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Minimal Time Control of Nonlinear Systems: Optimal Robust State-Feedback

Jacob Hammer

Department of Electrical and Computer Engineering, University of Florida, Gainesville, FL 32611-6130, USA (e-mail: hammer@mst.ufl.edu).

Abstract: Optimal robust state-feedback controllers for minimal-time control of nonlinear systems are developed under constraints on the input and output signal amplitudes of the controlled system. Simple to design and implement controllers that approximate optimal performance are presented.

Keywords: optimal control; state feedback; minimal time

1 Introduction

Traditionally, the design of optimal feedback controllers for nonlinear systems is via the Hamilton-Jacobi-Bellman equation (Sobolev (1950), Bellman (1954), Miranda (1955), Kruzkov (1960), Pontryagin et al. (1962)), a nonlinear partial differential equation. To solve this equation for nonlinear systems, one must defer to numerical solutions; these are encumbered by high computational complexity.

This note focuses on optimal robust feedback control of nonlinear input-affine systems, with constraints on input and output signal amplitudes. We prove existence of optimal robust state-feedback controllers that guide such systems to target in minimal time (Section 4). We develop simple state-feedback controllers that approximate optimal performance (Section 5).

As one might expect, optimal feedback controllers developed in the current note provide better performance than open-loop controllers considered in Yu and Hammer (2016), especially when there is significant uncertainty about the controlled system's model (Section 6). This comes, of course, at the cost of more involved implementation.

In Figure 1.1, the input signal $u(t)$ of the controlled system Σ is generated by the state-feedback function φ – a function of the time t and the state $x(t)$ of Σ. The objective is to design φ to guide Σ in minimal time from an initial state $x(0) = x_0$ to a target state x_{target}. This must be accomplished without exceeding specified amplitude bounds of $K > 0$ and $A > 0$ on the input and output signals of Σ, respectively. After shifting coordinates, we can take $x_{target} = 0$. To accommodate uncertainties, we allow a maximal deviation of $\ell > 0$ from the target, so our objective is to reach the ball $\rho(\ell) := \{x : x^\top x \leq \ell\}$ in minimal time. Section 4 shows that optimal robust feedback controllers that fulfill these requirements exist under a mild controllability condition on the controlled system Σ. Here, ℓ is the *operating error bound*.

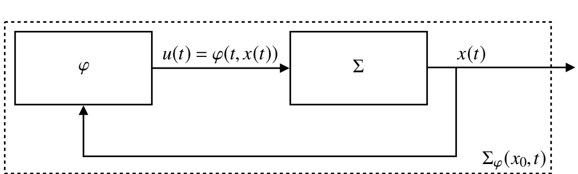

Fig. 1.1. State-feedback control

In Section 5, we show that optimal performance can be approximated by bang-bang feedback functions – functions whose components switch between the two values K and $-K$ as a function of time and state. Such feedback functions are much easier to calculate and implement than optimal feedback functions, since their values are in a finite discrete set. Our objectives can then be summarized as follows.

Problem 1.1. (i) *Derive conditions that guarantee the existence of optimal robust state-feedback functions φ that drive Σ in minimal time from x_0 to $\rho(\ell)$, while keeping input and output signals from exceeding the bounds K and A, respectively.*
(ii) *Derive easy to calculate and implement feedback functions that approximate optimal performance.* □

This note focuses on closed-loop optimal robust control of nonlinear systems. It draws on classical works on optimization, such as Kelendzheridze (1961), Pontryagin et al. (1962), Gamkrelidze (1965), Neustadt (1966, 1967), Luenberger (1969), Young (1969), Warga (1972), the references cited in these publications, and on earlier work by the author and coworkers on open-loop optimal control (Chakraborty and Hammer (2009, 2010), Chakraborty and Shaikshavali (2009), Yu and Hammer (2016), Choi and Hammer (2018, 2019), Hammer (2019)). Yet, the existence and approximation of optimal robust feedback controllers under input/output constraints have not been reported in the literature before.

The note is organized as follows. Sections 2 and 3 establish the mathematical framework; Section 4 proves the existence of optimal robust feedback controllers; Section 5 presents simple feedback functions that approximate optimal performance; Section 6 is an example; and Section 7 concludes the note.

2 Background

2.1 System equations

Denote by R the compactified set of real numbers; by R^n the set of n−dimensional real vectors; by R^+ the non-negative real numbers; by $|r|$ the absolute value of a number r; by $|V| := \max_{ij} |V_{ij}|$ the L^∞−norm of a constant matrix V; by $|W|_\infty := \sup_{t\geq 0} |W(t)|$ the L^∞−norm of a matrix function of time; by $|x|_2 := \left(x^\top x\right)^{1/2}$ the L^2−norm of a vector; and by $[-A, A]^n \subseteq R^n$ the set of all vectors x with $|x| \leq A$. The controlled system Σ is a nonlinear time-varying input-affine system

$$\Sigma : \dot{x}(t) = a(t, x(t)) + b(t, x(t))u(t), \quad x(0) = x_0; \tag{2.1}$$

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$x(t) \in R^n$ is the state; $u(t) \in R^m$ is the input; and $a : R^+ \times R^n \to R^n$ and $b : R^+ \times R^n \to R^{n\times m}$ satisfy the Lipchitz conditions

$$|a(t,y)-a(t,x)| \le \alpha^+|y-x|, |b(t,y)-b(t,x)| \le \alpha^+|y-x|, \quad (2.2)$$

where $\alpha^+ > 0$ is specified. To incorporate uncertainties, a and b are sums of nominal parts a_0, b_0 and unknown parts a_γ, b_γ:

$$a(t,x) = a_0(t,x)+a_\gamma(t,x), b(t,x) = b_0(t,x)+b_\gamma(t,x),$$

subject to the Lipschitz conditions

$$\begin{aligned}&|a_0(t,x')-a_0(t,x)| \le \alpha|x'-x|,\ a_0(t,0) = 0,\\ &|b_0(t,x')-b_0(t,x)| \le \alpha|x'-x|,\ |b_0(t,0)| \le \alpha;\end{aligned} \quad (2.3)$$

$$\begin{aligned}&|a_\gamma(t,x')-a_\gamma(t,x)| \le \gamma|x'-x|,\ a_\gamma(t,0) = 0,\\ &|b_\gamma(t,x')-b_\gamma(t,x)| \le \gamma|x'-x|, |b_\gamma(t,0)| \le \gamma;\end{aligned} \quad (2.4)$$

$\alpha,\gamma > 0$ are given and $\alpha^+ = \alpha+\gamma$. The *uncertainty parameter* γ quantifies the uncertainty of Σ. The nominal system is

$$\Sigma_0 : \dot{x}(t) = a_0(t,x(t))+b_0(t,x(t))u(t),\ x(0) = x_0. \quad (2.5)$$

2.2 Spaces

The Hilbert space $L_2^{\omega,m}$ consists of Lebesgue measurable functions $f,g : R^+ \to R^m$ with the inner product

$$\langle f,g\rangle := \int_0^\infty e^{-\omega s} f^\top(s)g(s)ds,\ \omega > 0.$$

The permissible input and output signals of Σ are, respectively, $U(K) := \{u \in L_2^{\omega,m} : |u|_\infty \le K\}, X(A) := \{x \in L_2^{\omega,n} : |x|_\infty \le A\}$

Notation 2.6. Given $\alpha,\gamma,K,A > 0$ and Σ_0 of (2.5), let $\mathcal{F}_\gamma(\Sigma_0)$ be the family of systems described by (2.1), (2.2), (2.3), and (2.4).
(i) Every member of $\mathcal{F}_\gamma(\Sigma_0)$ has the same initial state $x(0) = x_0$.
(ii) The same feedback function φ is used for all $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$.
(iii) All feedback functions are bounded by K.
(iv) States of all members of $\mathcal{F}_\gamma(\Sigma_0)$ are restricted to $X(A)$. □

Requirement *(ii)* of Notation 2.6 originates from fact that it is not known which member of $\mathcal{F}_\gamma(\Sigma_0)$ the controlled system Σ actually is, so feedback cannot be adjusted for each member.

2.3 State-feedback

State-feedback is provided by a Lebesgue measurable function $\varphi : R^+ \times R^n \to R^m$. The closed loop system yields the state $x(t) = \Sigma_\varphi(x_0,t)$ given by

$$\Sigma_\varphi : \dot{x}(t) = a(t,x(t))+b(t,x(t))\varphi(t,x(t)),\ x(0) = x_0.$$

State-feedback functions are in the Hilbert space $L_2^{\omega,n,m}$ of measurable functions $f,g : R^+ \times R^n \to R^m$ with inner product

$$\langle\langle f,g\rangle\rangle := \int_{R^+\times R^n} e^{-\omega(s+|z|_2)} f^\top(s,z)g(s,z)d(s,z),\ \omega > 0,$$

where $d(s,z)$ is the Lebesgue measure element in $R^+ \times R^n$. As input signals of Σ must be bounded by K, the class of permissible feedback functions is

$$\Phi(K) := \left\{\varphi \in L_2^{\omega,n,m} : |\varphi(t,x)| \le K \text{ for all } (t,x) \in R^+ \times R^n\right\}.$$

2.4 Convergence features

We need the following notions (e.g., Willard (2004), Zeidler (1985)).

Definition 2.7. *H is a Hilbert space with inner product $\langle\cdot,\cdot\rangle$.*
(i) A sequence $\{v_i\}_{i=1}^\infty \subseteq H$ *converges weakly* to a member $v \in H$ if $\lim_{i\to\infty}\langle v_i,y\rangle = \langle v,y\rangle$ for every $y \in H$.
(ii) A subset $W \subseteq H$ is *weakly compact* if every sequence in W has a subsequence that converges weakly to a member of W. □

The following is proved in Chakraborty and Hammer (2009).

Lemma 2.8. *The set of signals $U(K)$ is weakly compact.* □

The next statement's proof is similar (see Hammer (2019)).

Lemma 2.9. *The set of feedback functions $\Phi(K)$ is weakly compact in $L_2^{\omega,n,m}$.* □

2.5 Families of functions

With a feedback function φ, the input signal $u(t)$ of Figure 1.1 is the composition

$$u(t) = \varphi \circ x := \varphi(\cdot,x(\cdot)) : R^+ \to R^m : t \mapsto \varphi(t,x(t)),$$

where $\varphi \circ x \in L_2^{\omega,m}$ since $\varphi \in L_2^{\omega,n,m}$ and $x \in L_2^{\omega,n}$. Here, $x(t)$ represents a family of functions, since it is the response of any member of $\mathcal{F}_\gamma(\Sigma_0)$. We discuss next families of functions.

The *graph* $\Gamma(f)$ of a function $f \in L_2^{\omega,n}$ is all pairs $\Gamma(f) := \cup_{t\ge0}(t,f(t))$. The *graph* $\Gamma(F)$ of a family of functions $F \subseteq L_2^{\omega,n}$ is the union $\Gamma(F) := \cup_{f\in F}\Gamma(f)$. A *section* $\Gamma_\tau(F)$ at a time τ consists of all values at τ, i.e., $\Gamma_\tau(F) := \{x \in R^n : (\tau,x) \in \Gamma(F)\}$. All values of members of F are then $\Pi\Gamma(F) := \cup_{\tau\ge0}\Gamma_\tau(f)$.

Definition 2.10. A family $F \subseteq L_2^{\omega,n}$ is of *measure zero* if the section $\Gamma_\tau(F)$ is of measure zero for all $\tau \ge 0$. A statement is true for *almost every function* $f \in L_2^{\omega,n}$ if the family of functions for which the statement is untrue is of measure zero. □

Proposition 2.11. *Let $\{\varphi_i\}_{i=1}^\infty \subseteq \Phi(K)$ be a sequence weakly convergent to φ, let $F \subseteq L_2^{\omega,n}$ be a family of functions, and let $\varepsilon > 0$ be a real number. Then, for every $g \in U(K)$, there is an integer $N \ge 1$ for which $|\langle(\varphi_i-\varphi)\circ f,g\rangle| < \varepsilon$ for all $i \ge N$ and for almost all $f \in F$.*

Proof (sketch). By contradiction, assume that there is a family $F' \subseteq F$ of non-zero measure, a real number $\varepsilon > 0$, a subsequence $\{\varphi_{i_k}\}_{k=1}^\infty$, and a function $g \in U(K)$ for which $|\langle(\varphi_{i_k} - \varphi)\circ f,\ g\rangle| \ge \varepsilon$ for all $k \ge 1$ and all $f \in F'$. Then, $\lim_{k\to\infty}\langle(\varphi_{i_k}-\varphi)\circ f,g\rangle \ne 0$ for all $f \in F'$. One option is the case where $\lim_{k\to\infty}\langle(\varphi_{i_k}-\varphi)\circ f,g\rangle > 0$ for all $f \in F'$ and where, for some time $\tau \ge 0$, the section $\Gamma_\tau(F')$ includes a subset σ of non-zero measure. For a point $x \in \sigma$, let $f^x \in F'$ be a function satisfying $f^x(\tau) = x$ and set $\Gamma := \cup_{x\in\sigma}\Gamma(f^x)$. Build the function

$$\phi(t,x) := \begin{cases} g(t) & (t,x) \in \Gamma, \\ 0 & \text{otherwise.}\end{cases} : R^+ \times R^n \to R^m.$$

Then, for any function $h \in L_2^{\omega,n,m}$, we have $h^\top(t,x)\phi(t,x) = h^\top(t,f(t))g(t) = (h\circ f)^\top(t)g(t)$. Using $h := (\varphi_{i_k}-\varphi)$, we get

$$\lim_{k\to\infty}\langle\langle(\varphi_{i_k}-\varphi),\phi\rangle\rangle = \int_{\Pi\Gamma} e^{-\omega|x|_2}\left(\lim_{k\to\infty}\langle(\varphi_{i_k}-\varphi)\circ f^x,g\rangle\right)dx > 0,$$

contradicting the fact that $\{\varphi_i\}_{i=1}^\infty$ converges weakly to φ. Other options are analogous. □

2.6 Problem statement

Let $\Phi(x_0,K,A,\Sigma,t) \subseteq \Phi(K)$ be the class of state-feedback functions φ that keep the state of Σ_φ in $[-A,A]^n$ during the time interval $[0,t]$, i.e.,

$$\Phi(x_0,K,A,\Sigma,t) = \left\{\varphi \in \Phi(K) : \begin{array}{l}\Sigma_\varphi(x_0,\theta) \in [-A,A]^n \\ \text{for all } \theta \in [0,t].\end{array}\right\}$$

The class of state-feedback functions $\varphi \in \Phi(K)$ that keep the state of all members of $\mathcal{F}_\gamma(\Sigma_0)$ in $[-A,A]^n$ during $[0,t]$ is

$$\Phi(x_0,K,A,\gamma,t) = \left\{\varphi \in \Phi(K) : \sup_{\substack{\Sigma\in\mathcal{F}_\gamma(\Sigma_0)\\ 0\le\theta\le t}} |\Sigma_\varphi(x_0,\theta)| \le A\right\}.$$

For a system Σ with feedback function φ, the infimal time to $\rho(\ell)$ without violating constraints is

$$t(x_0,\ell,A,\Sigma,\varphi) = \inf_{t\ge0}\left\{\Sigma_\varphi(x_0,t) \in \rho(\ell), \varphi \in \Phi(x_0,K,A,\Sigma,t)\right\}.$$

When including almost all of $\mathcal{F}_\gamma(\Sigma_0)$, the infimal time to $\rho(\ell)$ is

$$t(x_0,\ell,A,\gamma,\varphi) := \inf_{t\geq 0}\left\{\begin{array}{l}\left(\operatorname*{ess\,sup}_{\Sigma\in\mathcal{F}_\gamma(\Sigma_0)}(|\Sigma_\varphi(x_0,t)|_2)^2\right)\leq \ell,\\ \varphi\in\Phi(x_0,K,A,\gamma,t),\end{array}\right\} \quad (2.12)$$

The infimal time among all state-feedback functions φ is

$$t^*(x_0,\ell,A,\gamma) = \inf_{\varphi\in\Phi(K)} t(x_0,\ell,A,\gamma,\varphi). \quad (2.13)$$

We show in Section 4 that $t^*(x_0,\ell,A,\gamma)$ is finite and there is an optimal state-feedback function $\varphi^*(x_0,\ell,A,\gamma)\in\Phi(K)$ satisfying

$$t^*(x_0,\ell,A,\gamma) = t(x_0,\ell,A,\gamma,\varphi^*(x_0,\ell,A,\gamma)), \quad (2.14)$$

if the nominal system Σ_0 satisfies a certain controllability condition. In these terms, Problem 1.1 becomes

Problem 2.15. (i) *State conditions that guarantee the existence of an optimal state-feedback function $\varphi^*(x_0,\ell,A,\gamma)\in\Phi(K)$.*
(ii) *Find easy to design and implement state-feedback functions that approximate optimal performance.* □

3 Feedback and continuity

3.1 Controllability

The following is a consequence of the fact that (2.1) has continuous solutions for bounded input functions (see Hammer (2019) for details).

Lemma 3.1. *For $T>0$ and a continuous function $c: R^+\times R^n \to R^p$, there is a bound $M_c(T)\geq 0$ such that $|c(t,\Sigma_\varphi(x_0,t))|\leq M_c(T)$ for all $t\in[0,T]$, all $\varphi\in\Phi(K)$, and all $\Sigma\in\mathcal{F}_\gamma(\Sigma_0)$.* □

As signals generated by feedback functions $\varphi\in\Phi(K)$ belong to $U(K)$, Problem 1.1*(i)* requires that there be an input signal $u\in U(K)$ that takes Σ from x_0 to the neighborhood of $x=0$, without violating the state amplitude bound. This leads to the following notion (Choi and Hammer (2018)).

Definition 3.2. A system Σ is (K,A)-*controllable* from an initial state x_0 if there is an input signal $u\in U(K)$ and a finite time $t_A\geq 0$ so that $\Sigma(x_0,u,t_A)=0$ and $|\Sigma(x_0,u,t)|\leq A$ for all $t\in[0,t_A]$. □

The next statement shows that (K,A)-controllability of the nominal system assures a finite optimal time $t^*(x_0,\ell,A,\gamma)$. It follows from Choi and Hammer (2018), since any input signal $u\in U(K)$ can be generated by a feedback function $\varphi(t,x)=u(t)$.

Proposition 3.3. *If the nominal system Σ_0 is (K,A_0)-controllable from initial state x_0, then, for every $A>A_0$, there is an uncertainty parameter $\gamma>0$ for which the time $t^*(x_0,\ell,A,\gamma)$ is finite.*

Thus, only one system (Σ_0) has to be tested to guarantee a finite optimal time. (K,A)-controllability can be determined via a relatively simple numerical search (Choi and Hammer (2018)).

We use the following notions (e.g., Zeidler (1985)).

Definition 3.4. H is a Hilbert space, S a subset of a H, and z is a member of S. A functional $F: S\to R$ is *weakly lower semi-continuous at z* if the following is true for every sequence $\{z_i\}_{i=1}^\infty\subseteq S$ that converges weakly to z: when $F(z)$ is bounded, there is, for every real number $\varepsilon>0$, an integer $N>0$ such that $F(z)-F(z_i)<\varepsilon$ for all $i\geq N$.
A function $G: S\times R^+\to R^n: (s,t)\mapsto G(s,t)$ is *weakly continuous* at z at a time t if the following is true for every sequence $\{z_i\}_{i=1}^\infty\subseteq S$ that converges weakly to z: for every real number $\varepsilon>0$, there is an integer $N>0$ such that $|G(z,t)-G(z_i,t)|<\varepsilon$ for all $i\geq N$.
A sequence of functions $\{G_i\}_{i=1}^\infty: S\times R^+\to R^n$ is *uniformly weakly convergent* to a function G over an interval $[t_1,t_2]$, $t_1<t_2$, if, for every real number $\varepsilon>0$, there is an integer $N>0$ such that $|G_i(z,t)-G(z,t)|<\varepsilon$ for all $z\in S$ and all $t\in[t_1,t_2]$. □

3.2 Feedback systems

Consider the feedback configuration of Figure 3.1, which has an external input signal v. With a system $\Sigma\in\mathcal{F}_\gamma(\Sigma_0)$, the differential equation is

$\Sigma_\varphi(x_0,v,t)$: $\dot{x}(t)=a(t,x(t))+b(t,x(t))[\varphi(t,x(t))+v(t)], x(0)=x_0$.

Then, $\Sigma_\varphi(x_0,t)=\Sigma_\varphi(x_0,0,t)$ and $\Sigma(x_0,v,t)=\Sigma_{\varphi=0}(x_0,v,t)$. By the next statement, $\Sigma_\varphi(x_0,v,t)$ is a continuous function of φ.

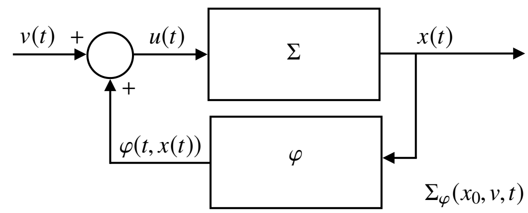


Fig. 3.1. Feedback system with external input

Theorem 3.5. *Let $\{\varphi_i\}_{i=1}^\infty\subseteq\Phi(K)$ be a sequence that converges weakly to $\varphi\in\Phi(K)$. Then, for almost every $\Sigma\in\mathcal{F}_\gamma(\Sigma_0)$, the sequence $\{\Sigma_{\varphi_i}(x_0,v,t)\}$ is uniformly weakly convergent to $\Sigma_\varphi(x_0,v,t)$ for all $v\in U(K)$ over any finite interval of time.* □

To prove Theorem 3.5, we need a few preliminary results. In the notation of the theorem, the next statement shows that the negative feedback $-\varphi$ nearly undoes the action of φ_i for large i.

Lemma 3.6. *Use the notation of Theorem 3.5. For almost every $\Sigma\in\mathcal{F}_\gamma(\Sigma_0)$, the sequence $\{\Sigma_{\varphi_i-\varphi}(x_0,v,t)\}_{i=1}^\infty$ converges weakly and uniformly to $\Sigma(x_0,v,t)$ for every input signal $v\in U(K)$ over every finite interval of time.*

Proof (sketch). Denote $x(t,i) := \Sigma_{(\varphi_i-\varphi)}(x_0,v,t)$, $x(t) := \Sigma(x_0,v,t)$, and $z(t,i) := x(t,i)-x(t)$. Then,

$$\dot{x}(t,i) = a(t,x(t,i))+b(t,x(t,i))[\varphi_i(t,x(t,i))-\varphi(t,x(t,i))+v(t)],$$

$x(0,i)=x_0$. Fix a time $T>0$; let $t_1<t_2\in[0,T]$, $t\in[t_1,t_2]$. Then,

$$\begin{aligned} z(t,i) &= z(t_1,i)+\int_{t_1}^t [a(s,x(s,i)-a(s,x(s))]ds \\ &+\int_{t_1}^t b(s,x(s,i))[\varphi_i(s,x(s,i))-\varphi(s,x(s,i))]ds \\ &+\int_{t_1}^t [b(s,x(s,i))-b(s,x(s))]v(s)ds. \end{aligned}$$

Using (2.2) and the fact that $v\in U(K)$ and $\varphi\in\Phi(K)$ yields

$$\begin{aligned} (1-\alpha^+(1+3K)(t_2-t_1))\sup_{s\in[t_1,t_2]}|z(s,i)| &\leq |z(t_1,i)| \\ &+\left|\int_{t_1}^{t_2} b(s,x(s))[\varphi_i(s,x(s,i))-\varphi(s,x(s,i))]ds\right|. \end{aligned}$$

Let $\eta>0$ be such that $\alpha^+(1+3K)\eta<1/2$; set $t_2:=t_1+\eta$. Then,

$$\begin{aligned} \sup_{s\in[t_1,t+\eta]}|z(s,i)| &\leq 2|z(t_1,i)| \\ &+2\left|\int_{t_1}^{t_1+\eta} b(s,x(s))[\varphi_i(s,x(s,i))-\varphi(s,x(s,i))]ds\right|. \end{aligned}$$

This becomes (3.7) by using the function

$$g(t) := \begin{cases} e^{\omega t}b^T(t,x(t)) & t\in[t_1,t_1+\eta], \\ 0 & \text{else.}\end{cases}$$

$$\begin{aligned} \sup_{s\in[t_1,t_1+\eta]}|z(s,i)| &\leq 2|z(t_1,i)| \\ &+2|\langle g,[\varphi_i\circ x(\cdot,i)-\varphi\circ x(\cdot,i)]\rangle|. \end{aligned} \quad (3.7)$$

Let $\varepsilon > 0$. By Proposition 2.11, for almost every $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$, there is an integer $N \geq 1$ for which $\langle g, [\varphi_i \circ x(\cdot,i) - \varphi \circ x(\cdot,i)]\rangle < \varepsilon$ for all $i \geq N$, so (3.7) becomes

$$\sup_{s\in[t_1,t_1+\eta]} |z(s,i)| \leq 2|z(t_1,i)| + 2\varepsilon \text{ for } i \geq N. \qquad (3.8)$$

Define $\zeta_{j,i} := \sup_{s\in[(j-1)\eta, j\eta]} |z(s,i)|$; let $q \geq t/\eta$ be an integer; build the partition $[0,T] \subseteq \{[0,\eta], [\eta,2\eta], \ldots, [(q-1)\eta, q\eta]\}$. Then, (3.8) yields $\zeta_{j+1,i} \leq 2\zeta_{j,i} + 2\varepsilon$, $\zeta_{0,i} = 0$, or

$$\zeta_{j,i} \leq \left(\sum_{r=1}^{j} 2^r\right)\varepsilon, \text{ for all } i \geq N,$$

so that

$$\sup_{s\in[0,T]} |z(s,i)| \leq \left(\sum_{r=1}^{q} 2^r\right)\varepsilon$$

for all $i \geq N$. Finally, select N so that $\varepsilon \leq \delta/\left(\sum_{r=1}^{q} 2^r\right)$. □

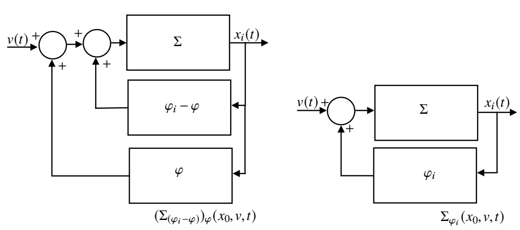
3.3 Equivalent control configurations

Define the signal

$$\varepsilon_i(t) := \Sigma_{\varphi_i - \varphi}(x_0, v, t) - \Sigma(x_0, v, t). \qquad (3.9)$$

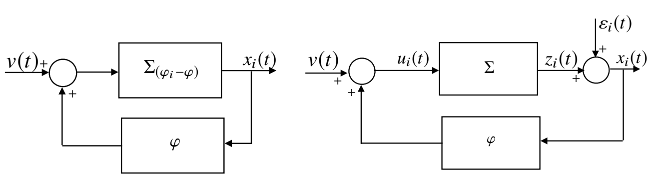
Figures 3.2(A), 3.2(B), 3.2(A), and 3.3(B) are equivalent, with

$$x_i(t) = z_i(t) + \varepsilon_i(t),\ i = 1,2,\ldots \qquad (3.10)$$



(a) (b)

Fig. 3.2. Equivalent configurations



(a) (b)

Fig. 3.3. Equivalent configurations

Proof of Theorem 3.5. The theorem follows from (3.10) since $\varepsilon_i \to 0$ as $i \to \infty$ by Lemma 3.6 and (3.9). □

4 Existence of Optimal feedback functions

The requirement for the existence of optimal feedback functions is (K,A)-controllability of the nominal system Σ_0; no need to test separately every member of $\mathcal{F}_\gamma(\Sigma_0)$. In fact, (K,A)-controllability is close to being necessary for the existence of optimal feedback functions as well, since Problem 1.1 requires taking the controlled system to the vicinity of the origin.

Theorem 4.1. *Given real numbers $A_0, A, \ell, \gamma > 0$, where $A > A_0$, assume that the nominal system Σ_0 is (K,A_0)-controllable from the initial state x_0 and that the uncertainty parameter γ satisfies Proposition 3.3. Using the notation of (2.12), (2.13), and (2.14), the following are true for almost every $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$.*

(i) *The time $t^*(x_0,\ell,A,\gamma)$ is finite.*

(ii) *There exists an optimal feedback function $\varphi^*(x_0,\ell,A,\gamma) \in \Phi(K)$ satisfying $t^*(x_0,\ell,A,\gamma) = t(x_0,\ell,A,\gamma,\varphi^*(x_0,\ell,A,\gamma))$.* □

Theorem 4.1 depends on the next statement, which is similar to a statement of Yu and Hammer (2016) (see Hammer (2019)).

Proposition 4.2. *Under the assumptions of Theorem 4.1, the functional $t(x_0,\ell,A,\gamma,\cdot) : \Phi(K) \to R : \varphi \mapsto t(x_0,\ell,A,\gamma,\varphi)$ of (2.12) is weakly lower semi-continuous.* □

Proof of Theorem 4.1 (sketch). Use the Generalized Weierstrass Theorem, (2.12), Proposition 4.2, and Lemma 2.9. □

5 An approximation of optimal performance

In this section, we examine simplified feedback functions.

5.1 Noises and disturbances

Errors introduced by simplifying feedback functions must be considered in context with other disturbances. Consider the disturbance $\upsilon(t)$ of Figure 5.1. Assume that $\upsilon(t)$ is uniformly distributed in the domain $[-\Delta,\Delta]^n$, where $\Delta > 0$ is a specified bound. Let $\Delta(x)$ be the hyper-square of edge 2Δ centered at the state x. The average feedback signal at a time t is then

$$\bar{\varphi}(t,x) := \frac{1}{(2\Delta)^n}\int_{\Delta(x)} \varphi(t,z)dz, \qquad (5.1)$$

where dz is the Lebesgue volume element in R^n.

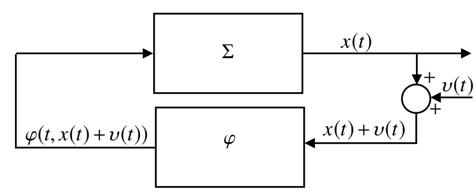


Fig. 5.1. A disturbance signal $\upsilon(t)$, $|\upsilon(t)| \leq \Delta$.

5.2 Simplified feedback functions

Recalling the input amplitude bound K of Σ, let $\mathbb{K}^m$ the set m-dimensional vectors with components of $-K$ or K. For example, $\mathbb{K}^2 = \{(-K,-K)^\top, (K,-K)^\top, (-K,K)^\top, (K,K)^\top\}$.

Definition 5.2. A *bang-bang feedback function* is a piecewise constant function $\varphi_\pm : R^+ \times R^n \to \mathbb{K}^m$, whose components switch among $-K$ and K as a function of time and state. □

Bang-bang feedback functions approximate optimal performance, when averaged over disturbances as in (5.1):

Theorem 5.3. *Let $A_0, A, \ell, \ell' > 0$ be numbers, where $A > A_0$ and $\ell' > \ell$. Assume that the nominal system Σ_0 is (K,A_0)-controllable from the initial state x_0. Then, in the notation of (2.12) and (2.13), there are an uncertainty parameter $\gamma > 0$ and a bang-bang feedback function $\varphi_\pm \in \Phi(K)$ for which $t(x_0,\ell',A,\gamma,\varphi_\pm) \leq t^*(x_0,\ell,A,\gamma)$, when feedback signals are averaged as in (5.1).* □

The conditions of Theorem 5.3 are similar to those of Theorem 4.1. Theorem 5.3 follows from the next statement (compare to the open-loop studies Chakraborty and Hammer (2009, 2010), Yu and Hammer (2016)).

Theorem 5.4. *Let $t' > 0$ be a finite time. For every $\varepsilon > 0$, there are a bang-bang feedback function $\varphi_\pm \in \Phi(K)$ and an uncertainty parameter $\gamma > 0$ for which the difference between the responses $x(t) := \Sigma_\varphi(x_0,t)$ and $x_\pm(t) := \Sigma_{\varphi_\pm}(x_0,t)$ satisfies $|x(t) - x_\pm(t)| < \varepsilon$ for all times $t \in [0,t']$ and almost all $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$, when feedback signals are averaged as in (5.1).* □

The proof of Theorem 5.4 uses the following statement, which is a consequence of a statement from Choi and Hammer (2018) (see Hammer (2019).

Lemma 5.5. *For a system $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ with initial state x_0 and feedback function $\varphi \in \Phi(K)$, denote $x(t) := \Sigma_\varphi(x_0,t)$. Let $t' > 0$ be a finite time and refer to (2.1). Then, for every $\zeta > 0$, there are numbers $\beta(x_0,\zeta,t') > 0$ and $\gamma > 0$ for which $|b(t_1,x(t_1)) - b(t_2,x(t_2))| < \zeta$ for all times $t_1,t_2 \in [0,t']$ satisfying $|t_1-t_2| < \beta(x_0,\zeta,t')$, for all $\varphi \in \Phi(K)$, and for all $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$.* □

Proof of Theorem 5.4 (sketch). (see Hammer (2019) for more details.) Let $\Delta > 0$ be the bound of Subsection 5.1. Let $t_1 < t_2 \in [0,t']$, where $(t_2-t_1)/\Delta$ is a rational number. Choose $\lambda > 0$ for which $p := (t_2-t_1)/\lambda$ and Δ/λ are integers. Recalling the state amplitude bound A, let $M \geq A$ be such that $r := M/\lambda$ is an integer. Denote by $q := (q_0,q_1,\ldots,q_n)$ a vector of integers. For each q, define the hyper-square

$$\chi(q) := [t_1+q_0\lambda, t_1+(q_0+1)\lambda]\times[-M+q_1\lambda,-M+(q_1+1)\lambda]\times \cdots\times[-M+q_n\lambda,-M+(q_n+1)\lambda].$$

These form a partition P of $[t_1,t_2]\times[-M,M]^n$:

$$P = \{\chi(q)\}_{q\in\{0,1,\ldots,p-1\}\times\{0,1,\ldots,2r-1\}^n}.$$

The components of $\varphi = \left(\varphi^1,\varphi^2,\cdots,\varphi^m\right)^\top$ are bounded by K, so

$$-K\lambda^{n+1} \leq \int_{\chi(q)} \varphi^j(s,x)d(s,x) \leq K\lambda^{n+1} \tag{5.6}$$

For a number $\mu(q) \in [0,\lambda]$, build the hyper-square of edge $\mu(q)$:

$$D(\mu(q)) := [t_1+q_0\lambda, t_1+q_0\lambda+\mu(q)]\times[-M+q_1\lambda, -M+q_1\lambda+\mu(q)]\times\cdots\times[-M+q_n\lambda,-M+q_n\lambda+\mu(q)].$$

Let $V(\mu(q))$ be the hyper-volume of $D(\mu(q))$. Then, by (5.6) there is a number $0 \leq \mu_j(q) \leq \lambda$ such that

$$K\left[2V(\mu_j(q)) - \lambda^{n+1}\right] = \int_{\chi(q)} \varphi^j(s,x)d(s,x).$$

Define the bang-bang feedback function $\varphi_\pm$ (\ is set difference):

$$\varphi^j_\pm(t,x) := \begin{cases} K & \text{for } (t,x) \in D(\mu_j(q)), \\ -K & \text{for } t \in \chi(q)\setminus D(\mu_j(q)), \end{cases}$$

$q \in \{0,1,\ldots,p-1\}\times\{0,1,\ldots,2r-1\}^n$, $j = 1,2,\ldots,m$. Then,

$$\int_{\chi(q)} \left(\varphi^j(s,x) - \varphi^j_\pm(s,x)\right) d(s,x) = 0 \text{ for all } q \text{ and } j. \tag{5.7}$$

Set $\xi(t) := x(t) - x_\pm(t)$. Then,

$$\bar\xi(t) = \bar\xi(t_1) + \frac{1}{(2\Delta)^n}\int_{t_1}^t \int_{\Delta(x(s))} \Big[a(s,x(s)) - a(s,x_\pm(s)) + b(s,x(s))\varphi(s,x(s)) - b(s,x_\pm(s))\varphi_\pm(s,x(s))\Big] d(s,x).$$

Using (2.2) and the fact that $\varphi,\varphi_\pm \in \Phi(K)$ yields

$$\left[1 - \frac{\alpha^+}{(2\Delta)^n}(1+K)(t_2-t_1)\right]\sup_{t\in[t_1,t_2]}\left|\bar\xi(t)\right| \leq \left|\bar\xi(t_1)\right| + \frac{1}{(2\Delta)^n}\times \sup_{t\in[t_1,t_2]}\left|\int_{t_1}^t\int_{\Delta(x(s))} b(s,x(s))\left(\varphi(s,x(s)) - \varphi_\pm(s,x(s))\right)d(s,x)\right|.$$

Let $\eta \in (0,t'-t_1]$ be an integer multiple of λ such that $\alpha^+(1+K)\eta/(2\Delta)^n \leq 1/2$, and set $t_2 := t_1+\eta$. Then,

$$\sup_{t\in[t_1,t_1+\eta]}\left|\bar\xi(t)\right| \leq 2\left|\bar\xi(t_1)\right| + \frac{2}{(2\Delta)^n}\times \sup_{t\in[t_1,t_1+\eta]}\left|\int_{t_1}^t\int_{\Delta(x(s))} b(s,x(s))\left(\varphi(s,x(s)) - \varphi_\pm(s,x(s))\right)d(s,x)\right|. \tag{5.8}$$

Denote by G the integration domain; then, the last integral is

$$z(t) := \int_G b(s,x(s))\left(\varphi(s,x(s)) - \varphi_\pm(s,x(s))\right)d(s,x). \tag{5.9}$$

Partition G into: *(i)* $G' \subseteq R^+\times R^n$ consists of all whole hyper-squares of P in G; and *(ii)* $G'' := G\setminus G'$. Then, letting $S_{G'}$ be the hyper-surface area of G', the hyper-volume of G'' satisfies

$$V(G'') \leq \lambda S_{G'}. \tag{5.10}$$

Split (5.9) into a sum of integrals over G' and over G''; using the fact that the integral over G' is zero due to (5.7), we get

$$\sup_{t\in[t_1,t_1+\eta]}|z(t)| \leq \sum_{q:\chi(q)\in G'}\int_{\chi(q)}\sup_{(s,x(s))\in\chi(q)} |b(s,x(s)) - b(t_1+q_0\lambda,x(t_1+q_0\lambda))|\,|\varphi(s,x(s))-\varphi_\pm(s,x(s))|\,d(s,x) + \sup_{t\in[t_1,t_2]}\left|\int_{G''} b(s,x(s))\left(\varphi(s,x(s))-\varphi_\pm(s,x(s))\right)d(s,x)\right|.$$

Now, given $\zeta > 0$, use $\beta(x_0,\zeta,t') > 0$ of Lemma 5.5 and choose $\lambda \leq \beta(x_0,\zeta,t')$. Thus, as $V(G') \leq \eta(2\Delta)^n$, we get

$$\sum_{q:\chi(q)\in G'}\int_{\chi(q)}\sup_{(s,x(s))\in\chi(q)} |b(s,x(s)) - b(t_1+q_0\lambda,x(t_1+q_0\lambda))|\times |\varphi(s,x(s))-\varphi_\pm(s,x(s))|\,d(s,x) \leq 2K\zeta\eta(2\Delta)^n.$$

From (5.10) and Lemma 3.1, we obtain

$$\sup_{t\in[t_1,t_1+\eta]}\left|\int_{G''} b(s,x(s))\left(\varphi(s,x(s))-\varphi_\pm(s,x(s))\right)ds\right| \leq \lambda S_G 2KM_b(t').$$

Substituting these inequalities into (5.8) yields

$$\sup_{t\in[t_1,t_1+\eta]}\left|\bar\xi(t)\right| \leq 2\left|\bar\xi(t_1)\right| + 4K\left[\zeta\eta + \lambda S_G M_b(t')/(2\Delta)^n\right].$$

Further, fix $\delta > 0$. Use $\zeta > 0$ so that $\zeta\eta 4K < \delta/2$; use $\lambda' > 0$ so that $\lambda' S_G M_b(t')4K/(2\Delta)^n < \delta/2$; and let $\kappa \geq t'/\eta$ be an integer. Then, for $\lambda > 0$ satisfying $\lambda \in (0,\min\{\lambda',\beta(x_0,\zeta,t')\})$, we get

$$\sup_{t\in[i\eta,(i+1)\eta]}\left|\bar\xi(t)\right| \leq 2\left|\bar\xi(i\eta)\right| + \delta,\ \bar\xi(0) = 0,$$

$i = 0,\ldots,\kappa-1$, so that

$$\sup_{t\in[0,t']}\left|\bar\xi(t)\right| \leq \delta\sum_{i=0}^{\kappa} 2^i = \delta(2^{\kappa+1}-1).$$

Therefore, the theorem holds for $0 < \delta < \varepsilon/(2^{\kappa+1}-1)$. □

Approximation of optimal performance through bang-bang feedback functions simplifies the design and implementation of optimal feedback, since bang-bang functions take values in a discrete set with only 2^m points. Thus, bang-bang feedback functions that approximate optimal performance can be obtained by relatively simple numerical search processes (Section 6).

6 Example

Consider the inverted pendulum (Choi and Hammer (2018))

$$\Sigma: \begin{aligned} \dot x_1(t) &= x_2(t), \\ \dot x_2(t) &= d_1\sin x_1(t) + d_2 x_2(t) + d_3\tanh u(t); \end{aligned}$$

here, d_1, d_2, and d_3 are constants with nominal values of $d_1^0 = 24.527$, $d_2^0 = -0.107$, $d_3^0 = 12.5$; and uncertainty ranges $d_1 \in [21,27]$, $d_2 \in [-0.3,-0.1]$, and $d_3 \in [10,14]$. The initial state is $x_0 = [\pi/8,-2]^\top$; the input amplitude bound is $K = 5$; the state amplitude bound is $A = 2$; and the operating error bound is

$\ell = 0.1$. By Choi and Hammer (2018), the nominal system is (5,2)-controllable and the minimal time does not exceed 0.3. Thus, our domain of interest is $[0,0.3]\times[-2,2]^2 \subseteq R^+ \times R^2$.

To search for a bang-bang function, we partition our domain into cubes of edge $\lambda = 0.01$. A numerical search for a bang-bang feedback function that guides Σ to $\rho(0.1)$ in minimal time without violating our constraints yields (see also Figure 6.1):

$$\varphi_\pm(t,x_1,x_2) := \begin{cases} 5 & \text{if } t \in [0,0.05); \\ -5 & \text{if } t \in [0.05,0.14); \\ 5 & \text{if } t \in [0.14,0.16) \text{ and } |x_2|\geq 1.12; \\ -5 & \text{if } t \in [0.14,0.15) \text{ and } 0.98 \leq |x_2| \leq 1.01; \\ -5 & \text{if } t \in [0.14,0.16) \text{ and } |x_2| \leq 0.86; \\ 5 & \text{else.} \end{cases}$$

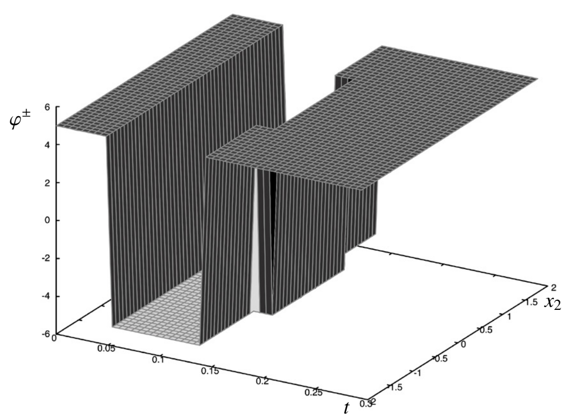


Fig. 6.1. The bang-bang feedback function $\varphi_\pm$

The performance of this feedback function is shown in Figure 6.2 for three representatives of our family of systems. The minimal time here is 0.229, an improvement of about 20% over the open-loop minimal time of 0.264 (Choi and Hammer (2018).

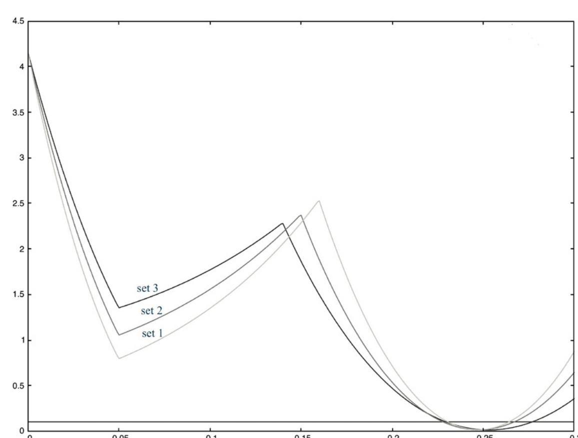


Fig. 6.2. the trajectories

7 Conclusion

We have shown that optimal robust feedback solutions exist and demonstrated a relatively simple implementation technique. The implementation technique is based on the use of bang-bang feedback functions; these functions can be calculated by a relatively simple numerical search process, since they have a discrete finite set of values.

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