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FASTEST RECOVERY FROM FEEDBACK LOSS: BOUNDED OVERSHOOT

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ABSTRACT. The problem of designing robust optimal controllers to reduce in minimal time operating errors that had accumulated during a period of feedback loss is revisited, with the objective of imposing a constraint on the maximal overshoot of the controlled system. It is shown that robust optimal controllers that satisfy this constraint exist under rather broad conditions. It is also shown that optimal performance can be closely approximated by bang-bang controllers – controllers that are relatively easy to design and implement.

1. INTRODUCTION

Sound operating policies of automatic control systems often require compliance with constraints on the maximal overshoot of the controlled system's response. Such overshoot constraints come to secure a safe operating environment by protecting the system and its operators from overload, over-stress, and over-strain. The present paper considers the design of controllers that guide a system toward a quick recovery from a period of feedback loss, while complying with a specified bound on overshoots of the controlled system. The objective is to reduce as quickly as possible operating errors that may have accumulated during a period of feedback service loss, without overloading the controlled system. The methodology developed in this paper is also applicable to other problems in minimal-time optimal control.

The configuration we consider is depicted in Figure 1.1, where Σ is the controlled system and C is a controller. As can be seen in the figure, feedback service to the controller C had been lost for some time, before having been restored at the time $t = 0$. During the period of feedback loss, operating errors may have increased. The goal of the controller C is to guide Σ so as to reduce operating errors to an acceptable level as quickly as possible, once feedback has been restored. Importantly, the controller C must achieve this goal without overloading the controlled system Σ. We show below that optimal controllers that achieve this goal exist under rather general conditions. We also show that optimal performance can be approximated as closely as desired by bang-bang controllers – controllers that are relatively easy to design and implement.

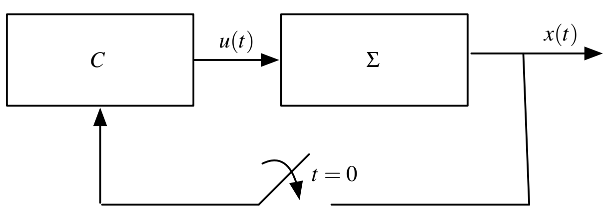

FIGURE 1.1. The control configuration

The current paper expands the work of Chakraborty and Hammer (2009b, 2010), Yu and Hammer (2016a,b), and Choi and Hammer (2019) by imposing a constraint on the maximal state amplitude the controlled system Σ may experience during the control process. This additional constraint prevents overload of the controlled system Σ at the cost of a potential increase in the time required to reduce operating errors to an acceptable level.

Loss of feedback service is not uncommon in the practice of control systems engineering. Loss of feedback may occur as a result of malfunctions or failures of components in the feedback channel; it may occur as a result of

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deteriorating operating conditions, such as the loss of line-of-sight to a satellite; or it may be a feature of a system's mode of operation, as in sampled-data systems, where no feedback is available between samples. Another example of an application where loss of feedback is an integral part of operating procedure is networked control systems, where feedback may be disrupted due to capacity limitations of feedback communication channels (Nair et al. (2007), Zhivogyladov and Middleton (2003), Montestruque and Antsaklis (2004)).

Feedback control theory reminds us that operating errors may increase during periods of feedback loss. Our objective is to develop controllers that reduce such operating errors to an acceptable level as quickly as possible, once feedback has been restored.

Referring to Figure 1.1 at a time t, the state of the controlled system Σ is $x(t)$, and the input signal of Σ is $u(t)$. The system Σ is controlled by the controller C. As depicted, the controller's feedback channel was open for some time until the time $t = 0$. At $t = 0$, feedback was momentarily restored, providing a reading of the state $x(0) = x_0$. The controller C must utilize this reading to guide Σ toward lower operating error. This must be accomplished without violating two constraints imposed by structural limitations of the controlled system Σ: *(1)* the input signal of Σ may not exceed an amplitude bound of K; and *(2)* the response of Σ may not exceed a signal amplitude bound of A. Here, K and A are specified bounds.

After possibly applying a shift transformation to the state coordinates of Σ, we assume that nominal operation of Σ is near its zero state $x = 0$. The disruption in feedback service may have increased operating errors, bringing Σ to the state $x(0) = x_0$ communicated by the restored feedback channel at $t = 0$. The controller C must guide Σ back from x_0 to the vicinity of the zero state.

Needless to say, inaccuracies, noises, and modeling errors prevent Σ from being driven exactly to the zero state. To accommodate such uncertainties, a deviation of ℓ from the zero state is permitted. The goal of the controller C is then to guide Σ as quickly as possible from the state x_0 to within ℓ of the zero state; specifically, C must guide Σ to a state x satisfying $x^\top x \leq \ell$. This process of guiding Σ must progress without violating the aforementioned input and output constraints imposed by structural limitations of Σ.

Optimal controllers are often hard to design and construct, since, in general, their design and construction involve the calculation and implementation of vector valued functions of time. An important goal of our discussion is to develop controllers that approximate optimal performance, while being relatively easy to design and implement.

We can summarize our objectives as follows.

Problem 1.1. In the control configuration of Figure 1.1, the controller C experienced a period of feedback loss that ended at the time $t = 0$. Feedback was restored momentarily at $t = 0$, providing C with a reading of the state $x(0) = x_0$ of the controlled system Σ. During the period of feedback loss, operating errors may have increased, possibly taking the state x of Σ out of the desired operating domain $\rho(\ell)$ given by

$$\rho(\ell) := \left\{x : x^\top x \leq \ell\right\}, \tag{1.2}$$

where $\ell > 0$ is a specified operating error bound.

The goal of the controller C is to guide Σ so as to reduce operating errors as quickly as possible by bringing Σ from x_0 to $\rho(\ell)$ in minimal time. During this process, C must comply with two constraints: *(1)* the input of Σ cannot exceed a signal amplitude bound of K, and *(2)* the state of Σ cannot exceed a signal amplitude bound of A; here, $K > 0$ and $A > 0$ are specified real numbers. To this end:
(i) Determine conditions under which there is an optimal controller C that guides Σ in minimal time from the state x_0 to the domain $\rho(\ell)$, without violating input and state constraints.
(ii) Find simple-to-calculate-and-implement controllers that approximate optimal performance. □

The current paper is a study of constrained optimization; it employs tools developed, referenced, and applied in earlier studies in the area of optimal control, including the studies of Kelendzheridze (1961), Pontryagin et al. (1962), Gamkrelidze (1965), Neustadt (1966, 1967), Luenberger (1969), Young (1969), Warga (1972), Chakraborty and Hammer (2007, 2008a,b,c, 2009a,b, 2010), Chakraborty and Shaikshavali (2009), Yu and Hammer (2016a,b), Choi and Hammer (2019, 2017), the references cited in these studies, and many others. A survey of more recent progress in the area of optimal control can be found in Tonon et al. (2017). Yet, to the best of our knowledge, there are no earlier reports that address existence and implementation of automatic controllers that reduce operating errors in minimal time under overshoot constraints.

The current paper is organized as follows. Section 2 introduces notation and setup, and Section 3 covers a few preliminary observations. The existence of optimal controllers is proved in Section 4, where we show that optimal controllers exist as long as the controlled system Σ satisfies a certain controllability condition. Section 5 shows that optimal performance can be approximated as closely as desired by bang-bang controllers – controllers that are relatively easy to calculate and implement. Section 6 consists of two examples. Finally, concluding remarks are offered in Section 7.

2. Mathematical framework

2.1. **Notation.** Denote by R the set of real numbers and by R^+ the set of non-negative real numbers. Here, R is the compactified set of real numbers, i.e., the real numbers augmented by the points $-\infty$ and ∞. The absolute value of a real number r is denoted, as usual, by $|r|$. The $L^\infty-$norm of a vector $x = (x_1, x_2, \ldots, x_n)^\top \in R^n$ is

$$|x| := \max_{i=1,2,\ldots,n} |x_i|.$$

For a real number $A > 0$, it is often convenient to use the notation

$$[-A,A]^n := \{x \in R^n : |x| \le A\}. \tag{2.1}$$

For a constant $n \times m$ matrix $G = (G_{ij}) \in R^{n\times m}$, the $L^\infty-$norm is

$$|G| := \max_{\substack{i=1,2,\ldots,n \\ j=1,2,\ldots,m}} |G_{ij}|,$$

and for a matrix function of time $g : R^+ \to R^{n\times m} : t \mapsto g(t)$, the $L^\infty-$norm is

$$|g|_\infty := \sup_{t\ge 0} |g(t)|.$$

We refer to $|g|_\infty$ as the *amplitude* of g. The L^2-norm of vector $x \in R^n$ is denoted by $|x|_2$, where

$$|x|_2^2 = x^\top x.$$

2.2. **Description of the system.** The controlled system Σ of Figure 1.1 is an input-affine time-varying nonlinear system described by a differential equation of the form

$$\Sigma : \begin{array}{l} \dot{x}(t) = a(t,x(t)) + b(t,x(t))u(t), \\ x(0) = x_0, \end{array} \tag{2.2}$$

where $x(t) \in R^n$ is the state of Σ at the time t, and $u(t) \in R^m$ is the input signal of Σ at the time t. The functions $a : R^+ \times R^n \to R^n : (t,x) \mapsto a(t,x)$ and $b : R^+ \times R^n \to R^{n\times m} : (t,x) \mapsto b(t,x)$ are continuous functions subject to the Lipchitz conditions

$$\begin{aligned} |a(t,x') - a(t,x)| &\le \alpha^+|x' - x|, \\ |b(t,x') - b(t,x)| &\le \alpha^+|x' - x|, \\ a(t,0) = 0, |b(t,0)| &\le \alpha^+, \end{aligned} \tag{2.3}$$

where $\alpha^+ > 0$ is a specified constant bound. We use the same bound α^+ in all inequalities to simplify notation; the qualitative results derived in this paper remain valid when a different bound is used in each inequality.

To incorporate modeling uncertainties that are prevalent in practice, and to build robustness into our results, we introduce uncertainty into the model (2.2) of Σ by decomposing the functions a and b into nominal and uncertain parts:

$$\begin{aligned} a(t,x) &= a_0(t,x) + a_\gamma(t,x), \\ b(t,x) &= b_0(t,x) + b_\gamma(t,x). \end{aligned} \tag{2.4}$$

Here, $a_0 : R^+ \times R^n \to R^n$ and $b_0 : R^+ \times R^n \to R^{n\times m}$ are specified continuous functions describing the nominal model of the controlled system Σ; they are subject to the Lipschitz conditions

$$\begin{aligned} |a_0(t,x') - a_0(t,x)| &\le \alpha|x' - x|, \\ |b_0(t,x') - b_0(t,x)| &\le \alpha|x' - x|, \\ a_0(t,0) = 0, |b_0(t,0)| &\le \alpha, \end{aligned} \tag{2.5}$$

where $\alpha \ge 0$ is a specified constant bound. The nominal controlled system Σ_0 is then

$$\Sigma_0 : \begin{array}{l} \dot{x}(t) = a_0(t,x(t)) + b_0(t,x(t))u(t),\ t \ge 0, \\ x(0) = x_0. \end{array} \tag{2.6}$$

The functions $a_\gamma : R^+ \times R^n \to R^n$ and $b_\gamma : R^+ \times R^n \to R^{n\times m}$ of (2.4) are unspecified continuous functions that represent modeling uncertainties; they are subject to the Lipschitz conditions

$$\begin{aligned} |a_\gamma(t,x') - a_\gamma(t,x)| &\le \gamma |x' - x|, \\ |b_\gamma(t,x') - b_\gamma(t,x)| &\le \gamma |x' - x|, \\ a_\gamma(t,0) = 0,\ |b_\gamma(t,0)| &\le \gamma, \end{aligned} \tag{2.7}$$

where $\gamma > 0$ is a specified constant bound that describes the uncertainty level. In many applications, γ is a relatively small number, as modeling uncertainties are often modest. Comparing (2.3), (2.4), (2.5) and (2.7), we can set

$$\alpha^+ = \alpha + \gamma.$$

2.3. **Input signals.** The space of input signals of the controlled system Σ of Figure 1.1 is the Hilbert space $L_2^{\omega,m}$ that consists of all Lebesgue measurable functions $f, g : R^+ \to R^m$ with the inner product

$$\langle f, g \rangle := \int_0^\infty e^{-\omega s} f^\top(s) g(s) ds,$$

where $\omega > 0$ is a real number (Chakraborty and Hammer (2009b, 2010)). Note that the exponential term inside the integral guarantees that the inner product is finite for all bounded functions f and g.

For a function $g \in L_2^{\omega,m}$ and an $n \times m$ matrix $D(t)$ with rows $D_1(t)$, $D_2(t)$, ..., $D_n(t) \in L_2^{\omega,m}$, we use the notation

$$\langle D, g \rangle := \sum_{j=1}^{n} \left\langle D_j^\top, g \right\rangle.$$

Like most practical systems, the system Σ of Figure 1.1 imposes a bound $K > 0$ on the largest input signal amplitude it permits. Thus, only input signals u that belong to the family

$$U(K) := \left\{ u \in L_2^{\omega,m} : |u|_\infty \le K \right\} \tag{2.8}$$

are allowed; $U(K)$ serves as the set of input signals throughout our discussion.

2.4. **Notation and formal statement of the problem.** In this subsection, we restate Problem 1.1 in formal terms. Recall that there are five bounds that are critical to our discussion: the input signal amplitude bound $K > 0$ of (2.8); the permissible operating error bound $\ell > 0$ of (1.2); the uncertainty bound $\gamma > 0$ of (2.7); the bound $\alpha > 0$ of the Lipschitz inequalities (2.5); and the state amplitude bound $A > 0$ that represents the largest permissible amplitude of the state $x(t)$ of the controlled system Σ. The latter enforces the constraint

$$|x(t)| \le A \text{ for all } t \ge 0. \tag{2.9}$$

Notation 2.10. Let Σ_0 be the nominal system of (2.6), and, given real numbers $K, \alpha, \gamma > 0$, denote by $\mathcal{F}_\gamma(\Sigma_0)$ the family of all systems of the form (2.2), subject to the requirements (2.3), (2.4), (2.5) and (2.7). All members of $\mathcal{F}_\gamma(\Sigma_0)$ share the same initial state $x(0) = x_0$, and their set of permissible input signals is $U(K)$ of (2.8). For a member $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$, the state $x(t)$ in response to an input signal u is denoted by $\Sigma(x_0, u, t) := x(t)$.

A state amplitude bound $A > 0$ is associated with the family $\mathcal{F}_\gamma(\Sigma_0)$; every member $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ must satisfy $|\Sigma(x_0, u, t)| \le A$ at all times t during the control process. In particular, the initial state satisfies $|x_0| \le A$. □

Consider a member $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ and refer to the specified state amplitude bound $A > 0$. Given a time $t \ge 0$, denote by $U(K, A, \Sigma, t)$ the set of all input signals $u \in U(K)$ for which the state of Σ remains bounded by A up to the time t, namely,

$$U(K, A, \Sigma, t) := \{ u \in U(K) : |\Sigma(x_0, u, \theta)| \le A \text{ for all } \theta \in [0, t] \}.$$

Now, the state of every member $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ must not exceed the amplitude bound A.

Since the functions a_γ and b_γ of (2.4) are not specified, it is not known which member of $\mathcal{F}_\gamma(\Sigma_0)$ serves as the active controlled system Σ of Figure 1.1. Therefore, the only input signals that are permitted are those for which the state of every member of $\mathcal{F}_\gamma(\Sigma_0)$ remains bounded by the specified state amplitude bound A. In explicit terms this means that, for the time interval $[0, t]$, the set of all permissible input signals is

$$U(K, A, \gamma, t) := \bigcap_{\Sigma \in \mathcal{F}_\gamma(\Sigma_0)} U(K, A, \Sigma, t). \tag{2.11}$$

Note that, since the initial state satisfies $|x_0| \le A$, it follows that $U(K, A, \gamma, 0) = U(K)$.

Further, it is required to bring the state of our system into the domain $\rho(\ell)$ of (1.2) as quickly as possible. The earliest time $t(x_0,\ell,A,\Sigma,u)$ at which a specific input signal $u \in U(K)$ can bring the state of a specific member $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ from the initial state x_0 to $\rho(\ell)$ without violating the state amplitude bound A is

$$t(x_0,\ell,A,\Sigma,u) = \inf_{t\geq 0}\left\{|\Sigma(x_0,u,t)|_2^2 \leq \ell, u \in U(K,A,\Sigma,t)\right\};$$

here, $t(x_0,\ell,A,\Sigma,u) := \infty$ if the infimum does not exist, i.e., if there is no time $t \geq 0$ at which both of the conditions $|\Sigma(x_0,u,t)|_2^2 \leq \ell$ and $u \in U(K,A,\Sigma,t)$ are met.

The earliest time $t(x_0,\ell,A,\gamma,u)$ at which the input signal $u \in U(K)$ can bring the state of every member of $\mathcal{F}_\gamma(\Sigma_0)$ from the initial state x_0 to the domain $\rho(\ell)$, without violating the state amplitude bound A, is

$$t(x_0,\ell,A,\gamma,u) := \inf_{t\geq 0}\left\{\left(\sup_{\Sigma\in\mathcal{F}_\gamma(\Sigma_0)}|\Sigma(x_0,u,t)|_2^2\right) \leq \ell, u \in U(K,A,\gamma,t)\right\}. \tag{2.12}$$

Here, $t(x_0,\ell,A,\gamma,u) := \infty$ if the infimum does not exist, namely, if there is no time $t \geq 0$ at which both conditions $\sup_{\Sigma\in\mathcal{F}_\gamma(\Sigma_0)}|\Sigma(x_0,u,t)|_2^2 \leq \ell$ and $u \in U(K,A,\gamma,t)$ are met. In particular, this includes cases where $\sup_{\Sigma\in\mathcal{F}_\gamma(\Sigma_0)}|\Sigma(x_0,u,t)|_2^2 \leq \ell$ for some time $t \geq 0$, but the input signal u is not in $U(K,A,\gamma,t)$.

The earliest time at which any input signal $u \in U(K)$ can bring the state of every member of $\mathcal{F}_\gamma(\Sigma_0)$ from the initial state x_0 into the domain $\rho(\ell)$, without violating the state amplitude bound A, is

$$t^*(x_0,\ell,A,\gamma) = \inf_{u\in U(K)} t(x_0,\ell,A,\gamma,u), \tag{2.13}$$

where $t^*(x_0,\ell,A,\gamma) := \infty$ if the infimum does not exist.

We show in Section 4 that $t^*(x_0,\ell,A,\gamma) < \infty$ under rather broad conditions, and that there is an optimal input signal $u^*(x_0,\ell,A,\gamma) \in U(K)$ that achieves the minimal time $t^*(x_0,\ell,A,\gamma)$, namely, that

$$t^*(x_0,\ell,A,\gamma) = t(x_0,\ell,A,\gamma,u^*(x_0,\ell,A,\gamma)).$$

In qualitative terms, the main condition under which an optimal input signal $u^*(x_0,\ell,A,\gamma))$ exists is a controllability-type condition on the nominal system Σ_0. It requires that there be an input signal in $U(K)$ that drives Σ_0 from the initial state x_0 to the zero state in finite time, without violating the state amplitude bound A. If the uncertainty parameter γ is not too large, then the existence of an optimal input signal can be determined from an inspection of a single system – the nominal system Σ_0; it is not necessary to check every member of the family $\mathcal{F}_\gamma(\Sigma_0)$.

An optimal input signal $u^*(x_0,\ell,A,\gamma)$ reduces operating errors as quickly as possible to a magnitude not exceeding ℓ, without violating the state amplitude bound A of the controlled system Σ. As $u^*(x_0,\ell,A,\gamma)$ is generally a vector valued function of time, it may be difficult to calculate it and implement it in practice. In Section 5, we show that an optimal input signal $u^*(x_0,\ell,A,\gamma)$ can be replaced by a bang-bang input signal $u^\pm \in U(K)$ without a significant departure from optimal performance. Bang-bang signals are relatively easy to calculate and implement, since they are determined by a finite string of scalars – their switching times.

For future reference, we summarize now our objectives in formal terms.

Problem 2.14. Let $K,A,\ell,\gamma > 0$ be specified real numbers. Using Notation 2.10, (2.12), and (2.13), address the following issues:
(i) Find conditions under which there is an optimal input signal $u^*(x_0,\ell,A,\gamma) \in U(K)$ satisfying $t^*(x_0,\ell,A,\gamma) = t(x_0,\ell,A,\gamma,u^*(x_0,\ell,A,\gamma))$.
(ii) If $u^*(x_0,\ell,A,\gamma)$ exists, find a simple-to-calculate-and-implement input signal that can replace $u^*(x_0,\ell,A,\gamma)$ without causing significant departure from optimal performance. □

3. Basic facts

3.1. **Preliminaries.** Our current discussion continues the work of Yu and Hammer (2016a,b) and Choi and Hammer (2019). We start by reproducing the following result from Yu and Hammer (2016a,b). It states that our systems generate a response that is bounded at all finite times.

Proposition 3.1. *Let $K,\gamma > 0$ be real numbers and let Σ be a member of the family $\mathcal{F}_\gamma(\Sigma_0)$. Then, for every finite time $T \geq 0$, there is a real number $M(T) \geq 0$ such that $|\Sigma(x_0,u,t)| \leq M(T)$ for all $t \in [0,T]$, for all input signals $u \in U(K)$, and for all members $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$.* □

The next statement, which is also reproduced here from Yu and Hammer (2016a,b), is a consequence of Proposition 3.1 and the fact that continuous functions are bounded over compact domains.

Corollary 3.2. *Let n and p be two positive integers, let $c : R \times R^n \to R^p : (t,x) \mapsto c(t,x)$ be a continuous function, let $K, \gamma > 0$ be real numbers, and let Σ be a member of $\mathcal{F}_\gamma(\Sigma_0)$. Then, for every finite time $T \geq 0$, there is a real number $M_c(T) \geq 0$ such that*

$$|c(t, \Sigma(x_0, u, t))| \leq M_c(T) \tag{3.3}$$

for all $t \in [0,T]$, for all input signals $u \in U(K)$, and for all $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$. □

According to Problem 2.14, our objective is to drive every member $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ from the initial state x_0 into the domain $\rho(\ell)$, without violating the state amplitude bound A. Whether this is possible or not depends on a number of factors, including the initial state x_0, the input amplitude bound K, and the state amplitude bound A. In this context, it is useful to introduce the following notion.

Definition 3.4. Let $K, A > 0$ be real numbers. A system $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ is $(K,A)-$*controllable* from the initial state x_0 if there is an input signal $u \in U(K)$ and a finite time $t_A \geq 0$ such that $\Sigma(x_0, u, t_A) = 0$ and $|\Sigma(x_0, u, t)| \leq A$ for all $t \in [0, t_A]$. □

The following statement shows that, if the nominal system Σ_0 is $(K,A)-$controllable, then the minimal time $t^*(x_0, \ell, A, \gamma)$ of (2.13) is finite, as long as the uncertainty parameter γ is not too large. In other words, if γ is not too large, then $(K,A)-$controllability of the nominal system guarantees that every member of the family $\mathcal{F}_\gamma(\Sigma_0)$ can be brought into the domain $\rho(\ell)$ in finite time, without violating specified constraints. This is an important fact, since it shows that by checking properties of a single system – the nominal system Σ_0 – one can assure proper performance of the entire family $\mathcal{F}_\gamma(\Sigma_0)$. In more precise terms, the following is true.

Proposition 3.5. *Let $K, A_0 > 0$ be two real numbers, and assume that the nominal system Σ_0 is $(K, A_0)-$controllable from the initial state x_0. Then, for every pair of real numbers $\ell > 0$ and $A > A_0$, there is an uncertainty parameter $\gamma > 0$ for which the minimal time $t^*(x_0, \ell, A, \gamma)$ of (2.13) is finite.*

Remark 3.6. Inequality (3.11), which forms part of the proof of Proposition 3.5, points out values of the uncertainty parameter γ that are compatible with the statement of Proposition 3.5; there may be additional values as well. □

Proof of Proposition 3.5. As the nominal system Σ_0 is $(K, A_0)-$controllable, there is a time $t_{A_0} \geq 0$ and an input signal $u_{A_0} \in U(K)$ such that $\Sigma_0(x_0, u_{A_0}, t_{A_0}) = 0$ and $|\Sigma_0(x_0, u_{A_0}, t)| \leq A_0$ for all $t \in [0, t_{A_0}]$. Now, let $\gamma > 0$ be a real number, let Σ be a member of $\mathcal{F}_\gamma(\Sigma_0)$, and, for a time $t \geq 0$, denote $x(t) := \Sigma_0(x_0, u_{A_0}, t)$, $x'(t) := \Sigma(x_0, u_{A_0}, t)$, and $\xi(t) = x'(t) - x(t)$. As both Σ and Σ_0 start from the same initial state x_0, we have

$$\xi(0) = 0. \tag{3.7}$$

Consider now two times $t_1, t_2 \in [0, t_{A_0}]$, $t_1 < t_2$, and examine a time $t \in [t_1, t_2]$. Using the same input signal $u_{A_0} \in U(K)$ for both Σ and Σ_0, and invoking (2.2), (2.4), (2.5), and (2.7), together with the facts that $a_\gamma(t,0,0) = 0$ and $b_\gamma(t,0,0) \leq \gamma$, we get

$$\begin{aligned} |\xi(t)| &= \left|\xi(t_1) + \int_{t_1}^{t} [a(s,x'(s)) - a_0(s,x(s))]\, ds + \int_{t_1}^{t} [b(s,x'(s)) - b_0(s,x(s))]\, u_{A_0}(s) ds\right| \\ &\leq |\xi(t_1)| + \int_{t_1}^{t} (\alpha|\xi(s)| + \gamma|x'(s)|)\, ds + \int_{t_1}^{t} (\alpha|\xi(s)| + \gamma|x'(s)| + \gamma)\, |u_{A_0}(s)| ds. \end{aligned}$$

Applying Proposition 3.1 and the fact that $u_{A_0} \in U(K)$, we obtain

$$\sup_{s \in [t_1, t_2]} |\xi(s)| \leq |\xi(t_1)| + \alpha(1+K)(t_2 - t_1) \sup_{s \in [t_1, t_2]} |\xi(s)| + \gamma[M(t_{A_0})(1+K) + K](t_2 - t_1). \tag{3.8}$$

Now, choose a real number $\mu > 0$ such that

$$\alpha(1+K)\mu < 1,$$

and set

$$t_2 := t_1 + \mu.$$

Substituting into (3.8) and rearranging terms, we get

$$(1 - \alpha(1+K)\mu) \sup_{s \in [t_1, t_1+\mu]} |\xi(s)| \leq |\xi(t_1)| + \gamma[M(t_{A_0})(1+K) + K]\mu.$$

Define the positive numbers

$$\begin{aligned} \eta &:= [1 - \alpha(1+K)\mu]^{-1}, \\ \eta_1 &:= [M(t_{A_0})(1+K) + K]\mu\eta. \end{aligned}$$

Then,

$$\sup_{s\in[t_1,t_1+\mu]} |\xi(s)| \le \eta|\xi(t_1)| + \gamma\eta_1. \tag{3.9}$$

Next, let q be an integer that satisfies the inequality $q \ge t_{A_0}/\mu$ and build the partition

$$[0,t_{A_0}] \subseteq \{[0,\mu],[\mu,2\mu],\ldots,[(q-1)\mu,q\mu]\}.$$

Then, letting $k \in [0,q-1]$ be an integer and choosing $t_1 := k\mu$, we obtain from (3.9) the recursion

$$\sup_{s\in[k\mu,(k+1)\mu]} |\xi(s)| \le \eta|\xi(k\mu)| + \gamma\eta_1, k = 0,1,2,\ldots,q-1.$$

Further, considering that $\xi(0) = 0$ by (3.7) and utilizing properties of linear recursions, we obtain

$$\sup_{s\in[0,(k+1)\mu]} |\xi(s)| \le \gamma\eta_1\eta^k, k = 0,1,2,\ldots,$$

so that

$$\sup_{s\in[0,t_{A_0}]} |\xi(s)| \le \gamma\eta_1\eta^{q-1}. \tag{3.10}$$

Finally, setting $\delta := \min\{(A-A_0),\ell\}$, it follows that the proposition is valid for any number γ satisfying

$$0 < \gamma < \frac{\delta}{\eta_1\eta^{q-1}}. \tag{3.11}$$

This concludes our proof. □

In view of Proposition 3.5, checking $(K,A_0)-$controllability of a single system – the nominal system Σ_0 – is sufficient to ensure that the objective of Problem 2.14*(i)* can be met in finite time, as long as the uncertainty parameter γ is not too large.

3.2. **Some basic features.** We start this subsection with a review of a few mathematical notions (e.g., Willard (2004), Zeidler (1985)).

Definition 3.12. Let H be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$.
(i) A sequence $\{v_i\}_{i=1}^{\infty}$ of members of H *converges weakly* to a member $v \in H$ if $\lim_{i\to\infty}\langle v_i, y\rangle = \langle v, y\rangle$ for every $y \in H$.
(ii) A subset W of H is *weakly compact* if every sequence of members of W has a subsequence that converges weakly to a member of W. □

The following statement is reproduced from Chakraborty and Hammer (2009b, 2010).

Lemma 3.13. *The set of signals $U(K)$ is weakly compact in $L_2^{\omega,m}$.* □

Next, we review a few more mathematical notions (Willard (2004), Zeidler (1985)).

Definition 3.14. Let S be a subset of a Hilbert space H, and let z be a member of S. A functional $F : S \to R$ is *weakly lower semi-continuous at* z if the following is true for every sequence $\{z_i\}_{i=1}^{\infty} \subseteq S$ that converges weakly to z: whenever $F(z)$ is bounded, there is, for every real number $\varepsilon > 0$, an integer $N > 0$ such that $F(z) - F(z_i) < \varepsilon$ for all $i \ge N$.

A function $G : S \times R \to R^n : (s,t) \mapsto G(s,t)$ is *weakly continuous* at z at a time t if the following is true for every sequence $\{z_i\}_{i=1}^{\infty} \subseteq S$ that converges weakly to z: for every real number $\varepsilon > 0$, there is an integer $N > 0$ such that $|G(z,t) - G(z_i,t)| < \varepsilon$ for all $i \ge N$.

Given two times $t_1 < t_2$, the function G is *uniformly weakly continuous* over the interval $[t_1,t_2]$ if the following is true for every sequence $\{z_i\}_{i=1}^{\infty} \subseteq S$ that converges weakly to z: for every real number $\varepsilon > 0$, there is an integer $N > 0$ such that $\sup_{t\in[t_1,t_2]} |G(z,t) - G(z_i,t)| < \varepsilon$ for all integers $i \ge N$. □

The following continuity feature of systems belonging to the family $\mathcal{F}_\gamma(\Sigma_0)$ is reproduced here from Yu and Hammer (2016a,b).

Lemma 3.15. *For a member $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$, the function $\Sigma(x_0,\cdot,\cdot) : U(K) \times R^+ \to R^n : (u,t) \mapsto \Sigma(x_0,u,t)$ is uniformly weakly continuous over every finite interval of time.* □

Returning to our discussion of Subsection 2.4, we rewrite (2.11) in the form

$$U(K,A,\gamma,t) = \left\{ u \in U(K) : \sup_{\substack{\Sigma\in\mathcal{F}_\gamma(\Sigma_0)\\ 0\le s\le t}} |\Sigma(x_0,u,s)| \le A \right\}. \tag{3.16}$$

A slight reflection shows that

$$U(K,A,\gamma,t_2) \subseteq U(K,A,\gamma,t_1) \text{ for all } t_2 \ge t_1, \tag{3.17}$$

so that the set $U(K,A,\gamma,t)$ is monotone decreasing as a function of t. Another feature of $U(K,A,\gamma,t)$ that is of interest is the following.

Proposition 3.18. *The set $U(K,A,\gamma,t)$ of (3.16) is weakly compact.*

Proof. Let $T \ge 0$ be a fixed time, and consider a sequence of signals $\{u_i\}_{i=1}^\infty \subseteq U(K,A,\gamma,T)$. As $U(K,A,\gamma,T) \subseteq U(K)$, it follows by Lemma 3.13 that the sequence $\{u_i\}_{i=1}^\infty$ has a subsequence $\{u_{i_k}\}_{k=1}^\infty$ that converges weakly to a signal $u \in U(K)$. We must show that, in fact, $u \in U(K,A,\gamma,T)$. Applying Lemma 3.15 over the finite time interval $[0,T]$, it follows that, for every real number $\varepsilon > 0$, there is an integer $N > 0$ such that $|\Sigma(x_0,u,t) - \Sigma(x_0,u_{i_k},t)| < \varepsilon$ for all $k \ge N$ and for all $t \in [0,T]$. In addition, since $u_{i_k} \in U(K,A,\gamma,T)$ for all $k \ge 1$, we have that $|\Sigma(x_0,u_{i_k},t)| \le A$ or, in the notation of (2.1), that $|\Sigma(x_0,u_{i_k},t)| \in [-A,A]^n$ for all $t \in [0,T]$ and for all $k \ge 1$. Now, at a particular time $t \in [0,T]$, this implies that the sequence of vectors $\{\Sigma(x_0,u_{i_k},t)\}_{k=1}^\infty \subseteq [-A,A]^n$ converges to the vector $\Sigma(x_0,u,t) \in R^n$, i.e., that $\lim_{k\to\infty} \Sigma(x_0,u_{i_k},t) = \Sigma(x_0,u,t)$. But then, since $[-A,A]^n$ is a compact subset of R^n, we must have that this limit belongs to $[-A,A]^n$, so that $\Sigma(x_0,u,t) \in [-A,A]^n$. As this is true for all $t \in [0,T]$, we conclude that $u \in U(K,A,\gamma,T)$, and the proposition follows. □

4. Existence of optimal solutions

We turn now to an examination of conditions under which Problem 2.14*(i)* has an optimal solution $u^*(x_0,\ell,A,\gamma)$. This examination depends on the following mathematical facts (e.g., Zeidler (1985), Willard (2004)).

Theorem 4.1. (i) *A weakly continuous functional is weakly lower semi-continuous.*
(ii) *Let S and A be topological spaces and assume that, for every member $a \in A$, there is a weakly lower semi-continuous functional $f_a : S \to R$. If $\sup_{a\in A} f_a(s)$ exists at each point $s \in S$, then the functional $f(s) := \sup_{a\in A} f_a(s)$ is weakly lower semi-continuous on S.* □

To continue, we adapt to our present framework a methodology used in Yu and Hammer (2016a,b). In these references, no state amplitude bound was imposed on the response of the controlled system Σ. To incorporate the state amplitude bound (2.9), we proceed as follows. For a time $t \ge 0$, define the functional $\psi(t,\cdot) : U(K) \to R : u \mapsto \psi(t,u)$ by setting

$$\psi(t,u) := \begin{cases} \sup_{\Sigma\in\mathcal{F}_\gamma(\Sigma_0)} |\Sigma(x_0,u,t)|_2^2 & \text{if } u \in U(K,A,\gamma,t), \\ \infty & \text{if } u \notin U(K,A,\gamma,t). \end{cases} \tag{4.2}$$

Then, the following is valid.

Lemma 4.3. *At every time $t \ge 0$, the functional $\psi(t,\cdot) : U(K) \to R$ of (4.2) is weakly lower semi-continuous over $U(K)$.*

Proof. By Lemma 3.15, the function $\Sigma(x_0,u,t)$ is weakly continuous over $U(K)$ at any finite time $t \ge 0$. Considering that a continuous function of a weakly continuous function is weakly continuous, it follows that the functional $|\Sigma(x_0,v,t)|_2^2$ is weakly continuous over $U(K)$ at every finite time $t \ge 0$. Then, Theorem 4.1*(i)* implies that the functional $|\Sigma(x_0,v,t)|_2^2$ is also weakly lower semi-continuous on $U(K)$ at every finite time $t \ge 0$. But then, in view of (4.2) and Theorem 4.1*(ii)*, we conclude that $\psi(t,\cdot)$ is weakly lower semi-continuous on $U(K)$ at every finite time $t \ge 0$. □

Based on Lemma 4.3, we can prove the following statement (compare to Yu and Hammer (2016a,b), where an analogous statement is proved for cases in which no amplitude bound is imposed on the response of the controlled system Σ).

Proposition 4.4. *Let $A_0, A, \ell, \gamma > 0$ be real numbers, where $A > A_0$. Assume that the nominal system Σ_0 is (K,A_0)-controllable from the initial state x_0, and that the uncertainty parameter γ is compatible with Proposition 3.5 for the current A, A_0, and ℓ. Then, the functional $t(x_0,\ell,A,\gamma,u)$ of (2.12) is weakly lower semi-continuous as a function of u over $U(K)$.*

Proof. Let $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ be a system, and let $u \in U(K)$ be an input signal. Using (4.2), we can write

$$t(x_0,\ell,A,\gamma,u) = \inf\{t \geq 0 : \psi(t,u) \leq \ell\}.$$

To temporarily simplify notation, set

$$\theta(u) := \inf_t \{t \geq 0 : \psi(t,u) \leq \ell\}. \tag{4.5}$$

By the proposition's assumption, γ is compatible with Proposition 3.5. Consequently, according to Proposition 3.5, there are input signals $u \in U(K)$ for which $\theta(u) < \infty$, and all such input signals u are included in $U(K,A,\gamma,t)$ for some $t \geq \theta(u)$. By (3.17), this implies that all such input signals u are members of $U(K,A,\gamma,\theta(u))$. Note that $\theta(u) \geq 0$ by (4.5). As the case $\theta(u) = 0$ is degenerate – the initial state x_0 already satisfies the requirement $|x_0| \leq \ell$, we assume below that $\theta(u) > 0$.

Let $u \in U(K)$ be an input signal for which $\theta(u) < \infty$, and consider a sequence of input signals $\{u_i\}_{i=1}^\infty \subseteq U(K)$ that converges weakly to u. Denote $\psi_i(t) := \psi(t,u_i)$, $i = 1,2,\ldots$, and $\psi_0(t) := \psi(t,u)$. Then, $\theta(u_i) := \inf\{t \geq 0 : \psi_i(t) \leq \ell\}$ and $\theta(u) := \inf\{t \geq 0 : \psi_0(t) \leq \ell\}$. We claim that $\theta(u)$ is a weakly lower semi-continuous functional of u over $U(K)$. For this, we need to show that, for every real number $\varepsilon > 0$, there is an integer $N > 0$ such that

$$\theta(u_i) > \theta(u) - \varepsilon \text{ for all } i \geq N. \tag{4.6}$$

To prove (4.6), choose $\varepsilon > 0$ satisfying $\varepsilon < \theta(u)$, and examine the following two cases:

Case 1: There is an integer $N > 0$ such that $\theta(u_i) \geq \theta(u)$ for all $i \geq N$.
Case 2: Case 1 is not valid.

In Case 1, inequality (4.6) is valid for all $i \geq N$. Since $\theta(u) < \infty$, this case includes all cases of sequences $\{u_i\}$ for which there is an integer $N > 0$ such that $\theta(u_i) = \infty$ for all $i \geq N$. This completes the discussion of Case 1.

Proceeding to Case 2, there is in this case a sequence of integers $j_1 < j_2 < j_3 < \cdots$ such that $\theta(u_{j_k}) < \theta(u)$ for all integers $k \geq 1$. As $\theta(u) < \infty$, the fact that $\theta(u_{j_k}) < \theta(u)$ entails that $\theta(u_{j_k}) < \infty$ for all $k \geq 1$. Now, the infimum (4.5) implies that $\psi_0(t) > \ell$ for all $t \in [0,\theta(u))$; in particular, any time

$$\bar{t} \in [\theta(u) - \varepsilon, \theta(u)) \tag{4.7}$$

satisfies $\psi_0(\bar{t}) > \ell$, so that

$$\psi_0(\bar{t}) - \ell > 0. \tag{4.8}$$

Now, $\psi(t,u)$ is weakly lower semi-continuous in u by Lemma 4.3. Consequently, there is, for every real number $\mu > 0$, an integer $N > 0$ for which

$$\psi_0(\bar{t}) - \psi_{j_k}(\bar{t}) < \mu \text{ for all } k \geq N. \tag{4.9}$$

Recalling (4.8), we can choose $\mu := (\psi_0(\bar{t}) - \ell)/2$. Substituting this value of μ into (4.9), we obtain that

$$\psi_0(\bar{t}) - \psi_{j_k}(\bar{t}) < (\psi_0(\bar{t}) - \ell)/2 \text{ for all } k \geq N,$$

so that

$$\psi_{j_k}(\bar{t}) > (\psi_0(\bar{t}) + \ell)/2 \text{ for all } k \geq N.$$

As $\psi_0(\bar{t}) > \ell$ by (4.8), this yields that $\psi_{j_k}(\bar{t}) > \ell$ for all $k \geq N$. Thus, $\theta(u_{j_k}) > \bar{t}$. In view of (4.7), we obtain that $\theta(u_{j_k}) > \theta(u) - \varepsilon$ for all $k \geq N$, so that weakly lower semi-continuity holds in Case 2. Combining this with the earlier discussion of Case 1, we conclude that θ is a weakly lower semi-continuous functional over $U(K)$. The proposition then follows by observing that $t(x_0,\ell,A,\gamma,u) = \theta(u)$. □

The following statement, which is the main result of the present section, follows from Proposition 4.4 and Lemma 3.13 through an application of the Generalized Weierstrass Theorem (e.g., Zeidler (1985)). Specifically, the Generalized Weierstrass Theorem states that a weakly lower semi-continuous function attains a minimum in a weakly compact set. Thus, the weakly lower semi-continuous function $t(x_0,\ell,A,\gamma,u)$ attains its minimum $t^*(x_0,\ell,A,\gamma)$ in the weakly compact set of input signals $U(K)$. In other words, there is an input signal $u^*(x_0,\ell,A,\gamma) \in U(K)$ that achieves this minimum, validating the following statement.

Theorem 4.10. *Let $A_0, A, \ell, \gamma > 0$ be real numbers, where $A > A_0$. Assume that the nominal system Σ_0 is (K,A_0)-controllable from the initial state x_0, and that the uncertainty parameter γ is compatible with Proposition 3.5 for the current A, A_0, and ℓ. Then, referring to the notation of Problem 2.14, the following hold.*
(i) There is a finite minimal time $t^*(x_0,\ell,A,\gamma)$.

(ii) There is an optimal input signal $u^*(x_0,\ell,A,\gamma) \in U(K)$ satisfying $t^*(x_0,\ell,A,\gamma) = t(x_0,\ell,A,\gamma,u^*(x_0,\ell,A,\gamma))$ without violating the state amplitude bound A. □

5. Approximating optimal performance

Optimal input signals $u^*(x_0,\ell,A,\gamma)$ of Problem 2.14, when they exist, are, in general, vector-valued functions of time. Needless to say, such functions may be difficult to calculate and implement. In the present section, we show that the performance of optimal input signals can be approximated as closely as desired by bang-bang input signals. As bang-bang input signals are relatively easy to calculate and implement, this section brings forward a relatively simple methodology of designing and implementing controllers that achieve close to optimal performance.

Recall that the task of the controller C of Figure 1.1 is to bring operating errors as quickly as possible down to the specified error bound ℓ, without violating the specified state amplitude bound A. The next statement indicates that, by increasing the operating error bound ℓ ever so slightly to ℓ', a bang-bang input signal can reduce the operating error to the bound ℓ' at least as quickly as the optimal time $t^*(x_0,\ell,A,\gamma)$ for the error bound ℓ; this is achieved without violating the specified state amplitude bound A. The accurate statement is as follows.

Theorem 5.1. *Let $A_0,A,\ell,\ell' > 0$ be real numbers, where $A > A_0$ and $\ell' > \ell$. Assume that the nominal system Σ_0 is $(K,A_0)-$controllable from the initial state x_0. Then, there are an uncertainty parameter $\gamma > 0$ and a bang-bang input signal $u^{\pm} \in U(K)$ such that $t(x_0,\ell',A,\gamma,u^{\pm}) \leq t^*(x_0,\ell,A,\gamma)$, where $u^{\pm}$ has a finite number of switchings.* □

As can be seen, Theorem 5.1 relies on two conditions: *(i) the nominal model Σ_0 must be* $(K,A_0)-$controllable, and *(ii)* the uncertainty parameter γ must not be excessively large. These two conditions are independent of each other, since the first is a property of the nominal model Σ_0, while the second refers to deviations of the active system Σ from the nominal model Σ_0. Further discussion of these conditions is provided following the proof of Theorem 5.1 (see Remarks 5.22 and 5.23).

The proof of Theorem 5.1 appears later in this section; it depends on the next statement, which shows that the response to any input signal can be approximated by the response to a bang-bang input signal (compare to Choi and Hammer (2018), where a similar result is derived for systems with delay; see also Yu and Hammer (2016a,b), where a slightly weaker statement is proved; and Chakraborty and Hammer (2009b, 2010), where a related statement is proved for linear systems).

Theorem 5.2. *Let Σ be a system of the form (2.2) with the initial state x_0, let $u \in U(K)$ be an input signal of Σ, and let $t' > 0$ be a finite time. Then, for every real number $\varepsilon > 0$, there is a bang-bang input signal $u^{\pm} \in U(K)$ (with a finite number of switchings) and an uncertainty parameter $\gamma > 0$ for which the following is true. The difference between the response $x(t) := \Sigma(x_0,u,t)$ of Σ to u and the response $x^{\pm}(t) := \Sigma(x_0,u^{\pm},t)$ of Σ to $u^{\pm}$ satisfies the inequality $|x(t)-x^{\pm}(t)| < \varepsilon$ at all times $0 \leq t \leq t'$ and for all members $\Sigma \in \mathcal{F}_{\gamma}(\Sigma_0)$.* □

To prove Theorem 5.2, we need the following statement (see also Choi and Hammer (2017), where a similar statement is proved for delay-differential systems; and Yu and Hammer (2016a), where a slightly weaker statement appears).

Lemma 5.3. *Let Σ be a system of the form (2.2) with functions $a(t,x)$ and $b(t,x)$ that are subject to (2.3). Let x_0 be the initial state of Σ, let $t' > 0$ be a finite time, and denote $x(t) := \Sigma(x_0,u,t)$. Then, for every real number $\zeta > 0$, there are real numbers $\beta(x_0,\zeta,t') > 0$ and $\gamma > 0$ such that the following is valid for all input signals $u \in U(K)$ and for all systems $\Sigma \in \mathcal{F}_{\gamma}(\Sigma_0)$: $|b(t_1,x(t_1))-b(t_2,x(t_2))| < \zeta$ for all times $t_1,t_2 \in [0,t']$ satisfying $|t_1-t_2| < \beta(x_0,\zeta,t')$.*

Proof. For an input signal $u \in U(K)$, the response $x(t) = \Sigma(x_0,u,t)$ is a continuous function of t, since it comes from the integration of bounded Lebesgue measurable functions over a finite time domain. As b is a continuous function as well, the composite function $b(t,x(t))$ is a continuous function of t, uniformly continuous on the compact interval $[0,t']$. As a result, there is, for every real number $\zeta > 0$, a real number $\beta(\zeta,u) > 0$ such that $|b(t_1,x(t_1))-b(t_2,x(t_2))| < \zeta$ for all $t_1,t_2 \in [0,t']$ satisfying $|t_1-t_2| < \beta(\zeta,u)$. We need to show that $\beta(\zeta,u)$ can be chosen to be the same for all $u \in U(K)$ and all $\Sigma \in \mathcal{F}_{\gamma}(\Sigma_0)$.

Choose a real number $\zeta' \in (0,\zeta)$ and a member $\Sigma \in \mathcal{F}_{\gamma}(\Sigma_0)$; denote

$$\beta'(\zeta',u) := \sup\{|t_1-t_2| : t_1,t_2 \in [0,t'] \text{ and } |b(t_1,x(t_1))-b(t_2,x(t_2))| < \zeta'\},$$

and set

$$\beta^*(\zeta') := \inf_{u \in U(K)} \beta'(\zeta',u).$$

Then, if there is a real number $\beta_0 > 0$ such that $\beta^*(\zeta') > \beta_0$ for all $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$, the lemma holds for $\beta(x_0, \zeta, t') = \beta_0$, and the proof concludes. To prove that this is the case, we show first that $\beta_0 = 0$ is not a valid option.

To that end, assume, by contradiction, that there is a system $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ for which $\beta^*(\zeta') = 0$. Then, there is a sequence $\{u_i\}_{i=1}^{\infty} \subseteq U(K)$ for which $\lim_{i\to\infty} \beta(\zeta', u_i) = 0$. In view of Lemma 3.13, there is a subsequence $\left\{u_{i_k}\right\}_{k=1}^{\infty}$ that converges weakly to a member $u \in U(K)$. By Lemma 3.15, there is, for every real number $\delta > 0$, an integer $N_\delta > 0$ such that

$$\sup_{t\in[0,t']} \left\{|\Sigma(x_0, u_{i_k}, t) - \Sigma(x_0, u, t)|\right\} < \delta \tag{5.4}$$

for all integers $k \geq N_\delta$. In view of Proposition 3.1, the response $x(t)$ is bounded: $|x(t)| \leq M(t')$ for all $t \in [0, t']$. Thus, the continuous function $b : R^+ \times R^n \to R^{n\times m}$ is uniformly continuous over the compact domain $[0, t'] \times [-M(t'), M(t')]^n$; consequently, there is, for every real number $\delta' > 0$, a real number $\zeta'' > 0$ such that $|b(t, y) - b(t, y')| < \delta'/4$ for all $|y|, |y'| \leq M(t')$ satisfying $|y - y'| < \zeta''$. Selecting $\delta = \zeta''$ in (5.4), we obtain for all integers $k \geq N_{\zeta''}$ that

$$|b(t, \Sigma(x_0, u_{i_k}, t)) - b(t, \Sigma(x_0, u, t))| < \delta'/4 \tag{5.5}$$

for all $t \in [0, t']$.

Next, by the uniform continuity of $b(t, x(t))$ as a function of t over the compact interval $t \in [0, t']$, there is a real number $\beta > 0$ such that

$$|b(t_1, \Sigma(x_0, u, t_1)) - b(t_2, \Sigma(x_0, u, t_2))| < \delta'/4$$

for all $t_1, t_2 \in [0, t']$ satisfying $|t_1 - t_2| < \beta$. Applying (5.5), this yields

$$\begin{aligned}
&\left|b(t_1, \Sigma(x_0, u_{i_k}, t_1)) - b(t_2, \Sigma(x_0, u_{i_k}, t_2))\right| \\
&\leq \left|b(t_1, \Sigma(x_0, u_{i_k}, t_1)) - b(t_1, \Sigma(x_0, u, t_1))\right| \\
&+ |b(t_1, \Sigma(x_0, u, t_1)) - b(t_2, \Sigma(x_0, u, t_2))| \\
&+ \left|b(t_2, \Sigma(x_0, u, t_2)) - b(t_2, \Sigma(x_0, u_{i_k}, t_2))\right| \\
&\leq \delta'/4 + \delta'/4 + \delta'/4 = 3\delta'/4
\end{aligned}$$

for all $t_1, t_2 \in [0, t']$ with $|t_1 - t_2| < \beta$ and for all $k \geq N_{\zeta''}$. As $\beta > 0$, it is not possible that $\beta^*(\zeta') = 0$ for all $\zeta' > 0$.

Further, replacing t_{A_0} by t' in the proof of Proposition 3.5, referring to (3.11), and using the notation of that proof, select a real number $\gamma > 0$ satisfying

$$\gamma < \frac{\delta'}{8\alpha^+ \eta_1 \eta^{q-1}}\ . \tag{5.6}$$

Then, using (2.3) and (3.10), we obtain for any member $\Sigma' \in \mathcal{F}_\gamma(\Sigma_0)$ the inequality

$$\begin{aligned}
&\left|b(t_1, \Sigma'(x_0, u_{i_k}, t_1)) - b(t_2, \Sigma'(x_0, u_{i_k}, t_2))\right| \\
&\leq \left|b(t_1, \Sigma'(x_0, u_{i_k}, t_1)) - b(t_1, \Sigma(x_0, u_{i_k}, t_1))\right| \\
&+ \left|b(t_1, \Sigma(x_0, u_{i_k}, t_1)) - b(t_2, \Sigma(x_0, u_{i_k}, t_2))\right| \\
&+ \left|b(t_2, \Sigma'(x_0, u_{i_k}, t_2)) - b(t_2, \Sigma(x_0, u_{i_k}, t_2))\right| \\
&\leq 2\alpha^+ \gamma \eta_1 \eta^{q-1} + 3\delta'/4 \\
&\leq \delta'
\end{aligned}$$

for all $|t_1 - t_2| < \beta$. Finally, selecting sufficiently small δ', this yields

$$\left|b(t_1, \Sigma'(x_0, u_{i_k}, t_1)) - b(t_2, \Sigma'(x_0, u_{i_k}, t_2))\right| < \zeta$$

for all $|t_1 - t_2| < \beta$, all $k \geq N_{\zeta''}$, and all members $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$. As $\beta > 0$, our proof concludes. □

We turn now to the proof of Theorem 5.2.

Proof of Theorem 5.2. Let $t_1, t_2 \in [0, t']$, $t_1 < t_2$, be two numbers to be selected later; let $\lambda > 0$ be another number to be selected later such that $p := (t_2 - t_1)/\lambda$ is an integer, and consider the partition

$$[t_1, t_2] = \{[t_1, t_1 + \lambda], [t_1 + \lambda, t_1 + 2\lambda], \cdots, [t_1 + (p-1)\lambda, t_2]\}\,. \tag{5.7}$$

Let $u=(u_1,u_2,\cdots,u_m)^\top\in U(K)$ be an input signal of the system Σ. The fact that each component u_i of u satisfies $|u_i|\le K$ implies that, for every $q=0,1,2,\ldots,p-1$, there are numbers $\theta_1^q,\theta_2^q,\ldots,\theta_m^q\in[t_1+q\lambda,t_1+(q+1)\lambda]$ satisfying

$$K[2(\theta_i^q-(t_1+q\lambda))-\lambda]=\int_{t_1+q\lambda}^{t_1+(q+1)\lambda}u_i(s)ds \tag{5.8}$$

for all $i=1,2,\ldots,m$ and all $q=0,1,2,\ldots,p-1$. We use $\{\theta_i^q\}$ as switching times for a bang-bang signal $u^\pm=\left(u_1^\pm,\cdots,u_m^\pm\right)^\top$ given by

$$u_i^\pm(t):=\begin{cases}+K & \text{for } t\in[t_1+q\lambda,\theta_i^q),\\ -K & \text{for } t\in[\theta_i^q,t_1+(q+1)\lambda),\ \text{if } \theta_i^q<t_1+(q+1)\lambda,\end{cases} \tag{5.9}$$

$i=1,2,\ldots,m$, $q=0,1,\ldots,p-1$. Then, by (5.8), we have

$$\int_{t_1+q\lambda}^{t_1+(q+1)\lambda}\left(u_i(s)-u_i^\pm(s)\right)ds=0, \tag{5.10}$$

$i=1,2,\ldots,m$, $q=0,1,2,\ldots,p-1$.

Next, for a member $\Sigma\in\mathcal{F}_\gamma(\Sigma_0)$, set $x(t):=\Sigma(x_0,u,t)$ and $x^\pm(t):=\Sigma(x_0,u^\pm,t)$, and examine the difference

$$\xi(t):=x(t)-x^\pm(t),t\in[0,t'].$$

By (2.2), we can write for a time $t\in[t_1,t_2]$ that

$$\begin{aligned}\xi(t)=\xi(t_1)+\int_{t_1}^{t}\Big[&a(s,x(s))-a(s,x^\pm(s))\\ &+b(s,x(s))u(s)-b(s,x^\pm(s))u^\pm(s)\Big]ds.\end{aligned}$$

As the initial state of Σ is x_0 for all inputs, it follows that

$$\xi(0)=0. \tag{5.11}$$

Invoking (2.3) together with the input signal bound K, we obtain

$$\begin{aligned}\sup_{t\in[t_1,t_2]}|\xi(t)|\le|\xi(t_1)|&+\alpha^+\left(\sup_{s\in[t_1,t_2]}|\xi(s)|\right)(t_2-t_1)\\ &+\sup_{t\in[t_1,t_2]}\left|\int_{t_1}^{t}b(s,x(s))\left(u(s)-u^\pm(s)\right)ds\right|\\ &+(\alpha^+)(t_2-t_1)\sup_{s\in[t_1,t_2]}|\xi(s)|\,K,\end{aligned}$$

or

$$\begin{aligned}&\left[1-\alpha^+(1+K)(t_2-t_1)\right]\sup_{t\in[t_1,t_2]}|\xi(t)|\\ &\le|\xi(t_1)|+\sup_{t\in[t_1,t_2]}\left|\int_{t_1}^{t}b(s,x(s))\left(u(s)-u^\pm(s)\right)ds\right|.\end{aligned} \tag{5.12}$$

To continue, let $\eta\in(0,t'-t_1]$ be such that $\alpha^+(1+K)\eta<1$. Setting

$$\mu(\eta):=\frac{1}{1-\alpha^+(1+K)\eta},$$

and

$$t_2:=t_1+\eta, \tag{5.13}$$

we get from (5.12) that

$$\begin{aligned}\sup_{t\in[t_1,t_2]}|\xi(t)|&\le\mu(\eta)\,|\xi(t_1)|\\ &+\mu(\eta)\sup_{t\in[t_1,t_1+\eta]}\left|\int_{t_1}^{t}b(s,x(s))\left(u(s)-u^\pm(s)\right)ds\right|.\end{aligned} \tag{5.14}$$

To estimate the integral, recall the partition (5.7), and denote by $q(t) \in \{0,1,2,\dots,p-1\}$ the integer for which $t \in [q(t)\lambda,(q(t)+1)\lambda]$. Then,

$$\begin{aligned}
&\sup_{t\in[t_1,t_1+\eta]}\left|\int_{t_1}^{t} b(s,x(s))\left(u(s)-u^{\pm}(s)\right)ds\right| \\
&= \sup_{t\in[t_1,t_1+\eta]}\left|\sum_{i=0}^{q(t)-1}\int_{t_1+i\lambda}^{t_1+(i+1)\lambda} b(s,x(s))\left(u(s)-u^{\pm}(s)\right)ds\right. \\
&\left.+\int_{t_1+q(t)\lambda}^{t} b(s,x(s))\left(u(s)-u^{\pm}(s)\right)ds\right| \\
&\le \sup_{t\in[t_1,t_1+\eta]}\left|\sum_{i=0}^{q(t)-1} b(t_1+i\lambda,x(t_1+i\lambda))\int_{t_1+i\lambda}^{t_1+(i+1)\lambda}\left(u(s)-u^{\pm}(s)\right)ds\right| \\
&+ \sup_{t\in[t_1,t_1+\eta]}\left|\sum_{i=0}^{q(t)-1}\int_{t_1+i\lambda}^{t_1+(i+1)\lambda}\left[b(s,x(s))-b(t_1+i\lambda,x(t_1+i\lambda))\right]\left(u(s)-u^{\pm}(s)\right)ds\right| \\
&+ \sup_{t\in[t_1,t_1+\eta]}\left|\int_{t_1+q(t)\lambda}^{t} b(s,x(s))\left(u(s)-u^{\pm}(s)\right)ds\right|.
\end{aligned}$$

Using (5.10) yields

$$\begin{aligned}
&\sup_{t\in[t_1,t_2]}\left|\int_{t_1}^{t} b(s,x(s))\left(u(s)-u^{\pm}(s)\right)ds\right| \\
&\le \sum_{i=0}^{q(t)-1}\int_{t_1+i\lambda}^{t_1+(i+1)\lambda}\sup_{s\in[t_1+i\lambda,t_1+(i+1)\lambda]}\left|b(s,x(s))-b(t_1+i\lambda,x(t_1+i\lambda))\right|\left|u(s)-u^{\pm}(s)\right|ds \\
&+ \sup_{t\in[t_1,t_2]}\left|\int_{t_1+q(t)\lambda}^{t} b(s,x(s))\left(u(s)-u^{\pm}(s)\right)ds\right|.
\end{aligned} \tag{5.15}$$

Further, let $\zeta > 0$ be a real number and let $\beta(x_0,\zeta,t') > 0$ be the corresponding number from Lemma 5.3. Then, using

$$\lambda \le \beta(x_0,\zeta,t') \tag{5.16}$$

in the partition (5.7), applying Corollary 3.2 with the function b as c, and noting that $0 \le q(t)\lambda \le t_2 - t_1 = \eta$ by (5.13), we obtain from (5.15) that

$$\begin{aligned}
&\sup_{t\in[t_1,t_2]}\left|\int_{t_1}^{t} b(s,x(s))\left(u(s)-u^{\pm}(s)\right)ds\right| \\
&\qquad \le 2K\zeta\eta + 2KM_b(t')\lambda.
\end{aligned}$$

Thus, (5.14) reduces to

$$\begin{aligned}
\sup_{t\in[t_1,t_2]}|\xi(t)| &\le \mu(\eta)\,|\xi(t_1)| \\
&+ \mu(\eta)\left[2K\zeta\eta + 2KM_b(t')\lambda\right].
\end{aligned} \tag{5.17}$$

Now, given a real number $\delta > 0$, choose $\zeta > 0$ to satisfy $\mu(\eta)K\zeta\eta < \delta/4$ and choose $\lambda > 0$ to satisfy $\mu(\eta)KM_b(t')\lambda < \delta/4$ in addition to satisfying (5.16). Substituting into (5.17), we get

$$\sup_{t\in[t_1,t_2]}|\xi(t)| \le \mu(\eta)\,|\xi(t_1)| + \delta. \tag{5.18}$$

Next, for an integer r satisfying $r \ge t'/\eta$, build the partition

$$[0,t'] \subseteq \{[0,\eta],[\eta,2\eta],\dots,[(r-1)\eta,r\eta]\}.$$

Then, (5.18) together with (5.11) yield the recursive relation

$$\begin{aligned}
\sup_{t\in[i\eta,(i+1)\eta]}|\xi(t)| &\le \mu(\eta)\,|\xi(i\eta)| + \delta, \\
\xi(0) &= 0,
\end{aligned}$$

$i = 0,\ldots,r-1$. Using properties of linear recursions, this yields

$$\sup_{t\in[0,t']} |\xi(t)| \le \delta \sum_{i=0}^{r} (\mu(\eta))^i.$$

Finally, referring to ε of the theorem's statement and choosing $\delta > 0$ to satisfy

$$\delta < \varepsilon \Big/ \left(\sum_{i=0}^{r} (\mu(\eta))^i \right) \tag{5.19}$$

validates the theorem. □

The following property of the minimal time $t^*(x_0,\ell,A,\gamma)$ of (2.13) is helpful.

Proposition 5.20. *The minimal time $t^*(x_0,\ell,A,\gamma)$ of (2.13) is a monotone decreasing function of the state amplitude bound A.*

Proof. Consider two state amplitude bounds $A' > A > 0$, and let $u^*(x_0,\ell,A,\gamma)$ be an optimal input signal that abides by the state amplitude bound A. Then, by (2.13), we have that $|\Sigma(x_0,u^*(x_0,\ell,A,\gamma),t^*(x_0,\ell,A,\gamma))|_2^2 \le \ell$ and that $|\Sigma(x_0,u^*(x_0,\ell,A,\gamma),t)| \le A$ at all times t satisfying $0 \le t \le t^*(x_0,\ell,A,\gamma)$. As $A < A'$, the input signal $u^*(x_0,\ell,A,\gamma)$ is also a member of the class of all input signals that take Σ into $\rho(\ell)$ and abide by the state amplitude bound A'. But then, since $u^*(x_0,\ell,A,\gamma)$ takes Σ into $\rho(\ell)$ at the time $t^*(x_0,\ell,A,\gamma)$, the minimal time achieved by an optimal input signal $u^*(x_0,\ell,A',\gamma)$ cannot be longer than $t^*(x_0,\ell,A,\gamma)$. Thus, $A' > A$ implies that $t^*(x_0,\ell,A',\gamma) \le t^*(x_0,\ell,A,\gamma)$, and our proof concludes. □

We arrive now at the proof of the main result of this section.

Proof of Theorem 5.1. Considering that $A > A_0$, there is a real number A' satisfying $A > A' > A_0$; choose such an A'. In view of Theorem 4.10 and the assumptions of the current theorem, there is an uncertainty parameter $\gamma > 0$ and an optimal input signal $u^* := u^*(x_0,\ell,A',\gamma) \in U(K)$ solving Problem 2.14*(i)* with the state amplitude bound A', the error bound ℓ, and the minimal time $t^* := t^*(x_0,\ell,A',\gamma)$. Then, $\Sigma(x_0,u^*,t^*) \in \rho(\ell)$ and $|\Sigma(x_0,u^*,t)| \le A'$ for all $t \in [0,t^*]$ and all $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$.

Further, according to Theorem 5.2, there is, for every real number $\varepsilon > 0$, a bang-bang input signal $u^\pm \in U(K)$ with a finite number of switchings for which $|\Sigma(x_0,u^*,t)-\Sigma(x_0,u^\pm,t)| < \varepsilon$ for all $t \in [0,t^*]$ and all $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$. Let us choose $0 < \varepsilon \le A - A'$. Then, $|\Sigma(x_0,u^\pm,t)| \le A$ for all $t \in [0,t^*]$ and all $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$, so that the output bound A is not violated.

Next, invoking the relation $z^\top z = y^\top y - 2y^\top(y-z) + (y-z)^\top(y-z) \le y^\top y + 2n|y||y-z| + n|y-z|^2$, which is valid for every pair of vectors $y,z \in R^n$, we can write

$$\begin{aligned}
&\Sigma^\top(x_0,u^\pm,t^*)\Sigma(x_0,u^\pm,t^*)\\
&\le \Sigma^\top(x_0,u^*,t^*)\Sigma(x_0,u^*,t^*)\\
&+2n\,|\Sigma(x_0,u^*,t^*)|\left|\Sigma(x_0,u^\pm,t^*)-\Sigma(x_0,u^*,t^*)\right|\\
&+n\left|\Sigma(x_0,u^\pm,t^*)-\Sigma(x_0,u^*,t^*)\right|^2\\
&\le \ell + 2n\sqrt{\ell}\varepsilon + n\varepsilon^2
\end{aligned}$$

for all $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$. In other words, we get $\Sigma(x_0,u^\pm,t^*) \in \rho(\ell')$ for all ℓ' satisfying

$$\ell + 2n\sqrt{\ell}\varepsilon + n\varepsilon^2 \le \ell'. \tag{5.21}$$

Denote

$$\varepsilon' := -\sqrt{\ell} + \sqrt{\ell + (\ell'-\ell)/n}\ ;$$

then, the quadratic inequality (5.21) and the previous paragraph show that, for every $\varepsilon > 0$ satisfying $\varepsilon \le \min\{\varepsilon', A-A'\}$, we have $\Sigma(x_0,u^\pm,t^*) \in \rho(\ell')$ and $|\Sigma(x_0,u^\pm,t| \le A$ for all $t \in [0,t^*]$ and all $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$. Thus, $t(x_0,\ell',A,\gamma,u^\pm) \le t^*$ and, since $t^* \le t^*(x_0,\ell,A,\gamma)$ by Proposition 5.20, we obtain that $t(x_0,\ell',A,\gamma,u^\pm) \le t^*(x_0,\ell,A,\gamma)$. This concludes our proof. □

Theorem 5.1 points to a relatively easy methodology for achieving close to optimal performance: replace optimal input signals by bang-bang input signals. Bang-bang signals are relatively easy to calculate and implement, since they are determined by a finite list of scalars – their switching times. In fact, experience shows that fairly simple bang-bang signals, i.e., bang-bang signals with a relatively small number of switching times, often yield performance that is almost indistinguishable from optimal performance (see Examples 6.1 and 6.2, for instance).

Remark 5.22. Testing $(K, A_0)-$controllability of the nominal system Σ_0: according to Theorem 5.2, the response to any input signal can be approximated by the response to a bang-bang input signal. As a result, $(K, A_0)-$controllability of the nominal system Σ_0 can be tested by a numerical search over a class of bang-bang input signals (see Example 6.1 for more details). Such a numerical search is relatively simple to implement, since a bang-bang signal is characterized by a finite list of scalars — its switching times. □

Remark 5.23. Values of the uncertainty parameter γ that are compatible with the requirements of Theorem 5.1 are described by the inequalities (3.11) and (5.6). Note that these inequalities describe sufficient conditions on values of γ; larger values of γ may be possible. □

Remark 5.24. Approximation of optimal performance by bang-bang input signals as described by Theorem 5.1 can be directly generalized to systems of the form

$$\Sigma^g : \begin{array}{l} \dot{x}(t) = a(t, x(t)) + b(t, x(t))g(u(t)), \\ x(0) = x_0, \end{array}$$

where $x : R^+ \to R^n$ is the state; $u : R^+ \to R^m$ is the input signal; $a : R^+ \times R^n \to R^n$ and $b : R^+ \times R^n \to R^{n\times m}$ are functions satisfying (2.3), (2.4), (2.5) and (2.7); and $g = (g_1, g_2, \ldots, g_m)^\top : R^m \to R^m$ is a continuous function that satisfies the following conditions: every component $g_i : R \to R : u_i \mapsto g_i(u_i)$ of g is a function of the component u_i of u only, and it includes the values K and $-K$ in its image.

Let K_i^+ and K_i^- be real numbers for which $g_i(K_i^+) = K$ and $g_i(K_i^-) = -K$, $i = 1, 2, \ldots, n$. Let $u^{g\pm}(t)$ be a piecewise constant input signal whose $i-th$ component switches between the values K_i^+ and K_i^-, $i = 1, 2, \ldots, n$. A slight reflection shows that this signal achieves for Σ^g an effect similar to the effect that the signal $u^\pm$ of Theorem 5.1 achieves for the system Σ. In other words, $u^{g\pm}$ drives Σ^g so as to achieve close to optimal performance. This feature is utilized in Example 6.2. □

6. Examples

In this section, we provide two examples, the second of which demonstrates Remark 5.24.

Example 6.1. Consider the system

$$\Sigma : \begin{array}{l} \dot{x}_1(t) = -x_1(t) - \sin(t)\sin x_1(t) - x_2(t) + u(t), \\ \dot{x}_2(t) = -x_1(t) + dx_2(t) - u(t), \end{array}$$

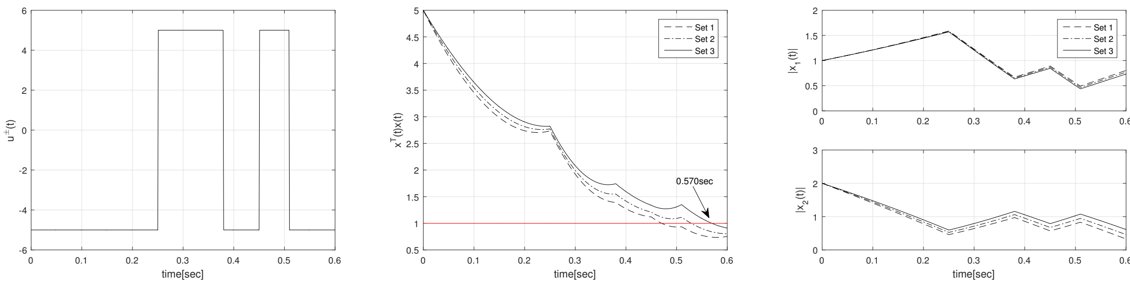
where the parameter d is an unspecified constant parameter in the range $0.1 \le d \le 0.5$. The initial state is $x_0 = [-1, -2]^\top$. The input signal bound of Σ is $K = 5$, so that only input signals satisfying $|u|_\infty \le 5$ are permitted. The state amplitude bound is $A = 2$, and the error bound is $\ell = 1$.

To check whether the system Σ is $(5,2)-$controllable, it is sufficient, according to Theorem 5.2, to test whether Σ can be taken from the initial state x_0 to the origin by a bang-bang input signal, without violating the input amplitude bound 5 and the state amplitude bound 2. To perform this test, we use a numerical search process similar to the one described in Choi and Hammer (2019) to search for a bang-bang input signal that satisfies these requirements. A result of this numerical search is shown in Figure 6.1, which demonstrates one appropriate bang-bang input signal.

The same numerical search process also shows that, under the specified input and state constraints, the shortest time within which the origin can be reached from x_0 is approximately $t^* = 0.57$ seconds. A similar time can be achieved by the bang-bang input signal of Figure 6.1(A), which, as can be seen from the figure, has only four switching times. The response to this input signal is shown in Figures 6.1(B) and 6.1(C) for three values of the parameter d:

$$\begin{array}{l} \text{Set 1:}\ \ d = 0.1, \\ \text{Set 2:}\ \ d = 0.3, \\ \text{Set 3:}\ \ d = 0.5. \end{array}$$

For comparison, the minimal time with no state amplitude bound (Yu and Hammer (2016a)) is 0.501 (see Figure 6.2, where the same values of d are used). Clearly, imposing a state amplitude bound may lengthen the minimal time to target.

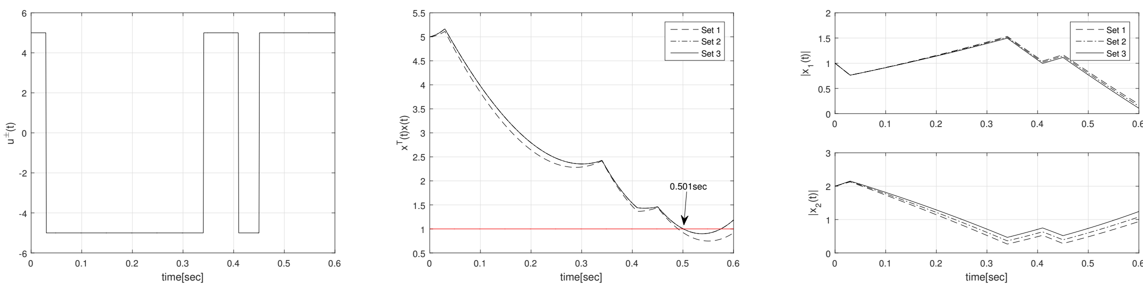


(A) Bang-bang input signal

(B) Trajectory of $x^\top(t)x(t)$

(C) Individual state trajectories

FIGURE 6.1. Control results with state amplitude bound



(A) Bang-bang input signal

(B) Trajectory of $x^\top(t)x(t)$

(C) Individual state trajectories

FIGURE 6.2. Control without state amplitude bound

Example 6.2. This example demonstrates Remark 5.24. Consider the inverted pendulum of Bian et al. (2014) (with a slight modification):

$$\Sigma: \begin{array}{l} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = d_1 \sin x_1(t) + d_2 x_2(t) + d_3 \tanh u(t); \end{array}$$

here, the nominal parameter values are $d_1^0 = 24.527$, $d_2^0 = -0.107$, $d_3^0 = 12.5$, and the initial state is $x_0 = [\pi/8, -2]^\top$. The state $x_1(t)$ is the pendulum's angle of deviation from the perpendicular axis. The input amplitude bound is $K = 5$, so we are looking for input signals u with amplitude $|u|_\infty \le 5$; the state amplitude bound is $A = 2$, and the error bound is $\ell = 0.1$. To account for modeling uncertainty, the parameters d_1, d_2, and d_3 are unspecified within the ranges $d_1 \in [21, 27]$, $d_2 \in [-0.3, -0.1]$, and $d_3 \in [10, 14]$. In this example, different uncertainty bounds are used for each parameter.

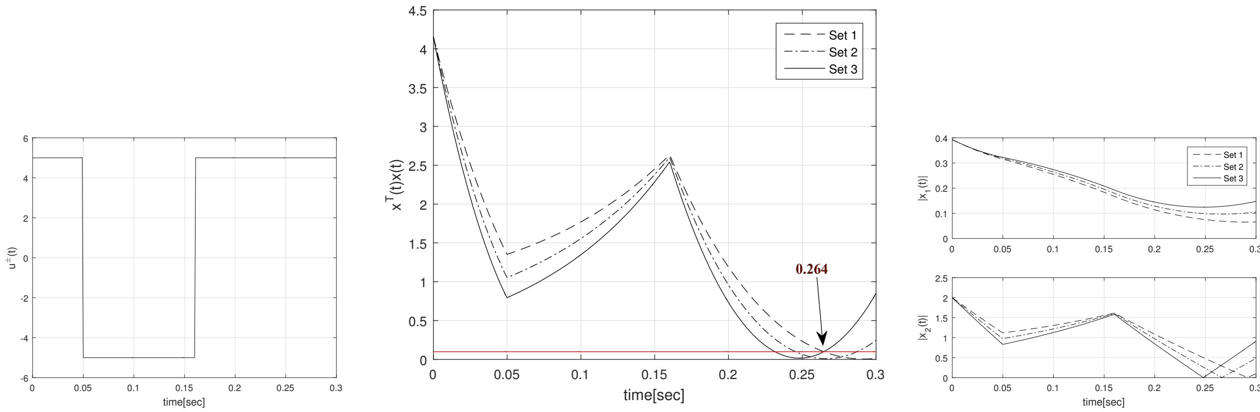
A numerical search process similar to the one described in Example 6.1 shows that this system is $(5, 2)-$controllable. It also shows that the minimal time within which the origin can be reached from the initial state x_0, without violating the input and state constraints, is $t^* = 0.246$ seconds. A simple bang-bang input signal that achieves a similar time without violating the constraints, is depicted in Figure 6.3(A). The response to this input signal is shown in Figures 6.3(B) and (C) for the following three sets of parameter values:

Set 1: $d_1 = 21$, $d_2 = -0.3$, $d_3 = 10$;

Set 2: $d_1 = 24$, $d_2 = -0.2$, $d_3 = 12$;

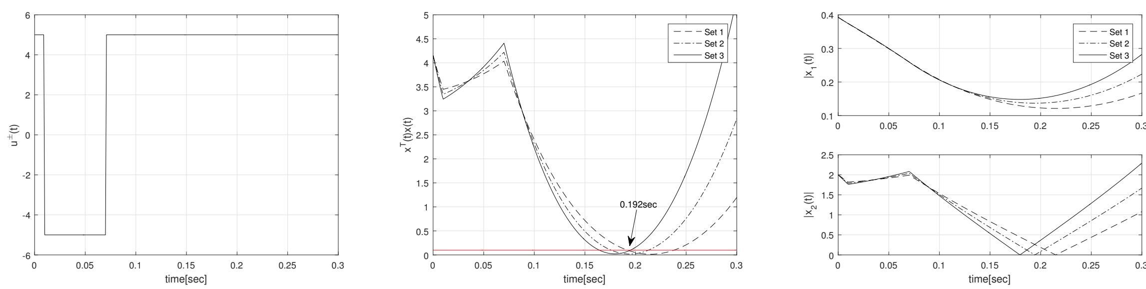
Set 3: $d_1 = 27$, $d_2 = -0.1$, $d_3 = 14$.

For comparison, Figure 6.4 demonstrates approximate optimal control for the case where no bound is imposed on the state amplitude. Releasing the state amplitude bound shortens the minimal time to 0.192 seconds (from 0.246 seconds).



(A) Bang-bang input signal (B) $x^\top(t)x(t)$ (C) Individual state trajectories

FIGURE 6.3. Control with state amplitude bound



(A) Bang-bang input signal (B) $x^\top(t)x(t)$ (C) Individual state trajectories

FIGURE 6.4. Control with no state amplitude bound

7. CONCLUSION

In this paper, we revisited the problem of reducing operating errors in minimal time in the aftermath of an episode of feedback loss, a problem first examined in (Yu and Hammer (2016a,b)). In the present paper, we imposed a new constraint: a bound on the maximal overshoot of the controlled system. We showed that optimal controllers that abide by this new constraint exist under rather general conditions. We also showed that optimal performance can be approximated as closely as desired by controllers that generate bang-bang input signals. The possibility of using bang-bang signals instead of optimal signals substantially simplifies design and implementation.

The results of the present paper are relevant to a wide range of applications. One such common application is the design and implementation of sampled-data control systems. Recall that sampled-data systems operate without feedback between samples. Controllers developed in this paper can be used to reduce inter-sample errors as quickly as possible after the arrival of a new sample, while preventing undesirable overshoots of the controlled system.

REFERENCES

Bian, T., Jiang, Y., and Jiang, Z.-P. (2014). Adaptive dynamic programming and optimal control of nonlinear nonaffine systems. *Automatica*, 50(10):2624–2632.

Chakraborty, D. and Hammer, J. (2007). Optimizing system performance in the event of feedback failure. In *Proceedings of the 6th International Congress on Industrial and Applied Mathematics*, pages 2060009– 2060010, Zurich, Switzerland.

Chakraborty, D. and Hammer, J. (2008a). Bang-bang functions: Universal approximants for the solution of min-max optimal control problems. In *Proceedings of the International Symposium On Dynamic Games and Applications*, Wroclaw, Poland.

Chakraborty, D. and Hammer, J. (2008b). Preserving system performance during feedback failure. In *Proceedings of the IFAC World Congress*, Seoul, Korea.

Chakraborty, D. and Hammer, J. (2008c). Robust optimal control: Maximum time of low-error operation. In *Proceedings of the Fifth International Conference of Applied Mathematics and Computing*, Plovdiv, Bulgaria.

Chakraborty, D. and Hammer, J. (2009a). Control during feedback failure: Characteristics of the optimal solution. In *Proceedings of the 17th Mediterranean Conference on Control and Automation*, Thessaloniki, Greece.

Chakraborty, D. and Hammer, J. (2009b). Optimal control during feedback failure. *International Journal of Control*, 82(8):1448–1468.

Chakraborty, D. and Hammer, J. (2010). Robust optimal control: low-error operation for the longest time. *International Journal of Control*, 83(4):731–740.

Chakraborty, D. and Shaikshavali, C. (2009). An approximate solution to the norm optimal control problem. In *Proceedings of the IEEE International Conference on Systems, Man, and Cybernetics*, pages 4490–4495, San Antonio, TX, USA.

Choi, H.-L. and Hammer, J. (2017). Quick recovery after feedback loss: delay-differential systems. In *Proceedings of the International Conference on Control, Automation, and Systems*, Jeju, Korea.

Choi, H.-L. and Hammer, J. (2018). Optimal robust control of nonlinear time-delay systems: Maintaining low operating errors during feedback outages. *International Journal of Control*, 91(2):297–319.

Choi, H.-L. and Hammer, J. (2019). Fastest recovery after feedback disruption: Nonlinear delay-differential systems. *International Journal of Control*, 92(4):717–733.

Gamkrelidze, R. (1965). On some extremal problems in the theory of differential equations with applications to the theory of optimal control. *SIAM Journal on Control*, 3:106–128.

Kelendzheridze, D. (1961). On the theory of optimal pursuit. *Soviet Mathematics Doklady*, 2:654–656.

Luenberger, D. G. (1969). *Optimization by Vector Space Methods*. Wiley, New York.

Montestruque, L. and Antsaklis, P. (2004). Stability of model-based networked control systems with time-varying transmission times. *IEEE Transactions on Automatic Control*, 49(9):1562–1572.

Nair, G., Fagnani, F., Zampieri, S., and Evans, R. J. (2007). Feedback control under data rate constraints: an overview. *Proceedings of the IEEE*, 95(1):108 – 137.

Neustadt, L. (1966). An abstract variational theory with applications to a broad class of optimization problems i, general theory. *SIAM Journal on Control*, 4:505–527.

Neustadt, L. (1967). An abstract variational theory with applications to a broad class of optimization problems ii, applications. *SIAM Journal on Control*, 5:90–137.

Pontryagin, L., Boltyansky, V., Gamkrelidze, R., and Mishchenko, E. (1962). *The Mathematical Theory of Optimal Processes*. Wiley, New York, London.

Tonon, D., Aronna, M. S., and Kalise, D., editors (2017). *Optimal Control: Novel Directions and Applications*, volume 2180 of *Lecture Notes in Mathematics*. Springer International Publishing, Cham, Switzerland.

Warga, J. (1972). *Optimal Control of Differential and Functional Equations*. Academic Press, New York.

Willard, S. (2004). *General Topology*. Dover Publications, Mineola. NY.

Young, L. (1969). *Lectures on the Calculus of Variations and Optimal Control Theory*. W. B. Saunders, Philadelphia.

Yu, Z. and Hammer, J. (2016a). Fastest recovery after feedback disruption. *International Journal of Control*, 89(10):2121–2138.

Yu, Z. and Hammer, J. (2016b). Recovering from feedback failure in minimal time. In *Proceedings of the 10th IFAC Symposium on Nonlinear Control Systems*, Monterey, California, USA.

Zeidler, E. (1985). *Nonlinear Functional Analysis and its Applications III*. Springer-Verlag, New York.

Zhivogyladov, P. and Middleton, R. (2003). Networked control design for linear systems. *Automatica*, 39:743–750.