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Optimal robust state-feedback control of nonlinear systems: minimal time to target

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ABSTRACT
The design of optimal robust state-feedback controllers that guide a system to a target in minimal time is considered under constraints on the maximal input amplitude and the maximal overshoot of the controlled system. It is shown that such robust feedback controllers exist for a rather broad family of nonlinear systems. It is also shown that optimal performance can be approximated by state-feedback controllers that are relatively easy to design and implement.

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Optimal control; state feedback; nonlinear control; minimal time

1. Introduction

The design of optimal feedback controllers for nonlinear systems has been an important topic of research for more than half a century. Many studies on this topic concentrate on the solution of a nonlinear partial differential equation – the Hamilton-Jacobi-Bellman equation (Bellman, 1954; Kruzkov, 1960; Miranda, 1955; Pontryagin, Boltyansky, Gamkrelidze, & Mishchenko, 1962; Sobolev, 1950). As one might imagine, analytical solutions of the Hamilton-Jacobi-Bellman equation for nonlinear systems are hard to come by; and the calculation of approximate numerical solutions is burdened by high computational complexity. In fact, attempts to use numerical techniques to calculate approximate solutions of the Hamilton-Jacobi-Bellman equation for higher dimensional systems encounter, in R. Bellman's words, the 'dimensionality curse' and are impractical even with today's digital computing technology. The implementation of such solutions imposes further difficulties. What's more, the incorporation of practical inequality constraints, such as input amplitude bounds and overshoot bounds, further complicate the process of deriving solutions of the Hamilton-Jacobi-Bellman equation.

The present paper investigates the existence and implementation of robust state-feedback controllers that drive a time-varying input-affine nonlinear system to a target state in minimal time, while abiding by input amplitude bounds and overshoot bounds imposed by the controlled system. In qualitative terms, our motivation is to develop a methodology for the design of 'best' feedback controllers under conditions of uncertainty about the controlled system's model, and while abiding by operational constraints imposed by the controlled system.

The discussion employs tools borrowed from the mathematical disciplines of functional analysis, topology, and measure theory. The current approach facilitates a thorough examination of robustness features of feedback controllers in the face of uncertainties about the controlled system's model. It allows us to take into consideration two common practical constraints on the operation of a control system: (i) a specified bound K on the maximal input amplitude the controlled system can tolerate; and (ii) a specified bound A on the maximal overshoot the controlled system may experience during the control process.

The current paper builds upon the work of Yu and Hammer (2016a, 2016b), which concentrates on the existence and the design of robust open-loop controllers that drive a system to a target in minimal time. As one might expect (see Section 6), optimal feedback controllers achieve better performance than open-loop controllers: they can guide an inaccurately described system to its target in a shorter time and may reduce other effects of inaccuracy. The benefits of feedback controllers become more prominent when the minimal time to target is longer. Of course, the cost of these benefits is an increase in the complexity of design and implementation. Still, in many applications, the benefits of better performance justify the additional effort.

The control configuration we consider is the classical state-feedback control configuration of Figure 1. Here, the controlled system Σ is a time-varying input-affine nonlinear system with input signal $u(t)$; its output signal is the state $x(t)$. The feedback controller is implemented by a state-feedback function φ that assigns to Σ an input signal $u(t) = \varphi(t, x(t))$ based on the time t and state $x(t)$ of Σ. As can be seen from the figure, the closed loop system is controlled solely by the feedback controller φ; it receives no external input signal.

As depicted in Figure 1, the feedback controllers discussed in this paper are static state-feedback controllers; they are formed by feedback functions $\varphi(t, x(t))$. Static controllers are simpler to implement than dynamic controllers that are described by differential equations. Generally speaking, many control objectives can be achieved by static state-feedback controllers; see, for example, Hammer (2013, 2014, 2015). Notwithstanding, if dynamic state-feedback controllers are desired, these can be derived from our present discussion by augmenting the controlled system Σ with 'dummy states' and considering static state-feedback controllers for the augmented system.

The control objective is to drive the controlled system Σ in minimal time from an initial state $x(0) = x_0$ to a specified target

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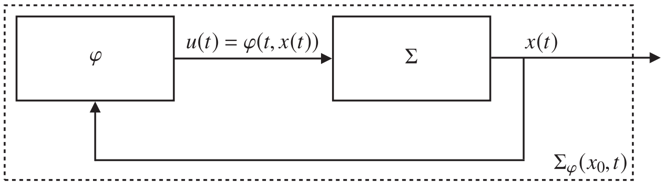


Figure 1. State-feedback control

state x_{target}, while abiding by constraints on the maximal input and output amplitudes of Σ. After appropriately shifting the state coordinates of Σ, we can assume that the target state of Σ is the zero state $x = 0$.

Due to uncertainties and errors prevalent in engineering systems, it is not possible to guide Σ exactly to the origin $x = 0$. Instead, we allow a small deviation from the target state $x = 0$, not to exceed a specified error bound of $\ell > 0$. We denote by

$$\rho(\ell) := \{x : x^\top x \le \ell\}$$

the corresponding vicinity of the origin. In these terms, our objective is to design a state-feedback function φ that guides Σ to the domain $\rho(\ell)$ in minimal time, subject to two constraints: (i) a bound of $K > 0$ on the maximal input signal amplitude of Σ; and (ii) a bound of $A > 0$ on the maximal state amplitude Σ. These constraints assure that the controlled system Σ will not be overloaded or overstressed during the control process; they reflect common constraints encountered by control engineers in practice.

The existence of optimal robust feedback controllers that guide a controlled system Σ in minimal time from an initial state x_0 to the vicinity $\rho(\ell)$ of its target state is discussed in Section 4, where the main result is Theorem 4.1. The theorem shows that optimal robust feedback controllers exist under rather broad conditions. The requirement for the existence of such optimal controllers is a certain controllability condition. In fact, this controllability condition is also close to being a necessary condition for the existence of controllers that guide the controlled system Σ to the vicinity of its target state (see discussion in Section 4).

The design and implementation of optimal feedback controllers is an involved process – it may require the derivation and implementation of intricate functions of the time and state: multivariable vector-valued functions of time. To address and resolve this difficulty, we introduce in Section 5 the class of bang-bang feedback functions. A bang-bang feedback function $\varphi_\pm$ is a piecewise-constant function of the time and state, whose components switch between two values: the values K or $-K$, where K is the input amplitude bound of the controlled system Σ (see Definition 5.3). When compared to general state-feedback functions, bang-bang state-feedback functions are relatively easy to calculate and implement. For a system Σ with an m-dimensional input space, bang-bang feedback function take at most 2^m discrete values of K and $-K$, while general feedback functions map into the continuum R^m. In Theorem 5.4 of Section 5 – the main result of the section – we show that optimal robust performance can be approximated as closely as desired by bang-bang state-feedback functions. This fact substantially simplifies the design and implementation of optimal robust feedback controllers.

For future reference, we summarise our objectives as follows.

Problem 1.1: Let Σ be an input/state system with a specified maximal input signal amplitude of $K > 0$ and a specified maximal response amplitude of $A > 0$. Let $\ell > 0$ be the maximal permissible deviation from the target state $x = 0$.

(i) Find conditions under which there is an optimal robust state-feedback function φ that takes Σ in minimal time from an initial state x_0 to $\rho(\ell)$, without violating the input amplitude bound K and without causing overshoots exceeding the response amplitude bound A.
(ii) Find a state-feedback function $\varphi_\pm$ that approximates optimal performance and is relatively easy to design and implement.

The current paper, which concentrates on optimal feedback control, builds upon the studies of Yu and Hammer (2016a, 2016b) and Choi and Hammer (2018a, 2018c), where open-loop minimal-time control is considered. The performance of optimal feedback controllers is never inferior to that of open-loop controllers. This observation is a consequence of the fact that the feedback function φ of Figure 1 is a function of the time and the state; open-loop control is the special case where φ is a function of the time only. Thus, optimisation over the class of feedback functions includes optimisation over open-loop controllers and, therefore, feedback performance is never inferior to open loop performance.

An example provided in Section 6 demonstrates the superiority of optimal feedback controllers. In this example, an optimal closed-loop controller achieves a significantly shorter time-to-target than an optimal open-loop controller. In addition, the outcome of optimal closed-loop control is less sensitive to uncertainties and errors present in the controlled system's model, thus providing more robust performance than the performance achieved by an optimal open-loop controller.

Notwithstanding, there are applications where feedback cannot be used, such as cases where feedback channels have been disrupted. Clearly, in such cases, open-loop control is the only option. In addition, there are applications where the performance of optimal open-loop controllers is adequate and a more complex closed-loop implementation may not be necessary.

Our discussion in this paper extends to optimal closed-loop control some of the techniques developed in the studies on optimal open-loop control reported by Chakraborty and Hammer (2007, 2008a, 2008b, 2008c, 2009a, 2009b, 2010), Chakraborty and Shaikshavali (2009), Yu and Hammer (2016a, 2016b), and Choi and Hammer (2017a, 2017b, 2017c, 2018a, 2018b, 2018c). In addition, the present paper draws on classical studies on optimal control, including Kelendzheridze (1961), Pontryagin et al. (1962), Gamkrelidze (1965), Neustadt (1966), Neustadt (1967), Luenberger (1969), Young (1969), and Warga (1972), the references cited in these publications, and many others. Yet, it seems that the problem considered in this paper – the existence, the design, and the bang-bang approximation of robust state-feedback controllers under input and output constraints – has not been resolved in the literature before.

Traditional approaches to the design of optimal feedback controllers are mostly based on the use of numerical techniques to derive approximate numerical solutions of the Hamilton-Jacobi-Bellman equation. A detailed exposition of existing techniques can be found in Bardi and Capuzzo-Dolcetta (1997). Generally speaking, these techniques do not address directly the issue of robustness: they do not lead to the derivation of feedback controllers that are optimal for a given level of uncertainty of the controlled system's model. The current paper develops new and efficient techniques that focus on the design and implementation of robust feedback controllers that comply with operational constraints imposed by the controlled system. The controllers derived in this paper – bang-bang feedback controllers – are also simpler to design and implement than classical controllers, as discussed in Sections 5 and 6.

The paper is organised as follows. Section 2 includes a formal formulation of our objectives, while Section 3 reviews the notion of conditional controllability, which forms the basic requirement for the existence of optimal state-feedback controllers. The existence of optimal state-feedback controllers is considered in Section 4, and the approximation of optimal feedback controllers by bang-bang feedback controllers is discussed in Section 5. A computational example that demonstrates the material presented in this paper is provided in Section 6. The paper concludes in Section 7 with a brief summary.

2. Background and statement of the problem

2.1 The controlled system

This paper concentrates on the control of time-varying input-affine nonlinear systems. In addition to time-varying linear systems, this class of systems includes models of certain common nonlinear engineering systems such as flexible robotic joints and advanced electrical motors (e.g. Spong, Hutchinson, & Vidyasagar, 2006).

Denote by R the compactified set of real numbers (i.e. the real numbers augmented by $\pm\infty$) and by R^n the compactified set of $n-$dimensional real vectors. By R^+ we denote the set of non-negative real numbers. The absolute value of a real number r is $|r|$; the $L^\infty-$norm of an $n \times m$ matrix $V = (V_{ij}) \in R^{n\times m}$ is $|V| := \max_{ij} |V_{ij}|$; and the $L^\infty-$norm of a matrix valued function $W : R^+ \to R^{n\times m} : t \mapsto W(t)$ is $|W|_\infty := \sup_{t\geq 0} |W(t)|$, where $|W|_\infty := \infty$ if the supremum does not exist. The L^2-norm of a vector $x \in R^n$ is $|x|_2 := (x^\top x)^{1/2}$.

The controlled system Σ of Figure 1 is a nonlinear time-varying input-affine system given by

$$\Sigma : \begin{array}{l} \dot{x}(t) = a(t, x(t)) + b(t, x(t))u(t), \\ x(0) = x_0; \end{array} \tag{2.1}$$

here, $x(t) \in R^n$ is the state and $u(t) \in R^m$ is the input signal. The functions $a : R^+ \times R^n \to R^n : (t, x) \mapsto a(t, x)$ and $b : R^+ \times R^n \to R^{n\times m} : (t, x) \mapsto b(t, x)$ are continuous functions satisfying the Lipchitz conditions

$$\begin{aligned} |a(t, y) - a(t, x)| &\leq \alpha^+|y - x|, \\ |b(t, y) - b(t, x)| &\leq \alpha^+|y - x| \end{aligned} \tag{2.2}$$

for all $(t, x) \in R^+ \times R^n$, where $\alpha^+ > 0$ is a specified real number.

To take modelling uncertainties into account, we split the functions a and b of (2.1) into a sum of nominal and uncertain parts:

$$\begin{aligned} a(t, x) &= a_0(t, x) + a_\gamma(t, x), \\ b(t, x) &= b_0(t, x) + b_\gamma(t, x), \end{aligned} \tag{2.3}$$

where $a_0 : R^+ \times R^n \to R^n$ and $b_0 : R^+ \times R^n \to R^{n\times m}$ are specified continuous functions describing the nominal model Σ_0 of Σ, while $a_\gamma : R^+ \times R^n \to R^n$ and $b_\gamma : R^+ \times R^n \to R^{n\times m}$ are unknown continuous functions that describe uncertainty about the model of Σ. All functions satisfy the Lipschitz conditions

$$\begin{aligned} &|a_0(t, x') - a_0(t, x)| \leq \alpha|x' - x|, \\ &|b_0(t, x') - b_0(t, x)| \leq \alpha|x' - x|, \\ &a_0(t, 0) = 0, \quad |b_0(t, 0)| \leq \alpha, \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} &|a_\gamma(t, x') - a_\gamma(t, x)| \leq \gamma|x' - x|, \\ &|b_\gamma(t, x') - b_\gamma(t, x)| \leq \gamma|x' - x|, \\ &a_\gamma(t, 0) = 0, \quad |b_\gamma(t, 0)| \leq \gamma \end{aligned} \tag{2.5}$$

for all $(t, x) \in R^+ \times R^n$. Here, $\alpha > 0$ and $\gamma > 0$ are specified real numbers, and $\alpha^+ = \alpha + \gamma$. The number γ is the *uncertainty parameter*; it quantifies the uncertainty about the model of Σ and is usually a small number. The nominal system Σ_0 is

$$\Sigma_0 : \begin{array}{l} \dot{x}(t) = a_0(t, x(t)) + b_0(t, x(t))u(t), \quad t \geq 0, \\ x(0) = x_0. \end{array} \tag{2.6}$$

The system Σ accepts only input signals of amplitude not exceeding a specified bound $K > 0$, and the response $x(t)$ of Σ is not permitted to overshoot a specified bound $A > 0$. In formal terms, these constrains are

$$|u|_\infty \leq K \quad \text{and} \quad |x|_\infty \leq A.$$

2.2 Spaces and notation

Denote by $L_2^{\omega,m}$ the Hilbert space of all Lebesgue measurable functions $f, g : R^+ \to R^m$ with the inner product

$$\langle f, g\rangle := \int_0^\infty e^{-\omega s} f^\top(s) g(s)\, ds, \tag{2.7}$$

where $\omega > 0$ is a real number (Chakraborty & Hammer, 2009b, 2010). The use of this inner product carries a number of benefits, the most obvious of which is the fact that it is bounded whenever the functions f and g are bounded. Later, we discuss other advantages of this inner product, including the fact that it turns certain sets of bounded functions into compact sets. We extend the definition of the inner product to cases where one of the factors is an $n \times m$ matrix $G(t)$. Let $G_1(t), G_2(t), \ldots, G_n(t)$ be the rows of $G(t)$, where $G_1^\top(t), G_2^\top(t), \ldots, G_n^\top(t) \in L_2^{\omega,m}$, and let $g \in L_2^{\omega,m}$; we use the notation

$$\langle G, g\rangle := \sum_{j=1}^{n} \left\langle G_j^\top, g\right\rangle.$$

Input signals of the controlled system Σ of Figure 1 are members of the Hilbert space $L_2^{\omega,m}$. As mentioned earlier, Σ is subject to two structural constraints: (i) its input signals may not exceeding the specified amplitude bound of $K > 0$; and (ii) its state $x(t)$ may not exceed a specified amplitude bound of $A > 0$. In view of (i), the set of input signals of Σ is constrained to the set

$$U(K) := \left\{u \in L_2^{\omega,m} : |u|_\infty \leq K\right\}.$$

Furthermore, by (ii), Σ must be controlled so that the state $x(t)$ of Σ remains constrained to the set $|x|_\infty \leq A$. As Σ is operated by input signals belonging to $U(K)$, its response $x(t)$ is a continuous function of time. Consequently, the set of responses that Σ may access is included in the set

$$X(A) := \left\{x \in L_2^{\omega,n} : |x|_\infty \leq A\right\}.$$

Denote by $[-A, A]^n$ the set of all vectors $x \in R^n$ satisfying $|x| \leq A$. Then, the requirement $|x|_\infty \leq A$ takes the form $x(t) \in [-A, A]^n$ for all $t \geq 0$.

As indicated in Problem 1.1, our objective is to guide the system Σ in minimal time from an initial state x_0 to the domain $\rho(\ell) = \{x \in R^n : (|x|_2)^2 \leq \ell\}$, where $\ell > 0$ is a specified error bound. For consistency, we require $\rho(\ell) \subseteq [-A, A]^n$.

Recall that the exact description of the controlled system Σ of Figure 1 is not known; any one of the models represented by (2.1), (2.3), (2.4) and (2.5) can be Σ. This fact has the two following implications. (i) All members of the family of systems represented by these equations share the same feedback function φ; this is so because it is not known which of the possible models of Σ represents the actual sample of Σ present in the loop. (ii) All members of the family of systems represented by these equations share the same initial state x_0, since the actual value of the state is provided by the feedback. The following notation incorporates these observations.

Notation 2.8: Let $\alpha, \gamma, K, A > 0$ be specified real numbers, and let Σ_0 be the nominal system of (2.6). Denote by $\mathcal{F}_\gamma(\Sigma_0)$ the family of systems described by (2.1), (2.3), (2.4), and (2.5).

(i) All members of $\mathcal{F}_\gamma(\Sigma_0)$ share the same initial state $x(0) = x_0$.
(ii) The initial state x_0 satisfies the constraint $|x_0| \leq A$.
(iii) All members of $\mathcal{F}_\gamma(\Sigma_0)$ use the same feedback function φ.
(iv) The input signal of Σ generated by the feedback function φ is in $U(K)$.
(v) The feedback function φ controls all members of $\mathcal{F}_\gamma(\Sigma_0)$ so that their response is in $X(A)$.

Remark 2.9: All equalities and inequalities in this paper are to be understood as valid almost everywhere in the Lebesgue sense.

2.3 State-feedback functions

As indicated in Problem 1.1, our focus is on the existence and the design of state-feedback functions φ that automatically guide the controlled system Σ in minimal time from an initial state x_0 to the domain $\rho(\ell)$. A state-feedback function for the system Σ of (2.1) is a Lebesgue measurable function $\varphi : R^+ \times R^n \to R^m : (t, x) \mapsto \varphi(t, x)$. With such a state-feedback function, the closed loop system Σ_φ of Figure 1 is described by the equation

$$\Sigma_\varphi : \begin{array}{l} \dot{x}(t) = a(t, x(t)) + b(t, x(t))\varphi(t, x(t)), \quad t \geq 0, \\ x(0) = x_0, \end{array} \tag{2.10}$$

where the functions a and b are subject to the requirements (2.3), (2.4), and (2.5). For a particular feedback function φ, the response $x(t)$ of Σ_φ is determined by the initial state x_0, so we use the notation $x(t) = \Sigma_\varphi(x_0, t)$. In view of the requirements listed in Notation 2.8, a feedback function φ is permissible only if $|\Sigma_\varphi(x_0, t)| \leq A$ at all times t during the control process.

As state-feedback functions are multivariable functions, we need to extend the inner product (2.7) to multivariable functions. To this end, let $\mathscr{L}(n, m)$ be the class of all Lebesgue measurable functions $: R^+ \times R^n \to R^m$, and let $\omega > 0$ a real number. For two members $f, g \in \mathscr{L}(n, m)$, define the inner product

$$\langle\!\langle f, g\rangle\!\rangle := \int_{R^+ \times R^n} \mathrm{e}^{-\omega(s+|z|_2)} f^\top(s, z) g(s, z)\, \mathrm{d}(s, z), \tag{2.11}$$

where $\mathrm{d}(s, z)$ represents an element of the Lebesgue measure on $R^+ \times R^n$. Note that the inner product (2.11) is bounded whenever the functions f and g are bounded. We denote by $L_2^{\omega,n,m}$ the Hilbert space of all members of $\mathscr{L}(n, m)$ with the inner product (2.11).

Now, recall that only input signals bounded by $K > 0$ may be used as input to the controlled system Σ. As the input signal of Σ in the feedback configuration of Figure 1 is produced by the feedback function φ, only feedback functions whose components are bounded by K are allowed. Denote by $\Phi(K)$ the family of all such members of $L_2^{\omega,n,m}$, namely,

$$\Phi(K) := \left\{\varphi \in L_2^{\omega,n,m} : |\varphi(t, x)| \leq K \text{ for all } (t, x) \in R^+ \times R^n\right\}. \tag{2.12}$$

The family $\Phi(K)$ includes all permissible feedback functions.

2.4 Convergence and compactness

We employ the following notions of convergence (e.g. Willard, 2004; Zeidler, 1985).

Definition 2.13: Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$.

(i) A sequence $\{v_i\}_{i=1}^\infty \subseteq H$ *converges weakly* to a member $v \in H$ if $\lim_{i\to\infty} \langle v_i, y\rangle = \langle v, y\rangle$ for every $y \in H$.
(ii) A subset W of H is *weakly compact* if every sequence of members of W has a subsequence that converges weakly to a member of W.

(iii) A sequence $\{v_i\}_{i=1}^{\infty} \subseteq H$ is *strongly convergent* if there is a member $v \in H$ such that $\lim_{i\to\infty}\langle(v_i - v), (v_i - v)\rangle = 0$.
(iv) A set $S \subseteq H$ is *strongly closed* if every strongly convergent sequence of members of S has its limit in S.

The following fact is reproduced here from Chakraborty and Hammer (2009b, 2010).

Lemma 2.14: *The set of input signals $U(K)$ is weakly compact in $L_2^{\omega,m}$.*

We also need the following.

Lemma 2.15: *The set of feedback functions $\Phi(K)$ is weakly compact in $L_2^{\omega,n,m}$.*

Proof: Given a sequence of feedback functions $\{\varphi_i\}_{i=1}^{\infty} \subseteq \Phi(K)$, we have to show that there is a subsequence $\{\varphi_{i_k}\}_{k=1}^{\infty}$ that converges weakly to a feedback function $\varphi \in \Phi(K)$. By (2.12) and (2.11), the set $\Phi(K)$ is a bounded set in $L_2^{\alpha,n,m}$. In view of Alaoglu's theorem (e.g. Halmos, 1982), every bounded sequence in a Hilbert space contains a weakly convergent subsequence. Thus, the sequence $\{\varphi_i\}_{i=1}^{\infty} \subseteq \Phi(K)$ contains a subsequence $\{\varphi_{i_k}\}_{k=1}^{\infty}$ that converges weakly to a function $\varphi \in L_2^{\omega,n,m}$. We have to show that $\varphi \in \Phi(K)$. To this end, we show that $\Phi(K)$ is a weakly closed set, i.e. that every weakly convergent sequence in $\Phi(K)$ has its weak limit in $\Phi(K)$. We use Mazur's theorem (e.g. Halmos, 1982), according to which a bounded and strongly closed convex set in Hilbert space is also weakly closed.

First, we show that $\Phi(K)$ is a convex set. Indeed, consider two feedback functions $\varphi', \varphi'' \in \Phi(K)$. Let $d \in [0, 1]$ be a real number and define the function $\varphi(t, x) := d\varphi'(t, x) + (1 - d)\varphi''(t, x)$. Then, φ is Lebesgue measurable and, since $|\varphi(t, x)| \leq d|\varphi'(t, x)| + (1 - d)|\varphi''(t, x)| \leq dK + (1 - d)K = K$ for all $(t, x) \in R^+ \times R^n$, it follows that $\varphi \in \Phi(K)$. Thus, $\Phi(K)$ is a convex set.

To show that $\Phi(K)$ is strongly closed, consider a strongly convergent sequence of functions $\{\psi_p\}_{p=1}^{\infty} \subseteq \Phi(K)$ with the strong limit ψ, i.e. $\lim_{p\to\infty} \langle\langle(\psi_p - \psi), (\psi_p - \psi)\rangle\rangle = 0$. We show that $\psi \in \Phi(K)$. Indeed, by contradiction, assume that $\psi \notin \Phi(K)$. Then, by (2.12), there is a real number $\varepsilon > 0$ and a Lebesgue measurable subset $\delta \subseteq R^+ \times R^n$ of non-zero measure for which $|\psi(t, x)| \geq K + \varepsilon$ for all $(t, x) \in \delta$. This implies that there is an integer $j \in \{1, 2, \ldots, m\}$ and a measurable subset of non-zero measure $\delta^j \subseteq \delta$ such that the jth component ψ^j of the vector ψ satisfies $|\psi^j(t, x)| \geq K + \varepsilon$ for all $(t, x) \in \delta^j$, namely,

$$|\psi^j(t,x)| - K \geq \varepsilon \quad \text{for all } (t,x) \in \delta^j.$$

Calculating the inner product and recalling that $|\psi_p^j| \leq K$ for all integers $p \geq 1$, we get

$$\begin{aligned}
&\langle\langle(\psi_p - \psi), (\psi_p - \psi)\rangle\rangle \\
&= \int_{R^+\times R^n} e^{-\omega(s+|z|_2)}[\psi_p(s,z) - \psi(s,z)]^T \\
&\quad \times [\psi_p(s,z) - \psi(s,z)]\, d(s,z) \\
&\geq \int_{\delta^j} e^{-\omega(s+|z|_2)}[\psi_p(s,z) - \psi(s,z)]^T \\
&\quad \times [\psi_p(s,z) - \psi(s,z)]\, d(s,z) \\
&\geq \int_{\delta^j} e^{-\omega(s+|z|_2)}(\psi_p^j(s,z) - \psi^j(s,z))^2\, d(s,z) \\
&\geq \int_{\delta_j} e^{-\omega(s+|z|_2)}\varepsilon^2\, d(s,z) > 0
\end{aligned}$$

for all integers $p \geq 1$, contradicting the fact that $\lim_{p\to\infty} \langle\langle(\psi_p - \psi), (\psi_p - \psi)\rangle\rangle = 0$. Thus, we must have $\psi \in \Phi(K)$, so that $\Phi(K)$ is strongly closed. As $\Phi(K)$ is convex and strongly closed, the lemma follows by Mazur's theorem. This concludes our proof. ■

2.5 Compositions and graphs

Being a feedback function, the function φ always appears in composition with the state function $x(t)$ of the controlled system Σ in the form $\varphi(t, x(t))$, as in (2.10). For this composition, we use the notation

$$\varphi \circ x := \varphi(\cdot, x(\cdot)) : R^+ \to R^m : t \mapsto \varphi(t, x(t)). \tag{2.16}$$

Note that the composite function $\varphi \circ x$ is a function of time only. A slight reflection shows that, when $\varphi \in L_2^{\omega,n,m}$ and $x \in L_2^{\omega,n}$, then $\varphi \circ x \in L_2^{\omega,m}$. In the feedback configuration of Figure 1, the function $\varphi \circ x$ serves as the input signal of the controlled system Σ.

Due to the uncertainty about the model of the controlled system Σ expressed by (2.3), (2.4), and (2.5), the response function $x(t)$ is not specifically known: $x(t)$ may be the state of any member $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$. Therefore, we must examine properties of compositions of the form (2.16) in which $x(t)$ is not specifically known. We introduce now tools to investigate properties of such compositions.

The *graph* $\Gamma(f)$ of a function $f \in L_2^{\omega,n}$ is the set of all pairs $(t, f(t))$, namely,

$$\Gamma(f) := \bigcup_{t\geq 0}(t, f(t)) \subseteq R^+ \times R^n.$$

For a family of functions $F \subseteq L_2^{\omega,n}$, the *graph* $\Gamma(F)$ is the union of all the graphs of members of F:

$$\Gamma(F) := \bigcup_{f\in F}\Gamma(f).$$

At a time $\tau \geq 0$, the *section* $\Gamma_\tau(F)$ of the graph $\Gamma(F)$ consists of all values of members of F at the time τ, i.e.

$$\Gamma_\tau(F) := \left\{x \in R^n : (\tau, x) \in \Gamma(F)\right\}.$$

The set of all values of members of F is denoted by $\Pi\Gamma(F)$, where

$$\Pi\Gamma(F) := \bigcup_{\tau\geq 0}\Gamma_\tau(f);$$

it is the union of images of all members of the family F.

Definition 2.17: A family of functions $F \subseteq L_2^{\omega,n}$ is of *non-zero measure* if there is a time $\tau \geq 0$ at which the section $\Gamma_\tau(F)$ includes a set of non-zero measure; otherwise, the family F is of *measure zero*. A statement is valid for *almost every function* $f \in L_2^{\omega,n}$ if the family of functions $F \subseteq L_2^{\omega,n}$ for which the statement is invalid is of measure zero.

We will need the following convergence feature.

Proposition 2.18: *Let $\{\varphi_i\}_{i=1}^{\infty} \subseteq \Phi(K)$ be a sequence of feedback functions that is weakly convergent to a feedback function φ. Let $F \subseteq L_2^{\omega,n}$ be a family of non-zero measure, and let $\varepsilon > 0$ be a real number. Then, the following is true for every $g \in U(K)$. There is an integer $N \geq 1$ such that $|\langle(\varphi_i - \varphi) \circ f, g\rangle| < \varepsilon$ for all $i \geq N$ and for almost all $f \in F$.*

Proof: Let $\{\varphi_i\}_{i=1}^{\infty} \subseteq \Phi(K)$ be a sequence that converges weakly to φ. Assume, by contradiction, that the proposition is invalid. Then, there are a real number $\varepsilon > 0$, a family $F' \subseteq F$ of non-zero measure, a subsequence $\{\varphi_{i_k}\}_{k=1}^{\infty}$, and a function $g \in U(K)$ such that $|\langle(\varphi_{i_k} - \varphi) \circ f, g\rangle| \geq \varepsilon$ for all $k \geq 1$ and all $f \in F'$. This means that

$$\lim_{k\to\infty} \left\langle(\varphi_{i_k} - \varphi) \circ f, g\right\rangle \neq 0 \quad \text{for all } f \in F'.$$

We partition the family F' into two subfamilies:

(a) the family F^+ that consists of all members $f \in F'$ for which $\lim_{k\to\infty}\langle(\varphi_{i_k} - \varphi) \circ f, g\rangle > 0$; and
(b) the family F^- that consists of all members $f \in F'$ for which $\lim_{k\to\infty}\langle(\varphi_{i_k} - \varphi) \circ f, g\rangle < 0$.

Now, as the family F' is of non-zero measure, there is a time $\tau \geq 0$ at which the section $\Gamma_\tau(F')$ includes a set of non-zero measure. But then, since $\Gamma_\tau(F') = \Gamma_\tau(F^+) \cup \Gamma_\tau(F^-)$, at least one of the sets $\Gamma_\tau(F^+)$ or $\Gamma_\tau(F^-)$ must include a measurable subset of non-zero measure. Let us assume that $\Gamma_\tau(F^+)$ includes a measurable subset σ of non-zero measure (the proof in the case where only $\Gamma_\tau(F^-)$ includes a measurable subset of non-zero measure is similar). Denote by F'' the family of all members $f \in F^+$ for which $\Gamma_\tau(f) \in \sigma$. Then, by construction,

$$\lim_{k\to\infty} \left\langle(\varphi_{i_k} - \varphi) \circ f, g\right\rangle > 0 \quad \text{for all } f \in F''. \tag{2.19}$$

Next, for a point $x \in \sigma$, denote by f^x a member of the family F'' for which $f^x(\tau) = x$. Define the set

$$\Gamma := \bigcup_{x\in\sigma} \Gamma(f^x).$$

Presently, build a function $\phi(t,x) : R^+ \times R^n \to R^m : (t,x) \mapsto \phi(t,x)$ by setting

$$\phi(t,x) := \begin{cases} g(t) & (t,x) \in \Gamma, \\ 0 & \text{otherwise.} \end{cases}$$

As $g \in L_2^{\omega,m}$, it follows that $\phi \in L_2^{\omega,n,m}$.

Further, let $h \in L_2^{\omega,n,m}$ be any function. Now, by the definition of the graph of a function $f \in L_2^{\omega,m}$, a point $(t,x) \in \Gamma(f)$ means that $x = f(t)$. Therefore, on a point $(t,x) \in \Gamma(f)$ we have $h^\top(t,x)\phi(t,x) = h^\top(t,f^x(t))g(t) = (h \circ f^x)^\top(t)g(t)$. Invoking the inner product (2.11), we can write

$$\begin{aligned} \langle\langle h, \phi\rangle\rangle &:= \int_{R^+\times R^n} \mathrm{e}^{-\omega(t+|x|_2)} h^\top(t,x)\phi(t,x)\,\mathrm{d}(t,x) \\ &= \int_{\Pi\Gamma} \mathrm{e}^{-\omega|x|_2} \left\langle h \circ f^x, g\right\rangle \mathrm{d}x. \end{aligned} \tag{2.20}$$

Inserting $h := (\varphi_{i_k} - \varphi)$ in (2.20) yields

$$\begin{aligned} &\lim_{k\to\infty} \left\langle\!\left\langle(\varphi_{i_k} - \varphi), \phi\right\rangle\!\right\rangle \\ &= \int_{\Pi\Gamma} \mathrm{e}^{-\omega|x|_2} \left(\lim_{k\to\infty} \left\langle(\varphi_{i_k} - \varphi) \circ f^x, g\right\rangle\right) \mathrm{d}x > 0, \end{aligned} \tag{2.21}$$

where the last inequality follows from the facts that the integrand is strictly positive by (2.19) and the integration domain, which includes σ, is of non-zero measure. However, as the sequence $\{\varphi_i\}_{i=1}^{\infty}$ is weakly convergent in $L_2^{\omega,n,m}$, so is its subsequence $\{\varphi_{i_k}\}_{k=1}^{\infty}$; this implies that $\lim_{k\to\infty}\langle\langle(\varphi_{i_k} - \varphi), \phi\rangle\rangle = 0$, contradicting (2.21). This contradiction implies that σ must be of measure zero, and our proof concludes. ■

We proceed now to restate Problem 1.1 in more formal terms.

2.6 Formal statement of objectives

According to Problem 1.1, our objective is to find a feedback function φ that guides the closed-loop system Σ_φ in minimal time from the initial state x_0 to $\rho(\ell)$, without violating the input amplitude bound K and the state amplitude bound A along the way. We concentrate first on the class of feedback functions that abide by the bounds K and A. Let $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ be a system with initial state $x_0 \in [-A, A]^n$. For a time $t \geq 0$, denote by $\Phi(x_0, K, A, \Sigma, t)$ the class of all state-feedback functions $\varphi \in \Phi(K)$ for which the state of the closed loop system Σ_φ stays within the domain $[-A, A]^n$ during the time interval $[0, t]$, namely,

$$\begin{aligned} &\Phi(x_0, K, A, \Sigma, t) \\ &\quad = \left\{\varphi \in \Phi(K) : \Sigma_\varphi(x_0, \theta) \in [-A, A]^n \text{ for all } \theta \in [0, t]\right\}. \end{aligned}$$

Then, the set of all state-feedback functions $\varphi \in \Phi(K)$ for which the states of all members of $\mathcal{F}_\gamma(\Sigma_0)$ stay within the domain $[-A, A]^n$ during the time interval $[0, t]$ is given by

$$\Phi(x_0, K, A, \gamma, t) = \bigcap_{\Sigma\in\mathcal{F}_\gamma(\Sigma_0)} \Phi(x_0, K, A, \Sigma, t). \tag{2.22}$$

Considering that $x_0 \in [-A, A]^n$, it follows that $\Phi(x_0, K, A, \gamma, 0) = \Phi(K)$. Note that (2.22) can be recast in the form

$$\Phi(x_0, K, A, \gamma, t) = \left\{\varphi \in \Phi(K) : \sup_{\substack{\Sigma\in\mathcal{F}_\gamma(\Sigma_0) \\ 0\leq\theta\leq t}} |\Sigma_\varphi(x_0, \theta)| \leq A\right\}. \tag{2.23}$$

It follows from (2.23) that any feedback function φ that is a member of $\Phi(x_0, K, A, \gamma, t_2)$ for a time t_2, is also a member of

$\Phi(K, A, \gamma, t_1)$ for any time $t_1 \le t_2$. Thus,

$$\Phi(x_0, K, A, \gamma, t_2) \subseteq \Phi(x_0, K, A, \gamma, t_1) \quad \text{whenever } t_1 \le t_2, \tag{2.24}$$

and $\Phi(x_0, K, A, \gamma, t)$ is monotone decreasing as a function of time.

Turning now to the time to target, let $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ be a system, and let $\varphi \in \Phi(K)$ be a feedback function. The infimal time to reach $\rho(\ell)$ under our constraints is

$$t(x_0, \ell, A, \Sigma, \varphi) = \inf_{t \ge 0} \left\{ \Sigma_\varphi(x_0, t) \in \rho(\ell), \varphi \in \Phi(x_0, K, A, \Sigma, t) \right\},$$

where $t(x_0, \ell, A, \Sigma, \varphi) := \infty$ if the infimum does not exist.

Further, the infimal time $t(x_0, \ell, A, \gamma, \varphi)$ at which the state-feedback function $\varphi \in \Phi(K)$ can bring almost every member $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ from the initial state x_0 to $\rho(\ell)$, without violating our constraints, is

$$t(x_0, \ell, A, \gamma, \varphi) := \inf_{t \ge 0} \left\{ \left(\operatorname*{ess\,sup}_{\Sigma \in \mathcal{F}_\gamma(\Sigma_0)} (|\Sigma_\varphi(x_0, t)|_2)^2 \right) \le \ell, \varphi \in \Phi(x_0, K, A, \gamma, t) \right\}, \tag{2.25}$$

where $t(x_0, \ell, A, \gamma, \varphi) := \infty$ if the infimum does not exist or if there is no $\varphi \in \Phi(x_0, K, A, \gamma, t)$ that guides Σ from x_0 to $\rho(\ell)$.

The infimal time at which any state-feedback function belonging to $\Phi(K)$ can take every member of $\mathcal{F}_\gamma(\Sigma_0)$ from the initial state x_0 to $\rho(\ell)$, without violating our constraints, is then

$$t^*(x_0, \ell, A, \gamma) = \inf_{\varphi \in \Phi(K)} t(x_0, \ell, A, \gamma, \varphi), \tag{2.26}$$

where, again, $t^*(x_0, \ell, A, \gamma) := \infty$ if there is no infimum.

In Section 4, we show that $t^*(x_0, \ell, A, \gamma) < \infty$ under rather broad conditions, and that, under those conditions, there is an optimal state-feedback function $\varphi^*(x_0, \ell, A, \gamma) \in \Phi(K)$ that achieves the minimal time $t^*(x_0, \ell, A, \gamma)$, so that

$$t^*(x_0, \ell, A, \gamma) = t(x_0, \ell, A, \gamma, \varphi^*(x_0, \ell, A, \gamma)). \tag{2.27}$$

In the meanwhile, we restate Problem 1.1 in present terms.

Problem 2.28: Let $K, A, \ell, \gamma > 0$ be specified real numbers and refer to Notation 2.8 and equations (2.25) and (2.26).

(i) Find conditions under which there is an optimal state-feedback function $\varphi^*(x_0, \ell, A, \gamma) \in \Phi(K)$ satisfying (2.27).
(ii) When optimal state-feedback functions exist, find feedback functions that approximate optimal performance and are relatively easy to calculate and implement.

2.7 Benefits of a feedback solution

Classical control theory teaches us that, generally, closed-loop systems are superior to open-loop systems in their performance and sensitivity to uncertainties. Yet, much of the considerations leading to this conclusion in classical control theory rely on the use of high-gain feedback controllers. This may be irrelevant in the current context, since optimal feedback controllers do not necessarily operate in high-gain environments.

One fact is quite obvious: a feedback function φ is a function of both the time t and the state $x(t)$ of the controlled system, whereas an open-loop controller is a function of the time t only. Thus, open-loop control represents a case where the function φ is restricted to time-dependence alone. As the result achieved through optimisation over the entire class of feedback functions cannot be worse than the result achieved through optimisation over a subclass of functions, optimal feedback control cannot be inferior to open-loop control. In our case, this means that the minimal time achieved by an optimal feedback function cannot be longer than the minimal time achieved by an open-loop controller.

Still, this argument does not show that an optimal feedback solution provides a significant improvement over an optimal open-loop solution. And, indeed, the improvement achieved by feedback control over open-loop control varies from case to case. One may imagine that feedback solutions have a significant advantage in cases where there is a large uncertainty about the parameters of the controlled system, especially when the minimal time to target is relatively long. In the next paragraph, we demonstrate the superiority of feedback on a simple example.

Consider the case where the controlled system is the linear single-state single-input system

$$\Sigma : \dot{x}(t) = (a_0 + a_\gamma)x(t) + 2u(t), \quad x(0) = x_0;$$

here, a_0 is a specified constant, a_γ is an unspecified constant that represents uncertainty about the model of Σ, and x_0 is the initial state of Σ. As this is a linear time-invariant system, the response is $x(t) = \mathrm{e}^{(a_0+a_\gamma)t}x_0$. Consequently, the feedback function $\varphi(t, x) := [(\log x - \log x_0)/t] - a_0$, $t > 0$, yields the value of the uncertainty parameter a_γ. Once a_γ is known, the system is completely specified – no more uncertainty; the system can then be controlled to any desired accuracy, irrespective of the uncertainty size. This simple example demonstrates that feedback control may allow us to reduce the impact of uncertainty and achieve better control outcomes. The example of Section 6 – a realistic example of an inverted pendulum – further demonstrates this point.

3. Feedback and continuity

3.1 Conditional controllability

In the present section, we setup the framework used later to prove the existence of solutions of Problem 2.28. We start with a statement reproduced from Choi and Hammer (2017a, 2018a); it shows that systems of the family $\mathcal{F}_\gamma(\Sigma_0)$ have continuous responses. Here, $\Sigma(x_0, u, t)$ is the response at the time t of the system Σ to the input signal u, starting from the initial state x_0.

Proposition 3.1: *Let $K, \gamma, T > 0$ be real numbers, and let Σ be a member of the family $\mathcal{F}_\gamma(\Sigma_0)$. Then, the following are true.*

(i) *There is a bound $M(T) \ge 0$ such that $|\Sigma(x_0, u, t)| \le M(T)$ at all times $t \in [0, T]$, for all input signals $u \in U(K)$, and for all members $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$.*

(ii) *The response* $\Sigma(x_0, u, \cdot) : R^+ \to R^n : t \mapsto \Sigma(x_0, u, t)$ *is a continuous function of time for every input signal* $u \in U(K)$ *and for every member* $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$.

Now, in the feedback configuration of Figure 1, the feedback function φ is a member of the family of functions $\Phi(K)$. As a result, the feedback signal $u(t) = \varphi(t, x(t))$ generated by φ belongs to the family of input signals $U(K)$. This directly yields the following consequence of Proposition 3.1.

Corollary 3.2: *Let* $K, \gamma, T > 0$ *be real numbers, let* Σ *be a member of the family* $\mathcal{F}_\gamma(\Sigma_0)$, *and let* $\varphi \in \Phi(K)$ *be a state-feedback function.*

(i) *There is a bound* $M(T) \geq 0$ *such that* $|\Sigma_\varphi(x_0, t)| \leq M(T)$ *at all times* $t \in [0, T]$, *for all state-feedback functions* $\varphi \in \Phi(K)$, *and for all members* $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$.
(ii) *The response* $\Sigma_\varphi(x_0, \cdot) : R^+ \to R^n : t \mapsto \Sigma_\varphi(x_0, t)$ *is a continuous function of time for every feedback function* $\varphi \in \Phi(K)$ *and for every member* $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$.

Considering that continuous functions are bounded over compact domains, we obtain the following consequence of Corollary 3.2.

Corollary 3.3: *Let* n *and* p *be two positive integers, let* $c : R^+ \times R^n \to R^p : (t, x) \mapsto c(t, x)$ *be a continuous function, let* $K, \gamma, T > 0$ *be real numbers, let* Σ *be a member of* $\mathcal{F}_\gamma(\Sigma_0)$, *and let* $\varphi \in \Phi(K)$ *be a state-feedback function. Then, there is a real number* $M_c(T) \geq 0$ *such that* $|c(t, \Sigma_\varphi(x_0, t))| \leq M_c(T)$ *at all times* $t \in [0, T]$, *for all state-feedback functions* $\varphi \in \Phi(K)$, *and for all* $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$.

Recall that we are looking for a state-feedback function $\varphi \in \Phi(K)$ that takes every system $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ from an initial state x_0 to $\rho(\ell)$ in minimal time, without violating the state amplitude bound A along the way. Now, as mentioned earlier, the input signal of Σ generated by our feedback function $\varphi \in \Phi(K)$ is a member of $U(K)$. Therefore, in order for such a feedback function to exist, there must be an input signal $u \in U(K)$ that takes Σ from x_0 to the vicinity $\rho(\ell)$ of $x = 0$, without violating the state amplitude bound A along the way. Whether or not such an input signal exists depends on controllability features of the controlled system Σ, on the input amplitude bound K, on the state amplitude bound A, and on the initial state x_0. These considerations motivate the following notion (Choi & Hammer, 2018a, 2018c).

Definition 3.4: Let K, $A > 0$ be real numbers. A system Σ is (K, A)-*controllable* from the initial state x_0 if there is an input signal $u \in U(K)$ and a finite time $t_A \geq 0$ such that $\Sigma(x_0, u, t_A) = 0$ and $|\Sigma(x_0, u, t)| \leq A$ for all $t \in [0, t_A]$.

A relatively simple numerical technique for determining (K, A)-controllability is outlined in Choi and Hammer (2018a, 2018c).

The next statement shows that, if the nominal system Σ_0 is (K, A)-controllable, then the minimal time $t^*(x_0, \ell, A, \gamma)$ of (2.26) is finite, as long as the uncertainty parameter γ is not too large. This is an important observation; it shows that by checking a single system – the nominal system Σ_0, one can verify proper performance of the entire family $\mathcal{F}_\gamma(\Sigma_0)$. Note that any open-loop input signal $u(t)$ can be generated by a feedback function by setting $\varphi(t, x) := u(t)$ (with no dependence on x). Thus, the following statement is a consequence of a statement derived in Choi and Hammer (2018a, 2018c).

Proposition 3.5: *Let* $K, A_0, \ell > 0$ *be real numbers, and assume that the nominal system* Σ_0 *is* (K, A_0)*-controllable from the initial state* x_0. *Then, for every bound* $A > A_0$, *there is an uncertainty parameter* $\gamma > 0$ *for which the infimal time* $t^*(x_0, \ell, A, \gamma)$ *of* (2.26) *is finite.*

Remark 3.6: An estimate of the uncertainty parameter γ that satisfies Proposition 3.5 can be found in Choi and Hammer (2018a, 2018c).

We show in Section 4 that Proposition 3.5 leads to the following important fact: (K, A)-controllability of a single member of the family $\mathcal{F}_\gamma(\Sigma_0)$ – that of the nominal system Σ_0 – guarantees the existence of an optimal feedback function of Problem 2.28(i).

3.2 Continuity and convergence

We review now a few mathematical notions that underlie our discussion (e.g. Willard, 2004; Zeidler, 1985).

Definition 3.7: Let S be a subset of a Hilbert space H, and let z be a member of S. A functional $F : S \to R$ is *weakly lower semi-continuous at* z if the following is true for every sequence $\{z_i\}_{i=1}^\infty \subseteq S$ that converges weakly to z: whenever $F(z)$ is bounded, there is, for every real number $\varepsilon > 0$, an integer $N > 0$ such that $F(z) - F(z_i) < \varepsilon$ for all $i \geq N$.

A function $G : S \times R \to R^n : (s, t) \mapsto G(s, t)$ is *weakly continuous* at z at a time t if the following is true for every sequence $\{z_i\}_{i=1}^\infty \subseteq S$ that converges weakly to z: for every real number $\varepsilon > 0$, there is an integer $N > 0$ such that $|G(z, t) - G(z_i, t)| < \varepsilon$ for all $i \geq N$.

The function G is *uniformly weakly continuous* over an interval $[t_1, t_2]$, $t_1 < t_2$, if the following is true for every sequence $\{z_i\}_{i=1}^\infty \subseteq S$ that converges weakly to z: for every real number $\varepsilon > 0$, there is an integer $N > 0$ such that $|G(z, t) - G(z_i, t)| < \varepsilon$ for all integers $i \geq N$ and for all times $t \in [t_1, t_2]$.

The next statement is reproduced from Yu and Hammer (2016a).

Proposition 3.8: *Let* Σ *be a system belonging to the family* $\mathcal{F}_\gamma(\Sigma_0)$. *Then, the response function* $\Sigma(x_0, \cdot, \cdot) : U(K) \times R^+ \to R^n : (u, t) \mapsto \Sigma(x_0, u, t)$ *is uniformly weakly continuous over every finite interval of time.*

In analogy to this, we show in the forthcoming discussion that the closed-loop system Σ_φ of Figure 1 is a weakly continuous function of the feedback function φ. To this end, we need a few auxiliary statements.

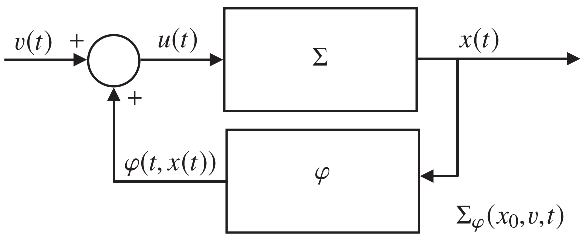


Figure 2. Feedback system with external input.

3.3 Feedback with input signal

Although the present paper concentrates on pure state-feedback configurations described by Figure 1, it is often useful to add an external input signal $v(t)$ to a feedback configuration, as depicted in Figure 2. The response $x(t)$ of the closed-loop system shown in the figure is denoted by $\Sigma_\varphi(x_0, v, t)$; it is described by the differential equation

$$\begin{aligned}\dot{x}(t) &= a(t, x(t)) + b(t, x(t))[\varphi(t, x(t)) + v(t)],\\ x(0) &= x_0.\end{aligned} \tag{3.9}$$

Two extreme cases of (3.9) are: (i) the case where $v = 0$, which yields the pure feedback configuration $\Sigma_\varphi(x_0, 0, t) = \Sigma_\varphi(x_0, t)$ of Figure 1; and (ii) the case where $\varphi = 0$, which yields the open-loop system $\Sigma_{\varphi=0}(x_0, v, t) = \Sigma(x_0, v, t)$.

Our main objective is to show that the closed-loop system depends continuously on the feedback function φ, as the following statement, which is the main result of this section, indicates.

Theorem 3.10: *Let $\{\varphi_i\}_{i=1}^\infty \subseteq \Phi(K)$ be a sequence of feedback functions that is weakly convergent to a feedback function $\varphi \in \Phi(K)$. Then, for almost every system $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$, the sequence of feedback systems $\Sigma_{\varphi_i}(x_0, v, t)$ is uniformly weakly convergent to the feedback system $\Sigma_\varphi(x_0, v, t)$ for every input signal $v \in U(K)$ and over every finite interval of time.*

Before stating the proof of Theorem 3.10, we need a number of auxiliary results. First, we show that, for a sequence of feedback functions $\{\varphi_i\}$ that converges φ, the negative feedback function $-\varphi$ nearly cancels the effect of feedback functions φ_i for large values of the integer i.

Lemma 3.11: *Let $\{\varphi_i\}_{i=1}^\infty \subseteq \Phi(K)$ be a sequence of feedback functions that is weakly convergent to a feedback function $\varphi \in \Phi(K)$. Then, for almost every system $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$, the sequence of feedback systems $\{\Sigma_{\varphi_i-\varphi}(x_0, v, t)\}_{i=1}^\infty$ converges weakly to the open-loop system $\Sigma(x_0, v, t)$ at every time $t \geq 0$ and for every input signal $v \in U(K)$. Moreover, this convergence is uniform over every finite interval of time.*

Proof: Let Σ be a member of the family of systems $\mathcal{F}_\gamma(\Sigma_0)$ with initial state x_0, and let $\{\varphi_i\}_{i=1}^\infty \subseteq \Phi(K)$ be a sequence of feedback functions that converges weakly to a feedback function $\varphi \in \Phi(K)$. Apply to Σ a feedback function given by the difference $\varphi_i - \varphi$, $i \in \{1, 2, \ldots\}$. For an input signal $v \in U(K)$ and a time $t \geq 0$, denote $x(t, i) := \Sigma_{(\varphi_i-\varphi)}(x_0, v, t)$, $x(t) := \Sigma(x_0, v, t)$, and $z(t, i) := x(t, i) - x(t)$. Note that $z(0, i) = x(0, i) - x(0) = x_0 - x_0 = 0$. The differential equation of $x(t, i)$ is

$$\begin{aligned}\dot{x}(t, i) = a(t, x(t, i)) &+ b(t, x(t, i))[\varphi_i(t, x(t, i))\\ &- \varphi(t, x(t, i)) + v(t)], \quad x(0, i) = x_0.\end{aligned} \tag{3.12}$$

Now, let $T > 0$ and $t_1, t_2 \in [0, T]$, $t_1 < t_2$, be times, and let $t \in [t_1, t_2]$. Then, using (2.1) and (3.12) yields

$$\begin{aligned}z(t, i) = z(t_1, i) &+ \int_{t_1}^t [a(s, x(s, i) - a(s, x(s))]\, ds\\ &+ \int_{t_1}^t b(s, x(s, i))[\varphi_i(s, x(s, i)) - \varphi(s, x(s, i))]\, ds\\ &+ \int_{t_1}^t [b(s, x(s, i)) - b(s, x(s))]v(s)\, ds\\ = z(t_1, i) &+ \int_{t_1}^t [a(s, x(s, i) - a(s, x(s))]\, ds\\ &+ \int_{t_1}^t [b(s, x(s, i)) - b(s, x(s))]\\ &\quad \times [\varphi_i(s, x(s, i)) - \varphi(s, x(s, i))]\, ds\\ &+ \int_{t_1}^t b(s, x(s))[\varphi_i(s, x(s, i)) - \varphi(s, x(s, i))]\, ds\\ &+ \int_{t_1}^t [b(s, x(s, i)) - b(s, x(s))]v(s)\, ds.\end{aligned}$$

Invoking the inequalities (2.2) and using the fact that all feedback functions and all input signals are bounded by K, we get

$$\begin{aligned}&\sup_{s\in[t_1,t_2]} |z(s, i)|\\ &\quad \leq |z(t_1, i)| + \alpha^+(1 + 3K) \sup_{s\in[t_1,t_2]} |z(s, i)|(t_2 - t_1)\\ &\quad + \left|\int_{t_1}^{t_2} b(s, x(s))[\varphi_i(s, x(s, i)) - \varphi(s, x(s, i))]\, ds\right|,\end{aligned}$$

or

$$\begin{aligned}&(1 - \alpha^+(1 + 3K)(t_2 - t_1)) \sup_{s\in[t_1,t_2]} |z(s, i)|\\ &\quad \leq |z(t_1, i)| + \left|\int_{t_1}^{t_2} b(s, x(s))[\varphi_i(s, x(s, i)) - \varphi(s, x(s, i))]\, ds\right|.\end{aligned}$$

Selecting a number $\eta > 0$ to satisfy $\alpha^+(1 + 3K)\eta < 1/2$ and setting $t_2 := t_1 + \eta$, yields

$$\begin{aligned}&\sup_{s\in[t_1,t+\eta]} |z(s, i)| \leq 2|z(t_1, i)|\\ &\quad + 2\left|\int_{t_1}^{t_1+\eta} b(s, x(s))[\varphi_i(s, x(s, i)) - \varphi(s, x(s, i))]\, ds\right|.\end{aligned} \tag{3.13}$$

Now, define the function

$$g(t) := \begin{cases} e^{\omega t} b^\top(t, x(t)) & t \in [t_1, t_1 + \eta], \\ 0 & \text{else.} \end{cases}$$

Then, with the inner product (2.7), we can rewrite (3.13) in the form

$$\sup_{s \in [t_1, t_1+\eta]} |z(s, i)| \leq 2|z(t_1, i)| + 2\left|\langle g, [\varphi_i \circ x(\cdot\,, i) - \varphi \circ x(\cdot\,, i)]\rangle\right|. \quad (3.14)$$

To continue, select a real number $\varepsilon > 0$. Using Proposition 2.18 and considering that the function $x(\cdot\,, i)$ depends on the system Σ, we conclude that, for almost every $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$, there is an integer $N \geq 1$ such that $\langle g, [\varphi_i \circ x(\cdot\,, i) - \varphi \circ x(\cdot\,, i)]\rangle < \varepsilon$ for all $i \geq N$. Then, inequality (3.14) can be rewritten in the form

$$\sup_{s \in [t_1, t_1+\eta]} |z(s, i)| \leq 2|z(t_1, i)| + 2\varepsilon \quad \text{for } i \geq N. \quad (3.15)$$

Next, define the scalar

$$\zeta_{j,i} := \sup_{s \in [(j-1)\eta, j\eta]} |z(s, i)|. \quad (3.16)$$

Let $q \geq 1$ be an integer satisfying $q \geq t/\eta$, and consider the partition

$$[0, T] \subseteq \{[0, \eta], [\eta, 2\eta], \ldots, [(q-1)\eta, q\eta]\}.$$

On this partition, (3.15) gives rise to the linear recursion

$$\zeta_{j+1,i} \leq 2\zeta_{j,i} + 2\varepsilon, \quad \zeta_{0,i} = 0, \ j = 0, 1, \ldots$$

By properties of linear recursions, we obtain

$$\zeta_{j,i} \leq \left(\sum_{r=1}^{j} 2^r\right) \varepsilon, \quad \text{for all } i \geq N.$$

Using (3.16), this yields

$$\sup_{s \in [0,T]} |z(s, i)| \leq \left(\sum_{r=1}^{q} 2^r\right) \varepsilon$$

for all $i \geq N$. Finally, given a real number $\delta > 0$, select the integer N so that $\varepsilon \leq \delta / (\sum_{r=1}^{q} 2^r)$. Then, we obtain

$$\sup_{s \in [0,t]} |z(s, i)| \leq \delta$$

for all integers $i \geq N$ and for almost every $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$. Thus, $\lim_{i \to \infty} z(s, i) = 0$ for almost every $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$, and our proof concludes. ■

3.4 Equivalent configurations and convergence

The next step in our preparations to prove Theorem 3.10 is an examination of the equivalence of certain feedback configurations. Specifically, we apply the feedback function φ to the closed-loop system $\Sigma_{(\varphi_i - \varphi)}(x_0, v, t)$, as depicted in Figure 3(A). A slight reflection shows that this configuration is equivalent to the configuration $\Sigma_{\varphi_i}(x_0, v, t)$ of Figure 3(B). To continue, define the signal

$$\varepsilon_i(t) := \Sigma_{\varphi_i - \varphi}(x_0, v, t) - \Sigma(x_0, v, t), \quad (3.17)$$

which, according to Lemma 3.11, is uniformly weakly convergent to the zero signal as $i \to \infty$. Now, the configuration of Figure 3(A) is the same as the configuration of Figure 4(A); the latter is equivalent to the configuration of Figure 4(B), where

$$x_i(t) = z_i(t) + \varepsilon_i(t), \quad i = 1, 2, \ldots \quad (3.18)$$

Proof of Theorem 3.10: In view of (3.18), we have $x_i(t) - z_i(t) = \varepsilon_i(t)$ for all integers $i \geq 1$. Combining this with Lemma 3.11 and (3.17), it follows that $x_i(t)$ is weakly uniformly convergent to $z_i(t)$ as $i \to \infty$, over finite intervals of time. This concludes our proof. ■

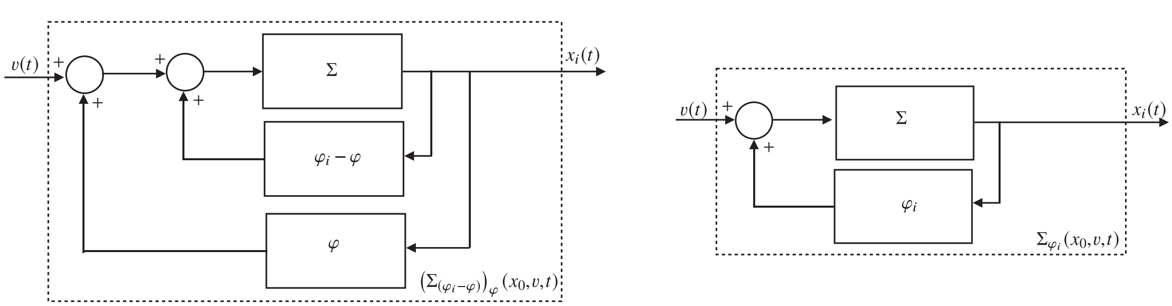
Considering that the pure feedback systems $\Sigma_{\varphi_i}(x_0, t)$ and $\Sigma_\varphi(x_0, t)$ are, respectively, $\Sigma_{\varphi_i}(x_0, v, t)$ and $\Sigma_\varphi(x_0, v, t)$ for $v = 0$, Theorem 3.10 yields the following.

Corollary 3.19: *Let $\{\varphi_i\}_{i=1}^\infty \subseteq \Phi(K)$ be a sequence of feedback functions that is weakly convergent to a feedback function $\varphi \in \Phi(K)$. Then, the sequence of feedback systems $\{\Sigma_{\varphi_i}(x_0, t)\}_{i=1}^\infty$ is uniformly weakly convergent to the feedback system $\Sigma_\varphi(x_0, t)$ over any finite interval of time and for almost every system $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$.*

Corollary 3.19 can be restated in the following form.

Corollary 3.20: *For almost every member $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$, the function $\Sigma_{(\cdot)}(x_0, \cdot) : \Phi(K) \times R^+ \to R^n : (\varphi, t) \mapsto \Sigma_\varphi(x_0, t)$ is uniformly weakly continuous over every finite interval of time.*

We turn now to the existence of optimal feedback functions.



(A) (B)

Figure 3. Equivalent configurations: (a) Double feedback and (b) Single feedback.

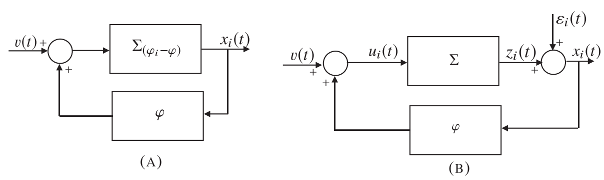


Figure 4. Equivalent configurations.

4. Existence of optimal feedback functions

In this section, we consider the existence of optimal robust feedback functions $\varphi^*(x_0, \ell, A, \gamma) \in \Phi(K)$ that achieve the minimal time $t^*(x_0, \ell, A, \gamma)$ of (2.26). The main result of this section is the following statement, which shows that optimal robust feedback functions exist under rather broad conditions. The primary requirement for the existence of such feedback functions is (K, A)-controllability of the nominal system, as follows.

Theorem 4.1: *Let $A_0, A, \ell, \gamma > 0$ be real numbers, where $A > A_0$. Assume that the nominal system Σ_0 is (K, A_0)-controllable from the initial state x_0, and that the uncertainty parameter γ fulfills the requirements of Proposition 3.5 for A, A_0, and ℓ. Then, referring to (2.25), (2.26), and (2.27), the following are true for almost every system $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$.*

(i) *There is a finite minimal time $t^*(x_0, \ell, A, \gamma)$.*
(ii) *There is an optimal feedback function $\varphi^*(x_0, \ell, A, \gamma) \in \Phi(K)$ that achieves the minimal time $t^*(x_0, \ell, A, \gamma)$, i.e. $t^*(x_0, \ell, A, \gamma) = t(x_0, \ell, A, \gamma, \varphi^*(x_0, \ell, A, \gamma))$.*

The proof of Theorem 4.1 appears near the end of this section, after preparing the tools needed for the proof. It is fair to say that computation and implementation of optimal feedback functions are often demanding tasks, since optimal feedback functions generally are intricate multi-variable vector-valued functions of time. To simplify computation and implementation of optimal feedback functions, we show in Section 5 that optimal performance can be approximated by bang-bang feedback functions that are relatively easy to design and implement.

As we can see from Theorem 4.1, the main condition for the existence of an optimal feedback function is (K, A_0)-controllability of the nominal system Σ_0. Thus, verifying the existence of an optimal solution requires testing of only one system – the nominal system Σ_0, assuming that the uncertainty parameter γ is not excessively large.

Note also that (K, A_0)-controllability is very close to being a necessary condition for the existence of an optimal feedback function. Indeed, the input signal created by an optimal feedback function φ^* must take the nominal system Σ_0 to the vicinity $\rho(\ell)$ of the origin, without violating the state amplitude bound A along the way. This is close to (K, A_0)-controllability, which requires the existence of an input signal that takes Σ_0 to the origin itself, without violating the state amplitude bound A_0 along the way. A relatively simple numerical technique for testing (K, A_0)-controllability of a system is described in Choi and Hammer (2017a).

Our discussion of the existence of optimal feedback functions depends on a few mathematical facts quoted next (e.g. Willard, 2004; Zeidler, 1985).

Theorem 4.2: (i) *A weakly continuous functional is weakly lower semi-continuous.*
(ii) *Let S and A be topological spaces and assume that, for every member $a \in A$, there is a weakly lower semi-continuous functional $f_a : S \to R$. If $\sup_{a\in A} f_a(s)$ exists at each point $s \in S$, then the functional $f(s) := \sup_{a\in A} f_a(s)$ is weakly lower semi-continuous on S.*

To discuss the existence of optimal feedback functions, we introduce a functional $\psi(t, \cdot) : \Phi(K) \to R : \varphi \mapsto \psi(t, \varphi)$ given, for $t \geq 0$, by

$$\psi(t,\varphi) := \begin{cases} \operatorname*{ess\,sup}_{\Sigma\in\mathcal{F}_\gamma(\Sigma_0)} |\Sigma_\varphi(x_0,t)|_2^2 & \text{if } \varphi \in \Phi(K,A,\gamma,t), \\ \infty & \text{if } \varphi \notin \Phi(K,A,\gamma,t) \end{cases} \tag{4.3}$$

(compare to Choi & Hammer, 2017a; Yu & Hammer, 2016a, 2016b). This functional has the following feature.

Lemma 4.4: *At every time $t \geq 0$, the functional $\psi(t, \cdot) : \Phi(K) \to R$ of (4.3) is weakly lower semi-continuous over $\Phi(K)$.*

Proof: By Proposition 3.20, the closed-loop system $\Sigma_\varphi(x_0, t)$ is a weakly continuous function of the feedback function $\varphi \in \Phi(K)$ at every time $t \geq 0$ for almost every system $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$. Using the fact that a continuous function of a weakly continuous function is weakly continuous, we conclude that $|\Sigma_{(\cdot)}(x_0, t)|_2^2 : \Phi(K) \to R : \varphi \mapsto |\Sigma_\varphi(x_0, t)|_2^2$ is a weakly continuous functional of $\varphi \in \Phi(K)$ at every time $t \geq 0$ for almost every $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$. Consequently, by Theorem 4.2(i), the functional $|\Sigma_\varphi(x_0, t)|_2^2$ is also weakly lower semi-continuous over $\Phi(K)$ at every finite time $t \geq 0$ for almost every $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$. The lemma is then a consequence of (4.3) and Theorem 4.2(ii). ■

The next statement brings us very close to proving the existence of optimal feedback functions (compare to Choi & Hammer, 2017a; Yu & Hammer, 2016a, 2016b).

Proposition 4.5: *Let $A_0, A, \ell, \gamma > 0$ be real numbers, where $A > A_0$. Assume that the nominal system Σ_0 is (K, A_0)-controllable from the initial state x_0, and that the uncertainty parameter γ fulfills the requirements of Proposition 3.5 with A, A_0, and ℓ. Then, the functional $t(x_0, \ell, A, \gamma, \cdot) : \Phi(K) \to R : \varphi \mapsto t(x_0, \ell, A, \gamma, \varphi)$ of (2.25) is weakly lower semi-continuous.*

Proof: In view of (4.3), we have

$$t(x_0, \ell, A, \gamma, \varphi) = \inf_t \{t \geq 0 : \psi(t, \varphi) \leq \ell\}.$$

To simplify notation, define the functional

$$\theta(\varphi) := \inf_t \{t \geq 0 : \psi(t, \varphi) \leq \ell\}. \tag{4.6}$$

According to the proposition's assumptions and Proposition 3.5, there is a feedback function $\varphi \in \Phi(K)$ satisfying $0 < \theta(\varphi) < \infty$. Also, by (4.6), we have that $\theta(\varphi) \geq 0$. Note that all feedback functions φ that satisfy $\theta(\varphi) < \infty$ are members of $\Phi(K, A, \gamma, t)$ for some time $t \geq \theta(\varphi)$. Thus, by (2.24), all such feedback functions φ are members of $\Phi(K, A, \gamma, \theta(\varphi))$. Now, the case

$\theta(\varphi) = 0$ means that the initial state x_0 already satisfies the requirement $x_0^\top x_0 \leq \ell$; in such case, the proposition is valid for every feedback function $\varphi \in \Phi(K)$. Next, consider the case $\theta(\varphi) > 0$.

Let $\varphi \in \Phi(K)$ be a feedback function satisfying $\theta(\varphi) < \infty$, and let $\{\varphi_i\}_{i=1}^{\infty} \subseteq \Phi(K)$ be a sequence of feedback functions that converges weakly to φ. Set $\psi_i(t) := \psi(t, \varphi_i)$, $i = 1, 2, \ldots$, and $\psi_0(t) := \psi(t, \varphi)$. By (4.6), we have $\theta(\varphi_i) = \inf_t\{t \geq 0 : \psi_i(t) \leq \ell\}$ and $\theta(\varphi) = \inf_t\{t \geq 0 : \psi_0(t) \leq \ell\}$. We show next that $\theta(\cdot)$ is a weakly lower semi-continuous functional of φ over $\Phi(K)$. To this end, fix a real number $\varepsilon > 0$; we claim that there is an integer $N > 0$ such that

$$\theta(\varphi_i) > \theta(\varphi) - \varepsilon \quad \text{for all } i \geq N. \tag{4.7}$$

Indeed, as $\theta(\varphi_i) \geq 0$ for all $i = 1, 2, \ldots$, the case $\varepsilon > \theta(\varphi)$ clearly satisfies (4.7). Consider then the case $\varepsilon \in (0, \theta(\varphi))$; here, we have the following two options:

Case 1. There is an integer $N > 0$ such that $\theta(\varphi_i) \geq \theta(\varphi)$ for all $i \geq N$.

Case 2. Case 1 is not valid.

In Case 1, the inequality (4.7) clearly holds for all $i \geq N$, so that $\theta(\cdot)$ is weakly lower semi-continuous in this case. Note that, since $\theta(\varphi) < \infty$, Case 1 includes all sequences $\{\varphi_i\}$ for which there is an integer $N \geq 1$ such that $\theta(\varphi_i) = \infty$ for all integers $i \geq N$.

Regarding Case 2, there is a sequence of integers $j_1 < j_2 < j_3 < \cdots$ for which

$$\theta(\varphi_{j_k}) < \theta(\varphi) \quad \text{for all integers } k \geq 1. \tag{4.8}$$

As $\theta(\varphi) < \infty$, inequality (4.8) implies that $\theta(\varphi_{j_k}) < \infty$ for all $k \geq 1$. Further, as (4.6) describes an infimum, we have that $\psi_0(t) > \ell$ for all $t \in [0, \theta(\varphi))$. Thus, at every time

$$\bar{t} \in [\theta(\varphi) - \varepsilon, \theta(\varphi)), \tag{4.9}$$

we have $\psi_0(\bar{t}) > \ell$, namely,

$$\psi_0(\bar{t}) - \ell > 0. \tag{4.10}$$

Now, by Lemma 4.4, the functional $\psi(t, \varphi)$ is weakly lower semi-continuous in φ over $\Phi(K)$. Thus, for every real number $\eta > 0$, there is an integer $N > 0$ satisfying

$$\psi_0(\bar{t}) - \psi_{j_k}(\bar{t}) < \eta \quad \text{for all } k \geq N. \tag{4.11}$$

In particular, by (4.10), we can take $\eta := (\psi_0(\bar{t}) - \ell)/2$. Inserting this η into (4.11) yields the inequality

$$\psi_0(\bar{t}) - \psi_{j_k}(\bar{t}) < (\psi_0(\bar{t}) - \ell)/2 \quad \text{for all } k \geq N.$$

Simplifying, we get

$$\psi_{j_k}(\bar{t}) > (\psi_0(\bar{t}) + \ell)/2 \quad \text{for all } k \geq N.$$

But then, since $\psi_0(\bar{t}) > \ell$ by (4.10), we obtain that $\psi_{j_k}(\bar{t}) > \ell$ for all $k \geq N$, or, referring to (4.6), we have $\theta(\varphi_{j_k}) > \bar{t}$. Recalling (4.9), we get $\theta(\varphi_{j_k}) > \theta(\varphi) - \varepsilon$ for all $k \geq N$. This shows that $\theta(\cdot)$ is weakly lower semi-continuous in Case 2 as well. The proposition follows from the fact that $t(x_0, \ell, A, \gamma, \varphi) = \theta(\varphi)$, and our proof concludes. ■

We can prove now the main result of the present section.

Proof of Theorem 4.1: By the Generalized Weierstrass Theorem (e.g. Zeidler, 1985), a weakly lower semi-continuous functional achieves a minimum in a weakly compact set. Now, the functional $t(x_0, \ell, A, \gamma, \varphi)$ is weakly lower semi-continuous by Proposition 4.5, and $\Phi(K)$ is weakly compact by Lemma 2.15. Therefore, $t(x_0, \ell, A, \gamma, \varphi)$ achieves a minimum $t^*(x_0, \ell, A, \gamma)$ in $\Phi(K)$. In other words, there is a feedback function $\varphi^*(x_0, \ell, A, \gamma) \in \Phi(K)$ at which the minimum occurs. This completes our proof. ■

In summary, we have shown in this section that there are robust feedback functions that fulfill the requirements of Problem 2.28(i). From a theoretical perspective, this is a critical fact. However, from a practical perspective, the derivation and the implementation of optimal feedback functions are often tedious tasks, requiring the computation and construction of intricate multivariable vector-valued functions of time. In the next section, we show that optimal performance can be approximated by using bang-bang feedback functions. These are functions of the time and state whose components switch between the values of $-K$ and K. For a system having an input signal with m components, such feedback functions take at most 2^m different values: one of the two values $-K$ or K for each input component; this compares very favourably to general feedback functions, which can take any value in the m-dimensional continuum $[-K, K]^m$. As a result, bang-bang feedback functions are much easier to calculate and implement than general feedback functions. The approximation of optimal performance by bang-bang feedback functions is the topic of the next section.

5. Approximating optimal performance

To make our results amenable to practical implementation, we show in this section that optimal performance can be approximated as closely as desired by bang-bang feedback functions – functions whose coordinates switch between the input bounds $\pm K$. Such feedback functions are relatively easy to calculate and implement. For example, for a single input controlled system Σ, a bang-bang state-feedback function switches only between the two values of $-K$ and K as a function of time and state. When discussing approximations, we must also take into account other sources of inaccuracies, such as noise and disturbances that affect the configuration, as discussed next.

5.1 Noise and disturbances

Replacing a feedback function by an approximate feedback function clearly introduces errors into the feedback loop. The impact of such errors must be considered in the broader context of the effects of noise and disturbances. We examine in this subsection the effects of a common noise or disturbance signal $\upsilon(t)$ that may appear as an additive signal at the input port of the feedback function φ, as depicted in Figure 5.

Assume that the disturbance signal $\upsilon(t)$ is a random signal with and amplitude bounded by a specified bound $\Delta > 0$ and with uniform probability distribution. Then, $|\upsilon(t)| \leq \Delta$ at all times $t \geq 0$ and the values of $\upsilon(t)$ are distributed uniformly in

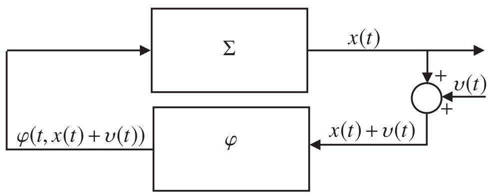


Figure 5. A disturbance signal $\upsilon(t)$, $|\upsilon(t)| \le \Delta$.

the hyper-square $\Delta(0) := [-\Delta, \Delta]^n$ of edge 2Δ. Let dz denote the Lebesgue element of hyper-volume in R^n. Then, the probability of obtaining a disturbance signal within dz is d$z/(2\Delta)^n$. Denote by $\Delta(x)$ the hyper-square of edge 2Δ centred at the state x in the state space of Σ. Then, the average value of the feedback function at a time t is

$$\bar{\varphi}(t, x) := \frac{1}{(2\Delta)^n} \int_{\Delta(x)} \varphi(t, z)\, \mathrm{d}z. \tag{5.1}$$

When approximating system performance, we take this average into consideration.

5.2 Approximating optimal performance

In this subsection, we address part (ii) of Problem 2.28: we derive robust feedback functions that provide close to optimal performance and are relatively easy to implement. We start with some notation. For a real number $K > 0$, denote by $\mathbb{K}^m$ the set consisting of the 2^m m-dimensional vectors with components of $-K$ or K.

Example 5.2: For $m = 2$, we have $\mathbb{K}^2 = \{(-K, -K)^\top, (K, -K)^\top, (-K, K)^\top, (K, K)^\top\}$.

We start with a formal definition of the term 'bang-bang feedback function'.

Definition 5.3: Let $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ be a system with input signal $u(t) \in R^m$ and state $x(t) \in R^n$, and let $T > 0$ be a finite time. Assume that Σ permits only input signals of amplitude bounded by $K > 0$, and that its state amplitude may not exceed $A > 0$. Then, a *bang-bang feedback function* for Σ over the time interval $[0, T]$ is a function $\varphi_\pm : [0, T] \times [-A, A]^n \to \mathbb{K}^m$ with the following property: there is a partition of the domain $[0, T] \times [-A, A]^n$ into a finite number $p \ge 1$ of hyper-rectangles $\sigma_1, \sigma_2, \ldots, \sigma_p$ so that each component of $\varphi_\pm$ takes a constant value of $-K$ or K in the interior of σ_j, $j = 1, 2, \ldots, p$.

Each component of a bang-bang feedback function $\varphi_\pm$ takes one of the values $-K$ or K, depending on the time and the state of the controlled system Σ.

The next statement is the main result of this section. In qualitative terms, the statement shows that, if the operating error bound ℓ is increased slightly to an error bound ℓ', then bang-bang feedback functions can drive the controlled system Σ to the ball $\rho(\ell')$ at least as fast as an optimal feedback function can drive Σ to the ball $\rho(\ell)$. The statement reflects average performance over the disturbance $\upsilon(t)$ of Figure 5, as described by (5.1).

Theorem 5.4: *Let $A_0, A, \ell, \ell' > 0$ be real numbers, where $A > A_0$ and $\ell' > \ell$, and assume that the nominal system Σ_0 is (K, A_0)-controllable from the initial state x_0. Then, referring to the notation of* (2.25) *and* (2.26), *there are a bang-bang feedback function $\varphi_\pm \in \Phi(K)$ and an uncertainty parameter $\gamma > 0$ such that $t(x_0, \ell', A, \gamma, \varphi_\pm) \le t^*(x_0, \ell, A, \gamma)$, when feedback is averaged over the disturbance signal $\upsilon(t)$ of Figure 5.*

Theorem 5.4 lists two sufficient conditions for the existence of a bang-bang feedback function that approximates optimal performance. The first condition requires the nominal system Σ_0 to be (K, A_0)-controllable; this condition also appeared earlier in Theorem 4.1 and was discussed in detail in the paragraphs following that theorem. The second condition requires that the uncertainty parameter γ not be excessively large. Even so, as one might expect, and as the example of Section 6 demonstrates, the uncertainty permitted by optimal feedback controllers is never smaller (and often quite larger) than the uncertainty permitted by optimal open-loop controllers (see Choi & Hammer, 2017a; Yu & Hammer, 2016a for a discussion of optimal open-loop controllers).

Theorem 5.4 is a consequence of the next statement, which shows that the response of a system with any feedback function can be approximated by the response of the same system with a bang-bang feedback function; here, feedback functions are averaged over the disturbance $\upsilon(t)$ of Figure 5. (Compare to Chakraborty & Hammer, 2009b, 2010; Choi & Hammer, 2018b; Yu & Hammer, 2016a, 2016b, where the approximation of open-loop performance is investigated.)

Theorem 5.5: *Let $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ be a system with initial state x_0, let $\varphi \in \Phi(K)$ be a feedback function for Σ, and let $t' > 0$ be a finite time. Then, for every real number $\varepsilon > 0$, there are a bang-bang feedback function $\varphi_\pm \in \Phi(K)$ and an uncertainty parameter $\gamma > 0$ such that the following holds. The difference between the response $x(t) := \Sigma_\varphi(x_0, t)$ of Σ with the feedback function φ and the response $x_\pm(t) := \Sigma_{\varphi_\pm}(x_0, t)$ of Σ with the feedback function $\varphi_\pm$ satisfies $|x(t) - x_\pm(t)| < \varepsilon$ at all times $0 \le t \le t'$ and for almost all members $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$, when feedback values are averaged over the disturbance signal $\upsilon(t)$ of Figure 5.*

The proof of Theorem 5.5 employs the following auxiliary statement reproduced here from Choi and Hammer (2018a).

Lemma 5.6: *Let Σ be a system of the form* (2.1) *with functions $a(t, x)$ and $b(t, x)$ that are subject to* (2.2). *Let x_0 be the initial state of Σ, let $t' > 0$ be a finite time, and denote by $x(t) := \Sigma(x_0, u, t)$ the response of Σ to an input signal u. Then, for every real number $\zeta > 0$, there are real numbers $\beta(x_0, \zeta, t') > 0$ and $\gamma > 0$ such that the following is valid for all input signals $u \in U(K)$ and for all systems $\Sigma \in \mathcal{F}_\gamma(\Sigma_0) : |b(t_1, x(t_1)) - b(t_2, x(t_2))| < \zeta$ for all times $t_1, t_2 \in [0, t']$ satisfying $|t_1 - t_2| < \beta(x_0, \zeta, t')$.*

Considering that feedback functions $\varphi \in \Phi(K)$ create input signals belonging to $U(K)$ for the controlled system Σ, the following statement is a direct consequence of Lemma 5.6.

Corollary 5.7: *Let Σ be a system of the form* (2.1) *with functions $a(t, x)$ and $b(t, x)$ that are subject to* (2.2). *Let x_0 be the* *initial state of* Σ, *let* $t' > 0$ *be a finite time, and, for a feedback function* $\varphi \in \Phi(K)$, *denote by* $x(t) := \Sigma_\varphi(x_0, t)$ *the response of the closed loop system. Then, for every real number* $\zeta > 0$, *there are real numbers* $\beta(x_0, \zeta, t') > 0$ *and* $\gamma > 0$ *such that the following is valid for all feedback functions* $\varphi \in \Phi(K)$ *and for all systems* $\Sigma \in \mathcal{F}_\gamma(\Sigma_0) : |b(t_1, x(t_1)) - b(t_2, x(t_2))| < \zeta$ *for all times* $t_1, t_2 \in [0, t']$ *satisfying* $|t_1 - t_2| < \beta(x_0, \zeta, t')$.

We turn now to the proof of Theorem 5.5.

Proof of Theorem 5.5: Recall the noise amplitude bound $\Delta > 0$ of Subsection 5.1. Let $t_1, t_2 \in [0, t']$, $t_1 < t_2$, be two times for which $(t_2 - t_1)/\Delta$ is a rational number. Let $\lambda > 0$ be a real number for which the ratios $p := (t_2 - t_1)/\lambda$ and Δ/λ are both integers. Recalling the state amplitude bound $A > 0$, let $M \geq A$ be a real number for which the ratio $r := M/\lambda$ is an integer.

It is convenient at this point to introduce the following notation: for $n+1$ integers $q_0, q_1, \ldots, q_n$, denote by q the vector

$$q := (q_0, q_1, \ldots, q_n).$$

For each vector q, define the hyper-square $\chi(q)$ of edge λ given by

$$\begin{aligned}\chi(q) := {} & [t_1 + q_0\lambda, t_1 + (q_0 + 1)\lambda] \\ & \times [-M + q_1\lambda, -M + (q_1 + 1)\lambda] \times \cdots \\ & \times [-M + q_n\lambda, -M + (q_n + 1)\lambda];\end{aligned}$$

this creates a partition of the domain $[t_1, t_2] \times [-M, M]^n \subseteq R^+ \times R^n$ into $p(2r)^n$ hyper-squares of edge λ given by

$$P := \{\chi(q)\}_{q \in \{0,1,\ldots,p-1\} \times \{0,1,\ldots,2r-1\}^n}. \tag{5.8}$$

Now, consider a feedback function $\varphi = (\varphi^1, \varphi^2, \ldots, \varphi^m)^\top \in \Phi(K)$. As each component φ^j of φ is bounded by K, integrating over the hyper-square $\chi(q)$ yields the relation

$$-K\lambda^{n+1} \leq \int_{\chi(q)} \varphi^j(s, x)\, \mathrm{d}(s, x) \leq K\lambda^{n+1} \tag{5.9}$$

for all $q \in \{0, 1, \ldots, p-1\} \times \{0, 1, \ldots, 2r-1\}^n$ and all $j = 1, 2, \ldots, m$; here, $\mathrm{d}(s, x)$ is the Lebesgue volume element in $R^+ \times R^n$.

Next, for a real number $\mu(q) \in [0, \lambda]$, let $D(\mu(q)) \subseteq R^+ \times R^n$ be the hyper-square of edge $\mu(q)$ given by

$$\begin{aligned}D(\mu(q)) := {} & [t_1 + q_0\lambda, t_1 + q_0\lambda + \mu(q)] \\ & \times [-M + q_1\lambda, -M + q_1\lambda + \mu(q)] \times \cdots \\ & \times [-M + q_n\lambda, -M + q_n\lambda + \mu(q)]\end{aligned}$$

and depicted for the case $n = 1$ as the shaded domain of Figure 6. Denote by $V(\mu(q))$ the hyper-volume of $D(\mu(q))$. Then, the inequality (5.9) implies that, for every $j = 1, 2, \ldots, m$, there is a real number $0 \leq \mu_j(q) \leq \lambda$ satisfying

$$K\left[2V(\mu_j(q)) - \lambda^{n+1}\right] = \int_{\chi(q)} \varphi^j(s, x)\, \mathrm{d}(s, x). \tag{5.10}$$

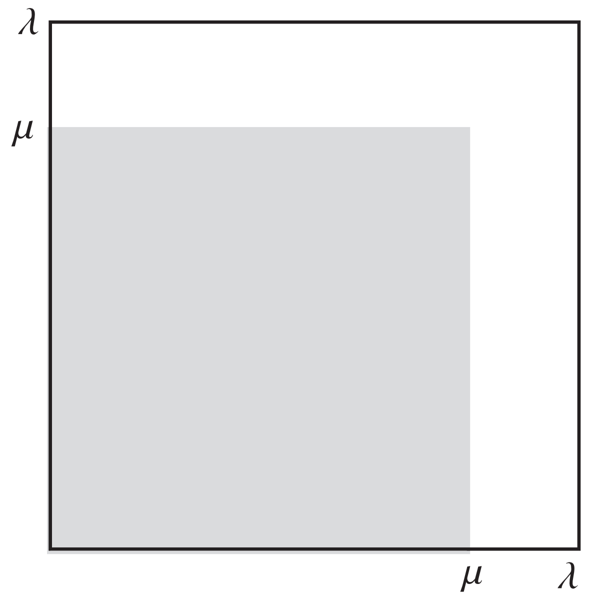


Figure 6. The domain $D(\mu(q))$ for $n = 1$.

Denoting set difference by \, define the jth component of a bang-bang feedback function $\varphi_\pm$ by setting

$$\varphi^j_\pm(t, x) := \begin{cases} K & \text{for } (t, x) \in D(\mu_j(q)), \\ -K & \text{for } t \in \chi(q) \setminus D(\mu_j(q)), \end{cases}$$

$q \in \{0, 1, \ldots, p-1\} \times \{0, 1, \ldots, 2r-1\}^n$, $j = 1, 2, \ldots, m$. Then, by (5.10), we get

$$\int_{\chi(q)} \left(\varphi^j(s, x) - \varphi^j_\pm(s, x)\right) \mathrm{d}(s, x) = 0 \tag{5.11}$$

for all $q \in \{0, 1, \ldots, p-1\} \times \{0, 1, \ldots, 2r-1\}^n$ and all $j = 1, 2, \ldots, m$.

Further, for a member $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$, denote $x(t) := \Sigma_\varphi(x_0, t)$ and $x_\pm(t) := \Sigma_{\varphi_\pm}(x_0, t)$, and consider the difference

$$\begin{aligned}\xi(t) :&= x(t) - x_\pm(t), \quad t \in [0, t'], \\ \xi(0) &= x_0 - x_0 = 0.\end{aligned}$$

Let $\Delta(x)$ be the hyper-square of side 2Δ centred at the point x. Then, averaging over the inaccuracy $\upsilon(t)$ of Figure 5, we obtain the relation

$$\begin{aligned}d\bar{\xi}(t) &= \overline{\mathrm{d}x(t) - \mathrm{d}x_\pm(t)} \\ &= \frac{1}{(2\Delta)^n} \int_{\Delta(x(t))} (\mathrm{d}x(t) - \mathrm{d}x_\pm(t))\, \mathrm{d}x.\end{aligned}$$

Combining this with (2.1) yields

$$\begin{aligned}\bar{\xi}(t) = {} & \bar{\xi}(t_1) + \frac{1}{(2\Delta)^n} \int_{t_1}^{t} \int_{\Delta(x(s))} \Big[a(s, x(s)) - a(s, x_\pm(s)) \\ & + b(s, x(s))\varphi(s, x(s)) - b(s, x_\pm(s))\varphi_\pm(s, x(s))\Big]\, \mathrm{d}(s, x) \\ = {} & \bar{\xi}(t_1) + \frac{1}{(2\Delta)^n} \int_{t_1}^{t} \int_{\Delta(x(s))} \big[[a(s, x(s)) - a(s, x_\pm(s))]\end{aligned}$$

$$+ [b(s,x(s))\varphi(s,x(s)) - b(s,x(s))\varphi_\pm(s,x(s))]$$

$$+ [b(s,x(s))\varphi_\pm(s,x(s)) - b(s,x_\pm(s))\varphi_\pm(s,x(s))]\Big]\, \mathrm{d}(s,x).$$

Using (2.2) and the fact that all feedback functions are bounded by K, we get

$$\sup_{t\in[t_1,t_2]} |\bar\xi(t)| \le |\bar\xi(t_1)| + \frac{\alpha^+}{(2\Delta)^n}\left(\sup_{s\in[t_1,t_2]} |\bar\xi(s)|\right)(t_2 - t_1) + \frac{1}{(2\Delta)^n}\sup_{t\in[t_1,t_2]}\left|\int_{t_1}^{t}\int_{\Delta(x(s))} b(s,x(s))\,(\varphi(s,x(s)) - \varphi_\pm(s,x(s)))\,\mathrm{d}(s,x)\right| + \frac{\alpha^+}{(2\Delta)^n}(t_2-t_1)\sup_{s\in[t_1,t_2]}|\bar\xi(s)|\,K.$$

This can be rewritten in the form

$$\left[1 - \frac{\alpha^+}{(2\Delta)^n}(1+K)(t_2-t_1)\right]\sup_{t\in[t_1,t_2]}|\bar\xi(t)| \le |\bar\xi(t_1)| + \frac{1}{(2\Delta)^n}\sup_{t\in[t_1,t_2]}\left|\int_{t_1}^{t}\int_{\Delta(x(s))} b(s,x(s))\,(\varphi(s,x(s)) - \varphi_\pm(s,x(s)))\,\mathrm{d}(s,x)\right|. \quad (5.12)$$

Next, select a real number $\eta \in (0, t' - t_1]$ that is an integer multiple of λ and satisfies $\alpha^+(1+K)\eta/(2\Delta)^n \le 1/2$ (note that $\lambda > 0$ can be selected as small as needed); then, set

$$t_2 := t_1 + \eta.$$

Inequality (5.12) then becomes

$$\sup_{t\in[t_1,t_2]}|\bar\xi(t)| \le 2|\bar\xi(t_1)| + \frac{2}{(2\Delta)^n}\sup_{t\in[t_1,t_1+\eta]}\left|\int_{t_1}^{t}\int_{\Delta(x(s))} b(s,x(s))\,(\varphi(s,x(s)) - \varphi_\pm(s,x(s)))\,\mathrm{d}(s,x)\right|. \quad (5.13)$$

Our next objective is to estimate the integral in the last expression. To simplify notation, denote by $G \subseteq R^+ \times R^n$ the domain of integration of this integral, and consider the expression

$$z(t) := \int_G b(s,x(s))\,(\varphi(s,x(s)) - \varphi_\pm(s,x(s)))\,\mathrm{d}(s,x). \quad (5.14)$$

Split the domain G into two parts: the domain $G' \subseteq R^+ \times R^n$ that forms the largest sub-domain of G consisting of a union of whole hyper-squares of the partition P of (5.8); and the remainder $G'' := G \setminus G'$ that includes no whole hyper-squares of the partition P. The domain G'' consists of incomplete hyper-squares of edge λ that may pad inside the boundary of G, as G may not synchronise with the partition P.

To calculate the hyper-volume $V(G'')$ of G'', let $S_{G'}$ be the hyper-area of the surface of G'; note that $S_{G'}$ is bounded, since G' is a union of a finite number of hyper-squares of edge λ. Now, by construction, G'' consists of less than one layer of hyper-squares of edge λ around the periphery of G'. Thus,

$$V(G'') \le \lambda S_{G'}. \quad (5.15)$$

Using this, the integral (5.14) can be rewritten in the form

$$\begin{aligned} z(t) &= \int_G b(s,x(s))\,(\varphi(s,x(s)) - \varphi_\pm(s,x(s)))\,\mathrm{d}(s,x) \\ &= \int_{G'} b(s,x(s))\,(\varphi(s,x(s)) - \varphi_\pm(s,x(s)))\,\mathrm{d}(s,x) + \int_{G''} b(s,x(s))\,(\varphi(s,x(s)) - \varphi_\pm(s,x(s)))\,\mathrm{d}(s,x) \\ &= \sum_{q:\chi(q)\in G'}\int_{\chi(q)} b(s,x(s))\,(\varphi(s,x(s)) - \varphi_\pm(s,x(s)))\,\mathrm{d}(s,x) + \int_{G''} b(s,x(s))\,(\varphi(s,x(s)) - \varphi_\pm(s,x(s)))\,\mathrm{d}(s,x) \\ &= \sum_{q:\chi(q)\in G'} b(t_1+q_0\lambda, x(t_1+q_0\lambda)) \times \int_{\chi(q)} (\varphi(s,x(s)) - \varphi_\pm(s,x(s)))\,\mathrm{d}(s,x) \\ &\quad + \sum_{q:\chi(q)\in G'}\int_{\chi(q)}\left[b(s,x(s)) - b(t_1+q_0\lambda, x(t_1+q_0\lambda))\right] \times (\varphi(s,x(s)) - \varphi_\pm(s,x(s)))\,\mathrm{d}(s,x) \\ &\quad + \int_{G''} b(s,x(s))\,(\varphi(s,x(s)) - \varphi_\pm(s,x(s)))\,\mathrm{d}(s,x). \end{aligned} \quad (5.16)$$

Note that the first integral after the last equal sign in (5.16) is zero by (5.11). As a result, we obtain

$$\sup_{t\in[t_1,t_1+\eta]}|z(t)| \le \sum_{q:\chi(q)\in G'}\int_{\chi(q)} \times \sup_{(s,x(s))\in\chi(q)}|b(s,x(s)) - b(t_1+q_0\lambda, \times x(t_1+q_0\lambda))|\,|\varphi(s,x(s)) - \varphi_\pm(s,x(s))|\mathrm{d}(s,x) + \sup_{t\in[t_1,t_2]}\left|\int_{G''} b(s,x(s))(\varphi(s,x(s)) - \varphi_\pm(s,x(s)))\mathrm{d}(s,x)\right|. \quad (5.17)$$

To continue, let $\zeta > 0$ be a real number and recall the number $\beta(x_0,\zeta,t') > 0$ of Corollary 5.7. Select

$$\lambda \le \beta(x_0,\zeta,t')$$

for the partition (5.8). Note that the hyper-volume of the domain G' is bounded by the hyper-volume of the domain G, and that the latter is bounded by $\eta(2\Delta)^n$. Then, recalling that feedback functions are bounded by K, we obtain the inequality

$$\sum_{q:\chi(q)\in G'} \int_{\chi(q)} \sup_{(s,x(s))\in\chi(q)} \big|b(s,x(s)) - b(t_1+q_0\lambda, x(t_1+q_0\lambda))\big| \, |\varphi(s,x(s)) - \varphi_\pm(s,x(s))| \, \mathrm{d}(s,x) \leq 2K\zeta\eta(2\Delta)^n.$$

Further, using (5.15) and the bound of Corollary 3.3 with b substituted for c, we get

$$\sup_{t\in[t_1,t_1+\eta]} \left| \int_{G''} b(s,x(s))\,(\varphi(s,x(s)) - \varphi_\pm(s,x(s)))\,\mathrm{d}s \right| \leq \lambda S_{G'} 2KM_b(t').$$

Inserting the last two inequalities into (5.17) yields

$$\sup_{t\in[t_1,t_1+\eta]} |z(t)| \leq 2K\left[\zeta\eta(2\Delta)^n + \lambda S_{G'}M_b(t')\right]$$

Substituting this into (5.13), we get

$$\sup_{t\in[t_1,t_1+\eta]} \left|\bar{\xi}(t)\right| \leq 2\left|\bar{\xi}(t_1)\right| + 4K\left[\zeta\eta + \lambda S_{G'}M_b(t')/(2\Delta)^n\right]. \tag{5.18}$$

Next, select a real number $\delta > 0$. Then, select $\zeta > 0$ to satisfy the inequality $\zeta\eta 4K < \delta/2$ and select a real number $\lambda' > 0$ satisfying the inequality $\lambda' S_{G'}M_b(t')4K/(2\Delta)^n < \delta/2$. Subsequently, select $\lambda > 0$ so that $\lambda \leq \min\{\lambda', \beta(x_0,\zeta,t')\}$. With these selections, inequality (5.18) becomes

$$\sup_{t\in[t_1,t_1+\eta]} \left|\bar{\xi}(t)\right| \leq 2\left|\bar{\xi}(t_1)\right| + \delta. \tag{5.19}$$

To explore the consequences of this inequality, let κ be an integer satisfying $\kappa \geq t'/\eta$ and construct the partition

$$[0,t'] \subseteq \{[0,\eta],[\eta,2\eta],\ldots,[(\kappa-1)\eta,\kappa\eta]\}.$$

Then, (5.19) yields the linear recursion

$$\sup_{t\in[i\eta,(i+1)\eta]} \left|\bar{\xi}(t)\right| \leq 2\left|\bar{\xi}(i\eta)\right| + \delta, \qquad \bar{\xi}(0) = 0,$$

$i = 0,\ldots,\kappa-1$. Using features of linear recursions, we conclude that

$$\sup_{t\in[0,t']} \left|\bar{\xi}(t)\right| \leq \delta\sum_{i=0}^{\kappa-1} 2^i = \delta(2^\kappa - 1).$$

Thus, the theorem is valid for any $\delta > 0$ satisfying

$$\delta < \varepsilon/(2^\kappa - 1).$$

This concludes our proof. ∎

Before continuing with our investigation of bang-bang feedback functions, we need the following statement, which shows that the minimal time $t^*(x_0,\ell,A,\gamma)$ of (2.26) is a monotone decreasing function of the state amplitude bound A.

Proposition 5.20: *The minimal time $t^*(x_0,\ell,A,\gamma)$ of (2.26) is a monotone decreasing function of the state amplitude bound A.*

Proof: Let $A' > A > 0$ be two state amplitude bounds, and consider an optimal feedback function $\varphi^*(x_0,\ell,A,\gamma) \in \Phi(K)$ that controls Σ so as to respect the state amplitude bound A. As $A < A'$, the feedback function $\varphi^*(x_0,\ell,A,\gamma)$ also controls Σ to comply with the state amplitude bound A'. Therefore, the time $t^*(x_0,\ell,A,\gamma)$ achieved by $\varphi^*(x_0,\ell,A,\gamma)$ cannot be shorter than the minimal time $t^*(x_0,\ell,A',\gamma)$ for the state amplitude bound A'. This concludes our proof. ∎

We can prove now Theorem 5.4.

Proof of Theorem 5.4: As $A > A_0$, we can select a real number A' such that $A > A' > A_0$. Then, by Theorem 4.1 and the current assumptions, there exist an uncertainty parameter $\gamma > 0$ and an optimal feedback function $\varphi^* := \varphi^*(x_0,\ell,A',\gamma) \in \Phi(K)$ satisfying part (i) of Problem 2.28 with the state amplitude bound A' and the error bound ℓ. The minimal time achieved by φ^* is then $t^* := t^*(x_0,\ell,A',\gamma)$, so that $\Sigma_{\varphi^*}(x_0,t^*) \in \rho(\ell)$ and $|\Sigma_{\varphi^*}(x_0,t)| \leq A'$ for all $t \in [0,t^*]$ and for almost all $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$.

Next, select a real number $\varepsilon > 0$. By Theorem 5.5, there is a bang-bang feedback function $\varphi_\pm \in \Phi(K)$ such that $|\Sigma_{\varphi^*}(x_0,t) - \Sigma_{\varphi_\pm}(x_0,t)| < \varepsilon$ for all $t \in [0,t^*]$ and almost all $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$, when averaged over the effects of the disturbance signal $\upsilon(t)$ of Figure 5. Now, select ε to satisfy $0 < \varepsilon \leq A - A'$. Then, $|\Sigma_{\varphi_\pm}(x_0,t)| \leq A$ for all $t \in [0,t^*]$ and for almost all $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$. Thus, $\varphi_\pm$ complies with the state bound A.

Finally, for every pair of vectors $y,z \in R^n$, we can write $z^\top z = y^\top y - 2y^\top(y-z) + (y-z)^\top(y-z) \leq y^\top y + 2n|y||y-z| + n|y-z|^2$. Applying this relation to the vectors $\Sigma_{\varphi_\pm}(x_0,t)$ and $\Sigma_{\varphi^*}(x_0,t)$, we obtain

$$\begin{aligned}
&\Sigma_{\varphi_\pm}^\top(x_0,t^*)\Sigma_{\varphi_\pm}(x_0,t^*) \\
&\leq \Sigma_{\varphi^*}^\top(x_0,t^*)\Sigma_{\varphi^*}(x_0,t^*) \\
&\quad + 2n\left|\Sigma_{\varphi^*}(x_0,t^*)\right|\left|\Sigma_{\varphi_\pm}(x_0,t^*) - \Sigma_{\varphi^*}(x_0,t^*)\right| \\
&\quad + n\left|\Sigma_{\varphi_\pm}(x_0,t^*) - \Sigma_{\varphi^*}(x_0,t^*)\right|^2 \\
&\leq \ell + 2n\sqrt{\ell}\varepsilon + n\varepsilon^2
\end{aligned} \tag{5.21}$$

for almost all $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$. Now, let $\ell' > 0$ be any real number for which

$$\ell + 2n\sqrt{\ell}\varepsilon + n\varepsilon^2 \leq \ell'. \tag{5.22}$$

Then, we have $\Sigma_{\varphi_\pm}(x_0,t^*) \in \rho(\ell')$. Denoting

$$\varepsilon' := -\sqrt{\ell} + \sqrt{\ell + (\ell' - \ell)/n},$$

we obtain from (5.22) and (5.21) that every $\varepsilon \in (0, \min\{\varepsilon', A - A'\}]$ guarantees that $\Sigma_{\varphi_\pm}(x_0,t^*) \in \rho(\ell')$ and $|\Sigma_{\varphi_\pm}(x_0,t| \leq A$ for all $t \in [0,t^*]$ and for almost all $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$, when averaged over the effects of the disturbance signal $\upsilon(t)$ of Figure 5. Consequently, $t(x_0,\ell',A,\gamma,\varphi_\pm) \leq t^*$. Also, by Proposition 5.20 we have that $t^* \leq t^*(x_0,\ell,A,\gamma)$. Thus, $t(x_0,\ell',A,\gamma,\varphi_\pm) \leq t^*(x_0,\ell,A,\gamma)$, and our proof concludes. ∎

Theorem 5.4 describes a relatively simple methodology for designing feedback functions that achieve close to optimal performance: use bang-bang feedback functions, instead of optimal feedback functions. In general, optimal feedback functions are measurable multi-variable vector-valued functions of time; they are harder to calculate and implement than bang-bang feedback functions. Indeed, at every time and state, a general measurable feedback function takes its values in the continuum $[-K, K]^m$, while a bang-bang function is restricted to taking only one of the values $-K$ or K in each component.

Bang-bang functions that approximate optimal performance can be derived through relatively simple numerical search algorithms (see the example of Section 6). Often, bang-bang feedback functions with a relatively small number of switchings can provide performance that is almost indistinguishable from optimal performance, as is the case in the example of Section 6. For comparison, classical methods for calculating optimal feedback functions involve numerical solutions of the Hamilton-Jacobi-Bellman partial differential equation; these are considerably harder to derive than simple bang-bang feedback functions. Note also that the approach presented in the current paper yields robust controllers that are optimised for a specified level of uncertainty about the model of the controlled system Σ.

6. Example

We use an example of an inverted pendulum taken from Choi and Hammer (2017a). Inverted pendulums appear in many applications, including missile control and walking robots (e.g. Boubaker & Iriarte, 2017). Using this example allows us to compare the performance achieved by optimal state-feedback controllers developed in the current paper to the performance optimal open-loop controllers developed in Choi and Hammer (2017a). The controlled system is described by the differential equation

$$\Sigma : \begin{array}{l} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = d_1 \sin x_1(t) + d_2 x_2(t) + d_3 \tanh u(t), \end{array}$$

where d_1, d_2, and d_3 are constant parameters that represent uncertainties. The nominal values of these parameters are $d_1^0 = 24.527$, $d_2^0 = -0.107$, $d_3^0 = 12.5$; their uncertainty ranges are $d_1 \in [21, 27]$, $d_2 \in [-0.3, -0.1]$, and $d_3 \in [10, 14]$. The initial state of Σ is $x_0 = [\pi/8, -2]^\top$. The input signal amplitude bound is $K = 5$; the state amplitude bound is $A = 2$; and the operating error bound is $\ell = 0.1$.

In Choi and Hammer (2017a), a numerical search process showed that the nominal system is (5, 2)-controllable and that the minimal time t^* required for reaching the ball $\rho(0.1)$ satisfies $t^* \leq 0.3$.

Our objective here is to use a search process to derive a bang-bang feedback function $\varphi_\pm$ that approximates optimal performance, driving Σ from the initial state x_0 to the domain $\rho(0.1)$ in close to minimal time. As the state amplitude bound is $A = 2$ and an upper bound on the minimal time is 0.3, our search domain is $[0, 0.3] \times [-2, 2]^2 \subseteq R^+ \times R^2$. We partition this domain into hyper-squares (cubes in this case) of edge $\lambda = 0.01$, and conduct a numerical search for bang-bang feedback functions that take the system to the target domain as quickly as possible. These functions are constant over each of these cubes, maintaining one of the values -5 or 5 on each cube.

The goal of the search is to find a feedback function $\varphi_\pm$ that guides Σ to $\rho(0.1)$ in the shortest time. The search is conducted by searching for the best feedback function with a specified number of switchings. It starts by searching for the best feedback function with no switchings; it proceeds by searching for the best feedback function with one switching, moving the switching point around all coordinates; next, it searches for the best feedback function with two switchings, again moving the switching points around all coordinates; and so on, until no significant further improvement in the minimal time is obtained by increasing the number of switchings.

Our search result shows that optimal performance can be approximated by the following simple bang-bang feedback function, which has a total of 6 switchings:

$$\varphi_\pm(t, x_1, x_2) := \begin{cases} 5 & \text{if } t \in [0, 0.05); \\ -5 & \text{if } t \in [0.05, 0.14); \\ 5 & \text{if } t \in [0.14, 0.16) \text{ and } |x_2| \geq 1.12; \\ -5 & \text{if } t \in [0.14, 0.15) \text{ and } 0.98 \leq |x_2| \leq 1.01; \\ -5 & \text{if } t \in [0.14, 0.16) \text{ and } |x_2| \leq 0.86; \\ 5 & \text{else.} \end{cases} \tag{6.1}$$

The function $\varphi_\pm$ is inserted for the feedback function φ in the control configuration of Figure 1. As can be seen from (6.1), this feedback function turns out to be independent of the first state variable x_1. Being a function of the two remaining variables (t and x_2), the function $\varphi_\pm$ can be depicted in the three-dimensional graph shown in Figure 7; here, the vertical coordinate represents the values of $\varphi_\pm$, while the two horizontal coordinates represent the variables t and x_2. As can be seen from the figure, the number of switchings – namely, the number of jumps of the function's value – is quite small. A low number of switchings makes the feedback function easy to implement.

Figure 8 demonstrates the performance of this feedback function on the following three representatives of our family of systems:

Set 1: $d_1 = 21, \ d_2 = -0.3, \ d_3 = 10$;

Set 2: $d_1 = 24, \ d_2 = -0.2, \ d_3 = 12$;

Set 3: $d_1 = 27, \ d_2 = -0.1, \ d_3 = 14$.

In the plot of Figure 8, the vertical axis represents the sum $x_1^2(t) + x_2^2(t)$, while the horizontal axis represents the time t. We can read from the plot the following times, which show how long it takes the system to reach the domain $\rho(0.1)$ for each parameter set (the initial state is $x_0 = [\pi/8, -2]^\top$):

Set 1: 0.226 s;

Set 2: 0.225 s;

Set 3: 0.229 s.

Compare these times to the minimal time of 0.264 s achieved by an open-loop controller in Choi and Hammer (2017a).

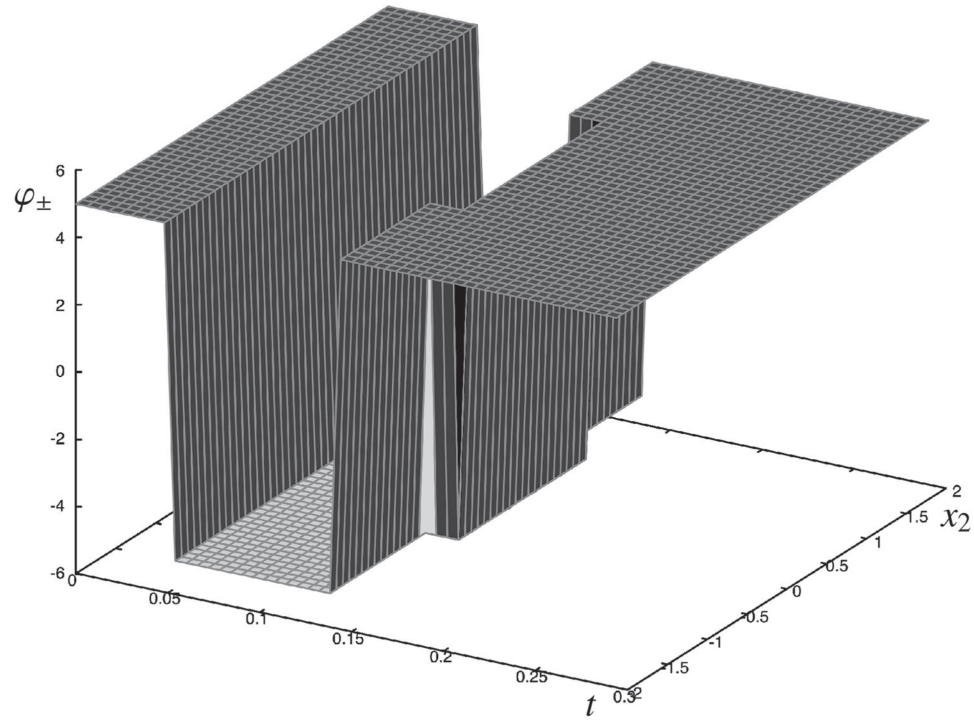


Figure 7. The bang-bang feedback function $\varphi_\pm$.

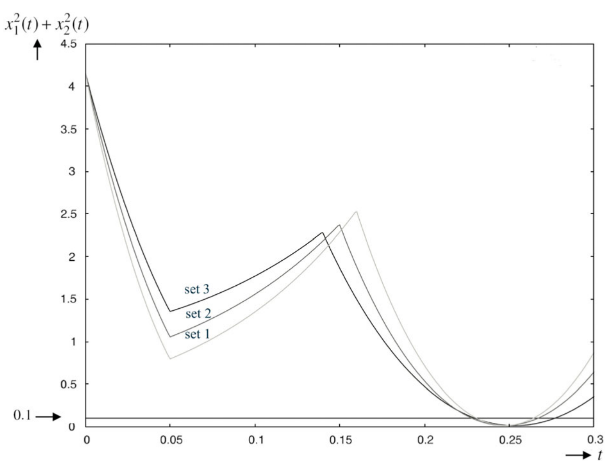


Figure 8. The closed-loop trajectories.

As expected, a closed loop system provides better performance; in this example, we obtained an improvement of about 20% over the performance of an open-loop controller.

In addition, the different models of the controlled system Σ reach the target very closely together; the discrepancy between the longest time and the shortest time is only 0.004 seconds; in other words, modelling uncertainty causes a dispersion of about 0.7% with the feedback controller. This compares to a discrepancy of 0.016 seconds in the open-loop case of Choi and Hammer (2017a)), which represents a dispersion of about 3%. Thus, as one would expect, feedback control can reduce the impact of uncertainties on performance; in this example, feedback achieves an improvement by a factor of about 4 in performance uniformity.

7. Conclusion

In this paper, we considered the existence and the implementation of optimal robust state-feedback controllers that take a system to a target state in minimal time. We showed that such controllers exist for a broad family of time-varying input-affine nonlinear systems. The main condition for the existence of such optimal robust feedback controllers is a certain controllability condition the nominal controlled system must satisfy. We also showed that optimal performance can be approximated by bang-bang feedback functions. Bang-bang feedback functions are piecewise-constant functions of time and state, whose components switch between the values $-K$ and K; here, K is the input signal amplitude bound of the controlled system. Bang-bang feedback functions are simpler to calculate and implement than general optimal feedback functions.

Many future research directions can be pursued in this area. One important effort would be to generalise the results of the current paper to classes of nonlinear systems that are more general than the class of time-varying nonlinear input-affine systems investigated here. Another important topic of future research is the development of dedicated high-speed numerical algorithms for the derivation of nearly optimal bang-bang feedback functions.

Disclosure statement

No potential conflict of interest was reported by the authors.

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