Minimal Time Control of Nonlinear Systems: Optimal Robust State-Feedback

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Abstract: Optimal robust state-feedback controllers for minimal-time control of nonlinear systems are developed under constraints on the input and output signal amplitudes of the controlled system. Simple to design and implement controllers that approximate optimal performance are presented.

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1 Introduction

Traditionally, the design of optimal feedback controllers for nonlinear systems is via the Hamilton-Jacobi-Bellman equation (Sobolev (1950), Bellman (1954), Miranda (1955), Kruzkov (1960), Pontryagin et al. (1962)), a nonlinear partial differential equation. To solve this equation for nonlinear systems, one must defer to numerical solutions; these are encumbered by high computational complexity.

This note focuses on optimal robust feedback control of nonlinear affine systems, with constraints on input and output signal amplitudes. We prove existence of optimal robust state-feedback controllers that guide such systems to target in minimal time (Section 4). We develop simple state-feedback controllers that approximate optimal performance (Section 5).

As one might expect, optimal feedback controllers developed in the current note provide better performance than open-loop controllers considered in Yu and Hammer (2016), especially when there is significant uncertainty about the controlled system’s model (Section 6). This comes, of course, at the cost of more involved implementation.

In Figure 1.1, the input signal $u(t)$ of the controlled system $\Sigma$ is generated by the state-feedback function $\varphi$ – a function of the time $t$ and the state $x(t)$ of $\Sigma$. The objective is to design $\varphi$ to guide $\Sigma$ in minimal time from an initial state $x(0) = x_0$ to a target state $x_{\text{target}}$. This must be accomplished without exceeding specified amplitude bounds of $K > 0$ and $A > 0$ on the input and output signals of $\Sigma$, respectively. After shifting coordinates, we can take $x_{\text{target}} = 0$. To accommodate uncertainties, we allow a maximal deviation of $\ell > 0$ from the target so our objective is to reach the ball $p(t) := \{x : x^T x \leq \ell\}$ in minimal time. Section 4 shows that optimal robust feedback controllers that fulfill these requirements exist under a mild controllability condition on the controlled system $\Sigma$. Here, $\ell$ is the operating error bound.

In Section 5, we show that optimal performance can be approximated by bang-bang feedback functions – functions whose components switch between the two values $K$ and $-K$ as a function of time and state. Such feedback functions are much easier to calculate and implement than optimal feedback functions, since their values are in a finite discrete set. Our objectives can then be summarized as follows.

**Problem 1.1.** (i) Derive conditions that guarantee the existence of optimal robust state-feedback functions $\varphi$ that drive $\Sigma$ in minimal time from $x_0$ to $p(t)$, while keeping input and output signals from exceeding the bounds $K$ and $A$, respectively. (ii) Derive easy to calculate and implement feedback functions that approximate optimal performance.

This note focuses on closed-loop optimal robust control of nonlinear systems. It draws on classical works on optimization, such as Kelendzheridze (1961), Pontryagin et al. (1962), Gamkrelidze (1965), Neustadt (1966, 1967), Luenberger (1969), Young (1969), Warga (1972), the references cited in these publications, and on earlier work by the author and coworkers on open-loop optimal control (Chakraborty and Hammer (2009, 2010), Chakraborty and Shaikshavali (2009), Yu and Hammer (2016), Choi and Hammer (2018, 2019), Hammer (2019)). Yet, the existence and approximation of optimal robust feedback controllers under input/output constraints have not been reported in the literature before.

The note is organized as follows. Sections 2 and 3 establish the mathematical framework; Section 4 proves the existence of optimal robust feedback controllers; Section 5 presents simple feedback functions that approximate optimal performance; Section 6 is an example; and Section 7 concludes the note.

2 Background

2.1 System equations

Denote by $R$ the compactified set of real numbers; by $R^n$ the set of $n$-dimensional real vectors; by $R^*$ the non-negative real numbers; by $|r|$ the absolute value of a number $r$; by $|V|$ the absolute value of a constant matrix $V$; by $W_{(x)} := \sup_{t \geq 0} |W(t)|$, the $L^\infty$-norm of a matrix function of time; by $|x|_2 := (x^T x)^{1/2}$ the $L^2$-norm of a vector; and by $[-A,A]^n \subseteq R^n$ the set of all vectors $x$ with $|x| \leq A$. The controlled system $\Sigma$ is a nonlinear time-varying input-affine system

\[ \dot{x}(t) = a(t,x(t)) + b(t,x(t))u(t), \quad x(0) = x_0; \]

where $\varphi$ is a measurable function such that $\varphi(t,x(t)) \in \{\pm K\}$ for all $t \geq 0$. The set of feedback functions $\Phi$ is of measure zero for all $t$.
2.2 Spaces

The Hilbert space $L_{2, m}^2$ consists of Lebesgue measurable functions $f, g : R^+ \rightarrow R^m$ with the inner product

$$\langle f, g \rangle = \int_0^\infty e^{-\omega s} f^T(s) g(s) ds, \omega > 0.$$  

The permissible input and output signals of $\Sigma$ are, respectively,

$$U(\Sigma) := \{ u \in L_{2, m}^2 : |u|_{\infty} \leq K \}$$

and

$$X(\Sigma) := \{ x \in L_{2, m}^m : |x|_{\infty} \leq A \}.$$  

\textbf{Notation 2.6.} Given $\alpha, \gamma, \nu, A > 0$ and $\Sigma_0$ of (2.5), let $F(\Sigma_0)$ be the family of systems described by (2.1), (2.2), (2.3), and (2.4).

(i) Every member of $F(\Sigma_0)$ has the same initial state $x(0) = x_0$.

(ii) The same feedback function $\varphi$ is used for all $\Sigma \in F(\Sigma_0)$.

(iii) All feedback functions are bounded by $K$.

(iv) States of all members of $F(\Sigma_0)$ are restricted to $X(\Sigma)$.

\textbf{Requirement (ii) of Notation 2.6 originates from the fact that it is not known which member of $F(\Sigma_0)$ the controlled system $\Sigma$ actually is, so feedback cannot be adjusted for each member.}

2.3 State-feedback

State-feedback is provided by a Lebesgue measurable function $\varphi : R^+ \times R^m \rightarrow R^r$. The closed loop system yields the state $x(t) = \Sigma_{\varphi}(x_0, t)$ given by

$$\Sigma_{\varphi} : \dot{x}(t) = a(t, x(t)) + b(t, x(t))\varphi(t, x(t)), x(0) = x_0.$$  

State-feedback functions are in the Hilbert space $L_{2, m}^2$ of measurable functions $f, g : R^+ \rightarrow R^m$ with inner product

$$\langle f, g \rangle = \int_0^\infty e^{-\omega s} f^T(s) g(s) ds, \omega > 0,$$

where $d(s, z)$ is the Lebesgue measure element in $R^r \times R^r$. As input signals of $\Sigma$ must be bounded by $K$, the class of permissible feedback functions is

$$\Phi(\Sigma) := \{ \varphi \in L_{2, m}^2 : |\varphi(x)| \leq K \text{ for all } (x, t) \in R^+ \times R^r \}.$$  

2.4 Convergence features

We need the following notions (e.g., Willard (2004), Zeidler (1985)).

\textbf{Definition 2.7.} $H$ is a Hilbert space with inner product $(\cdot, \cdot)$.

(i) A sequence $\{ v_i \}_{i=1}^\infty \subseteq H$ converges weakly to a member $v \in H$ if $\lim_{i \rightarrow \infty} \langle v_i, y \rangle = \langle v, y \rangle$ for every $y \in H$.

(ii) A subset $W \subseteq H$ is weakly compact if every sequence in $W$ has a subsequence that converges weakly to a member of $W$.

The following is proved in Chakraborty and Hammer (2009).

\textbf{Lemma 2.8.} The set of signals $U(\Sigma)$ is weakly compact.

\textbf{Lemma 2.9.} The set of feedback functions $\Phi(\Sigma)$ is weakly compact in $L_{2, m}^2$.
3 Feedback and continuity

3.1 Controllability

The following is a consequence of the fact that (2.1) has continuous solutions for bounded input functions (see Hammer (2019) for details).

Lemma 3.1. For $T > 0$ and a continuous function $c : R^+ \times R^d \rightarrow R^q$, there is a bound $M_c(T) \geq 0$ such that $|c(t, \Sigma)_{\Sigma}(x(t))| \leq M_c(T)$ for all $t \in [0, T]$, all $\Sigma \in \Phi(K)$, and all $x \in U(K)$. □

As signals generated by feedback functions $\varphi \in \Phi(K)$ belong to $U(K)$, Problem 1.1(i) requires that there be an input signal $u \in U(K)$ that takes $\Sigma$ from $x_0$ to the neighborhood of $x = 0$, without violating the state amplitude bound. This leads to the following notion (Choi and Hammer (2018)).

Definition 3.2. A system $\Sigma$ is $(K, A)$-controllable from an initial state $x_0$ if there is an input signal $u \in U(K)$ and a finite time $t_A \geq 0$ so that $\Sigma(t_0, u, t_A) = 0$ and $|\Sigma(t_0, u, t_A)| \leq A$ for all $t \in [0, t_A]$. □

The next statement shows that $(K, A)$-controllability of the nominal system assures a finite optimal time $t^*(x_0, R, A, \gamma)$. It follows from Choi and Hammer (2018), since any input signal $u \in U(K)$ can be generated by a feedback function $\varphi(t, x) = u(t)$. Prop. 3.3. If the nominal system $\Sigma_0$ is $(K, A_0)$-controllable from initial state $x_0$, then, for every $A > A_0$, there is an uncertainty parameter $\gamma > 0$ for which the time $t^*(x_0, A, \gamma)$ is finite. Thus, only one system $\Sigma_0$ has to be tested to guarantee a finite optimal time. $(K, A)$-controllability can be determined via a relatively simple numerical search (Choi and Hammer (2018)).

We use the following notions (e.g., Zeidler (1985)).

Definition 3.4. $H$ is a Hilbert space, $S$ a subset of a $H$, and $z$ is a member of $S$. A functional $F : S \rightarrow R$ is weakly lower semi-continuous at $z$ if the following is true for every sequence $(z_i)_{i=1}^{\infty} \subseteq S$ that converges weakly to $z$: when $F(z)$ is bounded, there is, for every real number $\varepsilon > 0$, an integer $N > 0$ such that $F(z) - F(z_i) < \varepsilon$ for all $i \geq N$. A function $G : S \times R^m \rightarrow R^n : (s, t) \rightarrow G(s, t)$ is weakly continuous at a time $t$ if the following is true for every sequence $(z_i)_{i=1}^{\infty} \subseteq S$ that converges weakly to $z$: for every real number $\varepsilon > 0$, there is an integer $N > 0$ such that $|G(z_i, t) - G(z, t)| < \varepsilon$ for all $i \geq N$.

A sequence of functions $\{G_t\}_{t=1}^{\infty} : S \times R^m \rightarrow R^n$ is uniformly weakly convergent to a function $G$ over an interval $[t_1, t_2]$, $t_1 < t_2$, if, for every real number $\varepsilon > 0$, there is an integer $N > 0$ such that $|G_t(z_i, t) - G(z, t)| < \varepsilon$ for all $i \in S$ and all $t \in [t_1, t_2]$. □

3.2 Feedback systems

Consider the feedback configuration of Figure 5.1, which has an external input signal $v$. With a system $\Sigma \in \mathcal{F}_Y(\Sigma_0)$, the differential equation is $\Sigma(\Sigma_0, v, t) \partial(t) = a(t, x(t)) + b(t, x(t))\varphi(t, x(t)) + v(t)], x(o) = x_0$. Then, $\Sigma_x \Sigma(\Sigma_0, v, t)$ and $\Sigma(\Sigma_0, v, t) = \Sigma_0(\Sigma_0, v, t)$. By the next statement, $\Sigma_x \Sigma(\Sigma_0, v, t)$ is a continuous function of $x$. □

Fig. 3.1. Feedback system with external input

Theorem 3.5. Let $\Sigma(\Sigma)_{i=1}^{\infty} \subseteq \Phi(K)$ be a sequence that converges weakly to $\varphi \in \Phi(K)$. Then, for almost every $\Sigma \in \mathcal{F}_Y(\Sigma_0)$, the sequence $\{\Sigma_x \Sigma(\Sigma_0, v, t)\}_{i=1}^{\infty}$ uniformly weakly convergent to $\Sigma_x \Sigma(\Sigma_0, v, t)$ for all $v \in U(K)$ over any finite interval of time. □

To prove Theorem 3.5, we need a few preliminary results. In the notation of the theorem, the next statement shows that the negative feedback $-\varphi$ nearly undoes the action of $\varphi$, for large $i$.

Lemma 3.6. Use the notation of Theorem 3.5. For almost every $\Sigma \in \mathcal{F}_Y(\Sigma_0)$, the sequence $\{\Sigma_x \Sigma(\Sigma_0, v, t)\}_{i=1}^{\infty}$ converges weakly and uniformly to $\Sigma(\Sigma_0, v, t)$ for every input signal $v \in U(K)$ over every finite interval of time.

Proof (sketch). Denote $x(t, i) := \Sigma_x \Sigma(\Sigma_0, v, t)$, $x(t) := \Sigma_x \Sigma(\Sigma_0, v, t)$, and $z(t) := x(t, i) - x(t)$. Then, $x(t, i) = a(t, x(t, i)) + b(t, x(t, i))(\varphi(t, x(t, i)) - \varphi(t, x(t, i)) + v(t))$, $x(0, i) = x_0$. Fix a time $T > 0$; let $t_1 < t_2 \in [0, T]$, $i \in [t_1, t_2]$. Then,

Using (2.2) and the fact that $v \in U(K)$ and $\varphi \in \Phi(K)$ yields

This becomes (3.7) by using the function

(3.7)
Let $\varepsilon > 0$. By Proposition 2.11, for almost every $\Sigma \in \mathcal{F}_r(\Sigma_0)$, there is an integer $N \geq 1$ for which $(g,[\varphi \circ x(\cdot,i) - \varphi \circ x(\cdot,i)]) < \varepsilon$ for all $i \geq N$, so (3.7) becomes

$$\sup_{x \in [t_i,t_{i+1}]} |z(s,i)| \leq 2|z(t_i,i)| + 2\varepsilon \quad \text{for all } i \geq N. \quad (3.8)$$

Define $\xi_{j,i} := \sup_{s \in [(j-1)\eta, j\eta]} |z(s,i)|$; let $q \geq t/\eta$ be an integer; build the partition $[0,T] \subseteq \{(0,\eta), (\eta, 2\eta), \ldots, ([q - 1]\eta, q\eta)\}$. Then, (3.8) yields $\xi_{j,i} \leq 2\xi_{j+1,i} + 2\varepsilon$, $\xi_{0,i} = 0$, or

$$\xi_{j,i} \leq \left( \sum_{r=1}^{j} \varepsilon \right) \varepsilon, \quad \text{for all } i \geq N,$$

so that

$$\sup_{s \in [0,T]} |z(s,i)| \leq \left( \sum_{r=1}^{q} \varepsilon \right) \varepsilon$$

for all $i \geq N$. Finally, select $N$ so that $\varepsilon \leq \delta/(\sum_{r=1}^{q} \varepsilon)$. \hfill \Box

### 3.3 Equivalent control configurations

Define the signal

$$\epsilon_i(t) := \sigma_{\varphi_i} - \varphi(x_0,v,t) - \varphi(x_0,v,t). \quad (3.9)$$

Figures 3.2(A), 3.2(B), 3.2(A), and 3.3(B) are equivalent, with $x_i(t) = z_i(t) + \epsilon_i(t)$, $i = 1, 2, \ldots$. \hfill \Box

### 5 An approximation of optimal performance

In this section, we examine simplified feedback functions.

#### 5.1 Noises and disturbances

Errors introduced by simplifying feedback functions must be considered in context with other disturbances. Consider the disturbance $v(t)$ of Figure 5.1. Assume that $v(t)$ is uniformly distributed in the domain $[-\Delta, \Delta]$ where $\Delta > 0$ is a specified bound. Let $\mathcal{D}(x)$ be the hyper-square of edge $2\Delta$ centered at the state $x$. The average feedback signal at a time $t$ is then

$$\varphi(t,x) := \frac{1}{(2\Delta)^n} \int_{\mathcal{D}(x)} \varphi(t,z)dz, \quad (5.1)$$

where $dz$ is the Lebesgue volume element in $\mathbb{R}^n$.

### 5.2 Simplified feedback functions

Recalling the input amplitude bound $K$ of $\Sigma$, let $\mathbb{R}^m$ the set $m$-dimensional vectors with components of $K$ or $K$. For example, $\mathbb{R}^2 = \{(K, -K)^T, (-K, K)^T, (K, K)^T\}$. \hfill \Box

**Definition 5.2.** A bang-bang feedback function is a piecewise constant function $\varphi_\pm : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$, whose components switch among $-K$ and $K$ as a function of time and state. \hfill \Box

Bang-bang feedback functions approximate optimal performance, when averaged over disturbances as in (5.1).

#### Theorem 5.3. Let $A_0, A, \ell > 0$ be numbers, where $A > A_0$ and $\ell > \ell$. Assume that the nominal system $\Sigma_0$ is $(K,A_0)$-controllable from the initial state $x_0$. Then, in the notation of (2.12) and (2.13), there are an uncertainty parameter $\gamma > 0$ and a bang-bang feedback function $\varphi_\pm \in \Phi(K)$ for which $t(x_0, \ell, A, \gamma, \varphi_\pm) \leq t'(x_0, \ell, A, \gamma)$, when feedback signals are averaged as in (5.1). \hfill \Box

The conditions of Theorem 5.3 are similar to those of Theorem 4.1. Theorem 5.3 follows from the next statement (compare to the open-loop studies Chakraborty and Hammer (2009, 2010), Yu and Hammer (2016)).

#### Theorem 5.4. Let $\ell > 0$ be a finite time. For every $\varepsilon > 0$, there are a bang-bang feedback function $\varphi_\pm \in \Phi(K)$ and an uncertainty parameter $\gamma > 0$ for which the difference between the responses $x(t) := \sigma_{\varphi}(x_0,t)$ and $x_\pm(t) := \sigma_{\varphi_\pm}(x_0,t)$ satisfies $|x(t) - x_\pm(t)| < \varepsilon$ for all times $t \in [0,\ell]$ and almost all $\Sigma \in \mathcal{F}_r(\Sigma_0)$, when feedback signals are averaged as in (5.1). \hfill \Box
The proof of Theorem 5.4 uses the following statement, which is a consequence of a statement from Choi and Hammer (2018) (see Hammer (2019)).

**Lemma 5.5.** For a system $\Sigma \in \mathcal{F}_\Sigma(\Sigma_0)$ with initial state $x_0$ and feedback function $\varphi \in \Phi(K)$, denote $x(t) := \Sigma(x_0,t)$. Let $t' > 0$ be a finite time and refer to (2.1). Then, for every $\zeta > 0$, there are numbers $\beta, \alpha(t'), \gamma > 0$ for which $|b(t_1,x(t_1)) - b(t_2,x(t_2))| < \zeta$ for all times $t_1, t_2 \in [0,t']$ satisfying $|t_1 - t_2| < \beta(x_0,t')$, for all $\varphi \in \Phi(K)$, and for all $\Sigma \in \mathcal{F}_\Sigma(\Sigma_0)$. □

**Proof of Theorem 5.4 (sketch).** (see Hammer (2019) for more details.) Let $\Delta > 0$ be the bound of Subsection 5.1. Let $t_1 < t_2 \in [0,t']$, where $(t_2 - t_1)/\Delta$ is a rational number. Choose $\lambda > 0$ for which $\rho := (t_2 - t_1)/\lambda$ and $\Delta/\lambda$ are integers. Recalling the state amplitude bound $A$, let $M \geq A$ be such that $r := M/\lambda$ is an integer. Denote by $q := (q_0,q_1,\ldots,q_n)$ a vector of integers. For each $q$, define the hyper-square

$$\chi(q) := [t_1 + q_0 \lambda, t_1 + (q_0 + 1) \lambda] \times [-M + q_1 \lambda, -M + (q_1 + 1) \lambda] \times \cdots \times [-M + q_n \lambda, -M + (q_n + 1) \lambda]$$

Then, this form a partition of $P = \{ \chi(q) \}_{q \in \{0,1,\ldots,p-1\} \times \{0,1,\ldots,2r-1\}^n}$.

The components of $\varphi = (\varphi^1, \varphi^2, \cdots, \varphi^m)$ are bounded by $K$, so

$$-K \lambda^{n+1} \leq \int_{\chi(q)} \varphi(x,s) d(s,x) \leq K \lambda^{n+1} \tag{5.6}$$

For a number $\mu(q) \in [0,1]$, build the hyper-square of edge $\mu(q)$:

$$D(\mu(q)) := [t_1 + q_0 \lambda, t_1 + q_0 \lambda + \mu(q)] \times [-M + q_1 \lambda, -M + q_1 \lambda + \mu(q)] \times \cdots \times [-M + q_n \lambda, -M + q_n \lambda + \mu(q)].$$

Let $V(\mu(q))$ be the hyper-volume of $D(\mu(q))$. Then, by (5.6) there is a number $0 < \mu(q) \leq \lambda$ such that

$$K \left[ 2V(\mu(q)) \lambda - \lambda^{n+1} \right] = \int_{\chi(q)} \varphi(x,s) d(s,x).$$

Define the bang-bang feedback function $\varphi_\ast(\lambda, i)$ as set difference:

$$\varphi_\ast^i(t,x) := \begin{cases} K & \text{for } (t,x) \in D(\mu(q)), \\ -K & \text{for } (t,x) \notin D(\mu(q)) \end{cases} \tag{5.7}$$

The following inequalities into (5.8) yields

$$\sup_{t \in [t_1,t_2]} \left[ \left| \int_{\chi(q)} b(s,x(s)) (\varphi_\ast(s,x(s)) - \varphi_\ast(s,x(s))) d(s,x) \right| \right] \leq 2K \zeta \eta (2\Delta)^n.$$
ℓ = 0.1. By Choi and Hammer (2018), the nominal system is (5,2)-controllable and the minimal time does not exceed 0.3. Thus, our domain of interest is $[0,0.3] \times [-2,2]^2 \subseteq \mathbb{R}^2 \times R^2$.

To search for a bang-bang function, we partition our domain into cubes of edge $\lambda = 0.01$. A numerical search for a bang-bang feedback function that guides $\Sigma$ to $\rho(0.1)$ in minimal time without violating our constraints yields (see also Figure 6.1):

$$\varphi_{\pm}(t,x_1,x_2) := \begin{cases} 
\pm 5 & \text{if } t \in (0,0.05); \\
-5 & \text{if } t \in [0.05,0.14); \\
5 & \text{if } t \in [0.14,0.16) \text{ and } |x_2| \geq 1.12; \\
-5 & \text{if } t \in [0.14,0.15) \text{ and } 0.98 \leq |x_2| \leq 1.01; \\
-5 & \text{if } t \in [0.14,0.16) \text{ and } |x_2| \leq 0.86; \\
5 & \text{else.} 
\end{cases}$$

Fig. 6.1. The bang-bang feedback function $\varphi_{\pm}$

The performance of this feedback function is shown in Figure 6.2 for three representatives of our family of systems. The minimal time here is 0.229, an improvement of about 20% over the open-loop minimal time of 0.264 (Choi and Hammer (2018)).

Fig. 6.2. The trajectories

7 Conclusion

We have shown that optimal robust feedback solutions exist and demonstrated a relatively simple implementation technique. The implementation technique is based on the use of bang-bang feedback functions; these functions can be calculated by a relatively simple numerical search process, since they have a discrete finite set of values.

References


