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Optimal robust control of nonlinear systems: inter-sample optimisation in sampled-data control

Jacob Hammer

Department of Electrical and Computer Engineering, University of Florida, Gainesville, FL, USA

ABSTRACT
A methodology is presented for the design and implementation of robust controllers that optimise inter-sample performance for a broad range of nonlinear systems operated within a sampled-data environment. The methodology applies to nonlinear continuous-time systems described by state equations, and it allows for modelling uncertainties and constraints on maximal control effort. It is shown that there exist optimal robust state-feedback controllers that minimise inter-sample tracking errors for such systems, as long as an appropriate controllability requirement is met. A relatively simple design and implementation procedure for such controllers is described.

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1. Introduction

The quest for control methodologies that facilitate robust control of broad families of nonlinear systems – and accede to relatively simple design and implementation – has been a feature of control engineering since its inception. The present paper revisits this quest by offering a methodology that establishes the existence of optimal robust controllers for a broad range of nonlinear systems and provides a relatively simple design and implementation technique. The focus is on nonlinear sampled-data control – the control of continuous-time systems by digital controllers via periodic sampling. The methodology presented in this paper is valid for a broad family of nonlinear continuous-time systems, including most systems of practical interest. It offers tools for proving existence and tools for design and implementation of sampled-data state-feedback controllers that robustly achieve minimal inter-sample errors for tracking control systems. The techniques introduced in this paper form a foundation for generalising earlier work (Chakraborty & Hammer, 2008, 2010; Choi & Hammer, 2019; Hammer, 2022; Yu & Hammer, 2016) from input-affine systems to a much broader class of nonlinear systems.

Motivated by low-cost and implementation convenience, current control engineering practice is largely based on the use of digital controllers that interface with continuous-time systems through a process of periodic sampling. Such control systems are often referred to as *sampled-data systems*. Their basic control configuration is depicted in Figure 1, where Σ is a nonlinear system controlled by the state-feedback controller C. The controller C links to the state $x(t)$ of Σ through a periodic sampler of period $T > 0$. Based on the state sample it receives, C generates the input signal $u(t)$ of Σ. This input signal is updated after every sample.

More specifically, the kth sample occurs at the time kT. At that time, the controller C receives the state $x(kT)$. Based on the state $x(kT)$, the controller generates a signal $u(t) = C(t, x(kT))$ that is applied as input to the controlled system Σ during the time interval $[kT, (k+1)T]$. This process repeats at every sample $k = 0, 1, \ldots$ The paper concentrates on optimising the signal $C(t, x(kT))$ to reduce inter-sample tracking errors. We assume that the sampling period T is specified based on independent considerations.

Most often, practical systems come with restrictions on the maximal control effort they can tolerate. These restrictions ensure safe operation and integrity of the controlled system Σ. To incorporate such restrictions into our framework, we impose input and state amplitude constraints on Σ. Specifically, the input amplitude of Σ may not exceed a specified bound of $K > 0$, while the state amplitude of Σ may not exceed a specified bound of $A > 0$.

An important aspect of control engineering is managing uncertainty, as the available description of Σ is susceptible to modelling uncertainty, disturbances, and noises. The controller C must cope with such uncertainties; namely, it must be *robust*.

Our objective is to design a controller C that guides the closed-loop system to track a specified state x_{target}. By shifting coordinates, we can take $x_{target} = 0$, the origin. It is convenient to use the square of the L^2-norm to quantify the deviation of the state $x(t)$ from the target state $x = 0$. In this way, the deviation, or the *tracking error*, at a time t is simply $|x(t)|_2^2 = x^\top(t)x(t)$. The supremal tracking error over the sampling interval $[0, T]$ is often referred to as the *inter-sample tracking error*, since it describes the highest error between the two consecutive samples received by the controller C, namely, the samples at $t = 0$ and $t = T$. Explicitly, the inter-sample tracking error is given by $\ell = \sup_{t\in[0,T]} |x(t)|_2^2$. Our objective is to design and implement robust controllers C that minimise ℓ, thus optimising inter-sample performance.

CONTACT Jacob Hammer hammer@mst.ufl.edu Department of Electrical and Computer Engineering, University of Florida, Gainesville, FL 32611-6130, USA

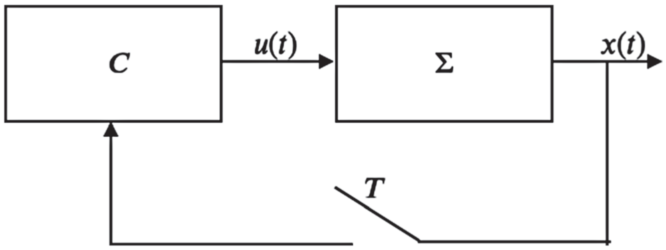


Figure 1. Control configuration.

1.1 The family of controlled systems

The controlled system Σ is described by the differential equation

$$\Sigma : \begin{array}{l} \dot{x}(t) = f(t, x(t), u(t)), \\ x(0) = x_0; \end{array} \tag{1}$$

here, $x(t) \in R^n$ is the state, and $u(t) \in R^m$ is the input. The only restriction is that the *recursion functionf* be continuously differentiable. The initial state x_0 is a member of the ball

$$\rho(\sigma) := \left\{ x : x^\top x \le \sigma \right\},$$

where $\sigma > 0$ is specified. The class of systems (1) is denoted by $\mathcal{S}$; it forms a broad class of nonlinear systems that includes most systems of practical interest.

1.2 Continuous signals

To facilitate some of the mathematical considerations that underlie our discussion, we impose a restriction on the permissible input signals $u(t)$ that the controlled system Σ may receive. We require that all input signals of Σ be continuous functions of time. From a practical perspective, this is hardly a consequential restriction, since truly discontinuous signals are not implementable in a continuous-time environment. Consider, for example, an electrical or electronic system. Here, the presence of spurious capacitance, inductance, and resistance prevents discontinuities in signals. Of course, we are all familiar with idealizations referring to 'switchings', or 'jumps', in signal values, but these are just idealizations. An examination on a sufficiently fine time scale would reveal that such signals are continuous functions of time. We show in Section 6 that, by restricting our attention to continuous input signals, we can prove existence of optimal robust controllers for members of the broad family $\mathcal{S}$ of controlled systems.

In this paper, we construct sampled-data state-feedback controllers that achieve optimal robust inter-sample tracking performance for systems belonging to the family $\mathcal{S}$. The existence of such controllers depends on a certain notion of controllability: the notion of *constrained controllability* of Choi and Hammer (2019). Basically, constrained controllability means that there is an input signal that drives the controlled system Σ from its initial state to the vicinity of the origin, without violating input and state amplitude constraints. As we discuss in Section 5, constrained controllability is very close to also being a necessary condition for the existence of optimal controllers. In other words, constrained controllability is a tight sufficient condition for the existence of optimal robust controllers.

Optimal controllers can be difficult to design and implement, since they require the computation and implementation of potentially involved multivariable vector-valued functions of time. We address this issue in Section 7, where we show that optimal performance can be approximated by controllers that are relatively easy to design and implement. Specifically, recall that a bang-bang signal switches between the values of K and $-K$; here, K is the input amplitude bound of the controlled system Σ. Bang-bang controllers – controllers that generate bang-bang input signals for the controlled system – are relatively simple to design and implement, and they can approximate optimal performance for input-affine systems (Chakraborty & Hammer, 2008, 2010; Choi & Hammer, 2019; Hammer, 2022; Yu & Hammer, 2016). Alas, bang-bang signals are not suitable for the general class $\mathcal{S}$ of systems due to certain mathematical requirements. Instead, we introduce the following class of bang-bang related signals.

1.3 Pseudo bang-bang signals

The class of *pseudo bang-bang signals* consists of differentiable functions of time that mimic bang-bang signals. Pseudo bang-bang signals, introduced in Section 7, have bounded slopes – instead of jumps – at their 'switching' times and somewhat rounded corners. Figure 2 compares a bang-bang signal to a related pseudo bang-bang signal. Pseudo bang-bang signals can be fairly similar to bang-bang signals, depending on the maximal permitted slope. They allow us to prove existence of relatively simple optimal controllers for a large family of nonlinear systems.

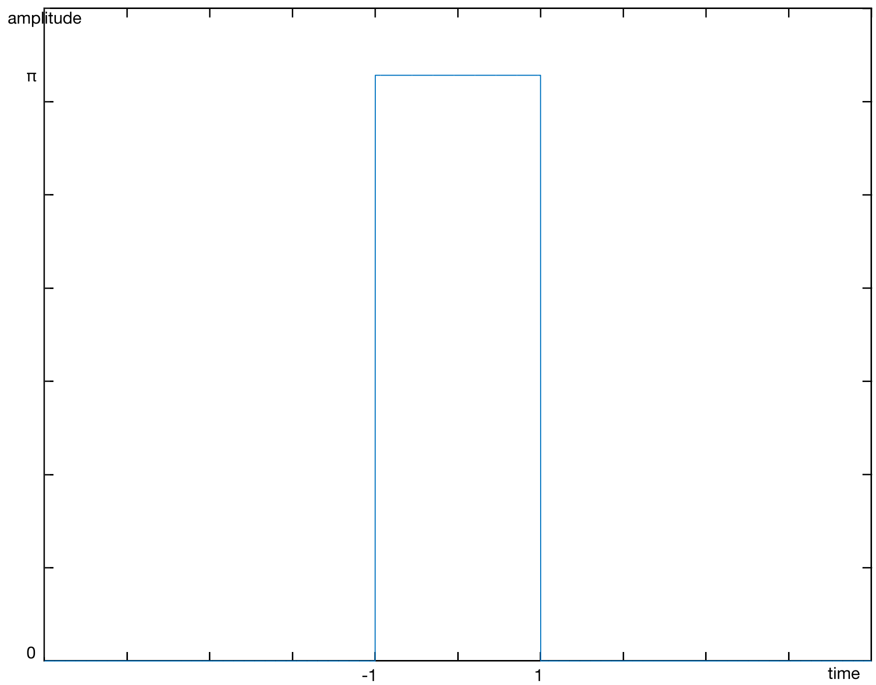
1.4 The main objectives

The objectives of our current discussion are as follows.

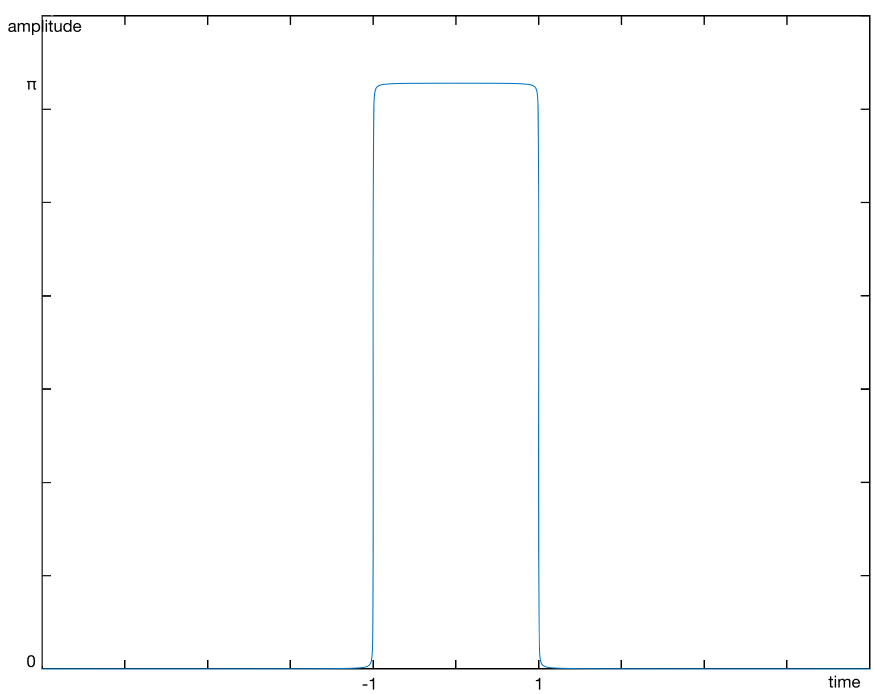
Problem 1.1: The controlled system Σ of Figure 1 is affected by modelling uncertainties, disturbances, and noises; it is subject to an input amplitude constraint of $K > 0$ and a state amplitude constraint of $A > 0$; it is operated in a sampled-data setting with a specified sampling period $T > 0$.

(i) Find conditions for the existence of optimal robust controllers C that minimise inter-sample tracking error.
(ii) Develop controllers that approximate optimal performance and are relatively easy to design and implement. □

The current paper expands the optimisation framework of Chakraborty and Hammer (2008, 2010), Yu and Hammer (2016), Choi and Hammer (2019) and Hammer (2022) from input-affine systems to a very broad family of nonlinear systems that encompasses most systems of practical interest. The material of this paper is within the general scope of optimal control theory, a broad area with a venerable history of a century or more. It is not possible in this limited space to provide fair credit to the innumerable researchers that contributed to the evolution of modern optimal control theory. In



(A) bang-bang signal



(B) pseudo bang-bang signal

Figure 2. Comparing bang-bang and pseudo bang-bang signals. (a) bang-bang signal and (b) pseudo bang-bang signal.

brief terms, the current paper generalises classical optimisation theory (Gamkrelidze, 1965; Kelendzheridze, 1961; Luenberger, 1969; Neustadt, 1966, 1967; Pontryagin et al., 1962; Warga, 1972; Young, 1969) by proving existence of optimal solutions for a very broad family of nonlinear systems; by proving the robustness of these solutions; and by proving the existence of simple-to-implement controllers that approximate optimal performance as closely as desired. To mention a sampling of recent publications, Allan et al. (2017) use model predictive control techniques to show the robustness of suboptimal predictive control solutions of optimal control problems with hard terminal constraints; Zhang et al. (2021) use the sliding-mode control framework to develop adaptive optimal controllers for a class of continuous-time switched nonlinear systems; and Zhao et al. (2023) study optimal decentralised control of interconnected nonlinear systems with stochastic dynamics by using event-triggered adaptive dynamic algorithms.

The material of the current paper builds a framework of optimal robust control that is valid for rather general families of nonlinear systems; it optimises performance and robustness of controllers. In addition, it yields simple controllers that approximate optimal performance as closely as desired. These approximating controllers are bang-bang controllers; as such, they are characterised by finite lists of scalars formed by their switching times, and thus require only sparse implementation resources.

The current paper concentrates on inter-sample optimisation of sampled-data systems, but its theoretical framework provides tools for proving the existence of optimal robust controllers for other classes of nonlinear optimisation problems as well. This framework also includes tools for creating simple-to-implement controllers that approximate optimal performance as closely as desired.

The paper is organised as follows. Section 2 introduces our framework of continuous input signals, and Section 3 introduces our robustness framework. A formal statement of Problem 1.1 is provided in Sections 4 and 5 reviews the notion of constrained controllability. Section 6 proves the existence of robust controllers that achieve minimal inter-sample tracking errors. Section 7 introduces the class of pseudo bang-bang controllers – controllers that are relatively easy to design and implement, and shows that such controllers can closely approximate optimal performance. Section 8 applies the methodology of this paper to an example and compares the results to classical sample-and-hold implementation. Finally, the paper concludes in Section 9 with a brief summary.

2. The class of input signals

In real-world continuous-time environments, abrupt changes in signal values, i.e. discontinuous jumps, are an idealisation and are not implementable. If the time scale is sufficiently refined, what appears like a discontinuity turns out to be a gradual – differentiable – change (see Figure 2). This fact leads us to restrict our attention to differentiable input signals; practically all signals in applications are differentiable. A restriction to differentiable input signals results in a simplification of the mathematical framework; it leads to more powerful results that apply to the broad family of nonlinear systems of (1).

First, some notation. We denote by $\mathbb{C}$ the set of complex numbers; by $\mathbb{C}^m$ the set of column vectors with m complex numbers as components, and by R^+ the set of non-negative real numbers. The absolute value of a complex number c is denoted by $|c|$. For a matrix $V \in \mathbb{C}^{n\times m}$, the L^∞-norm is $|V| := \max_{i,j} |V_{ij}|$. For a function of time $V : R^+ \to \mathbb{C}^{n\times m} : t \mapsto V(t)$, the L^∞-norm is $|V|_\infty := \sup_{t\geq 0} |V(t)|$.

The class of signals that underlies our discussion is introduced through the Fourier transform. Although we are dealing with nonlinear systems, and the Fourier transform is a linear tool, we will see that its use here is convenient and appropriate. To set our notation, the Fourier transform of a real vector-valued signal $u : R \to R^m : t \mapsto u(t)$ is $\upsilon = \mathscr{F}u : R \to \mathbb{C}^m : \omega \mapsto \upsilon(\omega)$, where ω is the Fourier variable (the frequency). Considering that input signals of the controlled system Σ are real vector-valued functions of time, we are interested only in functions $\upsilon : R \to \mathbb{C}^m$ whose inverse Fourier transform $\mathscr{F}^{-1}\upsilon$ is a real valued function of time. Recall that this is the case if and only if for each component υ^i of υ, $i \in \{1, 2, \dots, m\}$, the magnitude $|\upsilon^i(\omega)|$ is an even function of ω, while the phase $\angle\upsilon^i(\omega)$ is an odd function of ω. Thus, in the Fourier domain, we are interested only in complex vector-valued Lebesgue measurable functions υ that are members of the family

$$\Omega := \left\{ \upsilon : R \to \mathbb{C}^m \;\; \begin{array}{l} |\upsilon^i(\omega)| \text{is even and } \angle\upsilon^i(\omega) \text{ is odd} \\ \text{as a function of } \omega,\ i = 1, 2, \dots, m. \end{array} \right\} \tag{2}$$

According to Parseval's theorem, the 'energy' of a signal is given by the integral of the square magnitude of its Fourier transform. As only signals of finite energy are of interest to us, the square magnitude of the Fourier transform of our signals must be bounded and integrable over the entire axis. This leads us to the following subfamily of Ω, which plays a basic role in our discussion. Let $W, \kappa > 0$ be two real numbers. We introduce a family of complex valued exponentially-bounded functions of frequency, given by

$$\Omega(W, \kappa) := \left\{ \upsilon \in \Omega : |\upsilon(\omega)| \leq We^{-\kappa|\omega|} \text{ for all } \omega \in R \right\}. \tag{3}$$

Members of $\Omega(W, \kappa)$ have magnitude bounded by W, and their magnitude decays exponentially with increasing frequency. As a result, the square of their magnitude is integrable over the entire frequency axis. We show in a short while that the inverse Fourier transform of members of $\Omega(W, \kappa)$ are differentiable real vector-valued functions of time. The number κ is the *smoothing factor*; it determines the steepest slope of signals obtained by inverse Fourier transform of $\Omega(W, \kappa)$ members. Smaller values of κ lead to steeper slopes.

For example, Figure 2(b) was obtained by taking the Fourier transform of Figure 2(a); multiplying it by $e^{-\kappa|\omega|}$ with $\kappa = 0.001$; and applying the inverse Fourier transform to the result. As the figure indicates, inverse Fourier transforms of $\Omega(W, \kappa)$ members can closely mimic bang-bang signals by selecting a small value of $\kappa > 0$. This point is discussed in further detail in Section 7 below.

The class of time domain signals corresponding to the family $\Omega(W, \kappa)$ is given by

$$U(W, \kappa) = \left\{ \mathscr{F}^{-1}\upsilon : \upsilon \in \Omega(W, \kappa) \right\}; \tag{4}$$

it consists of functions $u : R \to R^m : t \mapsto u(t)$, where

$$u(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \upsilon(\omega) e^{j\omega t} \mathrm{d}\omega, \quad \upsilon \in \Omega(W, \kappa). \tag{5}$$

(We use the form of the inverse Fourier transform commonly used in engineering, where $j := \sqrt{-1}$.) Practically every signal that appears in real-world control systems belongs to the family $U(W, \kappa)$ for some values of W and κ. The next statement lists a few properties of members of $U(W, \kappa)$.

Lemma 2.1: *Functions $u \in U(W, \kappa)$ satisfy:*

(i) *All are bounded with the same bound.*
(ii) *All are uniformly continuous functions of time with the same uniformity.*

(iii) *All have bounded and uniformly continuous derivatives with respect to time.*

Proof: (i) By (4), $u(t) = \mathscr{F}^{-1}\upsilon$ for a $\upsilon \in \Omega(W,\kappa)$, so that

$$|u(t)| \le \frac{1}{2\pi}\int_{-\infty}^{\infty}|\upsilon(\omega)|\mathrm{d}\omega \le \frac{2}{2\pi}\int_{0}^{\infty}We^{-\kappa\omega}\mathrm{d}\omega = \frac{W}{\pi\kappa}.$$

(ii) For times $t_1 < t_2$ and a real number $\omega_0 > 0$, we have

$$\begin{aligned}
&|u(t_2) - u(t_1)|\\
&= \frac{1}{2\pi}\left|\int_{-\infty}^{\infty}\upsilon(\omega)(e^{j\omega t_2} - e^{j\omega t_1})\mathrm{d}\omega\right|\\
&\le \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0}We^{-\kappa|\omega|}\left|e^{j\omega t_2} - e^{j\omega t_1}\right|\mathrm{d}\omega + \frac{4}{2\pi}\int_{\omega_0}^{\infty}We^{-\kappa\omega}\mathrm{d}\omega\\
&\le \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0}\left|We^{-\kappa|\omega|}\right|\left|e^{j\omega t_2} - e^{j\omega t_1}\right|\mathrm{d}\omega + \frac{2W}{\pi\kappa}e^{-\kappa\omega_0}. \qquad (6)
\end{aligned}$$

To continue, regard ω as fixed for a moment; using the mean value theorem yields $e^{j\omega t_2} - e^{j\omega t_1} = -\omega\sin(\omega t')(t_2 - t_1) + j\omega\cos(\omega t'')(t_2 - t_1)$, where $t', t'' \in [t_1, t_2]$ are appropriate times. This implies that

$$\begin{aligned}
\left|e^{j\omega t_2} - e^{j\omega t_1}\right| &\le |\omega|\,(t_2 - t_1)\sqrt{\sin^2(\omega t') + \cos^2(\omega t'')}\\
&\le \sqrt{2}|\omega|(t_2 - t_1).
\end{aligned}$$

Substituting into (6), we get

$$\begin{aligned}
&|u(t_2) - u(t_1)|\\
&\le \frac{\sqrt{2}W}{\pi}(t_2 - t_1)\int_{0}^{\omega_0}e^{-\kappa\omega}\omega\mathrm{d}\omega + \frac{2W}{\pi\kappa}e^{-\kappa\omega_0}\\
&\le \frac{\sqrt{2}W}{\pi\kappa^2}\left[1 - e^{-\kappa\omega_0}(\kappa\omega_0 + 1)\right](t_2 - t_1) + \frac{2W}{\pi\kappa}e^{-\kappa\omega_0} \quad (7)
\end{aligned}$$

Now, let $\varepsilon > 0$ be a real number. Select the frequency ω_0 so that $\frac{2W}{\pi\kappa}e^{-\kappa\omega_0} < \varepsilon/2$; then, select a real number $\delta > 0$ such that $\frac{\sqrt{2}W}{\pi\kappa^2}[1 - e^{-\kappa\omega_0}(\kappa\omega_0 + 1)]\delta < \varepsilon/2$. Substituting these inequalities into (7), we get that $|u(t_2) - u(t_1)| < \varepsilon$ for all $|t_2 - t_1| < \delta$. This is true for any $u \in U(W,\kappa)$.

(iii) Differentiating (5), we get

$$\mathrm{d}u(t)/\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty}j\omega\upsilon(\omega)e^{j\omega t}\mathrm{d}\omega. \qquad (8)$$

Therefore,

$$|\mathrm{d}u(t)/\mathrm{d}t| \le \frac{1}{\pi}\int_{0}^{\infty}\omega We^{-\kappa\omega}\mathrm{d}\omega = \frac{W}{\pi\kappa^2}, \qquad (9)$$

so that the derivative is bounded independently of u.

Regarding uniform continuity, replace $\upsilon(\omega)$ in (6) by $j\omega\upsilon(\omega)$ to reflect (8). We obtain

$$\begin{aligned}
&\left|\frac{\mathrm{d}u(t_2)}{\mathrm{d}t} - \frac{\mathrm{d}u(t_1)}{\mathrm{d}t}\right|\\
&= \frac{1}{2\pi}\left|\int_{-\infty}^{\infty}\omega\upsilon(\omega)(e^{j\omega t_2} - e^{j\omega t_1})\mathrm{d}\omega\right|\\
&\le \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0}\left|W\omega e^{-\kappa|\omega|}\right|\left|e^{j\omega t_2} - e^{j\omega t_1}\right|\mathrm{d}\omega\\
&\quad + \frac{4}{2\pi}\int_{\omega_0}^{\infty}W\omega e^{-\kappa\omega}\mathrm{d}\omega\\
&\le \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0}\left|W\omega e^{-\kappa|\omega|}\right|\left|e^{j\omega t_2} - e^{j\omega t_1}\right|\mathrm{d}\omega\\
&\quad + \frac{2W(\kappa\omega_0 + 1)}{\pi\kappa^2}e^{-\kappa\omega_0}
\end{aligned}$$

Then, an argument similar to the one leading to (7) yields

$$\begin{aligned}
&\left|\frac{\mathrm{d}u(t_2)}{\mathrm{d}t} - \frac{\mathrm{d}u(t_1)}{\mathrm{d}t}\right|\\
&\le \frac{\sqrt{2}W}{\pi}(t_2 - t_1)\int_{0}^{\omega_0}e^{-\kappa\omega}\omega^2\mathrm{d}\omega + \frac{2W(\kappa\omega_0 + 1)}{\pi\kappa^2}e^{-\kappa\omega_0}\\
&\le \frac{\sqrt{2}W}{\pi\kappa^3}\left[2 - e^{-\kappa\omega_0}(\kappa^2\omega_0^2 + 2\kappa\omega_0 + 2)\right](t_2 - t_1)\\
&\quad + \frac{2W(\kappa\omega_0 + 1)}{\pi\kappa^2}e^{-\kappa\omega_0}. \qquad (10)
\end{aligned}$$

Given $\varepsilon > 0$, select ω_0 to satisfy $\frac{2W(\kappa\omega_0+1)}{\pi\kappa^2}e^{-\kappa\omega_0} < \varepsilon/2$; then, select $\delta > 0$ so that $\frac{\sqrt{2}W}{\pi\kappa^3}\,[2 - e^{-\kappa\omega_0}\,(\kappa^2\omega_0^2 + 2\kappa\omega_0 + 2)]\delta < \varepsilon/2$. Substituting into (10), we get $|\mathrm{d}u(t_2)/\mathrm{d}t - \mathrm{d}u(t_1)/\mathrm{d}t| < \varepsilon$ for all $|t_2 - t_1| < \delta$ and all $u(t) \in U(W,\kappa)$, concluding the proof. ■

A consequence of (9) is that the maximal slope magnitude achievable by a member of $U(W,\kappa)$ is $W/(\pi\kappa^2)$. Thus, any finite slope magnitude can be achieved by selecting W or κ.

As the controlled system Σ of Figure 1 permits only inputs bounded by $K > 0$, the class of permissible input signals is

$$U(K,W,\kappa) := \{u \in U(W,\kappa) : |u|_\infty \le K\}. \qquad (11)$$

We explore next some features of this class of input signals.

2.1 *Features of the family $U(K,W,\kappa)$*

Let $\mathcal{H}$ be the Hilbert space of Lebesgue measurable functions $: R \to \mathbb{C}^m$ with the usual L^2-inner product. Specifically, given two Lebesgue measurable functions $f, g : R \to \mathbb{C}^m$, and letting $\bar{f}$ be the complex conjugate of f, the inner product is

$$\langle f,g\rangle = \int_{-\infty}^{\infty}\bar{f}^\top(\omega)g(\omega)\mathrm{d}\omega. \qquad (12)$$

The inner product of members $f, g \in \Omega(W,\kappa)$ is bounded:

$$\begin{aligned}
|\langle f,g\rangle| &= \left|\int_{-\infty}^{\infty}\bar{f}^\top(\omega)g(\omega)\mathrm{d}\omega\right|\\
&\le mW^2\int_{-\infty}^{\infty}e^{-2\kappa|\omega|}\mathrm{d}\omega = mW^2/\kappa.
\end{aligned}$$

We review some mathematical notions (e.g. Lusternik & Sobolev, 1961).

Definition 2.2: Let H be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$. A sequence $\{\upsilon_n\}_{n=1}^{\infty} \subseteq H$ *is weakly convergent to* $\upsilon \in H$

if $\lim_{n\to\infty}\langle \upsilon_n, y\rangle = \langle \upsilon, y\rangle$ for every $y \in H$. A subset $G \subseteq H$ is *weakly compact* if every sequence of G members has a subsequence that weakly converges to a member of G. □

We claim that the set $\Omega(W,\kappa)$ of (3) is weakly compact in the Hilbert space $\mathcal{H}$, as follows.

Lemma 2.3: *The set $\Omega(W,\kappa)$ is weakly compact in the topology of the Hilbert space $\mathcal{H}$.*

Proof: This proof paraphrases a proof of Chakraborty and Hammer (2009, 2010). Consider a sequence $\{\upsilon_i\}_{i=1}^{\infty} \subseteq \Omega(W,\kappa)$. By (3), all members of $\Omega(W,\kappa)$ are bounded by W. By Alaoglu's theorem (e.g. Halmos, 1982), this implies that $\{\upsilon_i\}_{i=1}^{\infty}$ has a subsequence $\{\upsilon_{i_k}\}_{k=1}^{\infty}$ that converges weakly to a member $\upsilon \in \mathcal{H}$. To show that $\upsilon \in \Omega(W,\kappa)$, we use Mazur's theorem (e.g. Halmos, 1982).

First, we show that $\Omega(W,\kappa)$ is a convex set. Consider two members $\upsilon', \upsilon'' \in \Omega(W,\kappa)$, let $0 \le r \le 1$ be a real number, and let $\upsilon'''(\omega) := r\upsilon'(\omega) + (1-r)\upsilon''(\omega)$. Then, $|\upsilon'''(\omega)| \le r|\upsilon'(\omega)| + (1-r)|\upsilon''(\omega)| \le rWe^{-\kappa|\omega|} + (1-r)We^{-\kappa|\omega|} = We^{-\kappa|\omega|}$. In addition, since $\mathscr{F}^{-1}\upsilon''' = r\mathscr{F}^{-1}\upsilon' + (1-r)\mathscr{F}^{-1}\upsilon''$, and $\mathscr{F}^{-1}\upsilon'$ and $\mathscr{F}^{-1}\upsilon''$ are both real by (3), it follows that $\mathscr{F}^{-1}\upsilon'''$ is a real-valued function as well. Thus, $\upsilon''' \in \Omega(W,\kappa)$ and $\Omega(W,\kappa)$ is convex.

Next, we show that $\Omega(W,\kappa)$ is strongly closed. To do so, let $\{w_i\}_{i=1}^{\infty} \subseteq \Omega(W,\kappa)$ be a strongly convergent sequence with strong limit w, namely,

$$\lim_{i\to\infty} \langle (w-w_i), (w-w_i)\rangle = 0. \tag{13}$$

Note that w, as the limit of a sequence of Lebesgue measurable functions, is Lebesgue measurable. Also, since w is the limit of members of $\Omega(W,\kappa)$, its magnitude and phase satisfy the conditions of (2), so $w \in \Omega$. Now assume, by contradiction, that $w \notin \Omega(W,\kappa)$, namely, that w violates the bound (3). Then, there must be a real number $\varepsilon > 0$ and a Lebesgue measurable subset $\delta \subset (-\infty,\infty)$ of non-zero measure such that $|w(\omega)| > (W+\varepsilon)e^{-\kappa|\omega|}$ for all $\omega \in \delta$. This implies that w has a component, say component w^q, for which there is a subset of non-zero measure $\delta_q \subset \delta$ satisfying

$$|w^q(\omega)| - We^{-\kappa|\omega|} \ge \varepsilon e^{-\kappa|\omega|} \text{ for all } \omega \in \delta_q. \tag{14}$$

Using (12) and letting w_i^q be component q of w_i yields

$$\begin{aligned}
&\langle (w-w_i), (w-w_i)\rangle \\
&= \int_{-\infty}^{\infty} [\bar{w}(\omega) - \bar{w}_i(\omega)]^\top [w(\omega) - w_i(\omega)]\mathrm{d}\omega \\
&\ge \int_{\delta_q} [\bar{w}(\omega) - \bar{w}_i(\omega)]^\top [w(\omega) - w_i(\omega)]\mathrm{d}\omega \\
&\ge \int_{\delta_q} \left|w^q(\omega) - w_i^q(\omega)\right|^2 \mathrm{d}\omega.
\end{aligned} \tag{15}$$

As $w_i \in \Omega(W,\kappa)$, we have $|w_i(t)| \le We^{-\kappa|\omega|}$ for all $\omega \in (-\infty,\infty)$, so that $|w_i^q(t)| \le We^{-\kappa|\omega|}$ for all $\omega \in (-\infty,\infty)$. Thus, $|w^q(\omega) - w_i^q(\omega)| \ge |w^q(\omega)| - |w_i^q(\omega)| \ge |w^q(\omega)| - We^{-\kappa|\omega|}$. Applying (14), we get $|w^q(\omega) - w_i^q(\omega)| \ge \varepsilon e^{-\kappa|\omega|}$ for all $\omega \in \delta_q$. Inserting this into (15) yields

$$\begin{aligned}
\langle (w-w_i), (w-w_i)\rangle &\ge \int_{\delta_q} \left|w^q(\omega) - w_i^q(\omega)\right|^2 \mathrm{d}\omega \\
&\ge \int_{\delta_q} (\varepsilon e^{-\kappa|\omega|})^2 \mathrm{d}t
\end{aligned}$$

independently of i for all $i = 1, 2, \ldots$, thus contradicting (13). Therefore, $w \in \Omega(W,\kappa)$, so that $\Omega(W,\kappa)$ is strongly closed. Consequently, it is weakly compact by Mazur's theorem. This concludes our proof. ■

We need the following terms.

Definition 2.4: A family G of functions mapping $R^+ \to R^m$ is *pointwise compact* if every sequence of functions $\{g_i\}_{i=1}^{\infty} \subseteq G$ has a subsequence $\{g_{i_k}\}_{k=1}^{\infty}$ that is pointwise convergent to a member $g \in G$, namely, if $\lim_{k\to\infty} g_{i_k}(t) = g(t)$ for all $t \in R^+$. The family G is *uniformly pointwise compact* if, for every two times $t_1 \le t_2$ and for every real number $\varepsilon > 0$, there is an integer $N \ge 1$ such that $|g_{i_k}(t) - g(t)| < \varepsilon$ for all $k \ge N$ and all $t \in [t_1, t_2]$. □

The next statement highlights a property of the set $U(K, W, \kappa)$ of (11) that is critical to our discussion.

Lemma 2.5: *$U(K, W, \kappa)$ is uniformly pointwise compact.*

Proof: For a real number $a > 0$ and an integer $r \in \{1, 2, \ldots, m\}$, define the 'pulse' function $p_a(\omega, r) : R \to \mathbb{C}^m$:

$$p_a(\omega, r) = \begin{cases} (0,\ldots,0,1,0,\ldots,0)^\top (1 \text{ in entry } r) & \text{if } \omega \in [-a, a] \\ 0 & \textit{else}, \end{cases}$$

which has finite norm in the Hilbert space $\mathcal{H}$. Denote by w^r component r of a member $w \in \Omega(W, k)$; then,

$$\begin{aligned}
&\left|\int_{-\infty}^{\infty} w^r(\omega)\mathrm{d}\omega - \int_{-\infty}^{\infty} p_a^\top(\omega, r) w(\omega)\mathrm{d}\omega\right| \\
&\le \left|\int_{-\infty}^{-a} w^r(\omega)\mathrm{d}\omega\right| + \left|\int_{a}^{\infty} w^r(\omega)\mathrm{d}\omega\right| \\
&\le 2\int_{a}^{\infty} We^{-\kappa\omega}\mathrm{d}\omega = \frac{2W}{\kappa}e^{-\kappa a} =: \Delta(a)
\end{aligned} \tag{16}$$

Thus, for every real $\delta > 0$, there is an $a > 0$ such that $\Delta(a) < \delta$ for all $w \in \Omega(W,\kappa)$ and all $r \in \{1, 2, \ldots, m\}$.

Now, let $\{u_k\}_{k=1}^{\infty} \subseteq U(K, W, \kappa)$ be a sequence. In view of (4), (5), and (11), there is a corresponding sequence of functions $\upsilon_k = \mathscr{F}u_k \in \Omega(W,\kappa)$, $k = 1, 2, \ldots$, such that

$$u_k(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \upsilon_k(\omega)e^{j\omega t}\mathrm{d}\omega, \quad k = 1, 2, \ldots \tag{17}$$

By Lemma 2.3, the family $\Omega(W,\kappa)$ is weakly compact. Consequently, there is a subsequence $\{\upsilon_{k_i}\}_{i=1}^{\infty}$ that is weakly convergent to a member $\upsilon \in \Omega(W,\kappa)$. This means that, for every member $h \in \mathcal{H}$ with finite norm, we have

$$\lim_{i\to\infty} \int_{-\infty}^{\infty} \bar{h}^\top(\omega)(\upsilon_{k_i}(\omega) - \upsilon(\omega))\mathrm{d}\omega = 0.$$

Now, fix a time t, and consider the expression

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{h}^{\top}(\omega)[\upsilon_{k_i}(\omega)-\upsilon(\omega)]e^{j\omega t}\mathrm{d}\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{[h(\omega)e^{-j\omega t}]}^{\top}[\upsilon_{k_i}(\omega)-\upsilon(\omega)]\mathrm{d}\omega.$$

As $h(\omega)e^{-j\omega t}$ is also a finite norm member of $\mathcal{H}$, we conclude that, for any $h \in \mathcal{H}$ of finite norm,

$$\lim_{i\to\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{h}^{\top}(\omega)[(\upsilon_{k_i}(\omega)-\upsilon(\omega))e^{j\omega t}]\mathrm{d}\omega = 0.$$

To continue, choose $h(\omega) = p_a(\omega, r)$. Then, we get

$$\lim_{i\to\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{p}_a^{\top}(\omega,r)[(\upsilon_{k_i}(\omega)-\upsilon(\omega))e^{j\omega t}]\mathrm{d}\omega = \lim_{i\to\infty}\frac{1}{2\pi}\int_{-a}^{a}(\upsilon_{k_i}^{r}(\omega)-\upsilon^{r}(\omega))e^{j\omega t}\mathrm{d}\omega = 0. \tag{18}$$

Since $\{\upsilon_k\}_{k=1}^{\infty}$ and υ are all member of $\Omega(W,\kappa)$, so are $\{\upsilon_k e^{j\omega t}\}_{k=1}^{\infty}$ and $\upsilon e^{j\omega t}$; hence, their differences are members of $\Omega(2W,\kappa)$. Using (18) and (16), we get

$$\begin{aligned}\lim_{i\to\infty}&\left|\int_{-\infty}^{\infty}(\upsilon_{k_i}^{r}(\omega)-\upsilon^{r}(\omega))e^{j\omega t}\mathrm{d}\omega\right|\\ &\leq \lim_{i\to\infty}\left\{\left|\int_{-a}^{a}(\upsilon_{k_i}^{r}(\omega)-\upsilon^{r}(\omega))e^{j\omega t}\mathrm{d}\omega\right| + \left|\int_{-\infty}^{\infty}(\upsilon_{k_i}^{r}(\omega)-\upsilon^{r}(\omega))e^{j\omega t}\mathrm{d}\omega - \int_{-a}^{a}(\upsilon_{k_i}^{r}(\omega)-\upsilon^{r}(\omega))e^{j\omega t}\mathrm{d}\omega\right|\right\}\\ &\leq 2\Delta(a).\end{aligned}$$

Since $\Delta(a) \to 0$ as $a \to \infty$, it follows that

$$\lim_{i\to\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}(\upsilon_{k_i}^{r}(\omega)-\upsilon^{r}(\omega))e^{j\omega t}\mathrm{d}\omega = 0 \quad \text{for all } r = 1,\ldots,m.$$

Recalling (17) and setting $u(t) = \mathscr{F}^{-1}\upsilon(\omega)$, this yields

$$\lim_{i\to\infty} u_{k_i}(t) = u(t), \tag{19}$$

which proves pointwise convergence for every time t.

This convergence is uniform; indeed, by (19), for every $\varepsilon > 0$ and time t, there is an integer $N'(t) \geq 1$ such that

$$\left|u_{k_i}(t) - u(t)\right| < \varepsilon \quad \text{for all } i \geq N'(t).$$

Let $N(t)$ be the smallest integer among all values of $N'(t)$; then, $N(t)$ exists, since it belongs to the finite set $\{1,\ldots,N'(t)\}$.

Now, fix a time interval $[t', t'']$, $t' < t''$, and let $N^* := \sup_{t\in[t',t'']} N(t)$. To show that N^* is finite, assume, by contradiction, that N^* is not finite. Then, there is a sequence $\{t_p\}_{p=1}^{\infty} \subseteq [t', t'']$ for which the sequence of integers $N(t_1), N(t_2), \ldots$ is unbounded. As $[t', t'']$ is compact, there is a subsequence $\{t_{p_q}\}_{q=1}^{\infty}$ convergent to a time $t^* \in [t', t'']$. But then, by (19), there is an integer $N \geq 1$ such that $|u_{k_i}(t^*) - u(t^*)| < \varepsilon/2$ for all integers $i \geq N$. Fix an integer $i' \geq N$. By Lemma 2.1(ii), there is a real number $\delta_{i'} > 0$ such that $|u_{k_{i'}}(\theta_1) - u_{k_{i'}}(\theta_2)| \leq \varepsilon/4$ and $|u(\theta_1) - u(\theta_2)| < \varepsilon/4$ for all $\theta_1, \theta_2 \in (t^* - \delta_{i'}, t^* + \delta_{i'})$. Combining these inequalities, we obtain that $|u_{k_{i'}}(t) - u(t)| < \varepsilon$ for all $t \in (t^* - \delta_{i'}, t^* + \delta_{i'})$.

Further, by Lemma 2.1(ii), the number $\delta_{i'} > 0$ can be selected independently of i'. For such $\delta_{i'}$, the previous paragraph is valid for every integers $i' \geq N$. This implies that $N(t) \leq N$ for all $t \in (t^* - \delta_{i'}, t^* + \delta_{i'})$, contradicting the assumption that the sequence $\{N(t_p)\}_{p=1}^{\infty}$ is unbounded. Thus, there is a finite integer $N^* \geq 1$ for which $|u_{k_i}(t) - u(t)| < \varepsilon$ for all $i \geq N^*$ and all $t \in [t', t'']$. This concludes our proof. ■

Next, we study systems with inputs from $U(K, W, \kappa)$.

3. The family of nonlinear systems

The controlled system Σ of Figure 1 is subject to uncertainties that affect its recursion function f of (1). To represent these uncertainties, we decompose f into a sum

$$f(t,x,u) = f_0(x,u) + f_\gamma(t,x,u), \tag{20}$$

where f_0 is a specified *nominal recursion function*, while f_γ is an unknown *uncertainty function* representing uncertainties and disturbances. We invoke the following.

Assumption 3.1: *The functions f_0 and f_γ of (20) are continuously differentiable.* □

The nominal system is a nonlinear time-invariant system:

$$\Sigma_0 : \begin{aligned} \dot{x}(t) &= f_0(x(t), u(t)),\\ x(0) &= x_0. \end{aligned} \tag{21}$$

As mentioned earlier, to reflect conditions encountered in practice, we impose two constraints on Σ: (i) the input amplitude may not exceed a specified bound $K > 0$; and (ii) the state amplitude may not exceed a specified bound $A > 0$.

3.1 The recursion function

As input signals of the controlled system Σ are bounded by K and states are bounded by A, the functions f_0 and f_γ of (20) have the domain $(x,u) \in [-A,A]^n \times [-K,K]^m$. By Assumption 3.1, the mean value theorem (e.g. Hubbard & Hubbard, 2015) yields

$$f_0(x,u) - f_0(x',u') = \frac{\partial f_0(c)}{\partial \binom{x}{u}}\begin{pmatrix} x - x' \\ u - u' \end{pmatrix},$$

$$f_\gamma(t,x,u) - f_\gamma(t,x',u') = \frac{\partial f_\gamma(t,c'(t))}{\partial \binom{x}{u}}\begin{pmatrix} x - x' \\ u - u' \end{pmatrix}, \quad t \geq 0,$$

where $x, x' \in [-A,A]^n$, $u, u' \in [-K,K]^m$, and $c, c'(t) \in [-A,A]^n \times [-K,K]^m$. As $\partial f_0/\partial(x,u)$ and $\partial f_\gamma/\partial(x,u)$ are continuous by Assumption 3.1, and the domain $[-A,A]^n \times [-K,K]^m$ is compact, there are bounds $B > 0$ and $\gamma(t) > 0$:

$$\left|\frac{\partial f_0(c)}{\partial \binom{x}{u}}\right| \leq B \quad \text{and} \quad \left|\frac{\partial f_\gamma(t,c'(t))}{\partial \binom{x}{u}}\right| \leq \gamma(t), \quad t \geq 0, \tag{22}$$

for all $c, c'(t) \in [-A, A]^n \times [-K, K]^m$. Most often, the magnitude bound of uncertainties is constant:

Assumption 3.2: *Referring to* (22), *there is a known constant bound* $\gamma > 0$ *such that* $\gamma(t) \leq \gamma$ *for all* $t \geq 0$. □

Under this assumption, we rewrite (22) in the form

$$\left|\frac{\partial f_0(c)}{\partial \binom{x}{u}}\right| \leq B \quad \text{and} \quad \left|\frac{\partial f_\gamma(t, c'(t))}{\partial \binom{x}{u}}\right| \leq \gamma \tag{23}$$

for all $t \geq 0$ and for all $c, c'(t) \in [-A, A]^n \times [-K, K]^m$. As γ represents uncertainty, it is often small. For notational convenience, we use the same bounds for the values at the origin:

$$|f_0(0,0)| \leq B \quad \text{and} \quad |f_\gamma(t,0,0)| \leq \gamma, t \geq 0. \tag{24}$$

This leads us to the following family of systems.

Notation 3.3: Let $K, A, \sigma, \gamma > 0$ be specified. The family of systems $\mathcal{S}_\gamma(\Sigma_0, K, A)$ consists of all systems described by (1), (20), (23), and (24), where Σ_0 is the nominal system of (21). For all members of $\mathcal{S}_\gamma(\Sigma_0, K, A)$, input amplitude is bounded by K, state amplitude is bounded by A, and initial states are in $\rho(\sigma)$. In addition, the following apply.

(i) All input signals are members of $U(K, W, \kappa)$.
(ii) All members of $\mathcal{S}_\gamma(\Sigma_0, K, A)$ share the same initial state $x_0 \in \rho(\sigma)$.
(iii) All members of $\mathcal{S}_\gamma(\Sigma_0, K, A)$ share the same controller C of Figure 1. □

Item (ii) of Notation 3.3 reflects the fact that the actual initial state $x(0) = x_0$ is provided by the sampler of Figure 1. Item (iii) of Notation 3.3 guarantees robustness of the controller C, as it is not known which member of $\mathcal{S}_\gamma(\Sigma_0, K, A)$ the active model Σ is. So C must properly control every member of $\mathcal{S}_\gamma(\Sigma_0, K, A)$.

4. Inter-sample tracking

Over the last few decades, continuous-time control engineering has experienced a transition from pervasive use of analog controllers to almost exclusive use of sampled-data systems that employ digital controllers. Many sampled-data systems utilise periodic sampling, where the sampling period T is specified by separate technical considerations.

The design of controllers for sampled-data systems can be simplified by building a periodic framework based on the specified sampling period T. This periodic framework relies on time invariance of the nominal controlled system Σ_0 of (21). In this periodic framework, we design the controller C of Figure 1 based on the first sampling interval $[0, T]$. Then, by periodicity, the same controller C can be utilised during any sampling interval $[kT, (k+1)T]$, $k = 0, 1, \ldots$, after an appropriate time shift, as follows.

During a sampling interval $[kT, (k+1)T]$, $k \in \{0, 1, \ldots\}$, the response of the controller C is determined by the state $x(kT) = x$ of Σ at the start of that sampling interval, since no more samples arrive until $(k+1)T$. If Σ is at the same state $x(k'T) = x$ at the start of another sampling interval $[k'T, (k'+1)T]$, then, by periodicity, C can generate the same input to Σ, appropriately shifted in time. Thus, within a periodic framework, all we need to do is design the controller C for the initial sampling interval $[0, T]$. The action of the controller C during other sampling intervals is then obtained simply by shifting its response in time. This results in a substantial simplification of the process of designing the controller C. For this to be possible, we must guarantee that 'similar' states appear at the start of all sampling intervals. This leads to the following notion (Choi & Hammer, 2020).

Definition 4.1: Let $T > 0$ be the specified sampling period of the family $\mathcal{S}_\gamma(\Sigma_0, K, A)$. Denote by $x(t)$ the state of the controlled system Σ at a time t. A real number $\sigma \in (0, A)$ is a *sampling radius* for $\mathcal{S}_\gamma(\Sigma_0, K, A)$ if the following holds for all integers $k \geq 0$: for every state $x(kT) \in \rho(\sigma)$, there is an input signal $u_{x(kT)} \in U(K, W, \kappa)$ that takes every system $\Sigma \in \mathcal{S}_\gamma(\Sigma_0, K, A)$ to a state $x((k+1)T) \in \rho(\sigma)$, while maintaining $|x(t)| \leq A$ at all times $t \in [kT, (k+1)T]$. □

We discuss the existence of sampling radii in Section 5 below. For now, we turn to some technical considerations in preparation for proving existence of optimal robust controllers.

4.1 More on input signals

We proceed to examine the operation of the controller C of Figure 1 during the first sampling interval $[0, T]$. Let $\sigma > 0$ be a sampling radius of the family $\mathcal{S}_\gamma(\Sigma_0, K, A)$, and let $x(0) = x_0 \in \rho(\sigma)$ be the initial state. In response to x_0, the controller C generates a signal $u \in U(K, W, \kappa)$ as input to the controlled system Σ. Because σ is a sampling radius (Definition 4.1), the state of Σ must return to $\rho(\sigma)$ at the end of the sampling interval $[0, T]$, without exceeding the amplitude bound A along the way. Therefore, u must be in the family

$$U(A, \Sigma, x_0) := \left\{ u \in U(K, W, \kappa) : \begin{array}{l} \sup_{t \in [0,T]} |\Sigma(x_0, u, t)| \leq A \\ \text{and } \Sigma(x_0, u, T) \in \rho(\sigma) \end{array} \right\}. \tag{25}$$

As it is not known which member of $\mathcal{S}_\gamma(\Sigma_0, K, A)$ the controlled system Σ is, the input signal u must be compatible with all members $\Sigma \in \mathcal{S}_\gamma(\Sigma_0, K, A)$. Therefore, the class of signals that C may produce in response to x_0 is

$$U(A, \gamma, x_0) = \bigcap_{\Sigma \in \mathcal{S}_\gamma(\Sigma_0, K, A)} U(A, \Sigma, x_0). \tag{26}$$

4.2 Continuity and compactness

The next statement brings to light an important continuity feature of members of the family of systems $\mathcal{S}_\gamma(\Sigma_0, K, A)$.

Lemma 4.2: *Let* Σ *be a member of the family* $\mathcal{S}_\gamma(\Sigma_0, K, A)$ *with initial state* $x_0 \in \rho(\sigma)$. *Let* $\{u_i\}_{i=1}^\infty \subseteq U(A, \gamma, x_0)$ *be a sequence of input signals that is uniformly pointwise convergent to the input signal* $u \in U(K, W, \kappa)$. *Then, the sequence* $\{\Sigma(x_0, u_i, t)\}_{i=1}^\infty$

converges to $\Sigma(x_0, u, t)$ at every time $t \geq 0$. Moreover, this convergence is uniform in time.

Proof: Recall that members of $\mathcal{S}_\gamma(\Sigma_0, K, A)$ are restricted to inputs bounded by K and to states bounded by A, so we can use (20) and (23). Denoting $B' := B + \gamma$, we have

$$|\partial f(t, x, u)/\partial(x, u)| \leq B' \tag{27}$$

for all $(t, x, u) \in R^+ \times [-A, A]^n \times [-K, K]^m$. By the mean value theorem,

$$f(t, x, u) - f(0, 0, 0) = \partial f(t, c(t))/\partial(x, u) \begin{pmatrix} x \\ u \end{pmatrix}, \tag{28}$$

for some $c(t) \in [-A, A]^n \times [-K, K]^m$. Using (27), we get

$$|f(t, x, u) - f(0, 0, 0)| \leq (n + m)B' \left| \begin{pmatrix} x \\ u \end{pmatrix} \right|. \tag{29}$$

Denote $x(t, u_i) := \Sigma(x_0, u_i, t)$, $x(t, u) := \Sigma(x_0, u, t)$, and

$$x(t, i) := x(t, u) - x(t, u_i), \quad i \in \{1, 2, \ldots\},$$

where $x(0, i) = x_0 - x_0 = 0$ for all $i \geq 1$. Let $t_1, t_2 \geq 0$, $t_1 < t_2$, be times. Using (1) and (28), we get

$$\begin{aligned} x(t_2, i) &= x(t_1, i) + \int_{t_1}^{t_2} [f(s, x(s, u), u(s)) \\ &\quad - f(s, x(s, u_i), u_i(s))]\mathrm{d}s \\ &= x(t_1, i) + \int_{t_1}^{t_2} \partial f(s, c(s))/\partial(x, u) \\ &\quad \times \begin{pmatrix} x(s, u) - x(s, u_i) \\ u(s) - u_i(s) \end{pmatrix} \mathrm{d}s; \end{aligned}$$

here $c(s) \in [-A, A]^n \times [-K, K]^m$ for all s. Then, (29) yields

$$\begin{aligned} &\sup_{s\in[t_1,t_2]} |x(s, i)| \\ &\leq |x(t_1, i)| + \int_{t_1}^{t_2} (n + m)B' \left[\sup_{s\in[t_1,t_2]} |x(s, u) - x(s, u_i)| \right. \\ &\quad \left. + \sup_{s\in[t_1,t_2]} |u(s) - u_i(s)| \right] \mathrm{d}s \\ &\leq |x(t_1, i)| + (n + m)B'(t_2 - t_1) \left[\sup_{s\in[t_1,t_2]} |x(s, i)| \right. \\ &\quad \left. + \sup_{s\in[t_1,t_2]} |u(s) - u_i(s)| \right]. \end{aligned}$$

Rearranging terms and denoting $\eta := t_2 - t_1$, we obtain

$$\begin{aligned} &[1 - (n + m)B'\eta] \sup_{s\in[t_1,t_1+\eta]} |x(s, i)| \\ &\quad \leq |x(t_1, i)| + (n + m)B'\eta \sup_{s\in[t_1,t_1+\eta]} |u(s) - u_i(s)|. \end{aligned} \tag{30}$$

Now, set a time $\tau > 0$, and select η to satisfy $\eta > 0$, $\eta \leq \tau$, $(n + m)B'\eta \leq 1/2$, where $r := \tau/\eta$ is an integer. Then, pick times $t_1 \in [0, \tau - \eta]$, $t_2 := t_1 + \eta$. Inserting into (30) yields

$$\sup_{s\in[t_1,t_1+\eta]} |x(s, i)| \leq 2|x(t_1, i)| + \sup_{s\in[t_1,t_1+\eta]} |u(s) - u_i(s)|. \tag{31}$$

Next, partition $[0, \tau]$ into segments of length η:

$$\{[0, \eta], [\eta, 2\eta], \ldots, [(r - 1)\eta, r\eta]\}.$$

Introduce the scalar quantities

$$\begin{aligned} \zeta_p &:= \sup_{s\in[(p-1)\eta, \quad p\eta]} |x(s, i)|, p = 1, 2, \ldots, r, \\ \zeta_0 &= 0. \end{aligned} \tag{32}$$

Then, using (31) and setting $t_1 = p\eta$, we obtain

$$\begin{aligned} \zeta_{p+1} &\leq 2\zeta_p + \sup_{s\in[p\eta,(p+1)\eta]} |u(s) - u_i(s)|, \\ \zeta_0 &= 0, \quad p = 0, \ldots, r - 1. \end{aligned} \tag{33}$$

Further, by the lemma's statement, $\{u_i\}_{i=1}^\infty$ is uniformly pointwise convergent to u; so, for every $\varepsilon > 0$, there is an integer $N \geq 1$ such that $\sup_{s\in[0,\tau]} |u(s) - u_i(s)| < \varepsilon$ for all $i \geq N$. Then, for $i \geq N$, we obtain from (33) that

$$\begin{aligned} \zeta_{p+1} &\leq 2\zeta_p + \varepsilon, \quad p = 1, \ldots, r - 1, \\ \zeta_0 &= 0. \end{aligned}$$

This linear recursion yields

$$\zeta_p \leq \varepsilon \sum_{q=0}^{r-1} 2^q, \quad p = 0, \ldots, r - 1. \tag{34}$$

Then, since $\varepsilon > 0$ can be taken arbitrarily small by increasing N, it follows by (34) and (32) that $\sup_{s\in[0,\tau]} |x(s, i)| \to 0$ as $i \to \infty$. This concludes our proof. ■

Lemma 4.2 is an important step toward proving the existence of optimal robust controllers; we prove the existence of such controllers in Section 5. For now, we proceed to show that the set of input signals $U(A, \gamma, x_0)$ of (26) is compact.

Lemma 4.3: *The set of input signals $U(A, \gamma, x_0)$ of (26) is uniformly pointwise compact.*

Proof: Let $\Sigma \in \mathcal{S}_\gamma(\Sigma_0, K, A)$ be a system with initial state $x_0 \in \rho(\sigma)$, and refer to $U(A, \Sigma, x_0)$ of (25). Clearly, if $U(A, \Sigma, x_0)$ is empty, it is compact. Otherwise, $U(A, \Sigma, x_0)$ is not empty; let $\{u_i\}_{i=1}^\infty \subseteq U(A, \Sigma, x_0)$ be a sequence. By Lemma 2.5, there is a subsequence $\{u_{i_k}\}_{k=1}^\infty$ that is uniformly pointwise convergent to a member $u \in U(K, W, \kappa)$. Then, $\{\Sigma(x_0, u_{i_k}, t)\}_{k=1}^\infty$ converges to $\Sigma(x_0, u, t)$ uniformly in time by Lemma 4.2. Also, since $\{u_i\}_{i=1}^\infty \subseteq U(A, \Sigma, x_0)$, it follows by (25) that $\Sigma(x_0, u_{i_k}, T) \in \rho(\sigma)$ and $|\Sigma(x_0, u_{i_k}, t)| \leq A$ for all $t \in [0, T]$ and all $k \geq 1$. Considering that $[-A, A]^n$ and $\rho(\sigma)$ are closed in R^n, and $\Sigma(x_0, u_{i_k}, t) \to \Sigma(x_0, u, t)$, it follows that $|\Sigma(x_0, u, t)| \leq A$ for all $t \in [0, T]$ and $\Sigma(x_0, u, T) \in \rho(\sigma)$; whence, $u \in U(A, \Sigma, x_0)$ and $U(A, \Sigma, x_0)$ is uniformly pointwise compact. But then, $U(A, \gamma, x_0)$ is an intersection of compact sets by (26), and hence is compact. This concludes our proof. ■

4.3 Inter-sample tracking errors

Consider a controlled system Σ with initial state $x_0 \in \rho(\sigma)$, and recall that the target state is the origin $x = 0$. Then, for a specific input signal $u \in U(A, \gamma, x_0)$, the supremal inter-sample tracking error over the sampling interval $[0, T]$ and over all members $\Sigma \in \mathcal{S}_\gamma(\Sigma_0, K, A)$ is

$$\ell(\sigma, K, A, \gamma, T, x_0, u) := \sup_{\substack{\Sigma \in \mathcal{S}_\gamma(\Sigma_0, K, A)\\ t \in [0,T]}} |\Sigma(x_0, u, t)|_2^2 \qquad (35)$$

The infimal inter-sample tracking error over all inputs $u \in U(A, \gamma, x_0)$ is

$$\ell^*(\sigma, K, A, \gamma, T, x_0) = \inf_{u \in U(A,\gamma,x_0)} \ell(\sigma, K, A, \gamma, T, x_0, u) \quad (36)$$

where $\ell^*(\sigma, K, A, \gamma, T, x_0) := \infty$ if $U(A, \gamma, x_0)$ is empty. Now, there are two questions: (i) when is $U(A, \gamma, x_0)$ not empty; and (ii) if it is not empty, is there an input signal $u^*(x_0)$ that achieves the infimum (36), namely,

$$\ell^*(\sigma, K, A, \gamma, T, x_0) = \ell(\sigma, K, A, \gamma, T, x_0, u^*(x_0)). \qquad (37)$$

If a signal $u^*(x_0)$ exists, it forms an optimal robust solution to Problem 1.1(i), guiding every member $\Sigma \in \mathcal{S}_\gamma(\Sigma_0, K, A)$ from the initial state x_0 to the end of the sampling interval $[0, T]$ with minimal inter-sample tracking error. Accordingly, an optimal robust tracking controller C of Figure 1 delivers the signal $u^*(x_0)$ as input to the controlled system Σ, after having received the initial state x_0 from the sampler at $t = 0$. Such controller C is robust since $\mathcal{S}_\gamma(\Sigma_0, K, A)$ represents modelling errors and uncertainties that may affect the controlled system.

The quantity $\ell^*(\sigma, K, A, \gamma, T, x_0)$ represents the infimal inter-sample tracking error from the initial state $x_0 \in \rho(\sigma)$. The lower bound on inter-sample tracking errors over all permissible initial states $x_0 \in \rho(\sigma)$ is then

$$\ell^*(\sigma, K, A, \gamma, T) = \sup_{x_0 \in \rho(\sigma)} \ell^*(\sigma, K, A, \gamma, T, x_0).$$

These facts lead to the following restatement of Problem 1.1.

Problem 4.4: Refer to (36) and (37).

(i) Find conditions under which there are optimal robust input signals $u^*(x_0)$ for every initial state $x_0 \in \rho(\sigma)$.
(ii) If $u^*(x_0)$ exists, find signals that approximate optimal performance and are easy to design and implement. □

We show in Section 6 that optimal input signals $u^*(x_0)$ exist, as long as the nominal controlled system Σ_0 satisfies a certain controllability condition discussed in Section 5. We show in Section 7 that optimal performance can be approximated by pseudo bang-bang controllers – controllers that produce pseudo bang-bang signals. Such controllers are relatively easy to design and implement.

5. Constrained controllability

Constrained controllability affirms the existence of input signals that drive a system to the vicinity of the origin, without violating input and state amplitude constraints (Choi & Hammer, 2019).

Definition 5.1: Let $\Sigma \in \mathcal{S}_\gamma(\Sigma_0, K, A)$ be a system operated with sampling period $T > 0$, and let $\sigma > 0$ be a real number. Then, Σ is (K, A, σ, T)-*controllable* if there is a number $\sigma' \in (0, \sigma)$ such that, for every initial state $x \in \rho(\sigma)$, there is an input signal $u_x \in U(K, W, \kappa)$ satisfying $\Sigma(x, u_x, T) \in \rho(\sigma')$ and $|\Sigma(x, u_x, t)| \leq A$ for all $t \in [0, T]$. □

Note that (K, A, σ, T)-controllability is close to being a necessary condition for tracking in a periodic environment. Indeed, Definition 4.1 requires that it be possible to guide the controlled system Σ from a state in $\rho(\sigma)$ at $t = 0$ to a state in $\rho(\sigma)$ at $t = T$, without violating input and state amplitude constraints; (K, A, σ, T)-controllability adds a contractive requirement: at the end of the sampling interval, the state must be in $\rho(\sigma')$, where $\sigma' < \sigma$. Accordingly, (K, A, σ, T)-controllability is a slightly stronger requirement than the existence of a sampling radius. As shown later, this allows accommodation of uncertainties.

In practice, constrained controllability can often be deduced from physical characteristics of the controlled system and its performance limitations.

The following are established mathematical facts (e.g. Willard, 2004; Zeidler, 1985).

Theorem 5.2: (i) *A continuous functional is lower semi-continuous.*
(ii) *Let S and V be topological spaces. Assume that, for every member $a \in V$, there is a lower semi-continuous functional $f_a : S \to R$. If $\sup_{a \in V} f_a(s)$ exists at every point $s \in S$, then the functional $f(s) := \sup_{a \in V} f_a(s)$ is lower semi-continuous on S.*
(iii) *The Weierstrass Theorem: A lower semi-continuous functional attains a minimum in a compact set.* □

We have seen in Lemma 4.2 that the response $\Sigma(x_0, u, t)$ is a uniformly continuous function of the input signal u over the domain $U(A, \gamma, x_0)$. This implies that the functional $|\Sigma(x_0, u, t)|_2^2$ is similarly continuous as a function of u, since the square of a continuous function is continuous. Together with (35) and Theorem 5.2(ii), this yields the following.

Corollary 5.3: *The functional $\ell(\sigma, K, A, \gamma, T, x_0, u)$ of (35) is a lower semi-continuous functional of the input signal u over the domain $U(A, \gamma, x_0)$.* □

Next, we show that (K, A, σ, T)-controllability of the nominal controlled system Σ_0 entails (K, A, σ, T)-controllability of all members of the family $\mathcal{S}_\gamma(\Sigma_0, K, A)$, as long as the uncertainty parameter γ is not too large (Choi & Hammer, 2020 includes a similar result for input-affine systems).

Proposition 5.4: *Let $K, A_0, \sigma > 0$ be real numbers and let $T > 0$ be the sampling period. If the nominal system Σ_0 is (K, A_0, σ, T)-controllable, then, for every real number $A > A_0$, there is an uncertainty parameter $\gamma > 0$ for which the entire family $\mathcal{S}_\gamma(\Sigma_0, K, A)$ is (K, A, σ, T)-controllable.*

Proof: Let $x_0 \in \rho(\sigma)$ be an initial state of the nominal system Σ_0, and let $\gamma > 0$ be an uncertainty parameter.

As Σ_0 is (K, A_0, σ, T)-controllable, there is an input signal $u \in U(A_0, \Sigma_0, x_0)$ of (25) for which $\Sigma_0(x_0, u, T) \in \rho(\sigma')$, where $\sigma' \in (0, \sigma)$. Now, consider a member $\Sigma \in \mathcal{S}_\gamma(\Sigma_0, K, A)$ with the same initial state x_0 and the same input signal u. Denote $x'(t) := \Sigma(x_0, u, t)$, $x(t) := \Sigma_0(x_0, u, t)$, and $\xi(t) = x'(t) - x(t)$; then $\xi(0) = x_0 - x_0 = 0$. Further, let $t_1, t_2 \in [0, T]$, $t_1 < t_2$, be times, and let $t \in [t_1, t_2]$. By (1), we have

$$\xi(t) = \xi(t_1) + \int_{t_1}^{t} \left[f_0(x'(s), u(s)) + f_\gamma(s, x'(s), u(s))\right] \mathrm{d}s - \int_{t_1}^{t} f_0(x(s), u(s))\mathrm{d}s.$$

Then, since Σ_0 and Σ have the same u, x_0, and state dimension n, the mean value theorem together with (20), (23), and (24) yields

$$\begin{aligned}|\xi(t)| &\leq |\xi(t_1)| + \left|\int_{t_1}^{t} \left[(\partial f_0(c)/\partial x)\, \xi(s) + (\partial f_\gamma(s, c'(s))/\partial x)x'(s) + f_\gamma(s, 0, 0)\right] \mathrm{d}s\right| \\ &\leq |\xi(t_1)| + \int_{t_1}^{t} \left[nB|\xi(s)| + n\gamma|x'(s)| + \gamma\right] \mathrm{d}s,\end{aligned}$$

so that

$$\sup_{s\in[t_1,t_2]} |\xi(s)| \leq |\xi(t_1)| + nB \sup_{s\in[t_1,t_2]} |\xi(s)|(t_2 - t_1) + n\gamma A(t_2 - t_1) + \gamma(t_2 - t_1).$$

This leads to the inequality

$$[1 - nB(t_2 - t_1)] \sup_{s\in[t_1,t_2]} |\xi(s)| \leq |\xi(t_1)| + \gamma(nA + 1)(t_2 - t_1). \tag{38}$$

Next, select $\eta > 0$ satisfying $nB\eta \leq 1/2$ and for which $p := T/\eta$ is an integer. Taking $t_2 = t_1 + \eta$ in (38), we get

$$\sup_{t_1\leq s\leq t_1+\eta} |\xi(s)| \leq 2|\xi(t_1)| + 2\gamma\eta(nA + 1). \tag{39}$$

Now, create the partition

$$[0, T] = \{[0, \eta], [\eta, 2\eta], \ldots, [(p - 1)\eta, T]\},$$

and set $t_1 := i\eta, i \in \{0, 1, \ldots, p - 1\}$. Then, (39) yields

$$\sup_{i\eta\leq s\leq(i+1)\eta} |\xi(s)| \leq 2|\xi(i\eta)| + 2\gamma\eta(nA+1), \quad i = 0, \ldots, p-1$$

$$\xi(0) = 0.$$

From here, properties of linear recursions yield the inequality

$$\sup_{0\leq s\leq T} |\xi(s)| \leq \gamma\eta(nA + 1) \sum_{i=1}^{p} 2^i = \gamma\eta(nA + 1)2(2^p - 1). \tag{40}$$

Denote $\varepsilon := A - A_0$. Then, by (40), it follows that any uncertainty parameter $\gamma' > 0$ that satisfies the inequality

$$\gamma' < \varepsilon/\left[\eta(nA + 1)2(2^p - 1)\right]$$

assures that the response from x_0 is bounded by A for all $t \in [0, T]$.

Next, by Definition 5.1 of (K, A_0, σ, T)-controllability, at the end of the sampling interval, the state of Σ_0 satisfies $x(T) \in \rho(\sigma')$, where $0 < \sigma' < \sigma$. Let $\varepsilon' := (\sigma - \sigma')/2$, and consider an uncertainty parameter $\gamma > 0$ satisfying

$$0 < \gamma < \frac{\min\{\varepsilon, \varepsilon'\}}{\eta(nA + 1)2(2^p - 1)}. \tag{41}$$

Then, setting $\sigma'' := \sigma + \varepsilon'$, and noting that $\sigma'' < \sigma$, it follows by (40) that the response $x'(t)$ to the input signal u from the initial state x_0 satisfies $|x'(t)| \leq A$, $t \in [0, T]$, and $x'(T) \in \rho(\sigma'')$ for every member $\Sigma \in \mathcal{S}_\gamma(\Sigma_0, K, A)$. As the inequalities used above are valid for every initial state $x_0 \in \rho(\sigma)$, it follows that every member $\Sigma \in \mathcal{S}_\gamma(\Sigma_0, K, A)$ is (K, A, σ, T)-controllable. This concludes our proof. ■

Remark 5.5: Values of the uncertainty parameter γ that are compatible with Proposition 5.4 are indicated by (41). □

When Definition 5.1 of (K, A, σ, T)-controllability is combined with Proposition 5.4, we reach the following conclusion.

Corollary 5.6: *Under the conditions of Proposition 5.4, the class of input signals $U(A, \gamma, x_0)$ of (26) is not empty for any initial state $x_0 \in \rho(\sigma)$.* □

Corollary 5.6 forms another step toward proving the existence of optimal solutions to Problem 4.4(i), as discussed in the next section.

6. Existence of optimal robust controllers

In this section, we prove existence of controllers that achieve optimal robust tracking for the broad family of nonlinear systems $\mathcal{S}_\gamma(\Sigma_0, K, A)$. These controllers operate in a sampled-data setting, where access to the controlled system's state is available only at the sampling times $0, T, 2T, \ldots$ Between sampling times, the controller C operates in open-loop, providing an input signal u to the controlled system Σ, based on the state x received at the preceding sampling time. Our task is to prove existence of such an optimal robust input signal that guides all members $\Sigma \in \mathcal{S}_\gamma(\Sigma_0, K, A)$ to track the target state $x_{target} = 0$ as closely as possible during the sampling interval, without violating amplitude constraints.

The existence of optimal controllers is a consequence of two facts: (a) the set of input signals $U(A, \gamma, x_0)$ is compact and not empty (Lemma 4.3 and Corollary 5.6); and (b) the inter-sample tracking error $\ell(\sigma, K, A, \gamma, u, T, x_0)$ is a lower semi-continuous functional of the input signal u (Corollary 5.3). Combining with the Weierstrass theorem, we obtain the following confirmation of the existence of optimal solutions.

Theorem 6.1: *Assume the conditions and notation of Proposition 5.4, (35), and (36). Then, for every initial state $x_0 \in \rho(\sigma)$, there is an optimal robust input signal $u^*(x_0) \in U(A, \gamma, x_0)$ satisfying $\ell^*(\sigma, K, A, \gamma, T, x_0) = \ell(\sigma, K, A, \gamma, T, x_0, u^*(x_0))$.* □

The signal $u^*(x_0)$ of Theorem 6.1 is produced by the controller C of Figure 1 as an optimal input signal to the controlled system Σ during the sampling interval $[0, T]$. This is an optimal robust solution, since $u^*(x_0)$ is optimised over the entire family of systems $\mathcal{S}_\gamma(\Sigma_0, K, A)$. The initial state x_0 is provided to the controller by the sampler of Figure 1.

Next, we show that, after an appropriate shift in time, the optimal input signal $u^*(x_0)$ of Theorem 6.1 is suitable during any sampling interval that starts from the state x_0, not just the sampling interval $[0, T]$. In qualitative terms, this is a consequence of the facts that the nominal system Σ_0 of (21) is time-invariant, and that the bound γ of (23) is a constant; note that the uncertainty function f_γ can be time varying, as long as the bound γ of (23) holds.

In detail, let $x(kT)$ be the state communicated by the sampler at the start of the sampling interval $[kT, (k+1)T]$, $k \in \{0, 1, \ldots\}$. The next statement shows that the optimal input signal during this sampling interval is $u^*(x(kT))$, appropriately shifted in time. In this way, Theorem 6.1 provides a complete solution to the problem of optimal inter-sample tracking in sampled-data systems. We have the following statement, which is the main result of this section.

Theorem 6.2: *Let $\Sigma \in \mathcal{S}_\gamma(\Sigma_0, K, A)$ be the controlled system of Figure 1, and let $x(t)$ be its state at the time t. Under the assumptions and notation of Theorem 6.1, let $u^*(x_0, t)$, $t \in [0, T]$, be an optimal input signal, and let $k \geq 0$ be an integer. Then, $u^*(x(kT), t - kT)$, $t \in [kT, (k+1)T]$, is an optimal input signal during the sampling interval $[kT, (k+1)T]$. It achieves the minimal inter-sample tracking error $\ell^*(\sigma, K, A, \gamma, T, x(kT))$ in this sampling interval.*

Proof: Consider a sampling interval $[kT, (k+1)T]$, $k \in \{0, 1, \ldots\}$, and let $\Sigma \in \mathcal{S}_\gamma(\Sigma_0, K, A)$. Note that, according to Proposition 5.4, the system Σ is (K, A, σ, T)-controllable. As a result, for an initial state $x_0 \in \rho(\sigma)$, the input signal $u^*(x_0, t)$ of Theorem 6.1 takes Σ to a state $x(T) \in \rho(\sigma)$ at the end of the sampling interval $[0, T]$.

Further, according to (20), the recursion function f decomposes into $f = f_0 + f_\gamma$, where the nominal recursion function f_0 has no explicit dependence on the time t. The uncertainty function f_γ may depend on the time t, but, according to (23), the bound on the derivative $|\partial f_\gamma(t, x, u)/\partial(x, u)^\top| \leq \gamma$ is independent of the time t. Now, the arguments leading to Theorem 6.1 depend only on the nominal recursion function f_0, on the bound γ, and on the state at the start of the sampling interval. Consequently, the fact that $x(T) \in \rho(\sigma)$ implies that the time-shifted input signal $u^*(x(T), t - T)$, $t \in [T, 2T]$ is an optimal input signal during the sampling interval $[T, 2T]$, and that $x(2T) \in \rho(\sigma)$.

Induction based on the arguments of the previous paragraph leads to the conclusions that (i) $x(kT) \in \rho(\sigma)$ for all $k \in \{0, 1, \ldots\}$; and (ii) the shifted signal $u^*(x(kT), t - kT)$, $t \in [kT, (k+1)T]$, is an optimal input signal during the interval $[kT, (k+1)T]$. This concludes our proof. ■

Theorem 6.2 shows that the problem of tracking in sampled-data systems has an optimal robust solution under two conditions: first, the nominal system Σ_0 must be (K, A_0, σ, T)-controllable; and, second, the uncertainty parameter γ must not be too large. In the paragraph following Definition 5.1, we have seen that the requirement of (K, A_0, σ, T)-controllability is close to being a necessary condition for the existence of tracking controllers in a periodic sampled-data setting. Thus, (K, A_0, σ, T)-controllability of the nominal controlled system Σ_0 is a tight sufficient condition for the existence of optimal robust solutions to our tracking problem. Note that Theorem 6.2 applies to the general class of nonlinear systems described by differential equations of the form (1).

Theorem 6.2 leads to the following process of constructing optimal robust inter-sample controllers for the control configuration of Figure 1.

6.1 Controller operation (outline)

Assume the conditions and notation of Theorem 6.2.

- At a sampling time kT, $k \in \{0, 1, \ldots\}$, the feedback sampler provides the state $x(kT)$ to the controller C.
- During the sampling interval $[kT, (k+1)T]$, the controller C generates the signal $u^*(x(kT), t - kT)$ as input to the controlled system Σ. This robustly minimises inter-sample tracking errors. □

Generally, the signal $u^*(x(kT))$ of Theorem 6.2 is a vector valued function of the time t and the state vector $x(kT)$. Calculating and implementing such signals may be a challenge. In the next section, we show that optimal performance can be approximated by signals that are easier to calculate and implement.

7. Pseudo bang-bang controllers

By Theorem 6.2, there are optimal robust controllers that achieve minimal inter-sample tracking error for the sampled-data configuration of Figure 1. Yet, optimal controllers may be difficult to design and implement, since they require the calculation and implementation of potentially intricate signals. In the present section, we show that the performance of optimal robust controllers can be approximated by controllers that generate pseudo bang-bang input signals for the controlled system Σ. Such controllers are relatively easy to design and implement, since pseudo bang-bang signals are determined by a list of scalars – their switching times.

We start by listing a formal definition of pseudo bang-bang signals. Recall that a bang-bang signal is a signal whose components switch between the values of K and $-K$ a finite number of times in each finite time interval. Bang-bang signals are members of the family $U(K)$ of Lebesgue measurable functions with amplitude bounded by K, where

$$U(K) = \{u : R^+ \to R^m : |u|_\infty \leq K, u \text{ is measurable}\}. \quad (42)$$

Definition 7.1: A *pseudo-bang-bang signal* u_{pbb} is a member of $U(K, W, \kappa)$ that is derived from a bang-bang signal $u_{bb} \in U(K)$ in three steps: (i) obtain the Fourier transform $\upsilon_{bb}(\omega) := \mathscr{F} u_{bb}$; (ii) multiply $\upsilon_{bb}(\omega)$ by $e^{-\kappa|\omega|}$, where $\kappa > 0$ is a smoothing factor; and, finally, (iii) perform the inverse Fourier transform $u_{pbb}(t) = \mathscr{F}^{-1}(\upsilon_{bb}(\omega)e^{-\kappa|\omega|})$.

A *pseudo bang-bang controller* generates pseudo bang-bang input signals for the controlled system. □

Figure 2 shows a pseudo bang-bang signal and its associated bang-bang signal for smoothing factor $\kappa = 0.001$. Like a bang-bang signal, a pseudo bang-bang signal is determined by a string of scalars – its switching times. Hence, pseudo bang-bang signals are easy to implement. Next is the main result of this section; it shows that pseudo bang-bang signals can approximate the performance of any input signal. The proof of this theorem appears later.

Theorem 7.2: *Let Σ be a member of the family of systems $\mathcal{S}_\gamma(\Sigma_0, K, A)$ with initial state x_0 and sampling period $T > 0$, and let $\kappa, W > 0$ be real numbers. Then, for every input signal $u \in U(K, W, \kappa)$ and for every real number $\varepsilon > 0$, there is a real number $\kappa' > 0$ and a pseudo bang-bang input signal $u^\pm \in U(K+\varepsilon, W, \kappa')$ for which the following are true:*

(i) *$u^\pm$ has a finite number of switchings during $[0, T]$.*
(ii) *The discrepancy between the response to u and the response to $u^\pm$ satisfies the relation $|\Sigma(x_0, u, t) - \Sigma(x_0, u^\pm, t)| < \varepsilon$ for all $t \in [0, T]$.* □

By Theorem 7.2, the amplitudes of the pseudo bang-bang input signal $u^\pm$ and its response $\Sigma(x_0, u^\pm, t)$ may exceed by ε the input bound K and the state bound A of Σ. Yet, since $\varepsilon > 0$ can be selected as small as desired, this excess has minor practical implications. Before proving Theorem 7.2, we address a few preliminary issues, starting with the following comparison.

7.1 Comparing bang-bang and pseudo bang-bang signals

Figure 2 visually illustrates the difference between a bang-bang signal and its associated pseudo bang-bang signal. To further examine this difference, note the obvious fact that a bang-bang signal $u^\pm$ is a combination of pulses of the form depicted in Figure 2(A); the corresponding pseudo bang-bang signal $\tilde{u}^\pm$ is then a combination of signals of the form depicted in Figure 2(B). To obtain a quantitative estimate of the difference between bang-bang and pseudo bang-bang signals, let us examine first one pulse

$$h^\pm(t) = \pi\,[\text{heaviside}(1-t) - \text{heaviside}(-t-1)]\,.$$

The Fourier transform is

$$H^\pm(\omega) = \mathscr{F}h^\pm(t) = (\sin\omega)/\omega.$$

For a smoothing factor κ, the pseudo bang-bang signal is

$$\begin{aligned}\tilde{h}^\pm(t,\kappa) &= \mathscr{F}^{-1}\left(h^\pm(\omega)e^{-\kappa|\omega|}\right)\\ &= \arctan\left(\frac{t+1}{\kappa}\right) - \arctan\left(\frac{t-1}{\kappa}\right). \qquad (43)\end{aligned}$$

To gauge the discrepancy between the pulse and its corresponding pseudo bang-bang signal, we examine the following integral as a function of the smoothing factor κ. (Note that these signals are centred at the origin of the time axis.)

$$E_1(\kappa) := \int_{-5}^{5}\left|h^\pm(s) - \tilde{h}^\pm(s,\kappa)\right| \mathrm{d}s.$$

There seems to be no closed form available for this integral, so we examine it numerically. Figure 3 depicts $E_1(\kappa)$ as a function of $-log_{10}\kappa$ for $\kappa \in [10^{-6}, 10^{-3}]$, showing that $E_1(\kappa)$ tends to zero as κ tends to zero, namely,

$$\lim_{\substack{\kappa\to 0\\ \kappa>0}} E_1(\kappa) = 0. \qquad (44)$$

By (43) and the fact that arctangent is a monotone function in this domain, it follows that $\tilde{h}^\pm(t,\kappa)$ has no 'spikes'; this is also demonstrated in Figure 2(B). As a result, the fact that $E_1(\kappa)$ tends to zero also implies that the discrepancy between the signal amplitudes tends to zero as $\kappa \to 0$, namely, that

$$\lim_{\substack{\kappa\to 0\\ \kappa>0}}\left[\sup_{t\in[-5,5]}\left|h^\pm(t)\right| - \sup_{t\in[-5,5]}\left|\tilde{h}^\pm(t,\kappa)\right|\right] = 0. \qquad (45)$$

More generally, consider a bang-bang signal $u^\pm(t)$ with its associated pseudo bang-bang signal $\tilde{u}^\pm(t,\kappa)$. Focusing on the sampling interval $[0, T]$, define the *discrepancy*

$$E(\kappa) := \int_0^T \left|u^\pm(s) - \tilde{u}^\pm(s,\kappa)\right| \mathrm{d}s. \qquad (46)$$

As $u^\pm$ consists of a finite number of pulses similar to $h^\pm$, we heuristically reach from (44) and (45) the conclusions

$$\lim_{\substack{\kappa\to 0\\ \kappa>0}} E(\kappa) = 0, \quad \text{and}$$

$$\lim_{\substack{\kappa\to 0\\ \kappa>0}}\left[\sup_{t\in[0,T]}\left|u^\pm(t)\right| - \sup_{t\in[0,T]}\left|\tilde{u}^\pm(t,\kappa)\right|\right] = 0.$$

For future reference, we record these in the following.

Conclusion 7.3: Let $u^\pm(t) \in U(K)$ be a bang-bang signal, and let $\tilde{u}^\pm(t,\kappa)$ be the pseudo bang-bang signal associated with $u^\pm(t)$, using the smoothing factor $\kappa > 0$. Then, for every real number $\varepsilon > 0$, there are real numbers $\kappa, W > 0$ such that $E(\kappa) < \varepsilon$ and $\tilde{u}^\pm(t,\kappa) \in U(K+\varepsilon, W, \kappa)$. □

This helps us show in the next subsection that optimal performance is approximated by pseudo bang-bang signals.

7.2 Approximating optimal performance

We need the following auxiliary statement.

Lemma 7.4: *Let $f : R^+ \times R^n \times R^m \to R^n$ be a continuously differentiable function. Then, there is an $n \times (n+m)$ continuous matrix function $D(t, x, u, x', u')$ that satisfies*

$$f(t,x,u) - f(t,x',u') = D(t,x,u,x',u')\begin{pmatrix} x - x' \\ u - u' \end{pmatrix}$$

for all $x, x' \in R^n$ and $u, u' \in R^m$.

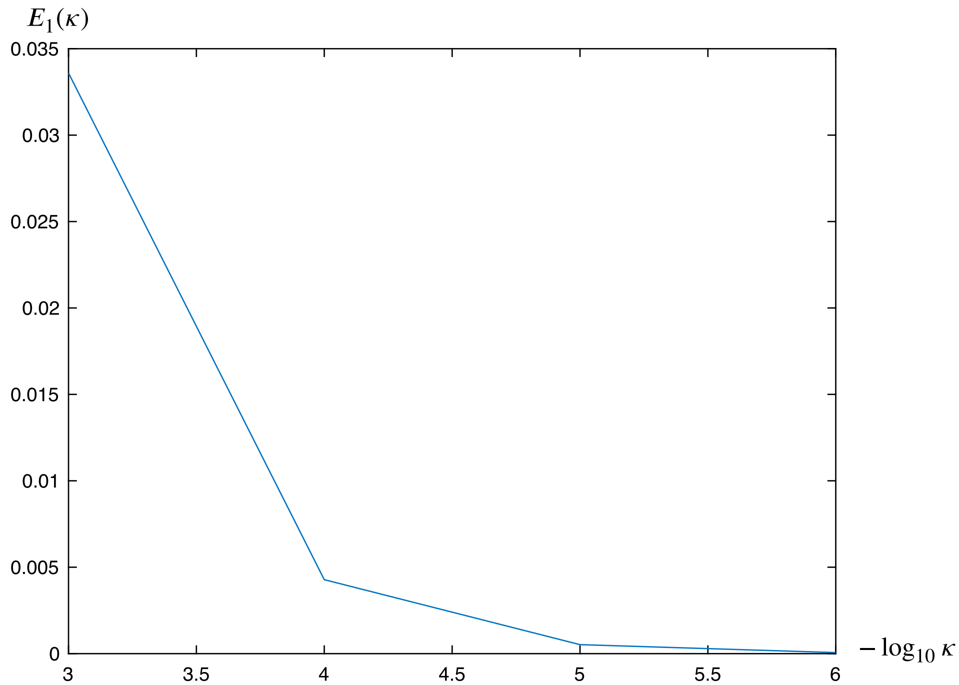


Figure 3. The effect of decreasing κ.

Proof: Consider a continuously differentiable function $g : R^+ \times R^q \to R^p : (t,z) \mapsto g(t,z)$, where $q, p \geq 1$. We show that $g(t,z) - g(t,z') = D(t,z,z')(z-z')$, where $D(t,z,z')$ is a continuous function of its variables. Denote by g^i the ith component of g; we proceed by induction on q. When $q = 1$, the variable z is a scalar, and, for $z \neq z'$ we have

$$D^i(t,z,z') = \frac{g^i(t,z) - g^i(t,z')}{z - z'}, \quad i = 1,2,\ldots,p.$$

Thus, $D^i(t,z,z')$ is a ratio of continuous functions; therefore, it is a continuous function (e.g. Hubbard & Hubbard, 2015). Furthermore, since g is continuously differentiable, this continuity is preserved as $z - z' \to 0$, validating our assertion for $q = 1$.

By induction, assume that $D(t,z,z')$ is continuous for $q \leq r$, where $r \geq 1$ is an integer, and consider the case $q = r+1$. Keep one component of z and z' constant, say component $z^j, z'^j, j \in \{1,2,\ldots,r+1\}$. Then, by the induction assumption, $D(t,z,z')$ is continuous over the remaining r variables, for any values of z^j, z'^j. As this is true for every $j \in \{1,2,\ldots,r+1\}$, and for any values of z^j, z'^j, it follows that $D(t,z,z')$ is a continuous function of z and z' for the dimension $q = r+1$. This proves that $D(t,z,z')$ is a continuous function of its variables for any dimension q. The lemma then follows by replacing g by f; z by $(x,u)^\top$; and z' by $(x',u')^\top$. This concludes our proof. ■

We turn now to a proof of Theorem 7.2.

Proof (of Theorem 7.2): The proof consists of three parts: Part I constructs a bang-bang signal $v^\pm(t)$ over a subinterval of $[0,T]$. Part II, extends $v^\pm(t)$ into a bang-bang signal over the entire sampling interval $[0,T]$. Finally, in Part III, we use this bang-bang signal to build a pseudo bang-bang signal $u^\pm$ that fulfils the requirements of the theorem.

7.2.1 Part I: building a basic bang-bang signal

Let $u = (u^1, u^2, \ldots, u^m)^\top \in U(K,W,\kappa)$ be an input signal of Σ. As $U(K,W,\kappa) \subseteq U(K)$, also $u \in U(K)$. Let $t_1 \in [0,T)$, and let $\eta \in (0, T - t_1]$ and $\lambda \in (0,\eta]$ be real numbers for which the following two ratios are integers:

$$p := T/\eta \quad \text{and} \quad r := \eta/\lambda. \tag{47}$$

Partition the interval $[t_1, t_1+\eta]$ into segments of length λ:

$$\{[t_1, t_1+\lambda], [t_1+\lambda, t_1+2\lambda], \ldots, [t_1+(r-1)\lambda, t_1+\eta\}. \tag{48}$$

Based on this partition, build a bang-bang signal $v^\pm = (v^{\pm,1}, v^{\pm,2}, \ldots, v^{\pm,m})^\top \in U(K)$ as follows. For each $k \in \{0,1,\ldots,r-1\}$ and for each component $v^{\pm,i}, i \in \{1,2,\ldots,m\}$, solve for a point $s_{k_i} \in [t_1+k\lambda, t_1+(k+1)\lambda]$ that satisfies

$$K[2(s_{k_i} - (t_1+k\lambda)) - \lambda] = \int_{t_1+k\lambda}^{t_1+(k+1)\lambda} u^i(\theta)\mathrm{d}\theta; \tag{49}$$

here, u^i is component i of u. There is a solution for s_{k_i}, since $u(t)$ is bounded by K. Then, use s_{k_i} as a switching point for component i of the bang-bang signal $v^\pm$ by setting

$$v^{\pm,i}(t) := \begin{cases} K & \text{for } t \in [t_1+k\lambda, s_{k_i}), \text{ and} \\ -K & \text{for } t \in [s_{k_i}, t_1+(k+1)\lambda), \end{cases} \tag{50}$$

$k = 0,1,\ldots,r-1$, $i = 1,2,\ldots,m$. By (49), the resulting signal $v^\pm$ satisfies the relation

$$\int_{t_1+k\lambda}^{t_1+(k+1)\lambda} [u^i(\theta) - v^{\pm,i}(\theta)]\mathrm{d}\theta = 0 \tag{51}$$

for all $i \in \{1,2,\ldots,m\}$ and all $k \in \{0,1,\ldots,r-1\}$.

To continue, denote $x(t) := \Sigma(x_0, u, t), x^{\pm}(t) = \Sigma(x_0, v^{\pm}, t)$, and

$$\xi(t) := x(t) - x^{\pm}(t). \tag{52}$$

Note that $\xi(0) = x_0 - x_0 = 0$. Also, referring to Lemma 7.4, we use the short-hand notation

$$\begin{aligned} &f(t, x(t), u(t) - f(t, x^{\pm}(t), v^{\pm}(t)) \\ &\quad := a(t)(x(t) - x^{\pm}(t)) + b(t)(u(t) - v^{\pm}(t)), \end{aligned} \tag{53}$$

where $a(t)$ is an $n \times n$ matrix consisting of the first n columns of $D(t, x(t), u(t), x^{\pm}(t), v^{\pm}(t))$, while $b(t)$ is an $n \times m$ matrix consisting of the last m columns of $D(t, x(t), u(t), x^{\pm}(t), v^{\pm}(t))$. By Lemma 7.4, the matrix $D(t, x(t), u(t), x^{\pm}(t), v^{\pm}(t))$ is a continuous function of its variables. As u is a member of $U(K, W, \kappa)$, it is continuous by Lemma 2.1. Also, since $x(t)$ and $x^{\pm}(t)$ are, by (1), integrals of bounded piecewise continuous functions, they are continuous functions of time as well.

We examine the dependence of $a(t)$ and $b(t)$ on $v^{\pm}$. Consider the interval $[t_1 + k\lambda, t_1 + (k+1)\lambda]$, $k \in \{0, 1, \ldots, r-1\}$. By (50), the component $v^{\pm,i}$ is constant over each of the subintervals $[t_1 + k\lambda, s_{k_i})$ and $[s_{k_i}, t_1 + (k+1)\lambda)$. This implies that the limits of $v^{\pm,i}$ from the interior exist at both endpoints of each of these subintervals (although these limits may not be equal on both sides of endpoints). Now, for an integer $k \in \{0, 1, \ldots, r-1\}$, consider the closed intervals

$$\Delta_{k,i,1} := [t_1 + k\lambda, s_{k_i}] \quad \text{and} \quad \Delta_{k,i,2} := [s_{k_i}, t_1 + (k+1)\lambda],$$

$i = 1, 2, \ldots, m$. Using these, fix an integer $k \in \{0, 1, \ldots, r-1\}$, and consider the family of closed subintervals

$$\Phi_k := \left\{ \bigcap_{i=1}^{m} \Delta_{k,i,\phi(i)} : \phi(i) \in \{1, 2\} \right\}.$$

Let α_k be the number of non-empty distinct members of Φ_k; denote these members by $\psi_{k,1}, \psi_{k,2}, \ldots, \psi_{k,\alpha_k}$. Then, $\psi_{k,1}, \psi_{k,2}, \ldots, \psi_{k,\alpha_k}$ have disjoint interiors, and

$$\bigcup_{d=1}^{\alpha_k} \psi_{k,d} = [t_1 + k\lambda, t_1 + (k+1)\lambda]. \tag{54}$$

As a bang-bang signal, all components of $v^{\pm}$ are constant inside $\psi_{k,d}$, $d \in \{1, 2, \ldots, \alpha_k\}$; hence, their limits from the interior of $\psi_{k,d}$ exist at the endpoints of $\psi_{k,d}$.

In view of the previous paragraph, the functions $a(t)$ and $b(t)$ are compositions of functions that are continuous and bounded over the interior of each subinterval $\psi_{k,d}$, $d \in \{1, 2, \ldots, \alpha_k\}$, and their limits from the interior exist at the endpoints of these subintervals. Consequently, $a(t)$ and $b(t)$ are uniformly continuous over $\psi_{k,d}$, $d \in \{1, 2, \ldots, \alpha_k\}$. As the entire sampling interval $[0, T]$ consists of a finite number of such subintervals, this has two implications:

(1) $a(t)$ and $b(t)$ are bounded over the entire time interval $[0, T]$, namely, there is a real number $N > 0$ satisfying

$$|a(t)| \leq N \quad \text{and} \quad |b(t)| \leq N \quad \text{for all } t \in [0, T]. \tag{55}$$

(2) For every real number $\delta > 0$, there is a real number $\beta(\delta) > 0$ such that

$$|a(t) - a(t')| < \delta \quad \text{and} \quad |b(t) - b(t')| < \delta \tag{56}$$

for all $t, t' \in \psi_{k,d}$ satisfying $|t - t'| < \beta(\delta)$, for all $d \in \{1, 2, \ldots, \alpha_k\}$ and all $k \in \{1, 2, \ldots, r-1\}$.

Now, returning to (52), letting $t \in [t_1, t_1 + \eta]$, substituting (53) into (1), and integrating, we get

$$\begin{aligned} \sup_{\theta \in [t_1, t_1+\eta]} |\xi(\theta)| \leq |\xi(t_1)| + \sup_{t \in [t_1, t_1+\eta]} \Bigg| \int_{t_1}^{t} \Big[a(\theta)(x(\theta) \\ - x^{\pm}(\theta)) + b(\theta)(u(\theta) - v^{\pm}(\theta)) \Big] \mathrm{d}\theta \Bigg|. \end{aligned} \tag{57}$$

Using (55), we obtain

$$\begin{aligned} \sup_{\theta \in [t_1, t_1+\eta]} |\xi(\theta)| &\leq |\xi(t_1)| + Nn\eta \sup_{\theta \in [t_1, t_1+\eta]} |\xi(\theta)| \\ &\quad + \sup_{t \in [t_1, t_1+\eta]} \left| \int_{t_1}^{t} b(\theta)(u(\theta) - v^{\pm}(\theta)) \mathrm{d}\theta \right| \end{aligned}$$

so that

$$\begin{aligned} &(1 - Nn\eta) \sup_{\theta \in [t_1, t_1+\eta]} |\xi(\theta)| \\ &\quad \leq |\xi(t_1)| + \sup_{t \in [t_1, t_1+\eta]} \left| \int_{t_1}^{t} b(\theta)(u(\theta) - v^{\pm}(\theta)) \mathrm{d}\theta \right|. \end{aligned}$$

Now, choose a value of η such that

$$Nn\eta \leq 1/2. \tag{58}$$

This yields the inequality

$$\begin{aligned} \sup_{\theta \in [t_1, t_1+\eta]} |\xi(\theta)| &\leq 2\, |\xi(t_1)| + 2 \sup_{t \in [t_1, t_1+\eta]} \\ &\quad \times \left| \int_{t_1}^{t} b(\theta)(u(\theta) - v^{\pm}(\theta)) \mathrm{d}\theta \right|. \end{aligned} \tag{59}$$

Consider the supremum on the right side of (59) and denote by $q(t) \in \{0, 1, \ldots, r-1\}$ the integer satisfying $t \in [q(t)\lambda, (q(t)+1)\lambda]$. From (50) we obtain

$$\begin{aligned} &\sup_{t \in [t_1, t_1+\eta]} \left| \int_{t_1}^{t} b(\theta)(u(\theta) - v^{\pm}(\theta)) \mathrm{d}\theta \right| \\ &= \sup_{t \in [t_1, t_1+\eta]} \Bigg| \sum_{i=0}^{q(t)-1} \int_{t_1+i\lambda}^{t_1+(i+1)\lambda} b(\theta)(u(\theta) - v^{\pm}(\theta)) \mathrm{d}\theta \\ &\quad + \int_{t_1+q(t)\lambda}^{t} b(\theta)(u(\theta) - v^{\pm}(\theta)) \mathrm{d}\theta \Bigg| \\ &\leq \sup_{t \in [t_1, t_1+\eta]} \Bigg| \Bigg[\sum_{i=0}^{q(t)-1} \int_{t_1+i\lambda}^{t_1+(i+1)\lambda} \{ b(t_1 + i\lambda) - b(t_1 + i\lambda) \\ &\quad + b(\theta) \} \left(u(\theta) - v^{\pm}(\theta) \right) \mathrm{d}\theta \Bigg] \Bigg| \end{aligned}$$

$$+ \sup_{t\in[t_1+q(t)\lambda, t_1+(q(t)+1)\lambda]} \left| \int_{t_1+q(t)\lambda}^{t} b(\theta)\left(u(\theta)-v^{\pm}(\theta)\right) \mathrm{d}\theta \right|$$

$$\leq \sup_{t\in[t_1,t_1+\eta]} \left| \sum_{i=0}^{q(t)-1} b(t_1+i\lambda) \int_{t_1+i\lambda}^{t_1+(i+1)\lambda} \left(u(\theta)-v^{\pm}(\theta)\right) \mathrm{d}\theta \right|$$

$$+ \sup_{t\in[t_1,t_1+\eta]} \left| \sum_{i=0}^{q(t)-1} \int_{t_1+i\lambda}^{t_1+(i+1)\lambda} [b(\theta)-b(t_1+i\lambda)] \right.$$

$$\left. \times \left(u(\theta)-v^{\pm}(\theta)\right) \mathrm{d}\theta \right|$$

$$+ \sup_{t\in[t_1+q(t)\lambda, t_1+(q(t)+1)\lambda]} \left| \int_{t_1+q(t)\lambda}^{t} b(\theta)\left(u(\theta)-v^{\pm}(\theta)\right) \mathrm{d}\theta \right|$$

Taking advantage of (51), we get

$$\sup_{t\in[t_1,t_1+\eta]} \left| \int_{t_1}^{t} b(\theta)(u(\theta)-v^{\pm}(\theta))\mathrm{d}\theta \right|$$
$$\leq \sum_{i=0}^{q(t)-1} \sup \left| \int_{t_1+i\lambda}^{t_1+(i+1)\lambda} [b(\theta)-b(t_1+i\lambda)][u(\theta)-v^{\pm}(\theta)]\mathrm{d}\theta \right|$$
$$+ \sup_{t\in[t_1+q(t)\lambda, t_1+(q(t)+1)\lambda]} \left| \int_{t_1+q(t)\lambda}^{t} b(\theta)\left(u(\theta)-v^{\pm}(\theta)\right) \mathrm{d}\theta \right| \tag{60}$$

Consider for a moment one of the integrals in the summation in (60). By (54), we can split this integral into a sum of integrals over the sub-intervals $\psi_{k,d}$, $d \in \{1, 2, \ldots, \alpha_k\}$:

$$\left| \int_{t_1+i\lambda}^{t_1+(i+1)\lambda} [b(\theta)-b(t_1+i\lambda)][u(\theta)-v^{\pm}(\theta)]\mathrm{d}\theta \right|$$
$$= \left| \sum_{d=1}^{\alpha_k} \int_{\psi_{k,d}} [b(\theta)-b(t_1+i\lambda)][u(\theta)-v^{\pm}(\theta)]\mathrm{d}\theta \right|$$
$$\leq \sum_{d=1}^{\alpha_k} \int_{\psi_{k,d}} m\,|[b(\theta)-b(t_1+i\lambda)]|\,\left|[u(\theta)-v^{\pm}(\theta)]\right| \mathrm{d}\theta. \tag{61}$$

In (56), choose $\lambda \leq \beta(\delta)$. Denote by $L(\psi_{k,d})$ $(\leq \lambda)$ the length of $\psi_{k,d}$. Then, the last integral satisfies

$$\int_{\psi_{k,d}} m\,|[b(\theta)-b(t_1+i\lambda)]|\,\left|[u(\theta)-v^{\pm}(\theta)]\right| \mathrm{d}\theta$$
$$\leq mL(\psi_{k,d})\delta 2K.$$

Substituting into (61) and considering that $\sum_{d=1}^{\alpha_k} L(\psi_{k,d}) = \lambda$, we get

$$\left| \int_{t_1+i\lambda}^{t_1+(i+1)\lambda} [b(\theta)-b(t_1+i\lambda)][u(\theta)-v^{\pm}(\theta)]\mathrm{d}\theta \right| \leq m\lambda\delta 2K. \tag{62}$$

Regarding the last integral of (60), it follows from (55) that

$$\left| \int_{t_1+q(t)\lambda}^{t} b(\theta)\left(u(\theta)-v^{\pm}(\theta)\right) \mathrm{d}\theta \right|$$
$$\leq m \int_{t_1+q(t)\lambda}^{t} |b(\theta)|\left|\left(u(\theta)-v^{\pm}(\theta)\right)\right| \mathrm{d}\theta \leq Nm2K\lambda. \tag{63}$$

Inserting (62) and (63) into (60), we obtain

$$\sup_{t\in[t_1,t_1+\eta]} \left| \int_{t_1}^{t} b(\theta)\left(u(\theta)-v^{\pm}(\theta)\right) \mathrm{d}\theta \right|$$
$$\leq q(t)m\lambda\delta 2K + Nm2K\lambda.$$

Further, since η is the entire length of the interval $[t_1, t_1+\eta]$, we have that $q(t)\lambda \leq \eta$, and we get

$$\sup_{t\in[t_1,t_1+\eta]} \left| \int_{t_1}^{t} b(\theta)\left(u(\theta)-v^{\pm}(\theta)\right) \mathrm{d}\theta \right| \leq 2mK\delta\eta + 2mKN\lambda.$$

Substituting this into (59) yields

$$\sup_{\theta\in[t_1,t_1+\eta]} |\xi(\theta)| \leq 2\,|\xi(t_1)| + 4mK(\delta\eta + N\lambda). \tag{64}$$

Let $\varepsilon' > 0$; select $\delta < \varepsilon'/(8mK\eta)$ and $\lambda < \min\{\beta(\delta), \varepsilon'/(8mKN)\}$. Inserting into (64), we get

$$\sup_{\theta\in[t_1,t_1+\eta]} |\xi(\theta)| \leq 2\,|\xi(t_1)| + \varepsilon'. \tag{65}$$

7.2.2 Part II: extending the bang-bang signal

We extend $v^{\pm}(t)$ from the interval $[t_1, t_1+\eta]$ to the interval $[0, T]$. First, referring to (47) and (58), create the partition

$$[0, T] = \left\{[0, \eta], [\eta, 2\eta], \ldots, [(p-1)\eta, T]\right\}.$$

Set $t_1 := i\eta$, $i \in \{0, \ldots, p-1\}$; using the process leading to (65), create over $[i\eta, (i+1)\eta]$ a bang-bang signal $v^{\pm}(i, t)$ satisfying (65). Assemble a bang-bang signal over $[0, T]$:

$$v^{\pm}(x_0, t) = \begin{cases} v^{\pm}(0, t) & \text{for } t \in [0, \eta], \\ v^{\pm}(1, t) & \text{for } t \in (\eta, 2\eta], \\ \vdots & \\ v^{\pm}(p-1, t) & \text{for } t \in ((p-1)\eta, T]. \end{cases}$$

As $\xi(0) = 0$, we obtain from (65) the relation

$$\sup_{\theta\in[i\eta,(i+1)\eta]} |\xi(\theta)| \leq 2\,|\xi(i\eta)| + \varepsilon',$$
$$\xi(0) = 0.$$

By properties of linear recursions, this yields

$$\sup_{\theta\in[0,T]} |\xi(\theta)| \leq \left(\sum_{i=0}^{p-1} 2^i\right) \varepsilon'.$$

Referring to the number $\varepsilon > 0$ of the theorem, select $\varepsilon' < \varepsilon/(2\sum_{i=0}^{p-1} 2^i)$, $\varepsilon' > 0$. As $\xi(t) = x(t) - x^{\pm}(t)$, this yields

$$\sup_{\theta\in[0,T]} \left|x(\theta) - x^{\pm}(\theta)\right| < \varepsilon/2. \tag{66}$$

7.2.3 Part III: deriving a pseudo bang-bang signal

We use $v^{\pm}(x_0, t)$ to derive a pseudo bang-bang signal $u^{\pm}(x_0, t)$ that fulfils the theorem's requirements, as follows. Let $\chi, \kappa' > 0$ be real numbers; set $V(\omega) = \mathscr{F}(v^{\pm}(x_0, t))$ and $u^{\pm}(x_0, t) = \mathscr{F}^{-1}(V(\omega)e^{-\kappa'|\omega|})$. According to Conclusion 7.3, κ' can be selected so that

$$\int_0^T |v^{\pm}(x_0, \theta) - u^{\pm}(x_0, \theta)| \mathrm{d}\theta < \chi, \tag{67}$$

where $u^{\pm}(x_0) \in U(K + \chi, W, \kappa')$.

To proceed, denote $z(t) := \Sigma(x_0, u^{\pm}, t)$, set

$$\zeta(t) := x^{\pm}(t) - z(t), \tag{68}$$

and note that $\zeta(0) = 0$. Let $\mu > 0$ be a number to be selected later for which $c := T/\mu$ is an integer. Build the partition

$$[0, T] = \{[0, \mu], [\mu, 2\mu], \ldots, [(c-1), T]\}. \tag{69}$$

Adapting (57) for our present situation yields

$$\begin{aligned}
\sup_{\theta \in [t_1, t_1+\mu]} |\zeta(\theta)| &\le |\zeta(t_1)| \\
&+ \sup_{t \in [t_1, t_1+\mu]} \left| \int_{t_1}^{t_1+\mu} \left[a(\theta)(x^{\pm}(\theta) - z(\theta)) + b(\theta)(v^{\pm}(\theta) \right.\right. \\
&\left.\left. -u^{\pm}(\theta)) \right] \mathrm{d}\theta \right|.
\end{aligned}$$

Employing (55) and (67), we get

$$\begin{aligned}
&\sup_{\theta \in [t_1, t_1+\mu]} |\zeta(\theta)| \\
&\le |\zeta(t_1)| + Nn \sup_{\theta \in [t_1, t_1+\mu]} |\zeta(\theta)| \mu \\
&\quad + Nm \sup_{t \in [t_1, t_1+\mu]} \int_{t_1}^{t_1+\mu} |(v^{\pm}(\theta) - u^{\pm}(\theta)| \mathrm{d}\theta \\
&\le |\zeta(t_1)| + Nn\mu \sup_{\theta \in [t_1, t_1+\mu]} |\zeta(\theta)| + Nm\chi,
\end{aligned}$$

or

$$(1 - Nn\mu) \sup_{\theta \in [t_1, t_1+\mu]} |\zeta(\theta)| \le |\zeta(t_1)| + Nm\chi. \tag{70}$$

Now, choose μ so that $Nn\mu \le 1/2$. Then, (70) becomes

$$\sup_{\theta \in [t_1, t_1+\mu]} |\zeta(\theta)| \le 2|\zeta(t_1)| + 2Nm\chi.$$

Referring to (69) and using $t_1 = i\mu$, the last inequality yields

$$\sup_{\theta \in [i\mu, (i+1)\mu]} |\zeta(\theta)| \le 2|\zeta(i\mu)| + 2Nm\chi, \quad i = 1, 2, \ldots, c-1,$$

$$\zeta(0) = 0.$$

Invoking properties of linear recursions, this yields

$$\sup_{\theta \in [0,T]} |\zeta(\theta)| \le \left(\sum_{i=1}^{c} 2^i \right) Nm\chi.$$

Next, choose the smoothing factor κ' so that $\chi < \varepsilon/(2Nm \sum_{i=1}^{c} 2^i)$. Then, the last inequality yields

$$\sup_{\theta \in [0,T]} |\zeta(\theta)| = \sup_{\theta \in [0,T]} |x^{\pm}(\theta) - z(\theta)| < \varepsilon/2. \tag{71}$$

Combining (66) with (71), we get $|x(t) - z(t)| \le |\xi(t)| + |\zeta(t)| < \varepsilon$, so that the pseudo bang-bang signal $u^{\pm}$ satisfies the theorem's requirements. This concludes our proof. ■

By Theorem 7.2, the response to any input signal in $U(K, W, \kappa)$ can be approximated as closely as desired by the response to a pseudo bang-bang signal. An argument similar to the one used to prove Theorem 6.2 yields the following consequence of Theorem 7.2.

Theorem 7.5: *Let $u^{\pm} \in U(K + \varepsilon, W, \kappa')$ be the signal of Theorem 7.2. Then, under the conditions of Theorem 7.2, the following is true. The discrepancy between the response to u and the response to $u^{\pm}$ satisfies $|\Sigma(x_0, u, t) - \Sigma(x_0, u^{\pm}(t - kT), t)| < \varepsilon$ for all $t \in [kT, (k+1)T]$, $k \in \{0, 1, 2, \ldots\}$.* □

In particular, Theorem 7.5 implies that optimal performance can be closely approximated by pseudo bang-bang signals, since u of the theorem can be taken to be the optimal input signal $u^*(x_0)$ of Theorem 6.2. Consequently, optimal performance $x^*(t)$ can be approximated within ε by the response generated by a pseudo bang-bang input signal, where $\varepsilon > 0$ can be as small as desired. Thus, optimal performance can be approximated by pseudo bang-bang signals. The next statement, which is the main result of this section, reflects this fact.

Corollary 7.6: *Assume the conditions and notation of Proposition 5.4 and Theorems 6.2 and 7.2. Let $x(kT) \in \rho(\sigma)$, $k \in \{0, 1, \ldots\}$, be the state of the controlled system Σ at the start of the sampling interval $[kT, (k+1)T]$. Then, for every real number $\varepsilon > 0$, there are real numbers $\kappa', W > 0$ and a pseudo bang-bang input signal $u^{\pm}(x(kT)) \in U(K + \varepsilon, W, \kappa')$ that satisfy the following. Compared to the optimal response $x^*(t)$, the response $x^{\pm}(t)$ to the time-shifted input signal $u^{\pm}(x(kT), t - kT)$ satisfies $|x^*(t) - x^{\pm}(t)| < \varepsilon$ for all $t \in [kT, (k+1)T]$. Furthermore, $x^{\pm}((k+1)T) \in \rho(\sigma)$.* □

When implementing the controller C of Figure 1, it is most convenient to implement it as a pseudo bang-bang controller, using the input signal $u^{\pm}$ of Corollary 7.6. This simplifies the design and implementation of C. Replacing the optimal controller by its pseudo bang-bang counterpart increases the tracking error by ε of Corollary 7.6. The value of ε can be made as small as desired at the cost of potentially increasing the number of switchings or reducing the smoothing factor κ'. The pseudo bang-bang controller C operates as follows.

7.3 Operation of a pseudo bang-bang controller C

In the notation of Corollary 7.6 and Theorem 6.2, let $k \in \{0, 1, \ldots\}$ be an integer.

- The state $x(kT) \in \rho(\sigma)$ is provided to the controller C at the time $t = kT$ by the feedback sampler of Figure 1.

- In response to $x(kT)$, the controller C generates the signal $u^{\pm}(x(kT), t-kT)$ as input to the controlled system Σ during the sampling interval $[kT,(k+1)T]$.
- By construction, the input signal $u^{\pm}(x(kT), t-kT)$ preserves the sampling radius σ, i.e. $x^{\pm}((k+1)T) \in \rho(\sigma)$. This results in a cyclical action that approximates minimal inter-sample tracking error over all sampling intervals $[kT,(k+1)T]$, $k \in \{0,1,\ldots\}$.
- The performance of a pseudo bang-bang controller can be as close as desired to optimal performance. □

8. Example

To demonstrate the techniques of the present paper, we use a modified version of the Michaelis-Menten equation, an equation that plays an important role in mathematical biology and in environmental science. Its many applications include the mathematical modelling of predator-prey processes in environmental biology (Michaelis & Menten, 1913); the mathematical modelling of certain enzymatic signalling chains in molecular biology (e.g. Cao, 2011); and the mathematical modelling of certain processes in pharmacokinetics (e.g. Wagner, 1973). Here,

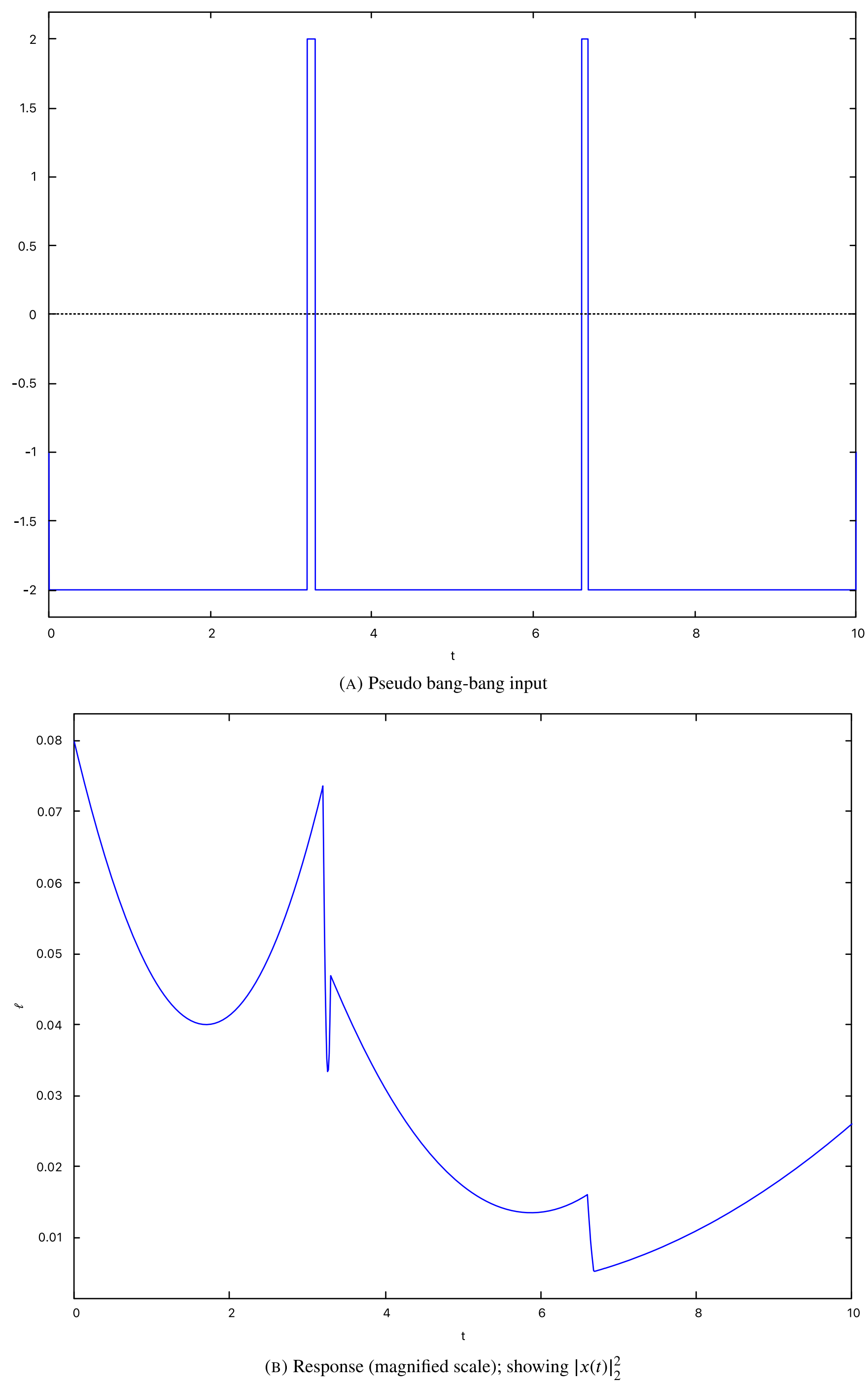


Figure 4. The input signal and its response. (A) Pseudo bang-bang input and (B) Response (magnified scale); showing $|x(t)|_2^2$.

we slightly modified the Michaelis-Menten equation to include a scalar input signal $u(t)$. In cases where the equation describes a predator-prey process, this input could represent control over the external food supply. The equation is of the following form.

$$\Sigma : \begin{aligned} \dot{x}^1(t) &= \frac{d(b+u(t))x^2(t)}{(b+x^2(t))}, \\ \dot{x}^2(t) &= \frac{(-a+u(t))x^1(t)}{(c+x^2(t))}, \end{aligned} \tag{72}$$

where the state is $x(t) = (x^1(t), x^2(t))^\top$, and the input is $u(t)$. The parameters a, b, c, and d are constants with nominal values $a_0 = 1$, $b_0 = 2$, $c_0 = 5$, and $d_0 = 8$ and with uncertainty ranges $0.95 \le a \le 1.05, 1.9 \le b \le 2.1, 4.8 \le c \le 5.2$, and $7.6 \le d \le 8.4$. The input signal amplitude bound is $K = 2$, and the state amplitude bound is $A = 0.5$. We use the sampling radius $\sigma = 0.08$, so that the domain of initial states is $\rho(0.08)$. The sampling period is specified as $T = 10$. We are looking for pseudo bang-bang signals $u^\pm$ that approximate minimal inter-sample tracking error, where the target state is $x = 0$. Referring to Theorem 7.2, we take the approximation error bound to be $\varepsilon = 0.005$.

In general, when designing a controller, it is easiest to follow the path used in the proof of Theorem 7.2: start with the derivation of a bang-bang signal, and then 'soften' it into a pseudo bang-bang signal. The latter is done by applying the Fourier transform to the bang-bang signal, multiplying the results by $e^{-\kappa|\omega|}$ for a suitable value of $\kappa > 0$, and then applying the inverse Fourier transform to the product. The process follows then along the following outline.

Procedure 8.1 (Building Pseudo Bang-Bang Controllers):

Step 1: Implement a search process similar to the one described in *Choi and Hammer (2019)* to find a bang-bang input signal for Σ that yields minimal inter-sample tracking error over the sampling interval $[0, T]$, starting from an initial state x_0. Briefly, the search is conducted numerically by searching over bang-bang input signals with an increasing number of switchings in $[0, T]$. For each number of switchings, search over all switching times combinations to find one that yields the lowest inter-sample tracking error. The process terminates when further increase of the number of switchings does not reduce inter-sample tracking error. Denote the resulting bang-bang signal by $v^\pm(x_0)$.

Step 2: Obtain the transform $V^\pm(x_0) = \mathscr{F} v^\pm(x_0)$.

Step 3: Multiply $V^\pm(x_0)$ by $e^{-\kappa|\omega|}$, where κ is a smoothing factor achieving a sufficiently low discrepancy $E(\kappa)$ of *(46)*.

Step 4: An appropriate pseudo bang-bang signal is then $u^\pm(x_0) = \mathscr{F}^{-1}(V^\pm(x_0, \omega)e^{-\kappa|\omega|})$.

Step 5: Repeat Steps 1 to 4 over a grid of permissible initial states to complete the controller's derivation. □

For the system Σ of (72), we followed the steps of Procedure 8.1, using $\kappa = 0.001$. This value of κ guarantees an adequately low discrepancy $E(\kappa)$ of (46). For demonstration, we use the initial state $x_0 = (0.2, 0.2)^\top$, with the parameter values a_0, b_0, c_0 and d_0 listed above. Note that this initial state is permissible, since $0.2^2 + 0.2^2 = 0.08 \in \rho(0.08)$; being on the boundary of $\rho(0.08)$, this initial state is, in a sense, a worst case example. The results are depicted in Figure 4, where Figure 4(A) shows the pseudo bang-bang input signal (which, in this scale, looks similar to a bang-bang signal), and Figure 4(B) shows the response.

As can be seen in Figure 4(B), the inter-sample tracking error is $\ell = 0.08$. In this case, the minimal inter-sample tracking error cannot be less than 0.08, since $\ell^* = \sup_{t\in[0,T]} |x(t)|_2^2 \ge |x_0|_2^2 = 0.08$, so this controller achieves the best tracking error possible. The additional amplitude error of $\varepsilon = 0.005$ that is permitted is not utilised in this case.

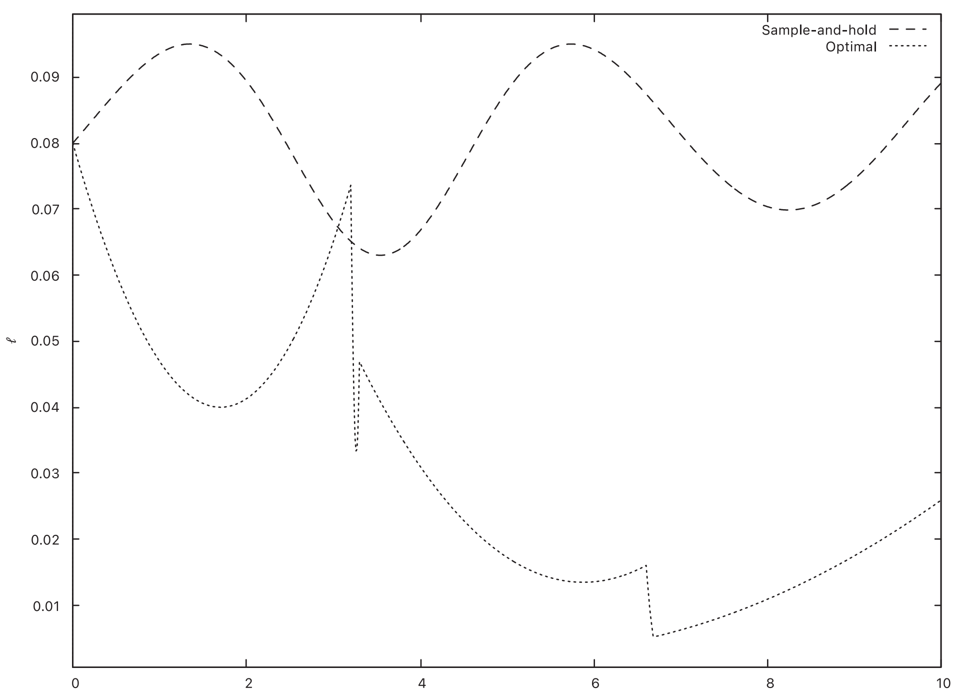


Figure 5. The best sample-and-hold response compared to the optimal response of Figure 4(B); showing $|x(t)|_2^2$. Best sample-and-hold input is -1.8 (constant).

At the end of the sampling interval, Figure 4(B) shows that $|x(10)|_2^2 = 0.025$, which is in $\rho(0.08)$. Thus, the state is back in the domain $\rho(0.08)$, as required by the sampling radius $\sigma = 0.08$. Consequently, the sampled-data control process can continue cyclically using the sampling period $T = 10$.

For comparison, we derive the lowest tracking error that can be achieved by a sample-and-hold controller in this case. A sample-and-hold controller provides a constant input signal during each sampling interval. For the current initial state, the lowest tracking error with constant input signal is obtained with input $u(t) = -1.8$. As can be seen in Figure 5, this input achieves a tracking error of 0.095, which is larger by about 20% than the optimal tracking error. Furthermore, at the end of the sampling interval, the sample-and-hold controller yields $|x(10)|_2^2 = 0.089$, which violates the sampling radius $\sigma = 0.08$. Consequently, this sample-and-hold controller is not compatible with cyclical operation.

9. Conclusion

The paper presents a methodology for the design and implementation of optimal robust sampled-data controllers that minimise inter-sample tracking errors for a wide range of nonlinear systems. The methodology is applicable to any nonlinear system described by a state representation with a continuously differentiable recursion function. It takes into consideration uncertainties and disturbances that may affect the controlled system, and it incorporates constraints on the control effort. Tools to simplify design and implementation are offered by showing that optimal performance can be approximated as closely as desired by pseudo bang-bang controllers – controllers that are relatively easy to implement.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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