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Quick Recovery after Feedback Loss: Delay-Differential Systems

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Abstract: Optimal controllers are developed for delay-differential systems. The objective is to reduce operating errors as quickly as possible after feedback loss, once feedback has been restored. It is shown that nearly optimal performance can be achieved by bang-bang controllers that are relatively simple to implement.

Keywords: Optimal control, Delay-differential systems, Feedback failure

1 INTRODUCTION

Feedback loss is commonplace. It occurs inevitably between every two samples in sampled data systems. Feedback loss is also common in many other systems, including systems that suffered component failures in their feedback channel and data-rate restricted networked control systems ([1], [2], [3]). As feedback loss may increase operating errors, it is important to develop controllers that reduce operating errors in minimal time, once feedback has been restored. In this note, we develop such controllers for nonlinear systems with delays. Examples of systems with delays include systems affected by telemetry delays, systems with long reaction times, and systems subject to real-time computing delays (e.g., [4], [5], [6], [7]).

The control configuration we consider is depicted in Figure 1. Here, the system Σ is controlled by the controller C. The controller experienced a feedback loss until the time $t = 0$, when state feedback was momentarily restored. Our objective is to design C to utilize the feedback data it received at $t = 0$ to produce a signal $u(t)$ that drives Σ so as to reduce operating errors in minimal time. Operating errors must be reduced to comply with a specified error bound $\ell > 0$.

This note expands [8, 9] to delay-differential systems. Background can be found in [10–26], the references cited in these papers, and many others. To the best of our knowledge, the problem discussed in this report has not been previously resolved for delay-differential systems.

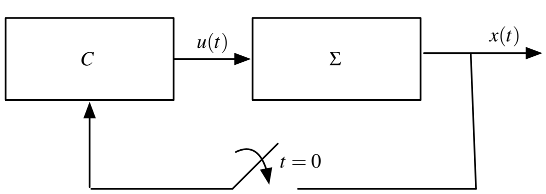

Fig. 1 Control System

The note is organized as follows. Background is covered in Sections 2, 3, and 4; existence of optimal controllers is proved in Section 5; Section 6 shows that easy-to-implement bang-bang controllers can achieve nearly optimal performance. Section 7 is an example, and Section 8 summarizes conclusions.

2 SYSTEMS AND SIGNALS

Let R be the real numbers, R^+ the non-negative real numbers, and $|r|$ the absolute value of a real number r. The L^∞−norm of a real matrix $A = (a_{ij})$ is $|A| = \max_{i,j} |a_{ij}|$. The L^∞−norm of a matrix-valued function $v : R^+ \to R^{n\times m} : t \mapsto v(t)$, often called the *amplitude* of v, is $|v|_\infty := \sup_{t\geq 0} |v(t)|$.

The Hilbert space $L_2^{\omega,m}$, where $\omega > 0$ is a real number and $m > 0$ is an integer, consists of Lebesgue measurable functions $f, g : R \to R^m$ that are zero for negative arguments; the inner product is $\langle f, g\rangle := \int_0^\infty e^{-\omega t} f^T(s)g(s)ds$ (see [24, 25]).

For a function $g \in L_2^{\omega,m}$ and an $n \times m$ matrix $D(t)$ with rows $D_1(t)$, $D_2(t)$, …, $D_n(t) \in L_2^{\omega,m}$, set $\langle D, g\rangle := \sum_{j=1}^n \left\langle D_j^T, g\right\rangle$.

The system Σ is an input-affine delay-differential system

$$\Sigma : \begin{aligned} \dot{x}(t) &= a(t, x(t), x(t-\tau)) + b(t, x(t), x(t-\tau))w(t) \\ &\quad + c(t, x(t), x(t-\tau))w(t-\tau), \end{aligned} \tag{1}$$

where $x(t) \in R^n$ is the state, $w(t) \in R^m$ is the input signal, $\tau > 0$ is the delay, and $a : R \times R^n \times R^n \to R^n$ and $b, c : R \times R^n \times R^n \to R^{n\times m}$ are continuous functions. The initial state $x(0) = x_0$ is provided by the feedback at $t = 0$ (see Figure 1).

The system Σ allows only input signals w that are bounded by $K > 0$. An input signal $w(t)$ has two parts: a *residual input signal* $v(t)$ received before control action started at $t = 0$; and a *control input signal* $u(t)$ – a signal we design to reduce operating errors in minimal time:

$$w(t) = \begin{cases} v(t) & t < 0, \\ u(t) & t \geq 0. \end{cases}$$

The nominal residual input signal is v_0. The actual residual input signal v has an uncertainty of γ; it is an unspecified member of the family

$$V_\gamma(v_0) := \left\{ v : R \to R^m : \begin{array}{l} |v - v_0|_\infty \le \gamma, \\ |v_0|_\infty \le K, |v(t)|_\infty \le K \end{array} \right\}.$$

The set of control input signals is

$$U(K) := \left\{ u \in L_2^{\omega,m} : |u|_\infty \le K \right\}.$$

The functions a, b, and c consist of nominal parts a_0, b_0, c_0 and unspecified parts a_γ, b_γ, c_γ that represent modeling uncertainties:

$$\begin{aligned} a(t,y,z) &= a_0(t,y,z) + a_\gamma(t,y,z), \\ b(t,y,z) &= b_0(t,y,z) + b_\gamma(t,y,z), \\ c(t,y,z) &= c_0(t,y,z) + c_\gamma(t,y,z). \end{aligned} \tag{2}$$

All are continuous functions subject to the Lipschitz conditions:

$$\begin{aligned} |a_0(t,y',z') - a_0(t,y,z)| &\le \alpha \max\{|y'-y|, |z'-z|\}, \\ |b_0(t,y',z') - b_0(t,y,z)| &\le \alpha \max\{|y'-y|, |z'-z|\}, \\ |c_0(t,y',z') - c_0(t,y,z)| &\le \alpha \max\{|y'-y|, |z'-z|\}, \\ a_0(t,0,0) = 0;\ |b_0(t,0,0)|, &|c_0(t,0,0)| \le \alpha; \end{aligned} \tag{3}$$

$$\begin{aligned} \left|a_\gamma(t,y',z') - a_\gamma(t,y,z)\right| &\le \gamma \max\{|y'-y|, |z'-z|\}, \\ \left|b_\gamma(t,y',z') - b_\gamma(t,y,z)\right| &\le \gamma \max\{|y'-y|, |z'-z|\}, \\ \left|c_\gamma(t,y',z') - c_\gamma(t,y,z)\right| &\le \gamma \max\{|y'-y|, |z'-z|\}, \\ a_\gamma(t,0,0) = 0;\ \left|b_\gamma(t,0,0)\right|, &\left|c_\gamma(t,0,0)\right| \le \gamma; \end{aligned} \tag{4}$$

here $\alpha, \gamma > 0$ are specified real numbers; γ describes uncertainty and is often small. Denote by $F_\gamma(\Sigma_0)$ the class of all systems compatible with (1), (2), (3), and (4).

Considering that systems are at rest before activation, we adopt:

Convention 1. *Σ has an* activation time $t_a < 0$, *prior to which Σ was at the zero state and had received the zero input signal.* □

3 STATEMENT OF THE PROBLEM

Let $x(t) = \Sigma(x_0, v, u, t)$ be the state at time t, where the initial state is x_0, the residual input is v, and the control input is u. To simplify notation, shift the state coordinates so that desired operation of Σ is near $x = 0$, with permissible deviation $x^T x$ not exceeding $\ell > 0$; here, ℓ is specified. We refer to $x^T x$ as the *operating error*. Then, we need to bring Σ in minimal time from x_0 into the domain

$$\rho(\ell) := \left\{ x \in R^n \middle| x^T x \le \ell \right\}.$$

The earliest Σ can reach $\rho(\ell)$ from x_0, given a residual input signal v and a control input signal u, is $t(x_0, \Sigma, v, u) = \inf_{t \ge 0}\{\Sigma^T(x_0, v, u, t)\Sigma(x_0, v, u, t) \le \ell\}$. The earliest u can take all members of $F_\gamma(\Sigma_0)$ into $\rho(\ell)$, for any $v \in V_\gamma(v_0)$, is

$$t(x_0, \gamma, \ell, u) = \inf_{t \ge 0} \left\{ \sup_{\substack{\Sigma \in F_\gamma(\Sigma_0) \\ v \in V_\gamma(v_0)}} \Sigma^T(x_0, v, u, t)\Sigma(x_0, v, u, t) \right] \le \ell \right\}$$

The earliest any control input signal $u \in U(K)$ can bring all members of $F_\gamma(\Sigma_0)$ into $\rho(\ell)$, no matter which $v \in V_\gamma(v_0)$ was used, is

$$t^*(x_0, \gamma, \ell) = \inf_{u \in U(K)} t(x_0, \gamma, \ell, u).$$

If $t^*(x_0, \gamma, \ell) < \infty$, we show in Section 5 that there is an optimal control input signal $u^*(x_0, \gamma, \ell) \in U(K)$ satisfying $t^*(x_0, \gamma, \ell) = t(x_0, \gamma, \ell, u^*(x_0, \gamma, \ell))$.

As $u^*(x_0, \gamma, \ell)$ is generally a vector valued function of time, it may be hard to implement. Section 6 shows that, with little sacrifice in performance, $u^*(x_0, \gamma, \ell)$ can be replaced by an easy-to-implement bang-bang signal. Our objectives are then as follows.

Problem 2. *(i)* Find conditions under which there is an optimal control input signal $u^*(x_0, \gamma, \ell)$.
(ii) Find simple-to-implement control input signals that closely approximate optimal performance. □

4 BASICS

It can be shown that the system Σ of (1) has no escape time (see [27]):

Lemma 3. *At every time $T \ge 0$, there is a bound $M(T) \ge 0$ such that $|\Sigma(x_0, v, u, t)| \le M(T)$ for all $t \in [0, T]$, all $\Sigma \in F_\gamma(\Sigma_0)$, all $v \in V_\gamma(v_0)$, and all $u \in U(K)$.* □

As a, b, and c of (1) are continuous functions, Lemma 3 implies

Corollary 4. *For every $T \ge 0$, there is $B(T) \ge 0$ such that $\sup_{0 \le t \le T} \{|a(t, x(t), x(t-\tau))|,\ |b(t, x(t), x(t-\tau))|,\ |c(t, x(t)|,\ x(t-\tau))|\} \le B(T)$ for all $t \in [0, T]$, all $\Sigma \in F_\gamma(\Sigma_0)$, all $v \in V_\gamma(v_0)$, and all $u \in U(K)$.* □

A solution of Problem 2 requires driving Σ from its initial state x_0 to the vicinity of the zero state. The possibility of doing so depends, among other factors, on the input bound K, on x_0, and on the residual input signal v.

Definition 5. A system Σ with residual input signal v and initial state x_0 is $K-$*controllable* if there is a control input signal $u_c \in U(K)$ and a time $t_c \ge 0$ such that $\Sigma(x_0, v, u_c, t_c) = 0$. □

$K-$controllability of the nominal system Σ_0 guarantees $K-$controllability of all $\Sigma \in F_\gamma(\Sigma_0)$, if γ is not too large. This follows from continuity of the functions a, b, and c of (1) (see [27] for details).

Proposition 6. *If the nominal system Σ_0 with the nominal residual input v_0 is K–controllable from x_0, then, for every $\ell > 0$, there is a $\gamma > 0$, a control input $u_\gamma \in U(K)$, and a time $t_\gamma \ge 0$ such that $\Sigma(x_0, v, u_\gamma, t_\gamma) \in \rho(\ell)$ for all $\Sigma \in F_\gamma(\Sigma_0)$ and all $v \in V_\gamma(v_0)$.*

We show next that $u^*(x_0, \gamma, \ell)$ exists when the nominal system Σ_0 is $K-$controllable and the uncertainty γ meets Proposition 6.

5 EXISTENCE OF OPTIMAL SOLUTIONS

Theorem 7. *If the nominal system Σ_0 with the nominal residual input v_0 is $K-$controllable from x_0, and if $\gamma > 0$ satisfies the condition of Proposition 6 for the error bound $\ell > 0$, then*
(i) There is a finite minimal time $t^*(x_0, \gamma, \ell)$, and
(ii) There is an optimal control input signal $u^*(x_0, \gamma, \ell) \in U(K)$. □

The proof of Theorem 7 uses the fact that continuous functions attain extrema in a compact domain (the Weierstrass Theorem). We show that the domain $U(K)$ is 'compact' and that $t(x_0, \gamma, \ell, u)$ is 'continuous' over $U(K)$. First, some terminology (e.g., [28]).

Definition 8. Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$.
(i) A sequence $u_1, u_2, \ldots \in H$ *converges weakly* to $u \in H$ if $\lim_{i\to\infty} \langle u_i, y\rangle = \langle u, y\rangle$ for every $y \in H$.
(ii) A subset $W \subseteq H$ is *weakly compact* if every sequence in W has a subsequence that converges weakly to a member of W. □

The following is taken from [24, 25].

Lemma 9. *The input set $U(K)$ is weakly compact in $L_2^{\omega,m}$.* □

We use the following notions of continuity (e.g., [29]).

Definition 10. Let S be a subset of a Hilbert space H. A functional $f : S \to R$ is *weakly lower semi-continuous* at a point $z \in S$ if the following holds whenever $f(z)$ is defined: for every sequence $\{z_i\}_{i=1}^{\infty} \subseteq S$ weakly convergent to z and for every $\varepsilon > 0$, there is an integer $N > 0$ such that $f(z) - f(z_i) < \varepsilon$ for all $i \geq N$.

A function $G : S \times R \to R^n : (s,t) \mapsto G(s,t)$ is *weakly continuous* at $z \in S$ at a time t if the following is true for every sequence $\{z_i\}_{i=1}^{\infty} \subseteq S$ that converges weakly to z: for every $\varepsilon > 0$, there is an integer $N > 0$ such that $|G(z,t) - G(z_i,t)| < \varepsilon$ for all $i \geq N$.

G is *uniformly weakly continuous* on S over a time interval $[t_1, t_2]$ if, for every $z \in S$, for every sequence $\{z_i\}_{i=1}^{\infty} \subseteq S$ that converges weakly to z, and for every $\varepsilon > 0$, there is an integer N such that $|G(z,t) - G(z_i,t)| < \varepsilon$ for all $i \geq N$ at all $t \in [t_1, t_2]$. □

We examine next the continuity of the response of Σ.

Lemma 11. *The function $\Sigma(x_0, v, u, t) : U(K) \to R^n$ is uniformly weakly continuous over $U(K)$ on every finite interval of time.*

Proof (sketch). For a sequence $\{u_i\}_{i=1}^{\infty} \subseteq U(K)$ that converges weakly to $u \in U(K)$, define the signals

$$w(t) = \begin{cases} v(t) & t \in [t_a, 0), \\ u(t) & t \geq 0; \end{cases} \quad w_i(t) = \begin{cases} v(t) & t \in [t_a, 0), \\ u_i(t) & t \geq 0. \end{cases}$$

Set $x(t,w) := \Sigma(x_0, v, u, t)$, $x(t, w_i) := \Sigma(x_0, v, u_i, t)$, $x(t,i) := x(t,w) - x(t,w_i)$. Then, $x(0,i) = 0$, $i = 1, 2, \ldots$ Denote $w_\tau(s) := w(s-\tau)$, $w_{\tau i}(s) := w_i(s-\tau)$; $x_\tau(t,w) := x(t-\tau, w)$. For $t_2 > t_1 \geq 0$, integrate (1) to a time $t \in [t_1, t_2]$, then take the supremum and use (3), (4), and the bounds $|w(t)| \leq K$, $|w_i(t)| \leq K$, to get

$$\begin{aligned} &\sup_{t_1 \leq t \leq t_2} |x(t,i)| \leq |x(t_1,i)| + (\alpha+\gamma) \sup_{t_1-\tau \leq t \leq t_2} |x(t,i)|(t_2 - t_1) \\ &+ 2(\alpha+\gamma) \sup_{t_1-\tau \leq t \leq t_2} |x(t,i)| K (t_2 - t_1) \\ &+ \sup_{t_1 \leq t \leq t_2} \left| \int_{t_1}^{t} b(s, x(s,w), x_\tau(s,w))(w(s) - w_i(s)) ds \right| \\ &+ \sup_{t_1 \leq t \leq t_2} \left| \int_{t_1}^{t} c(s, x(s,w), x_\tau(s,w))(w_\tau(s) - w_{\tau i}(s)) ds \right|. \end{aligned}$$

Choose $\mu > 0$ such that $(\alpha+\gamma)(1+2K)\mu < 1$; set $t_2 := t_1 + \mu$ and

$$\eta_4 := (1 - (\alpha+\gamma)(1+2K)\mu)^{-1}, \ \eta_5 := \eta_4(\alpha+\gamma)(1+2K)\mu.$$

Then, using $\sup_{t_1-\tau \leq t \leq t_2} |x(t,i)| \leq \sup_{t_1-\tau \leq t \leq t_1} |x(t,i)| + \sup_{t_1 \leq t \leq t_2} |x(t,i)|$ and $|x(t_1,i)| \leq \sup_{t_1-\tau \leq t \leq t_1} |x(t,i)|$, we get

$$\begin{aligned} &\sup_{t_1 \leq t \leq t_2} |x(t,i)| \leq (\eta_4 + \eta_5) \sup_{t_1-\tau \leq t \leq t_1} |x(t,i)| \\ &+ \eta_4 \sup_{t_1 \leq t \leq t_2} \left| \int_{t_1}^{t} b(s, x(s,w), x_\tau(s,w))(w(s) - w_i(s)) ds \right| \quad (5) \\ &+ \eta_4 \sup_{t_1 \leq t \leq t_2} \left| \int_{t_1}^{t} c(s, x(s,w), x_\tau(s,w))(w_\tau(s) - w_{\tau i}(s)) ds \right|. \end{aligned}$$

To investigate the integrals in (5), define

$$y_t^b(s) := \begin{cases} e^{\alpha(t-t_1)} b^T(s, x(s,w), x_\tau(s,w)) & t_1 \leq s \leq t, \\ 0 & \text{else;} \end{cases}$$

$$y_t^c(s) := \begin{cases} e^{\alpha(t-t_1)} c^T(s, x(s,w), x_\tau(s,w)) & t_1 \leq s \leq t, \\ 0 & \text{else.} \end{cases}$$

As $\{w_i\}_{i=1}^{\infty}$ converges weakly to w, it follows that, for every $\beta > 0$, there is an integer $N_t > 0$ such that $\max\{|\langle y_t^b, w - w_i\rangle|, |\langle y_t^c, w_\tau - w_{\tau i}\rangle|\} < \beta$ for all $i \geq N_t$. Using Corollary 4, (3), and (4), it can be verified that N_t can be chosen independently of $t \in [t_1, t_2]$ (see [27] for details), i.e., there is an N for which

$$\sup_{t_1 \leq t \leq t_1+\mu} \max\{|\langle y_t^b, w - w_i\rangle|, |\langle y_t^c, w_\tau - w_{\tau i}\rangle|\} < \beta, \ i \geq N \tag{6}$$

Now, choose $\varepsilon > 0$, and set $\beta = \varepsilon/(2\eta_4)$. Then, by (5) and (6) there is an integer $N_\varepsilon \geq 0$ such that for all $i \geq N_\varepsilon$

$$\sup_{t_1 \leq \theta \leq t_1+\mu} |x(\theta,i)| \leq \varepsilon + (\eta_4 + \eta_5) \sup_{t_1-\tau \leq \theta \leq t_1} |x(\theta,i)|. \tag{7}$$

From (7), the sequence $\zeta_k := \sup_{\theta \in [t_a+(k-1)\mu, t_a+k\mu]} |x(\theta,i)|$, $k = \ldots, -1, 0, 1, \ldots$ satisfies the linear recursion

$$\zeta_{k+1} \leq \varepsilon + (\eta_4 + \eta_5) \sum_{j=0}^{q-1} \zeta_{k-j}, k = \ldots, -1, 0, 1, \ldots;$$

$$\zeta_k = 0 \text{ for } k \leq 0.$$

Thus, there is a bound $H(k)$ satisfying $\zeta_k \leq H(k)\varepsilon$. The lemma follows by letting $\varepsilon \to 0$. □

To continue, we need to review the following facts (e.g., [30]).

Theorem 12. *(i) A weakly continuous functional is also weakly lower semi-continuous.*
(ii) Let S and A be topological spaces and assume that, for every $a \in A$, there is a weakly lower semi-continuous functional $f_a : S \to R$. If $\sup_{a\in A} f_a(s)$ exists at every $s \in S$, then the functional $f(s) := \sup_{a\in A} f_a(s)$ is weakly lower semi-continuous on S. □

At a fixed time t, construct the functional of $u \in U(K)$

$$\psi(t,u) := \sup_{(\Sigma,v)\in F_\gamma(\Sigma_0)\times V_\gamma(v_0)} \Sigma^T(x_0,v,u,t)\Sigma(x_0,v,u,t).$$

Using Lemma 11 and Theorem 12*(i)* and *(ii)*, we obtain

Lemma 13. *$\psi(t,u)$ is weakly lower semi-continuous.* □

As $t(x_0,\gamma,\ell,u) = \inf_{t\geq 0}\{\psi(t,u) \leq \ell\}$, Lemma 13 and Proposition 3.4 of [8] yield

Proposition 14. *$t(x_0,\gamma,\ell,u)$ is weakly lower semi-continuous.* □

We quote the Generalized Weierstrass Theorem (e.g., [30]):

Theorem 15. *A weakly lower semi-continuous functional attains its minimum in a weakly compact set.* □

Proof of Theorem 7. As $t^*(x_0,\gamma,\ell)$ is the minimum of $t(x_0,\gamma,\ell,u)$, it follows by Proposition 14, Lemma 9, and Theorem 15 that $u^*(x_0,\gamma,\ell) \in U(K)$ exists. □

Thus, an optimal solution exists when the nominal system Σ_0 is $K-$controllable and the uncertainty γ is not excessive. Considering that $u^*(x_0,\gamma,\ell)$ may be hard to calculate and implement, the next section shows that nearly optimal performance can be achieved by easy-to-calculate-and-implement bang-bang signals.

6 IMPLEMENTATION

We show that a bang-bang control input signal $u^\pm(x_0,\gamma,\ell)$ can replace an optimal signal $u^*(x_0,\gamma,\ell)$, with little impact on performance. Specifically, with an error bound ℓ' slightly larger than the specified error bound ℓ, the time $t(x_0,\gamma,\ell',u^\pm(x_0,\gamma,\ell))$ required by $u^\pm(x_0,\gamma,\ell)$ to reduce operating errors to within ℓ' is no longer than the optimal time $t^*(x_0,\gamma,\ell)$ for the error bound ℓ.

Theorem 16. *For $\ell' > \ell$, there is a bang-bang control input signal $u^\pm(x_0,\gamma,\ell) \in U(K)$ (with a finite number of switchings) and a $\gamma > 0$ such that $t(x_0,\gamma,\ell',u^\pm(x_0,\gamma,\ell)) \leq t^*(x_0,\gamma,\ell)$.* □

Theorem 16 follows from the fact that the response to every input signal can be approximated by the response to a bang-bang input signal, as follows.

Theorem 17. *For every pair of numbers $\sigma, t' > 0$ and for every input signal $u \in U(K)$, there is a bang-bang signal $u^\pm \in U(K)$ (with a finite number of switchings) and a number $\gamma > 0$ such that $|\Sigma(x_0,v,u,t) - \Sigma(x_0,v,u^\pm,t)| < \sigma$ for all $t \in [0,t']$, all $\Sigma \in F_\gamma(\Sigma_0)$, and all $v \in V_\gamma(v_0)$.* □

The next statement helps prove Theorem 17. It follows from the fact that continuous functions are uniformly continuous over compact domains (see [27] for details).

Lemma 18. *Let $x(t) = \Sigma(x_0,v,u,t)$, where Σ is given by (1) with the functions a, b and c, and let $t' > 0$ be a time. Then, for every $\varepsilon > 0$, there are $\beta(t',\varepsilon) > 0$ and $\gamma > 0$ such that*

$|b(t_1,x(t_1),x(t_1-\tau)) - b(t_2,x(t_2),x(t_2-\tau))| < \varepsilon$ and

$|c(t_1,x(t_1),x(t_1-\tau)) - c(t_2,x(t_2),x(t_2-\tau))| < \varepsilon$

for all $t_1, t_2 \in [0,t']$ satisfying $|t_1 - t_2| < \beta(t',\varepsilon)$, as long as $v \in V(v_0,\gamma)$, $u \in U(K)$, and $\Sigma \in F_\gamma(\Sigma_0)$. □

Proof of Theorem 17 (sketch). Denote $x(t) := \Sigma(x_0,v,u,t)$ and $x^\pm(t) := \Sigma(x_0,v,u^\pm,t)$; we construct $u^\pm$ below. For times $t_1 < t_2 \in [0,t']$, select $\lambda > 0$ for which $p := (t_2-t_1)/\lambda$ is an integer, and create the partition

$$[t_1,t_2] = \{[t_1,t_1+\lambda],[t_1+\lambda,t_1+2\lambda],\cdots,[t_1+(p-1)\lambda,t_2]\}.$$

Let u_i, $u_i^\pm$ be the $i-$th component of u and $u^\pm$, respectively. Calculate numbers $\theta_1^q,\theta_2^q,\cdots,\theta_m^q \in [t_1+q\lambda, t_1+(q+1)\lambda]$ to satisfy $K[2(\theta_i^q - (t_1+q\lambda)) - \lambda] = \int_{t_1+q\lambda}^{t_1+(q+1)\lambda} u_i(s)ds$, and build the bang-bang signal

$$u_i^\pm(t) := \begin{cases} +K & \text{for } t \in [t_1+q\lambda, \theta_i^q), \\ -K & \text{for } t \in [\theta_i^q, t_1+(q+1)\lambda), \\ & \text{if } \theta_i^q < t_1+(q+1)\lambda, \end{cases}$$

$q = 0,1,\ldots,p-1$, $i = 1,2,\ldots,m$; then,

$$\int_{t_1+q\lambda}^{t_1+(q+1)\lambda} \left(u_i(s) - u_i^\pm(s)\right) ds = 0, i = 1,...,m, q = 0,...,p-1. \tag{8}$$

With $t_a < 0$ of Convention 1, define the signals

$$w(t) := \begin{cases} v(t) & t \in [t_a,0), \\ u(t) & t \geq 0; \end{cases} \quad w^\pm(t) := \begin{cases} v(t) & t \in [t_a,0), \\ u^\pm(t) & t \geq 0. \end{cases}$$

Denote $\xi(t) := x(t) - x^\pm(t)$. Integrate (1) from t_1 to $t \in (t_1,t_2]$, and take the supremum while using (2), (3), (4), the bounds $|w(t)| \leq K$, $|w^\pm(t)| \leq K$, and the facts $|\xi(t_1)| \leq \sup_{t\in[t_1-\tau,t_1]}|\xi(t)|$, $\sup_{s\in[t_1-\tau,t_2]}|\xi(s)| \leq \sup_{t\in[t_1-\tau,t_1]}|\xi(t)| + \sup_{t\in[t_1,t_2]}|\xi(t)|$; choose a number $\mu \in (0,t'-t_1]$ satisfying $(\alpha+\gamma)(1+2K)\mu < 1$; set $t_2 := t_1 + \mu$; and define the constants

$$\eta_1 := \frac{1+(\alpha+\gamma)(1+2K)\mu}{1-(\alpha+\gamma)(1+2K)\mu}, \ \eta_2 := \frac{1}{1-(\alpha+\gamma)(1+2K)\mu}.$$

Then, we get the inequality

$$\sup_{t\in[t_1,t_1+\mu]}|\xi(t)| \leq \eta_1 \sup_{t\in[t_1-\tau,t_1]}|\xi(t)| + \eta_2 \sup_{t\in[t_1,t_1+\mu]}\left|\int_{t_1}^{t} b(s,x(s),x(s-\tau))\left(w(s)-w^\pm(s)\right)ds\right| \tag{9}$$

$$+\eta_2 \sup_{t\in[t_1,t_1+\mu]}\Bigg| \int_{t_1-\tau}^{t-\tau} c(s+\tau, x(s+\tau), x(s))\left(w(s)-w^{\pm}(s)\right) ds\Bigg|.$$

As $t_1 \geq 0$, we have $w(t) = u(t)$, $t \in [t_1, t_1+\mu]$. Let $q(t) \in \{0, 1, 2, \ldots, p-1\}$ be such that $t \in [q(t)\lambda, (q(t)+1)\lambda]$; then

$$\sup_{t\in[t_1,t_1+\mu]}\left|\int_{t_1}^{t} b(s, x(s), x(s-\tau))\left(w(s)-w^{\pm}(s)\right) ds\right|$$

$$\leq \sup_{t\in[t_1,t_1+\mu]}\Bigg|\sum_{i=0}^{q(t)-1} b(t_1+i\lambda, x(t_1+i\lambda), x(t_1-\tau+i\lambda))\times \int_{t_1+i\lambda}^{t_1+(i+1)\lambda}\left(u(s)-u^{\pm}(s)\right) ds$$

$$+ \sup_{t\in[t_1,t_1+\mu]}\Bigg|\sum_{i=0}^{q(t)-1}\int_{t_1+i\lambda}^{t_1+(i+1)\lambda}\big[b(s, x(s), x(s-\tau)) - b(t_1+i\lambda, x(t_1+i\lambda), x(s-\tau+i\lambda))\big]\left(u(s)-u^{\pm}(s)\right) ds\Bigg|$$

$$+ \sup_{t\in[t_1,t_1+\mu]}\left|\int_{t_1+q(t)\lambda}^{t} b(s, x(s), x(s-\tau))\left(u(s)-u^{\pm}(s)\right) ds\right|.$$

Let $\varepsilon > 0$. Using (8), Corollary 4, Lemma 18 with this ε, and setting $\lambda \leq \beta(t', \varepsilon)$, we get

$$\sup_{t\in[t_1,t_2]}\left|\int_{t_1}^{t} b(s, x(s), x(s-\tau))\left(u(s)-u^{\pm}(s)\right) ds\right| \leq 2K\varepsilon\mu + 2KB(t')\lambda. \tag{10}$$

Similarly,

$$\sup_{t\in[t_1,t_1+\mu]}\left|\int_{t_1-\tau}^{t-\tau} c(s+\tau, x(s+\tau), x(s))\left(w(s)-w^{\pm}(s)\right) ds\right| \leq 2K\varepsilon\mu + 2KB(t')\lambda. \tag{11}$$

Inserting (10) and (11) into (9) yields

$$\sup_{t\in[t_1,t_1+\mu]}|\xi(t)| \leq \eta_1 \sup_{t\in[t_1-\tau,t_1]}|\xi(t)| + 4K\eta_2\left[\varepsilon\mu + B(t')\lambda\right].$$

Now, set $\varepsilon > 0$ and $\lambda > 0$ so that $4K\eta_2\varepsilon\mu < \delta/2$ and $4K\eta_2 B(t')\lambda < \delta/2$. Then,

$$\sup_{t\in[t_1,t_1+\mu]}|\xi(t)| \leq \delta + \eta_1 \sup_{t\in[t_1-\tau,t_1]}|\xi(t)|.$$

Define $\zeta_k := \sup_{t\in[t_a+(k-1)\mu, t_a+k\mu]}|\xi(t)|$, $k = \ldots, -1, 0, 1, \ldots$; then, an argument similar to the one following (7) shows that there are constants d_k such that $|\zeta_k| \leq d_k\delta$, $k = 1, 2, \ldots$ Let $r \geq (t'-t_a)/\mu$ be an integer; as only $k \leq r$ is relevant, the bound $D := \max_{i=1,2,\ldots,r} d_k$ satisfies $\sup_{t\in[t_a,t']}|\xi(t)| \leq D\delta$; the theorem follows by selecting $\delta < \sigma/D$. □

Proof of Theorem 16 (sketch). By Theorem 7, $t^*(x_0, \gamma, \ell)$ and $u^*(x_0, \gamma, \ell) \in U(K)$ exist. By Theorem 17, there is, for every $\sigma > 0$, a bang-bang signal $u^{\pm}(x_0, \gamma, \ell) \in U(K)$ such that $|\Sigma(x_0, v, u^*(x_0, \gamma, \ell), t) - \Sigma(x_0, v, u^{\pm}(x_0, \gamma, \ell), t)| < \sigma$ for all $t \in [0, t^*(x_0, \gamma, \ell)]$, all $v \in V_\gamma(v_0)$, and all $\Sigma \in F_\gamma(\Sigma_0)$. Using the vector identity $z^T z = y^T y - 2y^T(y-z) + (y-z)^T(y-z) \leq y^T y + 2n|y||y-z| + n|y-z|^2$ with $z = \Sigma(x_0, v, u^{\pm}(x_0, \gamma, \ell), t^*(x_0, \gamma, \ell))$ and $y = \Sigma(x_0, v, u^*(x_0, \gamma, \ell), t^*(x_0, \gamma, \ell))$, we get $z^T z \leq \ell + 2n\sqrt{\ell}\sigma + n\sigma^2$. Finally, select $\sigma > 0$ to satisfy $2n\sqrt{\ell}\sigma + n\sigma^2 \leq \ell' - \ell$. □

Theorem 16 guarantees that optimal performance can be approximated as closely as desired by bang-bang control input signals – signals that are relatively easy to calculate and implement.

7 EXAMPLE

Consider the system Σ given by

$$\dot{x}_1(t) = (1 + d_1\sin t)x_2 + d_2 x_2(t-1) + (2 + d_3\cos t)u(t-1),$$

$$\dot{x}_2(t) = \frac{d_4 x_1^2(t)}{1 + x_1^2(t)}\cos 2t + x_1(t-1) + (2 + \cos x_2(t))u(t-1),$$

with the unspecified constant parameters $0.3 \leq d_1, d_2 \leq 0.5$ and $0.8 \leq d_3, d_4 \leq 1$; the initial state $x_0 = [-1, 4]^T$; the delay $\tau = 1$; and the constant residual input signal $v(t) \in [-0.1, 0.1]$, $t \leq 0$. The input bound is $K = 5$, and the operating error bound is $\ell = 1$.

Numerical optimization shows that $t^*(x_0, \gamma, \ell) \simeq 1.47$. As seen in Figure 2, a similar time is obtained by the simple bang-bang control signal $u^{\pm}(t)$ of Figure 2 (derived by numerical search). Figure 2 shows the response for three parameter sets:

Set 1: $v(t) = -0.1, d_1 = 0.5, d_2 = 0.5, d_3 = 1, d_4 = 1$;

Set 2: $v(t) = 0, d_1 = 0.4, d_2 = 0.4, d_3 = 0.9, d_4 = 0.9$;

Set 3: $v(t) = 0.1, d_1 = 0.3, d_2 = 0.3, d_3 = 0.8, d_4 = 0.8$.

8 CONCLUSION

We have seen that there are optimal controllers that reduce operating errors in minimal time after a feedback disruption, once feedback has been momentarily restored. We have also seen that such optimal controllers can be replaced by bang-bang controllers with little impact on performance. There are many possible applications of this methodology; one application is the quick reduction of inter-sample errors in sampled-data systems, when the next sample arrives.

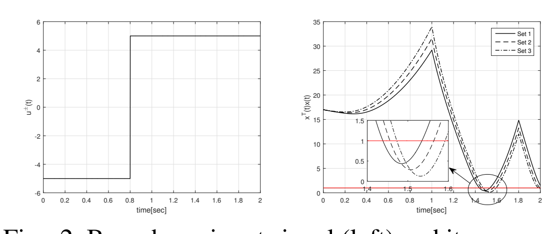

Fig. 2 Bang-bang input signal (left) and its response

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