-$controllability for each individual member of the family $\mathcal{F}_\gamma(\Sigma_0)$ of controlled systems could be quite a chore. The next statement shows, however, that this is unnecessary. As long as the uncertainty parameter γ is not too large and one is willing to accept a slightly larger error bound ℓ, then $(K,\ell,\sigma)-$controllability of the entire family $\mathcal{F}_\gamma(\Sigma_0)$ is implied by $(K,\ell,\sigma)-$controllability of the nominal system Σ_0. This is a consequence of the fact that solutions of the differential equation (1) depend in a continuous manner on the functions a and b (see [28] for detailed proof and for estimates of suitable uncertainty bounds γ).

Proposition 2.6. *If the nominal system Σ_0 is $(K,\ell_0,\sigma_0)-$controllable, then the entire family $\mathcal{F}_\gamma(\Sigma_0)$ is $(K,\ell,\sigma_0)-$controllable for every $\ell > \ell_0$, as long as the uncertainty parameter $\gamma > 0$ is not too large.* □

In view of Definition 2.5, the set $U(x_0,K,\ell,\sigma,\gamma,t')$ is not empty for some time $t' > 0$, whenever the family $\mathcal{F}_\gamma(\Sigma_0)$ is $(K,\ell,\sigma)-$controllable. As a result, the supremal time $T^*(\ell,\gamma)$ is not zero, and we obtain

Proposition 2.7. *If the nominal system Σ_0 is $(K,\ell_0,\sigma)-$controllable, then there is a real number $\gamma' > 0$ such that $T^*(\ell,\gamma) > 0$ for all $\gamma \in (0,\gamma']$.* □

III. Existence of optimal solutions

A. Main results

The next two statements, whose proofs are outlined later, show that Problem 2.4*(i)* has a solution under rather broad conditions. The first statement confirms the existence of optimal input signals for specific sample radii (see also [28]).

Theorem 3.1. *If the family of systems $\mathcal{F}_\gamma(\Sigma_0)$ is $(K,\ell,\sigma)-$controllable, then every initial condition $x_0 \in \rho(\sigma)$ has an optimal input signal $u^*(x_0,\ell,\gamma,\sigma) \in U(K)$ such that $T(x_0,\ell,\gamma,\sigma,u^*(x_0,\ell,\gamma,\sigma)) = T(x_0,\ell,\gamma,\sigma)$.* □

Theorem 3.1 shows that $(K,\ell,\sigma)-$controllability implies the existence of optimal input signals that attain maximal sampling periods. Note that, in cases where the uncertainty parameter γ is not excessive, Proposition 2.6 indicates that $(K,\ell,\sigma)-$controllability of the entire family $\mathcal{F}_\gamma(\Sigma_0)$ can be ascertained by checking only one system – the nominal system Σ_0. An estimate of appropriate values of γ can be found in [28].

Furthermore, the next statement shows that there are optimal sample radii attaining the maximal sampling period (see also [28]).

Theorem 3.2. *If the family $\mathcal{F}_\gamma(\Sigma_0)$ is $(K,\ell,\sigma)-$controllable, then there exists an optimal sample radius $\sigma^* \in [0,\ell]$ that yields the maximal sampling period $T^*(\ell,\gamma) = T(\ell,\gamma,\sigma^*)$.* □

Proposition 2.7 guarantees that the supremal time $T^*(\ell,\gamma)$ is not zero under the condition of Theorem 3.2; hence $T^*(\ell,\gamma)$ is an appropriate sampling period.

The remaining part of this section outlines the proofs of Theorems 3.1 and 3.2.

B. Basics

We start by reviewing a few notions from functional analysis (e.g., [29]–[31]).

Definition 3.3. Denote by $\langle\cdot,\cdot\rangle$ the inner product in a Hilbert space H.
(a) A sequence $\{x_i\}_{i=1}^\infty \subseteq H$ *converges weakly* to a member $x \in H$ if $\lim_{i\to\infty}\langle x_i,y\rangle = \langle x,y\rangle$ for all $y \in H$.
(b) A set $W \subseteq H$ is *weakly compact* if every sequence in W has a subsequence that is weakly convergent to an element of W.
Let $S \subseteq H$ be a set including an element z, let $\{z_i\}_{i=1}^\infty \subseteq S$ be any sequence that converges weakly to z, let $F : S \to R$ be a functional, and let $G : R^+ \times S \to R^n : (t,s) \mapsto G(t,s)$ be a function.
(c) F is *weakly upper semi-continuous at z if,* for every $\varepsilon > 0$, there is an integer $N(\varepsilon) \geq 1$ such that $F(z_i) - F(z) < \varepsilon$ for all $i \geq N(\varepsilon)$.
(d) G is *weakly continuous* at z at a time t if, for every $\varepsilon > 0$, there is an integer $N(\varepsilon) \geq 1$ such that $|G(z_i,t) - G(z,t)| < \varepsilon$ for all $i \geq N(\varepsilon)$. □

The notion of weak compactness is critical to our discussion, since the class of input signals $U(K)$ of (4) is weakly compact (see [12,17,18] for proof).

Lemma 3.4. *$U(K)$ is a weakly compact set in the Hilbert space $L_2^{\omega,m}$.* □

We will need the following classical facts (e.g., [30,31]).

Theorem 3.5. *S and A are two topological spaces, where each $a \in A$ is associated with a weakly upper semi-continuous functional $f_a : S \to R$, and $\inf_{a\in A} f_a(s)$ exists for all $s \in S$. Then, $f(s) := \inf_{a\in A} f_a(s)$ is a weakly upper semi-continuous functional on S.* □

Theorem 3.6 (The Generalized Weierstrass Theorem).
(i) *Every upper semi-continuous functional achieves a maximal value in a compact set.*
(ii) *Every weakly upper semi-continuous functional achieves a maximal value in a weakly compact set.* □

Theorems 3.5 and 3.6 provide the basic tools we use to prove the existence of maximal sampling periods. As a first step in this direction, we list the next statement, according to which the supremal time $T(x_0,\ell,\sigma,\Sigma,u)$ of (8) is upper semi-continuous as a function of the input signal u (see [20]–[22] for the proof of a similar result; see also [28]).

Lemma 3.7. *For a system $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ with initial condition $x_0 \in \rho(\sigma)$, let $T(x_0,\ell,\sigma,\Sigma,u)$ be the functional given by (8). Then, $T(x_0,\ell,\sigma,\Sigma,u)$ is weakly upper semi-continuous as a function of u over $U(K)$.* □

When Lemma 3.7 is taken together with Theorem 3.5 and applied to (9), we obtain the following stronger statement.

Lemma 3.8. *Let $T(x_0,\ell,\gamma,\sigma,u)$ be the functional of (9). Then, $T(x_0,\ell,\gamma,\sigma,u)$ is weakly upper semi-continuous as a function of u over $U(K)$.* □

At this point, we can prove Theorem 3.1.

Proof of Theorem 3.1 (sketch). The theorem follows from Lemmas 3.4 and 3.8 by Theorem 3.6*(ii)*. □

C. Optimal sample radii

We turn now to an examination of the sample radius σ (see also [28]).

Proposition 3.9. *For a system $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ with initial condition x_0 and input signal $u \in U(K)$, let $\ell > 0$ be a real number, and let $T(x_0,\ell,\sigma,\Sigma,u)$ be the functional given by (8), where σ is the sample radius. Then, $T(x_0,\ell,\sigma,\Sigma,u)$ is upper semi-continuous as a function of σ over the domain $[0,\ell]$.*

Proof (sketch). The function $x(t) := \Sigma(x_0,u,t)$ is a continuous function of time, and, therefore, so is $|x(t)|_2^2 = x^\top(t)x(t)$. Set $T(\sigma) := T(x_0,\ell,\sigma,\Sigma,u)$; we examine two cases.

Case 1: Assume that $|x(t)|_2^2$ is strictly increasing on an interval $[t_1,t_2]$, $0 \leq t_1 < t_2$, and the following hold: *(a)* $|x(t)|_2^2 < \ell$ for all $t \leq t_2$; *(b)* $|x(t)|_2^2 < |x(t_1)|_2^2$ for all $t < t_1$; and *(c)* $|x(t)|_2^2 > |x(t_2)|_2^2$ for all $t > t_2$. Denote $\sigma_1 := |x(t_1)|_2^2$ and $\sigma_2 := |x(t_2)|_2^2$. Then, $T(\sigma)$ increases as σ increases due to the supremum in (8), and it follows that $T(\sigma)$ is monotone increasing on (σ_1,σ_2).

Case 2: Assume that *(c)* of Case 1 is not valid and $|x(t)|_2^2$ has a local maximum at t_2; it declines after t_2 and later resumes its increase to pass σ_2 at a time $t_3 > t_2$. Then, the supremum in (8) causes the value of $T(\sigma)$ to jump at $\sigma = \sigma_2$ from t_2 to t_3, taking the value $T(\sigma_2) = t_3$. Thus, $T(x_0,\ell,\sigma,\Sigma,u)$ takes the higher value at a jump. Note that if $|x(t)|_2^2$ never exceeds σ_2 after the time t_2, then $T(\sigma)$ jumps to ∞ at σ_2. In either case, $T(\sigma)$ is monotone increasing. The discontinuity of Case 2 is the only type of discontinuity of $T(\sigma)$; this implies the lemma. □

Combining Proposition 3.9*(iii)* with (9) and Theorem 3.5, we obtain the stronger statement:

Corollary 3.10. *The functional $T(x_0,\ell,\gamma,\sigma,u)$ is an upper semi-continuous functional of the sample radius σ.* □

We examine next the functional $T(x_0,\ell,\gamma,\sigma)$ of (10), showing that it is upper semi-continuous as a function of the sample radius σ.

Proposition 3.11. *Let $T(x_0,\ell,\gamma,\sigma)$ be given by (10). Then, $T(x_0,\ell,\gamma,\sigma)$ is an upper semi-continuous functional of the sample radius σ.*

Proof (sketch). Consider a sequence $\{\sigma_i\}_{i=1}^{\infty}$ converging to σ and let $\varepsilon > 0$ be a real number. By Theorem 3.1, there is an input signal $u = u^*(x_0,\sigma)$ that maximizes $T(x_0,\ell,\gamma,\sigma,u)$. By Corollary 3.10, there is an integer $N > 0$ for which $T(x_0,\ell,\gamma,\sigma_i,u^*(x_0,\sigma_i)) - T(x_0,\ell,\gamma,\sigma,u^*(x_0,\sigma_i)) < \varepsilon$ for all $i \geq N$. In addition, since $u^*(x_0,\sigma)$ induces a maximal value, we get $T(x_0,\ell,\gamma,\sigma,u^*(x_0,\sigma_i)) - T(x_0,\ell,\gamma,\sigma,u^*(x_0,\sigma)) \leq 0$. The proposition follows by substituting these two facts into the equality $T(x_0,\ell,\gamma,\sigma_i,u^*(x_0,\sigma_i)) - T(x_0,\ell,\gamma,\sigma,u^*(x_0,\sigma)) = [T(x_0,\ell,\gamma,\sigma_i,u^*(x_0,\sigma_i)) - T(x_0,\ell,\gamma,\sigma,u^*(x_0,\sigma_i))]+[T(x_0,\ell,\gamma,\sigma,u^*(x_0,\sigma_i)) - T(x_0,\ell,\gamma,\sigma,u^*(x_0,\sigma))]$. □

Combining Theorem 3.5 with Proposition 3.11 yields:

Corollary 3.12. *Let $T(\ell,\gamma,\sigma)$ be given by (11). Then, $T(\ell,\gamma,\sigma)$ is an upper semi-continuous functional of the sample radius σ.* □

We can prove now Theorem 3.2.

Proof of Theorem 3.2 (sketch). The theorem follows from the Generalized Weierstrass Theorem (Theorem 3.6*(i)*) by (12) and Corollary 3.12. □

By Proposition 2.7, we have $T^*(\ell,\gamma) > 0$. This implies that optimal sample radii cannot be zero, since the uncertainty about the controlled system Σ creates a scattering of system trajectories over time. Thus, the following is true (see [28] for more details).

Proposition 3.13. *An optimal sample radius σ^* cannot be zero under the conditions of Proposition 2.7.*

In summary, we have seen in this section that there are robust optimal controllers that facilitate maximal sampling periods in sampled-data systems, without violating specified operating error bounds and specified input amplitude bounds. Utilizing maximal sampling periods is advantageous in many applications, since longer sampling periods reduce energy use and cluttering of communication links. A longer time between samples also presents opportunities for deploying in real time more advanced and more accurate algorithms to improve performance.

IV. Approximating an optimal response

The implementation of optimal controllers may burden the engineer with the calculation and the implementation of complicated vector-valued functions of time. In the present section, we show that this burden can be substantially reduced: we show that optimal performance can be approximated by using bang-bang controllers. Bang-bang controllers produce bang-bang signals as input signals to the controlled system Σ. Bang-bang signals are input signals whose components switch between the values of K and $-K$, where K is the input bound of the controlled system Σ. As a result, bang-bang controllers are relatively easy to calculate and implement. The use of bang-bang controllers requires a slight increase of the operating error bound from ℓ to ℓ' (see also [28]). The following is the main result of this section.

Theorem 4.1. *Assume that the maximal sampling period $T^*(\ell,\gamma)$ is bounded, and denote by σ^* an optimal sample radius. Then, for every $\ell' > \ell$, there is a value $\gamma > 0$ of the uncertainty parameter that satisfies the following: for every initial state $x_0 \in \rho(\sigma^*)$, there is a bang-bang input signal $u_{x_0}^{\pm} \in U(K)$ such that $T(x_0,\ell',\gamma,\sigma^*,u_{x_0}^{\pm}) \geq T^*(\ell,\gamma)$.* □

As can be seen from the theorem, a bang-bang input signal can approximate optimal performance. To prove Theorem 4.1, we need the following statement reproduced here from [27]; it shows that the response to any signal can be approximated by the response to a bang-bang signal.

Theorem 4.2. *Assume that a system Σ is operated by an input signal $u \in U(K)$ from an initial state x_0 over a finite interval of time $[0,t']$, $t' > 0$, and let $\varepsilon > 0$ be a real number. Then, there exist a bang-bang input signal $u^{\pm} \in U(K)$ and an uncertainty parameter $\gamma > 0$ such that the difference between the response $x(t) := \Sigma(x_0,u,t)$ of Σ to u and the response $x^{\pm}(t) := \Sigma(x_0,u^{\pm},t)$ of Σ to $u^{\pm}$ satisfies $|x(t) - x^{\pm}(t)| < \varepsilon$ for all $0 \leq t \leq t'$ and for all $\Sigma \in \mathcal{F}_{\gamma}(\Sigma_0)$.* □

We can state now the proof of the main result of this section (see [28] for more details).

Proof of Theorem 4.1 (sketch). Referring to Theorem 3.2, denote by σ^* an optimal sample radius, and note that $\sigma^* > 0$ by Proposition 3.13. Let u^* be an optimal input signal of Theorem 3.1, and let $x^*(t) := \Sigma(x_0,u^*,t)$ be the response to u^*. Denote $\ell'' := \ell + (\ell' - \ell)/2$. Then, Proposition 2.6 indicates the existence of an uncertainty parameter $\gamma' > 0$ such that $\mathcal{F}_{\gamma'}(\Sigma_0)$ is (K,ℓ'',σ^*)–controllable. Now, select a real number $\varepsilon \in (0,(\ell' - \ell''))$. Theorem 4.2 implies the existence of an uncertainty parameter $\gamma_\varepsilon > 0$ and a bang-bang input signal $u^{\pm} \in U(K)$ for which $|x^*(t) - x^{\pm}(t)| < \varepsilon$ for all $0 \leq t \leq T^*(\ell,\gamma_\varepsilon)$ and for all $\Sigma \in \mathcal{F}_{\gamma_\varepsilon}(\Sigma_0)$. Take $\gamma'' := \min\{\gamma',\gamma_\varepsilon\}$.

Finally, Definition 2.5 implies the existence of $\sigma \in (0,\sigma^*)$ such that $\Sigma(x_0,u^*,T^*(\ell,\gamma)) \in \rho(\sigma)$ for all $x_0 \in \rho(\sigma^*)$ and all

$\Sigma \in \mathcal{F}_{\gamma''}(\Sigma_0)$. Set $\varepsilon < \min\{\ell' - \ell'', \sigma^* - \sigma\}$. Then, every $\Sigma \in \mathcal{F}_{\gamma''}(\Sigma_0)$ satisfies $\Sigma(x_0, u^{\pm}, T^*(\ell_0, \gamma_0)) < \sigma^*$ and $\Sigma(x_0, u^{\pm}, t) \in \rho(\ell')$ for all $t \in [0, T^*(\ell_0, \gamma_0)]$. This verifies the theorem. □

V. EXAMPLE

Consider an inverted pendulum with motion not limited to small angles:

$$\Sigma : \begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= c \sin x_1(t) + bu(t); \end{aligned} \qquad (13)$$

here, c and b are constants with nominal values $c_0 = 40.95$ and $b_0 = 14.30$. This system and its parameter values are reproduced here from [32]. The input signal amplitude bound is $K = 2$, and we require an operating error magnitude not exceeding $\ell = 3$. The uncertainty ranges are

$$29.11 \le c \le 47.84 \text{ and } 13 \le b \le 15.88. \qquad (14)$$

Our aim is to devise a bang-bang input signal that yields a sampling period that approximates the maximal sampling period.

A. Finding bang-bang controllers that approximate optimal performance

By Theorem 4.1, we can approximate optimal performance by using bang-bang controllers. At the level of generality of this note, no theoretical tools are available to find such controllers; numerical techniques must be used. In fact, even if theoretical tools were available, one would still need to employ numerical calculations to evaluate such tools. Therefore, it seems that numerical techniques are the best approach.

Before starting our numerical calculations, we note that qualitative examination of – and numerical experimentation with – the family of systems (13), (14) show that the maximal sampling period does not exceed $T = 1$. Therefore, all calculations can be limited to the time interval $[0,1]$.

As indicated by Theorem 4.1, maximal sampling periods for a given sample radius can be achieved (approximately) by bang-bang input signals. The process of finding bang-bang controllers that approximate optimal performance can be divided into several steps, as follows.

Algorithm 5.1 (sketch)**.**

- Step 1: Select a number of potential sample radii $\sigma_1, \sigma_2, \dots, \sigma_p \in [0, \ell]$; for the present calculation, we selected the potential sample radii $\sigma_1 = 0.5$; $\sigma_2 = 1.0$; $\sigma_3 = 1.5$; $\sigma_4 = 2.0$; $\sigma_5 = 2.5$; and $\sigma_6 = 3.0$.
- Step 2: Perform the steps below for each one of the sampling radii of Step 1; suppose the current sample radius is σ_j.
- Step 3: Recall that the time interval of interest in this case is the interval $[0,1]$; partition this interval into 100 equal sub-intervals, and denote their endpoints by $t_1, t_2, \dots, t_{100}$.
- Step 4: Check system performance with bang-bang signals, to find the longest sampling period for the sample radius σ_j. To this end, find a bang-bang signal that drives the controlled system from an initial condition $x_0 \in \rho(\sigma_j)$ so as to obtain the longest time before the system's response returns to $\rho(\sigma_j)$, without exiting $\rho(\ell)$ along the way.
  - Specifically, recall that our input signal bound in this case is 2. Using the notation of Step 3, a bang-bang signal with k switching times $t_{i_1}, t_{i_2}, \dots, t_{i_k}$ switches from -2 to 2, or vice-versa, at each one of these points in time. Start with $k = 0$ (no switching), and continue with $k = 1$, $k = 2$, ..., moving in each case the switching points around the times $t_1, t_2, \dots, t_{100}$. Increase the number of switching points until a further increase in the number of switching points does not lead to a significantly longer sampling period.
- Step 5: Still using the sample radius σ_j, repeat Step 4 for each member of a finite representative ensemble of the family of systems (13), (14), to estimate the longest sampling period achieved for the given family of systems. In our case, the representative ensemble of the three members (15) was used. Record the bang-bang signal that achieved the longest sampling period.
- Step 6: Repeat Steps 4 and 5 for the next sample radius σ_{j+1}.
- Outcome: Record the sample radius that achieves the longest sampling period for the given family of systems as well as the corresponding bang-bang input signal. □

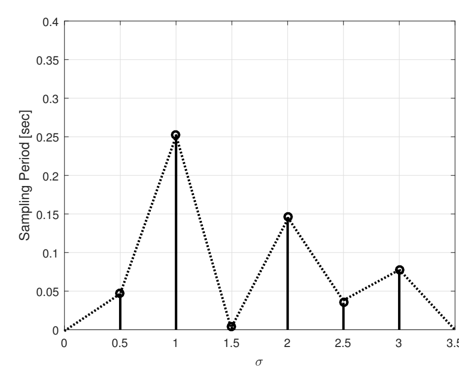


Fig. 2: Maximal sampling period vs. sampling radii

The results of Algorithm 5.1 for the present example are depicted in Figure 2. We can see from the figure that the best sample radius in this case is $\sigma^* = 1$. With this sample radius, maximal sampling period is $T^* = 0.257$. Figures 3 and 4, respectively, depict the bang-bang input signal and the corresponding output signal for the initial condition $x_0 = [0.1, 0.9]^T$. The figure demonstrates the response for the following three representatives of the family (13), (14):

$$\begin{aligned} &\text{Parameter Set 1: } c = 29.11, b = 15.88; \\ &\text{Parameter Set 2: } c = 40.93, b = 14.30; \\ &\text{Parameter Set 3: } c = 47.84, b = 13. \end{aligned} \qquad (15)$$

The response achieved by bang-bang controllers is almost indistinguishable from the optimal response.

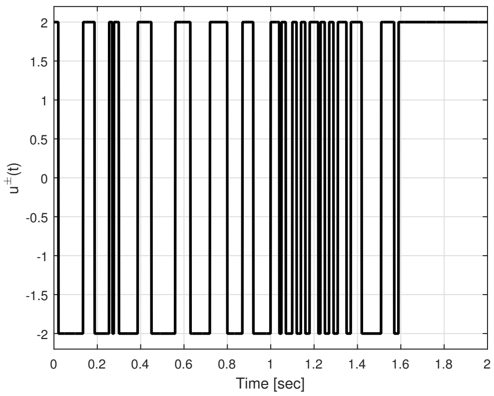
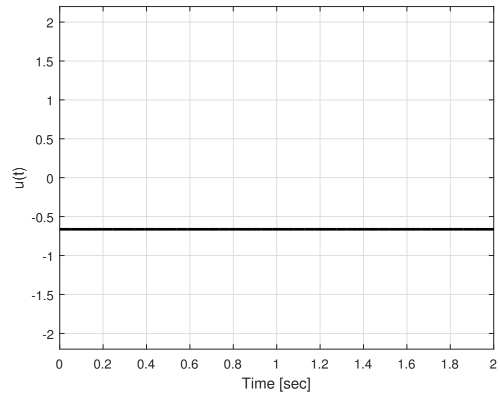


Fig. 3: Bang-bang control signal: optimization approach



(a) Control input: sample-and-hold

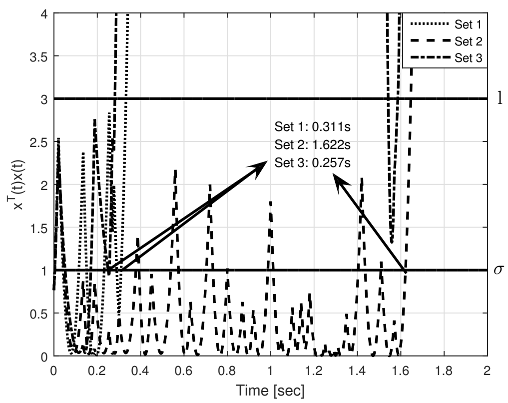
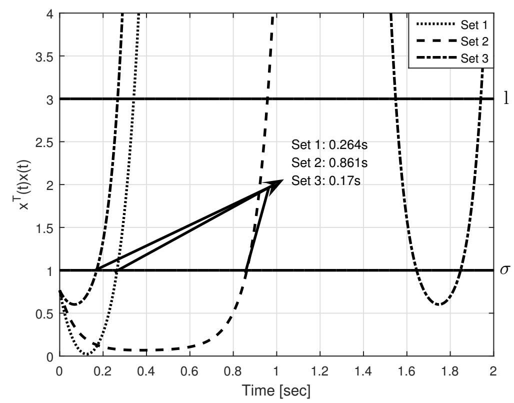


Fig. 4: State trajectories: optimization approach



(b) State trajectories: sample-and-hold

Fig. 5: Results of the sample-and-hold method

B. Comparing to sample-and-hold

The sample-and-hold technique is commonly used in engineering applications involving sampled-data systems. As per this technique, constant input signals are used between samples, instead of the bang-bang signals used in our approach. For the initial state $x_0 = [0.1, 0.9]^T$ used in the previous subsection, numerical optimization shows that the maximal sampling period that can be achieved with the sample-and-hold technique is $T = 0.17$. This sampling period is achieved by the constant input $u \approx -0.67$ shown in Figure 5. As we can see, in this case, the maximal sampling period $T^* = 0.257$ achieved by our optimization method is by more than 50% longer than the maximal sampling period that can be achieved by the sample-and-hold technique.

VI. Conclusion

This note presents a design methodology that achieves maximal sampling periods for sampled-data systems, subject to specified error bounds, input amplitude constraints, and system uncertainty. It is shown that optimal controllers exist for this problem for a rather general family of input-affine nonlinear systems. Furthermore, it is shown that optimal performance – namely, maximal sampling periods – can be closely approximated by bang-bang controllers. Such controllers are relatively simple to design and implement, since they generate bang-bang signals as input to the controlled system. A maximal sampling period carries many advantages, including reduced energy use and reduced cluttering of communication links, as well as the potential for using more advanced control and signal processing algorithms in real-time between samples.

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