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Decentralized Control of Large-Scale Nonlinear Systems: Optimal Robust Tracking

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Abstract— **The design and implementation of decentralized controllers for large-scale nonlinear systems is considered. The objective is to track a specified target state over the infinite time horizon. Simple optimal robust decentralized controllers that comply with specified constraints are developed. Potential applications include electrical power distribution networks and the internet of things.**

I. Introduction

Large-scale systems have become more prevalent in recent decades. Systems such as electrical power distribution networks, the internet of things, cloud computing hardware assemblies, and vehicle brigades on automated highways are examples of large-scale systems. Central control of such systems is impractical, as it requires massively complex controllers. Instead, decentralized control makes it possible to control such systems by multiple relatively simple controllers, as is depicted in Figure 1.

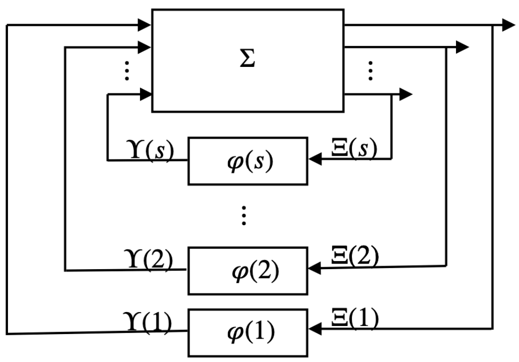

Fig. 1. Decentralized state feedback

A common goal in the control of large-scale systems is tracking: keeping the the state $x(t)$ of the system in the vicinity of a target state x_{target} at all times $t \geq 0$. For example, in an electrical power distribution network, one must maintain voltages and phases close to specifications to ensure power quality.

Like all systems, large-scale systems are affected by uncertainties, disturbances, and errors, whose probabilistic descriptions are often not available. Accordingly, we adopt here the robust control approach. Robust control achieves proper response irrespective of probabilistic features, as long as uncertainties do not exceed specified bounds.

To protect system integrity, the controlled system Σ of Figure 1 imposes a bound $K > 0$ on its input amplitude and a bound of $A > 0$ on its state amplitude.

In many large-scale systems, state coordinates have physical significance. For instance, in an electrical power distribution network, state variables include voltages and currents at nodes (or buses). These variables are associated with geographical locations. When designing decentralized controllers, it is important to limit each controller to variables from a small geographical region. To to avoid combining variables from distant geographical regions, we forbid coordinate transformations (other than shifts).

Often in large-scale systems, a particular state x is affected mainly by a relatively small collection $C(x)$ of states and inputs. Other states and inputs have only small effects on the response of x and may be considered disturbances. We call $C(x)$ the *'influence set'* of x. An influence set is characterized by a specified *'influence error bound'* $\zeta > 0$. States and inputs outside $C(x)$ affect x by no more than ζ in magnitude. That being so, we design a robust controller for x that involves only states and inputs from $C(x)$. Being robust, the controller handles states and inputs outside $C(x)$ as if they were disturbances. In this way, a large-scale system is controlled by a number of relatively simple robust controllers.

Influence sets are constructed in Section III; they partition the state and input components of Σ into a family of s disjoint influence sets $C(1)$, $C(2)$, ..., $C(s)$. Here, $C(i)$ consists of a set $C^x(i)$ of $n(i)$ state components spanning the subspace $\Xi(i)$ of state space; and a set $C^u(i)$ of $m(i)$ input components spanning the subspace $\Upsilon(i)$ of the input space. This partitions a large-scale system Σ into a family $\Sigma(1), \Sigma(2), \ldots, \Sigma(s)$ of 'nearly decoupled' simple subsystems. Each $\Sigma(i)$ is then controlled by an independent robust state-feedback controller $\varphi(i)$, as is depicted in Figure 1. The overall feedback controllers is then $\varphi = \varphi(1) \times \varphi(2) \times \cdots \times \varphi(s)$.

The control objective is tracking over the infinite time horizon $t \geq 0$, to keep the large-scale system Σ in the vicinity of a specified target state x_{target}. By shifting state coordinates, we can take $x_{target} = 0$. For a feedback controller φ, the deviation of the closed-loop system from the target state $x = 0$ is the *tracking error* $\ell(\varphi) := \sup_{t \geq 0} x^\top(t)x(t)$. We are seeking an optimal robust decentralized feedback controller φ^* that minimizes the tracking error, so that $\ell(\varphi^*) = \inf_\varphi \ell(\varphi)$. Section IV shows that such a controller exists when Σ has the property of *constrained controllability* ([1]–[3]), as long as errors are not too large. In Section V

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we show that optimal performance can be approximated by bang-bang controllers – controllers that are relatively easy to implement. Our objectives are then as follows.

Problem 1. For a large-scale system Σ:
(i) Partition Σ into nearly-decoupled subsystems for decentralized control.
(ii) Find decentralized optimal robust state-feedback controllers that achieve minimal tracking error in the presence of modeling uncertainties and influence errors.
(iii) Find simple to implement decentralized state-feedback controllers that approximate optimal performance. □

The material of this paper relies on [1], [3]–[8]; on classical studies on optimal control [9]–[16]; on the references cited in these publications; and on many others. For background on decentralized control, see [17], [18], the references cited in these publications, and many others. Yet, it seems that the topic of this paper — optimal robust decentralized state-feedback control of large-scale input-affine nonlinear systems — has not been settled in the literature.

The paper is organized as follows. Section II introduces notation; Section III develops influence sets and their use in decentralization of large-scale systems; Section IV deals with the existence of decentralized state-feedback controllers; Section V handles approximation of optimal performance by bang-bang controllers; Section VI is an example; and Section VII is a brief summary of the paper.

II. The system model

A. Notation

Regarding notation, the compactified set of real numbers is R; the set of non-negative real numbers is R^+; the compactified set of n-dimensional real vectors is R^n; the absolute value of a real number r is $|r|$; the L^∞-norm of an $n\times m$ matrix $V=(V_{ij})\in R^{n\times m}$ is $|V|:=\max_{ij}|V_{ij}|$; the L^∞-norm (the *amplitude*) of a matrix-valued function of time $W: R^+\to R^{n\times m}$ is $|W|_\infty := \sup_{t\geq 0}|W(t)|$; the L^2-norm of a vector x is $|x|_2=(x^\top x)^{1/2}$; the closed ball of square radius $\sigma>0$ around the origin is $\rho(\sigma):=\{x\in R^n : |x|_2^2\leq\sigma\}$; and $[-A,A]^n$ is the set of all vectors $x\in R^n$ with $|x|\leq A$.

B. The system equation

The system Σ of Figure 1 is nonlinear input-affine:

$$\Sigma: \begin{array}{l}\dot{x}(t)=a(t,x(t))+b(t,x(t))u(t), t\geq 0,\\ x(0)=x_0,\end{array} \tag{1}$$

where $x(t)\in R^n$ is the state and $u(t)\in R^m$ is the input. The functions $a: R^+\times R^n\to R^n$ and $b: R^+\times R^n\to R^{n\times m}$ are continuously differentiable. The initial state x_0 is any vector in $\rho(\sigma)$, where $\sigma>0$ is given. We often use $\Sigma(x_0,u,t)$ instead of $x(t)$.

To represent uncertainties and disturbances, we decompose the functions into sums

$$a(t,x)=a_0(x)+a_\delta(t,x),\ b(t,x)=b_0(x)+b_\delta(t,x), \tag{2}$$

where $a_0: R^n\to R^n$, $b_0: R^n\to R^{n\times m}$ are specified nominal functions, and $a_\delta: R^+\times R^n\to R^n$, $b_\delta: R^+\times R^n\to R^{n\times m}$ are unknown functions representing uncertainties; all functions are continuously differentiable. The nominal system

$$\Sigma_0: \begin{array}{l}\dot{x}(t)=a_0(x(t))+b_0(x(t))u(t), t\geq 0,\\ x(0)=x_0,\end{array} \tag{3}$$

is time-invariant, but the controlled system Σ may be time varying due to a_δ and b_δ.

Convention 2. Two systems Σ and Σ' are *equivalent* if they have the same response to the same input signal from the same initial states. □

The input and state amplitudes of Σ are constrained:

$$|u|_\infty\leq K \text{ and } |x|_\infty\leq A; \tag{4}$$

K and A are specified. As $x_0\in\rho(\sigma)$, we require $\sqrt{\sigma}<A$.

Remark 3. The mean value theorem (e.g., [19]) implies the following inequalities for all $|x|,|x'|\leq 2A$, $t\geq 0$ (see [1]):

$$\begin{array}{c}|a_0(x)-a_0(x')|\leq\beta|x-x'|,\ a_0(0)=0,\\ |b_0(x)-b_0(x')|\leq\beta|x-x'|,\ |b_0(0)|\leq\beta,\\ |a_\delta(t,x)-a_\delta(t,x')|\leq\delta|x-x'|,\ |a_\delta(t,0)|\leq\delta,\\ |b_\delta(t,x)-b_\delta(t,x')|\leq\delta|x-x'|,\ |b_\delta(t,0)|\leq\delta;\end{array} \tag{5}$$

here, $\beta>0$ and $\delta>0$ are given. We assume that δ is constant; δ is the system's *uncertainty parameter*. □

Notation 4. $\mathcal{F}_\delta(\Sigma_0)$ denotes the family of all systems represented by (3), (1), (2), and (5), with the following restrictions: *(i)* input amplitude is bounded by K; *(ii)* state amplitude is bounded A; *(iii)* initial state $x_0\in\rho(\sigma)$; *(iv)* all members of $\mathcal{F}_\delta(\Sigma_0)$ have the same initial state and the same state-feedback function. □

The family $\mathcal{F}_\delta(\Sigma_0)$ describes uncertainty about the controlled system Σ. As uncertainties are involved, the feedback function φ cannot be adjusted separately to each member of $\mathcal{F}_\delta(\Sigma_0)$, so all members share the same feedback function.

III. Decentralization

In many cases, the response of a state x of a large-scale systems is affected mainly by a small number of states and inputs; other states and inputs contribute little to the response of x, and may be considered as disturbances. The latter states and inputs can be ignored when designing robust controllers. Based on this, we partition the large-scale system Σ into a family of nearly independent systems.

As an example, consider an electrical power distribution network. Such network forms a large-scale system with thousands of states and inputs. Notwithstanding, the voltages and phases of a bus are mostly affected by nearby buses; the effects of remote buses are relatively small. Robust controllers based only on local buses can provide adequate control, avoiding the need of an unwieldy central controller.

A. Influence sets

1) Notation: For a set of integers $\lambda=\{\lambda_1,\lambda_2,\ldots,\lambda_k\}\subseteq\{1,2,\ldots,n\}$, the *cardinality* is $\#\lambda=k$, and the *complement* is the difference set $\lambda^c := \{1,\ 2,\ \ldots,\ n\}\setminus\lambda$. For a vector x

$= (x^1,\ x^2,\ \ldots,\ x^n)^\top \in R^n$, denote by $x_{|\lambda}$ the vector in R^k given by $x_{|\lambda} = (x^{\lambda_1}, x^{\lambda_2}, \ldots, x^{\lambda_k})^\top$, i.e., by the components of x that correspond to the integers in λ. For example, for $n = 4$, $x = (x^1, x^2, x^3, x^4)^\top$ and $\lambda = \{1,3\}$, we have $x_{|\lambda} = (x^1, x^3)^\top$.

For two vectors $x, d \in R^n$, denote by $x_{|d(\lambda)}$ the vector in R^n obtained by merging components of x and d; component i of the result is

$$x^i_{|d(\lambda)} := \begin{cases} x^i & \text{if } i \in \lambda, \\ d^i & \text{if } i \in \lambda^c; \end{cases}$$

set $x_{|d(\varnothing)} = d$, where $\varnothing$ denotes the empty set. For example, for $x = (x^1, x^2, x^3, x^4)^\top$, $d = (d^1,\ d^2,\ d^3\ d^4)^\top$, and $\lambda = \{1,3\}$, we get $x_{|d(\lambda)} = (x^1, d^2, x^3, d^4)^\top$.

For two sets of integers $\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_k\} \subseteq \{1,2,\ldots,n\}$ and $\mu = \{\mu_1, \mu_2, \ldots, \mu_r\} \subseteq \{1,2,\ldots,m\}$, and an $n \times m$ matrix V, the matrix $V_{|\lambda\times\mu}$ is a $k \times r$ matrix consisting of entries $\mu_1, \mu_2, \ldots, \mu_r$ of rows $\lambda_1, \lambda_2, \ldots, \lambda_k$ of V.

Example 5. For $\lambda = \{2,3\}$, $\mu = \{1,2\}$, and $V = \left(\begin{smallmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{smallmatrix}\right)$, we get $V_{|\lambda\times\mu} = \left(\begin{smallmatrix} 2 & 5 \\ 3 & 6 \end{smallmatrix}\right)$. □

2) Lipschitz-type bounds: Consider a function $h : R^n \to R$ satisfying for all $x, y \in [-A, A]^n$ the Lipschitz condition

$$|h(x) - h(y)| \le \beta |x - y|, \tag{6}$$

where $\beta > 0$. Let $\lambda \subseteq \{1,2,\ldots,n\}$ be a set of integers. The following quantity gauges the impact on h of the λ^c components of x:

$$\Delta(h,\lambda) := \begin{cases} \sup_{d\in[-A,A]^n} \sup_{\substack{x\in[-A,A]^n \\ x \ne x_{|d(\lambda)}}} \frac{|h(x) - h(x_{|d(\lambda)})|}{|x - x_{|d(\lambda)}|} & \text{if } \lambda \subsetneq \{1,\ldots,n\}, \\ 0 & \text{if } \lambda = \{1,\ldots,n\}. \end{cases} \tag{7}$$

Here, $\Delta(h,\lambda) \le \beta$ by (6). Moreover, (7) yields the Lipschitz inequality

$$\left|h(x) - h(x_{|d(\lambda)})\right| \le \Delta(h,\lambda)|x - x_{|d(\lambda)}| \text{ for all } x, d \in [-A, A]^n.$$

3) The influence error bound: Consider the nominal large-scale system Σ_0 of (3). Let a^i_0 be entry i of a_0, and let b^i_0 be row i of b_0. Let $\zeta > 0$ be a specified real number to serve as the *influence error bound*. Assemble the family $S^1(i, a_0, \zeta)$ of integer sets $\lambda \subseteq \{1,2,\ldots,n\}$ for which the influence of the corresponding state coordinates may exceed ζ:

$$S^1(i, a_0, \zeta) := \left\{ \lambda \subseteq \{1,2,\ldots,n\} : \begin{array}{l} \Delta(a^i_0, \lambda) \le \zeta \text{ and} \\ |(a^i_0(0))_{|\lambda^c}| \le \zeta \end{array} \right\}. \tag{8}$$

If $S^1(i, a_0, \zeta)$ is empty, select a larger value of ζ.

We show in Section V that smaller members of $S^1(i, a_0, \zeta)$ yield simpler controllers. The family of members of $S^1(i, a_0, \zeta)$ with lowest cardinality is

$$S^2(i, a_0, \zeta) := \left\{ \lambda \in S^1(i, a_0, \zeta) : \#\lambda = \min_{\#\lambda}\{\lambda \in S^1(i, a_0, \zeta)\} \right\}$$

Regarding the impact of inputs on $\dot{x}^i$, let $b^{i1}_0, b^{i2}_0, \ldots, b^{im}_0$ be the entries b^i_0. The class of subsets of $\{1,2,\ldots,m\}$ that represent entries of b^i_0 with influence not exceeding ζ is

$$S^3(i, b_0, \zeta) := \left\{ \begin{array}{l} \mu \subseteq \{1,2,\ldots,m\} : \\ \sum_{j\in\mu^c} \Delta(b^{ij}_0, \varnothing) \le \zeta \text{ and } \sum_{j\in\mu^c} |b^{ij}_0(0)| \le \zeta \end{array} \right\}. \tag{9}$$

The lowest cardinality members of $S^3(i, b_0, \zeta)$ form

$$S^u(i, b_0, \zeta) := \left\{ \mu \in S^3(i, b_0, \zeta) : \#\mu = \min_{\#\mu}\{\mu \in S^3(i, b_0, \zeta)\} \right\}$$

Let $\mu(i,1),\ \mu(i,2),\ \ldots,\ \mu(i,k(i))$ be the members of $S^u(i, b_0, \zeta)$; each is a subset of $\{1,2,\ldots,m\}$. Influential state coordinates for components $\mu(i,p)$ of b^i_0 are members of

$$S^4(i, b_0, \zeta, p) := \left\{ \begin{array}{l} \lambda \subseteq \{1,2,\ldots,n\} : \sum_{j\in\mu(i,p)} \{\Delta(b^{ij}_0, \lambda)\} \le \zeta \\ \qquad\qquad \text{and } \sum_{j\in(\mu(i,p))^c} |b^{ij}_0(0)| \le \zeta \end{array} \right\}.$$

Now, combine the influential state components of $S^2(i, a_0, \zeta)$ and $S^4(i, b_0, \zeta, p)$:

$$S^5(i, a_0, b_0, \zeta, p) := \left\{ s_1 \cup s_2 : \begin{array}{l} s_1 \in S^2(i, a_0, \zeta), \\ s_2 \in S^4(i, b_0, \zeta, p) \end{array} \right\}.$$

Then, select the lowest cardinality sets from $S^5(i, a_0, b_0, \zeta, p)$:

$$S^x(i, a_0, b_0, \zeta, p) := \left\{ \begin{array}{l} \lambda \in S^5(i, a_0, b_0, \zeta, p) : \\ \#\lambda = \min_{\#\lambda}\{\lambda \in S^5(i, a_0, b_0, \zeta, p)\} \end{array} \right\}.$$

All members of $S^x(i, a_0, b_0, \zeta, p)$ have the same cardinality, denoted $c(i, a_0, b_0, \zeta, p)$. The integers $p \in \{1,2,\ldots,k(i)\}$ that yield the smallest such cardinality form the family

$$M(i, a_0, b_0, \zeta) = \left\{ \begin{array}{l} p \in \{1,2,\ldots,k(i)\} : \\ c(i, a_0, b_0, \zeta, p) = \min_{1\le q\le k(i)} c(i, a_0, b_0, \zeta, q) \end{array} \right\}$$

Each integer $p_i \in M(i, a_0, b_0, \zeta)$ points to members $\lambda(i, p_i) \in S^x(i, a_0, b_0, \zeta, p)$ and $\mu(i, p_i) \in S^u(i, b_0, \zeta)$ representing influential state and input components for $\dot{x}^i$, respectively. We list this data for all components $\dot{x}^1, \dot{x}^2, \ldots, \dot{x}^n$ as a vector $\theta := (p_1, p_2, \ldots, p_n)$. All options of θ are in the cross product

$$\Theta := M(1, a_0, b_0, \zeta) \times M(2, a_0, b_0, \zeta) \times \cdots \times M(n, a_0, b_0, \zeta).$$

B. A disjoint partition

For an integer $i \in \{1,2,\ldots,n\}$ and a member $\theta \in \Theta$, let $\eta_1(i,\theta)$ be the family of all components of $\dot{x}(t)$ that share influential states or inputs with $\dot{x}^i$:

$$\eta_1(i,\theta) := \left\{ k \in \{1,2,\ldots,n\} : \begin{array}{c} \lambda(k, q(k)) \cap \lambda(i, q(i)) \ne \varnothing \\ \text{or} \\ \mu(k, q(k)) \cap \mu(i, q(i)) \ne \varnothing. \end{array} \right\} \tag{10}$$

Next, state components in $\eta_1(i,\theta)$ may share influential state or input components with states that are not in $\eta_1(i,\theta)$. These form the family $\eta_2(i,\theta) := \cup_{k\in\eta_1(i,\theta)} \eta_1(k,\theta)$. Building up recursively, we obtain the set

$$\eta_{r+1}(i,\theta) := \bigcup_{k\in\eta_r(i,\theta)} \eta_r(k,\theta), r = 1,2,\dots$$

We obtain a monotone sequence $\eta_r(i,\theta) \subseteq \eta_{r+1}(i,\theta)$; as $\eta_r(i,\theta) \subseteq \{1,2,\dots,n\}$, it becomes constant equal to

$$\eta^x(i,\theta) := \bigcup_{k\geq 1} \eta_k(i,\theta);$$

$\eta^x(i,\theta)$ is called the *state influence set*. It consists of all state components significantly influencing, directly or indirectly, $\dot{x}^i$. Based on our construction:

Lemma 6. *If $\eta^x(i,\theta)\cap\eta^x(j,\theta) \neq \varnothing$ for integers $i,j \in \{1, 2, \dots, n\}$, then $\eta^x(i,\theta) = \eta^x(j,\theta)$.* □

By Lemma 6, state influence sets partition the state components of Σ_0 into a family of, say s, disjoint sets. For input components, the *input influence set* associated with $\dot{x}^i$ is

$$\eta^u(i,\theta) := \bigcup_{j\in\eta^x(i,\theta)} \mu(j,\theta).$$

The following is a consequence of construction.

Lemma 7. *If $\eta^u(i,\theta)\cap\eta^u(j,\theta) \neq \varnothing$ for $i,j \in \{1,2,\dots,n\}$, then $\eta^u(i,\theta) = \eta^u(j,\theta)$. The number of distinct input influence sets equals the number s of distinct state influence sets.*

Thus, we obtained disjoint partitions $\eta_1^x(\theta)$, $\eta_2^x(\theta)$, ..., $\eta_s^x(\theta)$ of state components and $\eta_1^u(\theta)$, $\eta_2^u(\theta)$, ..., $\eta_s^u(\theta)$ of input components of Σ_0. Values of θ that yield the lowest cardinalities form the class

$$\Theta^* := \left\{ \theta\in\Theta : \begin{array}{l} \max\limits_{1\leq j\leq s} \{\#\eta_j^x(\theta), \#\eta_j^u(\theta)\} \\ = \min\limits_{\theta\in\Theta} \max\limits_{1\leq j\leq s} \{\#\eta_j^x(\theta), \#\eta_j^u(\theta)\} \end{array} \right\}.$$

Notation 8. Select a member $\theta^* \in \Theta^*$, and use the notation:

$$\begin{aligned} \eta_j^x &:= \eta_j^x(\theta^*), \\ \eta_j^u &:= \eta_j^u(\theta^*), \text{and} \\ \eta(j) &:= (\eta_j^x, \eta_j^u), j\in\{1,2,\dots,s\}. \end{aligned} \tag{11}$$

The cardinalities: $n_j = \#\eta_j^x$ and $m_j = \#\eta_j^u$.
The states associated with η_j^x are $\Xi(j) := \{x_{|\eta_j^x}, x \in R^n\}$;
The inputs associated with η_j^u is $\Upsilon(j) := \{u_{|\eta_j^u}, u \in R^m\}$.
Renumber state and input components:
The first n_1 components are $\Xi(1)$; components $n_1+n_2+\dots+n_{j-1}+1$ to $n_1+n_2+\dots+n_j$ are $\Xi(j)$.
The first m_1 input components are $\Upsilon(1)$; components $m_1+m_2+\dots+m_{j-1}+1$ to $m_1+m_2+\dots+m_j$ are $\Upsilon(j)$. □

A smaller influence error bound ζ may increase n_j or m_j.

C. A nearly independent decomposition

We build an approximation of Σ_0 as a family of independent systems $\Sigma_0(1)$, $\Sigma_0(2)$, ..., $\Sigma_0(s)$, where $\Sigma_0(i)$ has state $x(i) \in \Xi(i)$ and input $u(i) \in \Upsilon(i)$. Select a constant vector $d \in [-A,A]^n$ and define functions

$$a_0(i,x(i)) := a_{0|\eta_i^x}(x_{|d(\eta_i^x)}),\ \ b_0(i,x(i)) := b_{0|\eta_i^x\times\eta_i^u}(x_{|d(\eta_i^x)}),$$

$i = 1,2,\dots,s$. Then, we obtain the independent systems

$$\Sigma_0(i) : \begin{array}{l} \dot{x}(i,t) = a_0(i,x(i,t)) + b_0(i,x(i,t))u(i,t), \\ x(i,0) = x_{0|\eta_i^x}, i = 1,2,\dots,s. \end{array} \tag{12}$$

We assemble a large-scale system by cross product

$$\Sigma_0^\times := \Sigma_0(1)\times\Sigma_0(2)\times\dots\times\Sigma_0(s).$$

This system has the state $z(t) := (x(1,t), x(2,t), \dots, x(s,t))^\top$; the input $w(t) := (u(1,t), u(2,t), \dots, u(s,t))^\top$; the functions

$$\begin{aligned} a_0^\times(z(t)) &:= \begin{pmatrix} a_0(1,x(1,t)) & 0 & 0 & 0 \\ 0 & a_0(2,x(2,t)) & 0 & 0 \\ & & \ddots & \\ 0 & 0 & & 0 \\ 0 & 0 & 0 & a_0(s,x(s,t)) \end{pmatrix}, \\ b_0^\times(z(t)) &:= \begin{pmatrix} b_0(1,x(1,t)) & 0 & 0 & 0 \\ 0 & b_0(2,x(2,t)) & 0 & 0 \\ & & \ddots & \\ 0 & 0 & & 0 \\ 0 & 0 & 0 & b_0(s,x(s,t)) \end{pmatrix}; \end{aligned} \tag{13}$$

and the equation

$$\Sigma_0^\times :\ \dot{z}(t) = a_0^\times(z(t)) + b_0^\times(z(t))w(t).$$

This systems is the basis of our decentralization process.

IV. Decentralized control

A. Combining error bounds

Using (3) and (13), define

$$a_\zeta(x) := a_0(x) - a_0^\times(x) \text{ and } b_\zeta(x) := b_0(x) - b_0^\times(x). \tag{14}$$

By (8) and (9), we obtain

$$\begin{aligned} &|a_\zeta(x) - a_\zeta(x')| \leq \zeta|x-x'|,\ |a_\zeta(0)| \leq \zeta, \\ &|b_\zeta(x) - b_\zeta(x')| \leq \zeta|x-x'|,\ |b_\zeta(0)| \leq \zeta \end{aligned}$$

for all $x,x' \in [-A,A]^n$. Define the combined error functions

$$a_\gamma(t,x) := a_\delta(t,x) + a_\zeta(x) \text{ and } b_\gamma(t,x) := b_\delta(t,x) + b_\zeta(x).$$

Then, by (14), (1), (2), (5), and (14), we can write

$$a(t,x) = a_0^\times(x) + a_\gamma(t,x) \text{ and } b(t,x) = b_0^\times(x) + b_\gamma(t,x).$$

Define the bounds

$$\gamma := \delta + \zeta \text{ and } \alpha := \beta + \zeta. \tag{15}$$

Then, we obtain the Lipschitz conditions

$$\begin{aligned} &|a_0^\times(x) - a_0^\times(x')| \leq \alpha|x-x'|,\ a_0^\times(0) = 0, \\ &|b_0^\times(x) - b_0^\times(x')| \leq \alpha|x-x'|,\ |b_0^\times(0)| \leq \alpha, \\ &|a_\gamma(t,x) - a_\gamma(t,x')| \leq \gamma|x-x'|,\ |a_\gamma(t,0)| \leq \gamma, \\ &|b_\gamma(t,x) - b_\gamma(t,x')| \leq \gamma|x-x'|,\ |b_\gamma(t,0)| \leq \gamma \end{aligned}$$

for all $x,x' \in [-2A,2A]^n$ and all $t \geq 0$.

We compare now two families of systems: $\mathcal{F}_\gamma(\Sigma_0^\times)$ – all systems deviating by γ or less from $\Sigma_0^\times$; and $\mathcal{F}_\delta(\Sigma_0)$ – the original family of systems. Our construction implies:

Proposition 9. $\mathcal{F}_\delta(\Sigma_0) \subseteq \mathcal{F}_\gamma(\Sigma_0^\times)$. □

Next, define the cross product of families

$$\mathcal{F}_\gamma^\times(\Sigma_0^\times) := \mathcal{F}_\gamma(\Sigma_0(1))\times\mathcal{F}_\gamma(\Sigma_0(2))\times\dots\times\mathcal{F}_\gamma(\Sigma_0(s)).$$

Proposition 10. $\mathcal{F}_\gamma(\Sigma_0^\times) \subseteq \mathcal{F}_\gamma^\times(\Sigma_0^\times)$ *under Convention 2.*

Proof (sketch). Constituent i of $\Sigma \in \mathcal{F}_\gamma(\Sigma_0^\times)$ has the equation

$$\Sigma(i): \begin{aligned} \dot{x}(i,t) &= a_0(i,x(i,t)) + a_{\gamma|\eta_i^x}(z(t)) \\ &+ (b_0(i,x(i,t)) + b_{\gamma|\eta_i^x}(z(t))u(i,t), i = 1,2,\dots,s. \end{aligned}$$

For each i, replace all coordinates of $z(t)$ that are outside $x(i,t)$ by an equal time function. This yields the system

$$\Sigma'(i): \begin{aligned} \dot{x}(i,t) &= [a_0(i,x(i,t)) + a'_{\gamma|\eta_i^x}(t,x(i,t))] \\ &+ [b_0(i,x(i,t)) + b'_{\gamma|\eta_i^x}(t,x(i,t))]u(i,t). \end{aligned}$$

By Convention 2, the system $\Sigma'(i)$ is equivalent to $\Sigma(i)$, so $\Sigma' := \Sigma'(1) \times \Sigma'(2) \times \cdots \times \Sigma'(s) \in \mathcal{F}_\gamma^\times(\Sigma_0^\times)$ is equivalent to Σ (see [20] for details). □

B. Optimal robust decentralized feedback

Assume that there are optimal robust state-feedback functions $\varphi^*(i)$ for the family $\mathcal{F}_\gamma(\Sigma_0(i))$, $i \in \{1,2,\dots,s\}$. Then, the parallel combination

$$\varphi = \varphi^*(1) \times \varphi^*(2) \times \cdots \times \varphi^*(s)$$

is a decentralized robust state-feedback function that properly controls the family $\mathcal{F}_\gamma^\times(\Sigma_0^\times)$. But then, since $\mathcal{F}_\delta(\Sigma_0) \subseteq \mathcal{F}_\gamma^\times(\Sigma_0^\times)$ by Propositions 9 and 10, φ also properly controls the original family $\mathcal{F}_\delta(\Sigma_0)$. The existence of $\varphi^*(i)$ depends on the notion of constrained controllability ([1], [2], [3]):

Definition 11. A system Σ is (K,A,σ)-*controllable* if there are times $T_2 > T_1 > 0$ and a real number $\sigma' < \sigma$ such that, for every state $x \in \rho(\sigma)$, there are a time $t(x) \in [T_1,T_2]$ and an input signal $u_x \in U(K)$ for which $|\Sigma(x,u_x,t)| \le A$ for all $t \in [0,t(x)]$ and $\Sigma(x,u_x,t(x)) \in \rho(\sigma')$. □

The following follows from (14) and [2].

Proposition 12. *Let $A > A_0$ be real numbers. If the nominal system Σ_0 is (K,A_0,σ)-controllable, then there is an influence error bound $\zeta > 0$ such that all systems $\Sigma_0(1), \Sigma_0(2), \dots, \Sigma_0(s)$ of (12) are (K,A,σ)-controllable.* □

The family of state-feedback functions that prevent the state amplitude of $\Sigma(i)$ from exceeding the bound A is

$$\Phi(i,\sigma,K,A,\gamma) = \left\{ \begin{matrix} \varphi(i): R^+ \times [-A,A]^{n_i} \to [-K,K]^{m_i}: \\ \sup\limits_{\substack{\Sigma(i) \in \mathcal{F}_\gamma(\Sigma_0(i)) \\ x_0(i) \in \rho(\sigma),\ t \ge 0}} \left|\Sigma_{\varphi(i)}(i,x_0(i),t)\right| \le A \end{matrix} \right\}.$$

The tracking error achieved by one of these functions over the family $\mathcal{F}_\gamma(\Sigma_0(i))$ is as follows:
for a finite time interval $[0,\tau]$, $\tau > 0$:

$$\ell(i,A,\sigma,\gamma,\varphi(i),\tau) := \operatorname*{ess\,sup}_{\substack{\Sigma(i) \in \mathcal{F}_\gamma(\Sigma_0(i)) \\ x_0(i) \in \rho(\sigma),\ t \in [0,\tau]}} \left|\Sigma_{\varphi(i)}(i,x_0(i),t)\right|_2^2;$$

and over the infinite time horizon $t \ge 0$:

$$\ell(i,A,\sigma,\gamma,\varphi(i)) := \operatorname*{ess\,sup}_{\substack{\Sigma(i) \in \mathcal{F}_\gamma(\Sigma_0(i)) \\ x_0(i) \in \rho(\sigma),\ t \ge 0}} \left|\Sigma_{\varphi(i)}(i,x_0(i),t)\right|_2^2.$$

The corresponding infimal deviations are

$$\ell^*(i,A,\sigma,\gamma,\tau) = \inf_{\varphi(i) \in \Phi(i,\sigma,K,A,\gamma)} \ell(i,\sigma,\gamma,\varphi(i),\tau), \text{ and}$$

$$\ell^*(i,A,\sigma,\gamma) = \inf_{\varphi(i) \in \Phi(i,x_0,K,A,\gamma)} \ell(i,\sigma,\gamma,\varphi(i)).$$

Optimal state-feedback functions $\varphi_\tau^*(i)$ and $\varphi^*(i)$, if they exist, achieve these infima:

$$\ell(i,A,\sigma,\gamma,\varphi_\tau^*(i),\tau) = \ell^*(i,A,\sigma,\gamma,\tau), \text{ and} \tag{16}$$

$$\ell(i,A,\sigma,\gamma,\varphi^*(i)) = \ell^*(i,A,\sigma,\gamma). \tag{17}$$

The existence of such optimal state-feedback functions is assured by Proposition 12 and [1], as long as the combined uncertainty parameter γ of (15) is not too large:

Theorem 13. *Assume that the nominal system Σ_0 is (K,A_0,σ)-controllable, and let $A > A_0$ and $\tau > 0$ be real numbers. Then, there is a combined uncertainty parameter $\gamma > 0$ for which there exist optimal robust state-feedback functions $\varphi_\tau^*(i)$ and $\varphi^*(i)$ satisfying (16) and (17), respectively, $i = 1,2,\dots,s$.* □

Using Theorem 13, build the decentralized state-feedback

$$\varphi_\tau = \varphi_\tau^*(1) \times \varphi_\tau^*(2) \times \cdots \times \varphi_\tau^*(s), \text{ and}$$
$$\varphi = \varphi^*(1) \times \varphi^*(2) \times \cdots \times \varphi^*(s).$$

As we use an L^2-tracking error over orthogonal coordinates, the overall errors are

$$\ell(A,\sigma,\gamma,\tau) = \sum_{i=1}^{s} \ell^*(i,A,\sigma,\gamma,\tau), \text{ and}$$
$$\ell(A,\sigma,\gamma) = \sum_{i=1}^{s} \ell^*(i,A,\sigma,\gamma).$$

Due to influence errors, these errors may be somewhat bigger than the minimal tracking errors achievable by centralized controllers. Yet, this is a small price to pay for significant controller simplification achieved by decentralization.

V. Bang-bang approximation

Bang-bang state-feedback functions are piecewise-constant; their components switch between $-K$ and K as function of state and time; they are easier to calculate and implement than optimal functions and can approximate optimal performance. The next statement is from [8], [20]:

Theorem 14. *Assume that the nominal system Σ_0 is (K,A_0,σ)-controllable, and let $A > A_0$ be a real number. Then, for every $\varepsilon > 0$, there are a bang-bang feedback function $\varphi_\pm(i)$ and a combined uncertainty parameter $\gamma > 0$ such that $|\ell(i,A,\sigma,\gamma,\varphi_\pm(i),\tau) - \ell^*(i,A_0,\sigma,\gamma,\tau)| < \varepsilon$ (feedback signals are averaged over noise).* □

A smaller ε may require more switchings in $\varphi_\pm(i)$.

VI. Example

Consider a fourth order system consisting of two coupled inverted pendulums. Coupled pendulums are used to model bipedic and quadrapedic locomotion (e.g., [21]).

$$\Sigma: \begin{aligned} \dot{x}_1(t) &= x_2(t) + 0.1x_3(t), \\ \dot{x}_2(t) &= d_1 \sin x_1(t) + d_2 x_2(t) + d_3 \tanh u_1(t), \\ \dot{x}_3(t) &= x_4(t) + 0.1x_1(t), \\ \dot{x}_4(t) &= d_4 \sin x_3(t) + d_5 x_4(t) + d_6 \tanh u_2(t); \end{aligned}$$

here, $d_1, d_4 \in [11.5, 12.5]$, $d_2, d_5 \in [-0.105, -0.095]$, and $d_3, d_6 \in [6.7, 7.3]$ are constants with nominal values $d_1^0 = d_4^0 = 12$, $d_2^0 = d_5^0 = -0.1$, $d_3^0 = d_6^0 = 7$. The input amplitude bound is $K = 1$, the state amplitude bound is $A = 0.5$, and the initial state is $x_0 \in \rho(0.2)$.

As in Section IV, we represent the interaction terms $0.1x_3(t)$ and $0.1x_1(t)$ by disturbance functions $a_\zeta(1,t)$ and $a_\zeta(2,t)$:

$$\Sigma(1): \begin{array}{l} \dot{x}_1(1,t) = x_2(1,t) + a_\zeta(1,t), \\ \dot{x}_2(1,t) = d_1 \sin x_1(1,t) + d_2 x_2(1,t) + d_3 \tanh u(1,t), \end{array}$$

$$\Sigma(2): \begin{array}{l} \dot{x}_1(2,t) = x_1(2,t) + a_\zeta(2,t), \\ \dot{x}_2(2,t) = d_4 \sin x_1(2,t) + d_5 x_2(2,t) + d_6 \tanh u(2,t). \end{array}$$

A system similar to $\Sigma(1)$ and $\Sigma(2)$ was an example in [1], where a state-feedback function was derived. We use that feedback function for each decentralize arm:

$$\varphi(1) = \begin{cases} 1 & \text{if } x_1 \le 0 \text{ and } x_2 < 0.19 \\ -1 & \text{if } x_1 \le 0 \text{ and } x_2 > 0.21 \\ 1 - 100(x_2 - 0.19) & \text{if } x_1 \le 0 \text{ and } x_2 \in [0.19, 0.21] \\ 1 & \text{if } x_1 > 0 \text{ and } x_2 < -0.21 \\ -1 & \text{if } x_1 > 0 \text{ and } x_2 > -0.19 \\ -1 - 100(x_2 + 0.19) & \text{if } x_1 > 0 \& x_2 \in [-0.21, -0.19] \end{cases}$$

$$\varphi(2) = \begin{cases} 1 & \text{if } x_3 \le 0 \text{ and } x_4 < 0.19 \\ -1 & \text{if } x_3 \le 0 \text{ and } x_4 > 0.21 \\ 1 - 100(x_4 - 0.19) & \text{if } x_3 \le 0 \text{ and } x_4 \in [0.19, 0.21] \\ 1 & \text{if } x_3 > 0 \text{ and } x_4 < -0.21 \\ -1 & \text{if } x_3 > 0 \text{ and } x_4 > -0.19 \\ -1 - 100(x_4 + 0.19) & \text{if} x_3 > 0 \& x_4 \in [-0.21, -0.19] \end{cases}$$

The response is depicted in Figure 2 for the time interval $[0, 0.5]$. These feedback functions can be extended to the entire time axis $t \ge 0$ by shifting to achieve tracking over the infinite time horizon. The tracking error is $\ell = 0.16$, almost indistinguishable from optimal centralized control.

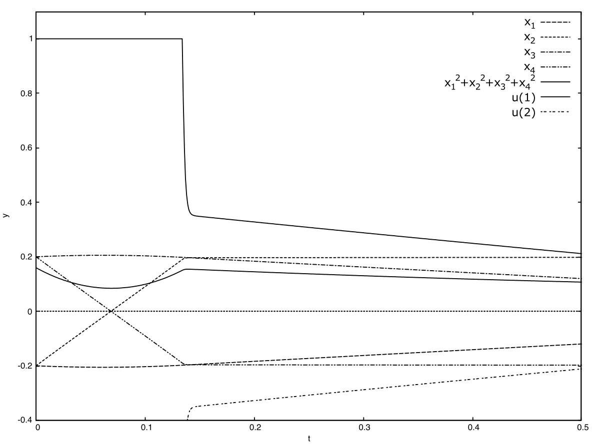

Fig. 2. The response

VII. Conclusion

A methodology for the design and implementation of decentralized optimal robust controllers for large-scale nonlinear systems was presented. It is based on approximating large-scale systems by a families of independent systems, based on the influence error bound. The paper provides a systematic approach for design and implementation of decentralized optimal robust controllers for large-scale nonlinear systems, such as electrical power distribution networks.

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