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Nonlinear Tracking on the Infinite Horizon: Optimal Robust State Feedback

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Abstract: Optimal robust state-feedback controllers that minimize tracking errors over all times $t \geq 0$ are considered. It is shown that such controllers exist for nonlinear input-affine systems that satisfy a certain controllability condition. Easy-to-implement controllers that approximate optimal performance are presented.

Keywords: optimal control, tracking, nonlinear control, state feedback, bang-bang control.

1. INTRODUCTION

Many applications in modern control engineering require tracking over extended periods of time. Examples include airplane, missile, and spacecraft autopilot and guidance systems; self-driving cars; certain robots; power distribution systems; biomedical systems such as heart pacers and aids for diabetes patients; and many other systems encountered in modern control engineering practice. The present note explores the existence and the implementation of state-feedback controllers that achieve optimal robust tracking for nonlinear input-affine systems. The note shows that such optimal robust controllers exist under broad conditions. It also shows that optimal performance can be approximated by a class of state-feedback controllers whose implementation is undemanding.

The control configuration considered in the note is the classical configuration of Figure 1.1, where the controlled system Σ receives the input signal $u(t)$ and provides its state $x(t)$ as output. Control is achieved by the state-feedback function φ, which generates the input signal $u(t) = \varphi(t, x(t))$. The resulting closed-loop system is Σ_φ, and its response $x(t)$ is often written as $\Sigma_\varphi(x_0, t)$ to point out the dependence on φ and on the initial state x_0. The controlled system Σ is a nonlinear input-affine system that is beset by modeling inaccuracies and is constrained by control effort limitations: its input amplitude may not exceed a specified bound $K > 0$, and its state amplitude may not exceed a specified bound $A > 0$. The purpose of the controller is to keep the response $x(t)$ as close as possible to a target state x_{target} at all times $t \geq 0$. After possibly shifting the state coordinates of Σ, we can take the target state to be the origin $x = 0$.

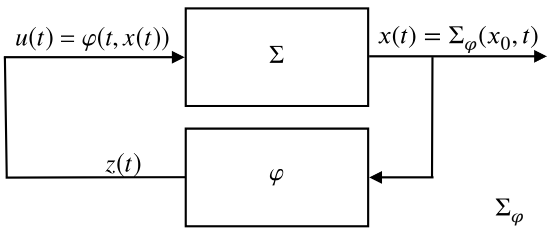

Fig. 1.1. State-feedback configuration

Due to uncertainty about the model of the controlled system Σ, it is not possible to maintain the state $x(t)$ exactly at the target state $x = 0$; some deviation is unavoidable. We use the square of the L^2-norm to quantify the deviation ℓ from the target state $x = 0$; for a specific feedback function φ, this deviation is then

$$\ell(\varphi) := \sup_{t \geq 0} x^\top(t)x(t). \tag{1.1}$$

The infinite time horizon $t \geq 0$ is an important aspect of our study; it implies that the optimal robust state-feedback functions φ we derive also attain, in a certain sense, robust stabilization of the controlled system Σ. The objectives can the be stated as follows.

Problem 1.1.

(i) Characterize the infimal deviation $\ell^* = \inf_\varphi \ell(\varphi)$ and determine under what conditions there are optimal robust state-feedback functions φ^* that achieves the infimal deviation

$$\ell(\varphi^*) = \ell^*. \tag{1.2}$$

(ii) As optimal state-feedback functions may be difficult to implement, find simple-to-implement state-feedback functions that approximate optimal performance. □

We show in Section 6 that optimal robust state-feedback functions φ^* that fulfill the requirements of Problem 1.1*(i)* exist for nonlinear input-affine systems Σ that meet the requirement of constrained controllability (Choi and Hammer (2019b, 2018b), Hammer (2019c)). Constrained controllability is a certain form of controllability: it simply requires that it be possible to drive the nominal system to the target state, without violating control effort constraints. Constrained controllability is close to being a necessary condition for the existence of φ^*.

Regarding Problem 1.1*(ii)*, we show in Section 7 that optimal performance can be approximated by bang-bang state-feedback functions; these are piecewise-constant functions that are easy to implement. The remaining sections of the note introduce background material (Sections 2, 3, 4, and 5); provide an example (Section 8); and a summary (Section 9).

The current note continues the work of Chakraborty and Hammer (2007, 2008a,b,c, 2009a,b, 2010), Chakraborty and Shaikshavali (2009), Yu and Hammer (2016a,b), Choi and Hammer (2019a, 2017c, 2018a, 2017a, 2018b, 2019b) and Hammer (2019b,a,c). It draws on classical investigation in optimal control, including Kelendzheridze (1961), Pontryagin et al. (1962), Gamkrelidze (1965), Neustadt (1966, 1967), Luenberger (1969), Young (1969), Warga (1972), the references cited in these studies, and many others. It seems that the issues examined in this note – existence and implementation of nonlinear optimal robust state-feedback controllers that achieve minimal tracking error over the infinite horizon – have not been previously settled in the literature.

2. SYSTEM MODELS AND PROPERTIES

2.1 The system model

The system Σ of Figure 1.1 is a nonlinear input-affine system given by

$$\Sigma: \quad \begin{aligned} &\dot{x}(t) = a(t, x(t)) + b(t, x(t))u(t), t \geq 0, \\ &x(0) = x_0. \end{aligned} \tag{2.1}$$

Here, $u(t) \in R^m$ is the input and $x(t) \in R^n$ is the state at time t; the initial state x_0 is a member of the ball $\rho(\sigma) := \left\{x \in R^n : |x|_2^2 \leq \sigma\right\}$, where $\sigma > 0$ is given:

$$x_0 \in \rho(\sigma). \tag{2.2}$$

The functions $a : R^+ \times R^n \to R^n$ and $b : R^+ \times R^n \to R^{n\times m}$ are continuously differentiable. Often, we denote the state $x(t)$ by $\Sigma(x_0, u, t)$ to bring forth the dependence on the initial state, the input signal, and the time.

Remark 2.1. Note that by (1.1), Problem 1.1*(i)*, and (2.2), the minimal deviation satisfies $\ell^* \geq \sigma$. □

As modeling uncertainties are a constant presence in practice, we separate the functions a and b of (2.1) into sums

$$\begin{aligned} a(t, x) &= a_0(x) + a_\gamma(t, x), \\ b(t, x) &= b_0(x) + b_\gamma(t, x), \end{aligned} \tag{2.3}$$

where $a_0 : R^n \to R^n$ and $b_0 : R^n \to R^{n\times m}$ are known nominal functions; and $a_\gamma : R^+ \times R^n \to R^n$ and $b_\gamma : R^+ \times R^n \to R^{n\times m}$ are unknown functions representing uncertainties and noise. The *nominal system*

$$\Sigma_0: \quad \begin{aligned} &\dot{x}(t) = a_0(x(t)) + b_0(x(t))u(t), t \geq 0, \\ &x(0) = x_0, \end{aligned} \tag{2.4}$$

is time-invariant; yet, as a_γ and b_γ may be time dependent, the actual system Σ is time-varying. It is convenient to set $a_0(0) = 0$ to simplify some expressions; this has no impact on the nature of the results.

Adopting standard notation, let $|r|$ be the absolute value of a real number r. The L^∞-norm of a matrix $V = (V_{ij}) \in R^{n\times m}$ is then $|V| := \max_{ij} |V_{ij}|$; the L^∞-norm (also called the *amplitude*) of a matrix-valued function $W : R^+ \to R^{n\times m} : t \mapsto W(t)$ is $|W|_\infty := \sup_{t\geq 0} |W(t)|$. The L^2-norm of a vector $x \in R^n$ is $|x|_2 = \left(x^\top x\right)^{1/2}$.

The controllers we develop must respect restrictions on the maximal control effort in which the controlled system Σ may engage. In this regard, we respect the following amplitude bounds on the input u and the state x of the controlled system Σ:

$$|u|_\infty \leq K, \tag{2.5}$$

$$|x|_\infty \leq A. \tag{2.6}$$

To respect the bounds, the domain $\rho(\sigma)$ of the initial state satisfies $\sqrt{\sigma} < A$.

Remark 2.2. Using the fact that the functions a_0, b_0, a_γ, and b_γ are continuously differentiable, it follows by the mean value theorem (e.g., Hubbard and Hubbard (2006)) that, for any states $x, x' \in [-2A, 2A]^n$, the following Lipschitz conditions are valid:

$$\begin{aligned} |a_0(x) - a_0(x')| &\leq \alpha|x - x'|, \ a_0(0) = 0, \\ |b_0(x) - b_0(x')| &\leq \alpha|x - x'|, \ |b_0(0)| \leq \alpha, \\ |a_\gamma(t, x) - a_\gamma(t, x')| &\leq \gamma|x - x'|, \ |a_\gamma(0)| \leq \gamma, \\ |b_\gamma(t, x) - b_\gamma(t, x')| &\leq \gamma|x - x'|, \ |b_\gamma(0)| \leq \gamma, \end{aligned} \tag{2.7}$$

where $\alpha, \gamma > 0$ are specified; we assume that γ, called the *uncertainty parameter*, is a constant. □

Notation 2.3. The class of systems descried by (2.1), (2.3), and (2.7) and associated with the bounds $K, A, \sigma, \gamma > 0$ is denoted by $\mathcal{F}_\gamma(\Sigma_0)$. The following apply.

(i) Input amplitude cannot exceed K.
(ii) State amplitude cannot exceed A.
(iii) All members of $\mathcal{F}_\gamma(\Sigma_0)$ share the initial state $x_0 \in \rho(\sigma)$.
(iv) All members of $\mathcal{F}_\gamma(\Sigma_0)$ share the state-feedback function φ. □

The family $\mathcal{F}_\gamma(\Sigma_0)$ represents uncertainty about the controlled system Σ of Figure 1.1. Therefore, the state-feedback function φ cannot be adjusted individually to each member of $\mathcal{F}_\gamma(\Sigma_0)$; hence item *(iv)* of Notation 2.3.

2.2 The basic framework

Employing the framework of Chakraborty and Hammer (2009b, 2010), denote by $L_2^{\vartheta,m}$ the Hilbert space of Lebesgue measurable functions $v, w : R^+ \to R^m$ with the inner product

$$\langle v, w\rangle := \int_0^\infty e^{-\vartheta s} v^\top(s)w(s)ds,$$

where $\vartheta > 0$. This inner product is bounded for all bounded v and w. By (2.5), the class of input signals of Σ is then

$$U(K) := \left\{u \in L_2^{\vartheta,m} : |u|_\infty \leq K\right\}. \tag{2.8}$$

Similarly, by (2.6), the class of responses of Σ is included in

$$X(A) := \left\{x \in L_2^{\vartheta,n} : |x|_\infty \leq A\right\}.$$

We use a few concepts from mathematical analysis (e.g., Willard (2004), Zeidler (1985)).

Definition 2.4. In a Hilbert space H with inner product $\langle\cdot,\cdot\rangle$:

(i) A sequence $\left\{v_i\right\}_{i=1}^\infty \subseteq H$ *converges weakly* to a member $v \in H$ if $\lim_{i\to\infty} \langle v_i, y\rangle = \langle v, y\rangle$ for every $y \in H$.
(ii) A subset W of H is *weakly compact* if every sequence in W has a subsequence that converges weakly to a member of W. □

We need the following statement taken from Chakraborty and Hammer (2009b, 2010).

Lemma 2.5. The set $U(K)$ of (2.8) is weakly compact in the topology of the Hilbert space $L_2^{\vartheta,m}$. □

For an initial state $x_0 \in \rho(\sigma)$ and a time $\tau > 0$, let $U(x_0, K, A, \gamma, \tau)$ be the set of input signals for which the response of every member of $\mathcal{F}_\gamma(\Sigma_0)$ is bounded by A during the time $[0, \tau]$, i.e.,

$$U(x_0, K, A, \gamma, \tau) := \left\{u \in U(K) : \begin{array}{l} \left|\Sigma(x_0, u, t)\right| \leq A \text{ for all } \Sigma \in \mathcal{F}_\gamma(\Sigma_0) \\ \text{and for all } t \in [0, \tau]. \end{array}\right\} \tag{2.9}$$

Based on the fact that $\sqrt{\sigma} < A$, it can be shown that, for some $\tau > 0$, this class of input signals is not empty (see Hammer (2019c) for details):

Lemma 2.6. Assume that $\sqrt{\sigma} < A$. Then, for every initial state $x_0 \in \rho(\sigma)$, there is a time $\tau > 0$ such that $U(K) = U(x_0, K, A, \gamma, \tau)$. $\square$

2.3 State feedback

A state-feedback function for a system $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ is a Lebesgue measurable function $\varphi : R^+ \times R^n \to R^m$. Employing the framework of Hammer (2019b,a), let $L_2^{\vartheta,n,m}$ be the Hilbert space of measurable functions $v, w : R^+ \times R^n \to R^m$ with the inner product

$$\langle\langle v, w \rangle\rangle := \int_{R^+ \times R^n} e^{-\vartheta(s+|z|_2)} v^\top(s,z) w(s,z) d(s,z),$$

where $\vartheta > 0$. This inner product is bounded for bounded v and w.

As the feedback function generates the input of the controlled system Σ in Figure (1.1), it must be bounded by K. Thus, feedback functions can come only from the family

$$\Phi(K) := \left\{ \varphi \in L_2^{\vartheta,n,m} : |\varphi(t,x)| \le K \text{ for all } (t,x) \in R^+ \times R^n \right\}.$$

The following fact is taken from Hammer (2019b,a).

Lemma 2.7. The set of state-feedback functions $\Phi(K)$ is weakly compact in the topology of the Hilbert space $L_2^{\vartheta,n,m}$. $\square$

When a state-feedback function φ is applied to a system $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$, we obtain the system Σ_φ of Figure 1.1; it is described by

$$\Sigma_\varphi : \begin{array}{l} \dot{x}_\varphi(t) = a(t, x_\varphi(t)) + b(t, x_\varphi(t))\varphi(t, x_\varphi(t)), \\ x_\varphi(0) = x_0. \end{array}$$

We use $\Sigma_\varphi(x_0, t)$ to refer to $x_\varphi(t)$.

2.4 Admissible feedback functions

Denote by $\Phi(x_0, K, A, \Sigma, \tau)$ the family of state-feedback functions that keep the closed-loop response bounded by A during the time $[0, \tau]$ for the system Σ, namely,

$$\Phi(x_0, K, A, \Sigma, \tau) = \left\{ \varphi \in \Phi(K) : |\Sigma_\varphi(x_0, t)| \le A \text{ for all } t \in [0, \tau] \right\}.$$

Feedback functions that keep the closed-loop response bounded by A at all times $t \ge 0$ are members of the family

$$\Phi(x_0, K, A, \Sigma) = \left\{ \varphi \in \Phi(K) : \sup_{t \ge 0} |\Sigma_\varphi(x_0, t)| \le A \right\}.$$

As the family $\mathcal{F}_\gamma(\Sigma_0)$ describes uncertainty, it is not possible to adjust the feedback function separately to each member $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$. Thus, we can use only feedback functions that keep the response bounded by A for (almost) all members, namely, feedback functions of the family

$$\Phi(x_0, K, A, \gamma) = \left\{ \varphi \in \Phi(K) : \operatorname*{ess\,sup}_{\Sigma \in \mathcal{F}_\gamma(\Sigma_0),\, t \ge 0} |\Sigma_\varphi(x_0, t)| \le A \right\}.$$

To encompass every initial condition $x_0 \in \rho(\sigma)$, we must confine to the family of feedback functions

$$\Phi(\sigma, K, A, \gamma) = \left\{ \varphi \in \Phi(K) : \operatorname*{ess\,sup}_{\Sigma \in \mathcal{F}_\gamma(\Sigma_0),\, x_0 \in \rho(\sigma),\, t \ge 0} |\Sigma_\varphi(x_0, t)| \le A \right\}. \tag{2.10}$$

If we are interested only in a finite interval of time $[0, \tau]$, $\tau > 0$, then the corresponding class of feedback functions is

$$\Phi(\sigma, K, A, \gamma, \tau) = \left\{ \varphi \in \Phi(K) : \operatorname*{ess\,sup}_{\Sigma \in \mathcal{F}_\gamma(\Sigma_0),\, x_0 \in \rho(\sigma),\, t \in [0,\tau]} |\Sigma_\varphi(x_0, t)| \le A \right\}.$$

In Section 4, we show that $\Phi(\sigma, K, A, \gamma)$ and $\Phi(\sigma, K, A, \gamma, \tau)$ are not empty, if the nominal system Σ_0 has a certain controllability property.

Now, feedback functions in $\Phi(K)$ create input signals in $U(K)$, and, conversely, an input signal $u \in U(K)$ is produced by a 'pseudo' feedback function $\varphi(t, x) = u(t) \in \Phi(K)$. On this account, we obtain the following consequence of Lemma 2.6, which shows that $\Phi(\sigma, K, A, \gamma, \tau)$ is not always empty.

Lemma 2.8. Assume that $\sqrt{\sigma} < A$. Then, there is a time $\tau > 0$ such that $\Phi(K) = \Phi(\sigma, K, A, \gamma, \tau)$. $\square$

3. FORMAL PROBLEM STATEMENT

Referring to Figure 1.1, recall that our intent is to find a state-feedback function that minimizes the deviation $\ell = x^\top x$ of the closed-loop system over all times $t \ge 0$. Recall also that, by construction, only feedback functions belonging to the family $\Phi(\sigma, K, A, \gamma)$ of (2.10) can be used. For a function $\varphi \in \Phi(\sigma, K, A, \gamma)$, the supremal deviation is

$$\ell(\sigma, K, A, \gamma, \varphi) = \operatorname*{ess\,sup}_{\Sigma \in \mathcal{F}_\gamma(\Sigma_0),\, x_0 \in \rho(\sigma),\, t \ge 0} \left| \Sigma_\varphi(x_0, t) \right|_2^2. \tag{3.1}$$

When $\Phi(\sigma, K, A, \gamma) \ne \emptyset$, the potentially minimal deviation over all permissible state-feedback functions is

$$\ell^*(\sigma, K, A, \gamma) = \inf_{\varphi \in \Phi(\sigma, K, A, \gamma)} \ell(\sigma, K, A, \gamma, \varphi). \tag{3.2}$$

The question we are facing is: under what conditions is there an optimal robust state-feedback function $\varphi^* \in \Phi(\sigma, K, A, \gamma)$ that achieves this infimum:

$$\ell(\sigma, K, A, \gamma, \varphi^*) = \ell^*(\sigma, K, A, \gamma). \tag{3.3}$$

In these terms, Problem 1.1 can be restated as follows.

Problem 3.1. Refer to (3.2) and (3.3).

(i) Under what conditions is there an optimal robust state-feedback function φ^*?

(ii) Are there simple-to-implement state-feedback functions that approximate optimal performance? $\square$

Answers to the questions of Problem 3.1*(i)* and *(ii)* are provided, respectively, in Sections 6 and 7 below. We show that the existence of φ^* is guaranteed by a certain controllability requirement on the nominal controlled system Σ_0, while bang-bang state-feedback functions (Hammer (2019b,a) provide effective and easy to implement approximations of optimal performance.

4. CONSTRAINED CONTROLLABILITY

The existence of optimal robust state-feedback functions depends on the following variant of the notion of constrained controllability (Choi and Hammer (2017b), Choi and Hammer (2019b, 2018b), Hammer (2019c).

Definition 4.1. The system Σ is (K, A, σ)*-controllable* if there are times $\tau_2 > \tau_1 > 0$ and a positive number $\sigma' < \sigma$ for which the following holds: for every state $x \in \rho(\sigma)$, there is a time $t_x \in [\tau_1, \tau_2]$ and an input signal $u_x \in U(K)$ such that $\Sigma(x, u_x, t_x) \in \rho(\sigma')$, and $\left|\Sigma(x, u_x, t)\right| \le A$ for all $t \in [0, t_x]$. $\square$

We show later that (K, A, σ)-controllability guarantees the existence of optimal robust state-feedback functions φ^* that fulfill the requirements of Problem 3.1*(i)*. A slight reflection shows that, when omitting the contractive requirement $\sigma' < \sigma$, the remaining parts of Definition 4.1 are, in fact, necessary conditions for the existence of φ^*. Thus, (K, A, σ)-controllability is a tight sufficient condition for the existence of optimal robust state-feedback functions φ^*.

4.1 Model uncertainty

Recall that the family $\mathcal{F}_\gamma(\Sigma_0)$ describes modeling uncertainty of the controlled system Σ of Figure 1.1. The next statement, which is taken from Choi and Hammer (2017b), shows that the entire family $\mathcal{F}_\gamma(\Sigma_0)$ is (K, A, σ)-controllable, if so is the nominal system Σ_0 (as long as the uncertainty parameter γ is not too large).

Proposition 4.2. Let the nominal system Σ_0 be (K, A_0, σ)-controllable and assume that $\sqrt{\sigma} < A_0$. Then, for every $A > A_0$, there is an uncertainty parameter $\gamma > 0$ for which all members of the family $\mathcal{F}_\gamma(\Sigma_0)$ are (K, A, σ)-controllable. □

The next statement states that, in case the nominal system Σ_0 is (K, A, σ)-controllable, the class of permissible feedback functions $\Phi(\sigma, K, A, \gamma)$ is not empty.

Proposition 4.3. Assume that $\sqrt{\sigma} < A$ and that the nominal system Σ_0 is (K, A, σ)-controllable. Then, there are an uncertainty parameter $\gamma > 0$ and a bound $A' > A$ for which $\Phi(\sigma, K, A', \gamma)$ is not empty.

Proof (sketch). Let $x_0 \in \rho(\sigma)$ be an initial state. The definition of (K, A, σ)-controllability implies the existence of an input $u_{x_0} \in U(K)$, $t_1 \in [\tau_1, \tau_2]$, and an number $\sigma' \in (0, \sigma)$ such that $x(t_1) \in \rho(\sigma')$ and $|\Sigma_0(x_0, u_{x_0}, t)| \le A$ for all $t \in [0, t_1]$. As $\rho(\sigma') \subset \rho(\sigma)$, the same argument implies the existence of $u_{x(t_1)} \in U(K)$ and $t_2 \in [t_1 + \tau_1, t_1 + \tau_2]$, $t_2 \ge t_1 + \tau_1 \ge 2\tau_1$, satisfying $x(t_2) \in \rho(\sigma')$ and $|\Sigma_0(x(t_1), u_{x(t_1)}, t)| \le A$ for all $t \in [t_1, t_2]$. Using the time-invariance of Σ_0, we can continue in this way for every integer $k \ge 2$ to obtain an input signal $u_{x(t_{k-1})} \in U(K)$ and a time $t_k \ge k\tau_1$ satisfying $x(t_k) \in \rho(\sigma')$ and $|\Sigma_0(x(t_{k-1}), u_{x(t_{k-1})}, t)| \le A$ for all $t \in [t_{k-1}, t_k]$. Setting $t_0 := 0$, we obtain a state-feedback function

$$\varphi(t, x(t)) := u_{x(t_{k-1})}(t), t \in [t_{k-1}, t_k), k = 1, 2, \ldots \tag{4.1}$$

This state feedback functions keeps the nominal system bounded by A at all times. It can then be shown that, for every $\varepsilon > 0$, there is an uncertainty parameter $\gamma > 0$ such that $\varphi \in \Phi(\sigma, K, A', \gamma)$ for $A' = A + \varepsilon$ (see Hammer (2019c) for details). □

We list now two properties of the set of inputs $U(x_0, K, A, \gamma, \tau)$ of (2.9). These properties relate to continuity and compactness; they are critical to proving the existence of optimal robust state-feedback functions.

Theorem 4.4. *(i)* Let $x_0 \in \rho(\sigma)$ and $\tau > 0$, and assume that the sequence of inputs $\{u_i\}_{i=1}^\infty \subseteq U(x_0, K, A, \gamma, \tau)$ converges weakly to the input u. Then, for any system $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$, the sequence $\{\Sigma(x_0, u_i, t)\}_{i=1}^\infty$ converges to $\Sigma(x_0, u, t)$ uniformly over $[0, \tau]$.
(ii) $U(x_0, K, A, \gamma, \tau)$ is a weakly compact set.

Proof (sketch). Considering Remark 2.2, Part *(i)* was proved in Yu and Hammer (2016a). Regarding Part *(ii)*, it follows by Lemma 2.5 that $u \in U(K)$. By Part *(i)*,

$$\lim_{j\to\infty} \Sigma(x_0, u_i, t) = \Sigma(x_0, u, t) \text{ for all } t \in [0, \tau]. \tag{4.2}$$

As $\Sigma(x_0, u_i, t) \in [-A, A]^n$ for all $i = 1, 2, \ldots$, it follows by the compactness of $[-A, A]^n$ that $\Sigma(x_0, u, t) \in [-A, A]^n$ as well, so that $u \in U(x_0, K, A, \gamma, \tau)$. Hence, $U(x_0, K, A, \gamma, \tau)$ is weakly compact. □

5. STATE-FEEDBACK AS A PRECOMPENSATOR

It is generally recognized that, from an input/output perspective, feedback controllers can be represented as precompensators (e.g., Hammer (1984, 1986)). Needless to say, feedback has many advantages over precompensation: if properly designed, feedback can lessen the impact of modeling uncertainties and disturbances, and it can stabilize unstable systems. Nonetheless, representing feedback controllers as precompensators is sometimes instrumental in simplifying mathematical expressions.

5.1 External input and precompensation

Figure 5.1 describes a feedback configuration with an external input signal $w(t)$. The controlled system is $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ and the additive state-feedback function is φ_i; the closed-loop system is marked by $\Sigma_{(+)\varphi_i}$.

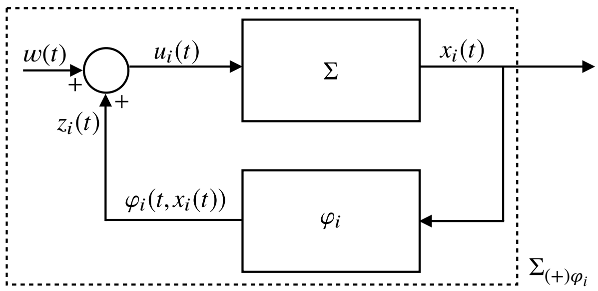

Fig. 5.1. Feedback with external input

Denoting by $\circ$ system composition, Figure 5.1 indicates the following relations:

$$\begin{aligned} z_i &= \varphi_i \circ \Sigma u_i, \\ x_i &= \Sigma u_i, \\ u_i &= w + z_i. \end{aligned} \tag{5.1}$$

Denoting by I the identity, these relations yield

$$(I - \varphi_i \circ \Sigma) u_i = w. \tag{5.2}$$

The next statement shows that $(I - \varphi_i \circ \Sigma)$ is a set isomorphism.

Proposition 5.1. When all signals in the configuration of Figure 5.1 are bounded, then $(I - \varphi_i \circ \Sigma)$ is an isomorphism in measure.

Proof (sketch). The continuous-time system of Figure 5.1 is approximated by a discrete-time system through sampling; then, the result follows from known facts about discrete-time systems (e.g., Hammer (1986)). For a detailed proof, see Hammer (2019c). □

Using Proposition 5.1, equation (5.2) can be rewritten as

$$u_i = (I - \varphi_i \circ \Sigma)^{-1} w. \tag{5.3}$$

Together with (5.1), we obtain

$$x_i = \Sigma u_i = \Sigma \circ (I - \varphi_i \circ \Sigma)^{-1} w.$$

As $x_i = \Sigma_{(+)\varphi_i} w$ in Figure 5.1, we get the input/output relation

$$\Sigma_{(+)\varphi_i} = \Sigma \circ (I - \varphi_i \circ \Sigma)^{-1}, \tag{5.4}$$

showing that the input/output effect of the state feedback function φ_i is equivalent to the effect of the precompensator $(I - \varphi_i \circ \Sigma)^{-1}$.

5.2 Feedback and continuity

In this subsection we show that the response of the closed-loop system Σ_φ of Figure 1.1 depends in a continuous way on the state-feedback function φ. We need the following notions from mathematical analysis (e.g., Zeidler (1985), Willard (2004)).

Definition 5.2. Let H be a Hilbert space, S a subset of H, and z a point of S.

(i) A functional $F : S \to R$ is *weakly lower semi-continuous* at z if, for every sequence $\left\{z_i\right\}_{i=1}^{\infty} \subseteq S$ that converges weakly to z and for every $\varepsilon > 0$, there is an integer $N > 0$ such that $F(z) - F(z_i) < \varepsilon$ for all $i \geq N$.

(ii) A function $G : S \to R^n$ is *weakly continuous* at z if, for every $\varepsilon > 0$, there is an integer $N > 0$ such that $\left|G(z) - G(z_i)\right| < \varepsilon$ for all $i \geq N$. □

In the process of examining continuity of Σ_{φ_i}, we consider first the system $[I - \varphi_i \circ \Sigma]$ of (5.2).

Lemma 5.3. Assume that all signals in the configuration of Figure 5.1 are bounded. If $\left\{\varphi_i\right\}_{i=1}^{\infty} \subseteq \Phi(K)$ is a sequence of state-feedback functions that converges weakly to the state-feedback function φ, then the sequence $\left\{[I - \varphi_i \circ \Sigma]\right\}_{i=1}^{\infty}$ converges weakly to $[I - \varphi \circ \Sigma]$ for almost every $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$.

Proof (sketch). Let $g \in L_2^{\vartheta,n,m}$ be a function. The inner products $\langle g \circ \Sigma u, \varphi_i \circ \Sigma u\rangle$, $\langle g \circ \Sigma u, \varphi \circ \Sigma u\rangle$, and $\langle g \circ \Sigma u, (\varphi_i - \varphi) \circ \Sigma u\rangle$ are restrictions to Σu of the inner products $\langle\langle g, \varphi_i\rangle\rangle$, $\langle\langle g, \varphi\rangle\rangle$, and $\langle\langle g, (\varphi_i - \varphi)\rangle\rangle$, respectively. Consequently, since $\langle\langle g, (\varphi_i - \varphi)\rangle\rangle$ converges to 0 as $i \to \infty$, the sequence $\langle g \circ \Sigma u, (\varphi_i - \varphi) \circ \Sigma u\rangle$ converges to 0 as $i \to \infty$ for almost every $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$. This shows that $\left\{\varphi_i \circ \Sigma u\right\}_{i=1}^{\infty}$ converges weakly to $\varphi \circ \Sigma u$ for almost every $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$. As the addition of the identity does not impair convergence, the lemma follows. □

The next statement shows that the signal u of Figure 1.1 depends continuously on the state-feedback function φ. A similar statement was derived in Hammer (2019b,a) via a different approach. Note that the configuration of Figure 5.1 reduces to the configuration of Figure 1.1 upon setting $w = 0$.

Lemma 5.4. Referring to Figure 1.1, assume that all signals are bounded. Then, u is a weakly continuous function of $\varphi \in \Phi(K)$ for almost every $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$.

Proof The sequence $u_i = [I - \varphi_i \circ \Sigma]^{-1} w$, $i = 1, 2, \ldots$, of (5.3) is bounded by the lemma's assumption, say $u_i \in U(B)$, $i = 1, 2, \ldots$, $B > 0$. By Lemma 2.5, this sequence has a weakly convergent subsequent $\{u_{i_k}\}_{k=1}^{\infty}$ that converges weakly, say, to $u \in U(B)$. For an integer $j \in \{1, 2, \ldots\}$, introduce the set $S_j := \cup_{k \geq j} u_{i_k} = \cup_{k \geq j} [I - \varphi_{i_k} \circ \Sigma]^{-1} w$. Recalling that u is the weak limit of $\{u_{i_k}\}_{k=1}^{\infty}$, the weak closure $\bar{S}_j$ of S_j satisfies

$$\cap_{j \geq p} \bar{S}_j = \{u\}, p = 1, 2, \ldots \tag{5.5}$$

Let $q \geq j$ be an integer and define the set $V_{q,j} := [I - \varphi_{i_q} \circ \Sigma] \bar{S}_j$. In view of Lemma 5.3, the weak limit $\lim_{q \to \infty} V_{q,j}$ $= [I - \varphi \circ \Sigma] \bar{S}_j$ for almost every $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$. As the external input signal w is fixed, $w = [I - \varphi_{i_k} \circ \Sigma] u_{i_k}$ for all $k = 1, 2, \ldots$ Therefore, $w \in [I - \varphi_{i_q} \circ \Sigma] \bar{S}_j$ for all $q \geq j$, and we obtain

$$w \in [I - \varphi \circ \Sigma] \bar{S}_j. \tag{5.6}$$

Invoking Proposition 5.1 in combination with (5.6) and (5.5) yields

$$\begin{aligned} w &\in \cap_{j \geq 1} [I - \varphi \circ \Sigma] \bar{S}_j \\ &= [I - \varphi \circ \Sigma] \left[\cap_{j \geq 1} \bar{S}_j\right] \\ &= [I - \varphi \circ \Sigma] \{u\}, \end{aligned}$$

so that $w = [I - \varphi \circ \Sigma] u$ for almost every $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$. Proposition 5.1 then shows that $u = [I - \varphi \circ \Sigma]^{-1} w$, so that u is the weak limit of the sequence $[I - \varphi_{i_k} \circ \Sigma]^{-1} w$ for almost every $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$. □

The class of feedback functions $\Phi(\sigma, K, A', \gamma)$ is not empty under the conditions of Proposition 4.3; combining this with Lemma 5.4 yields the following (see also Hammer (2019a,c)).

Theorem 5.5. Let Σ_φ be the closed-loop system of Figure 1.1. Under the conditions of Proposition 4.3, Σ_φ is a weakly continuous function of $\varphi \in \Phi(\sigma, K, A', \gamma)$ for almost every $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$. □

The existence of optimal state-feedback functions hinges on the following property of compactness (compare to Hammer (2019c)).

Lemma 5.6. The family $\Phi(\sigma, K, A, \gamma)$ of (2.10) is weakly compact in the topology of the Hilbert space $L_2^{\vartheta,n,m}$.

Proof (sketch). Given a sequence $\{\varphi_k\}_{k=1}^{\infty} \subseteq \Phi(\sigma, K, A, \gamma)$, it follows by Lemma 2.7 that there is a subsequence $\{\varphi_{k_i}\}_{i=1}^{\infty}$ that converges weakly to a function $\varphi \in \Phi(K)$. Then, by Theorem 5.5, we have $\lim_{i \to \infty} \Sigma_{\varphi_{k_i}}(x_0, t) = \Sigma_\varphi(x_0, t)$. The compactness of $[-A, A]^n$ then implies that $\Sigma_\varphi(x_0, t) \in [-A, A]^n$, so that $\varphi \in \Phi(\sigma, K, A, \gamma)$. □

6. EXISTENCE OF OPTIMAL SOLUTIONS

We use the following facts from mathematical analysis (e.g., Zeidler (1985), Willard (2004)).

Theorem 6.1.

(i) A weakly continuous functional is weakly lower semi-continuous.

(ii) Let S and A be topological spaces. Assume that, for every member $a \in A$, there is a weakly lower semi-continuous functional $f_a : S \to R$. If $\sup_{a \in A} f_a(s)$ exists at every point $s \in S$, then the functional $f(s) := \sup_{a \in A} f_a(s)$ is weakly lower semi-continuous on S.

(iii) The Generalized Weierstrass Theorem: A weakly lower semi-continuous functional attains a minimum in a weakly compact set. □

Over the time interval $[0, \tau]$, the supremal deviation of Σ_φ from the target state $x = 0$ is

$$\ell(\sigma, K, A, \gamma, \varphi, \tau) := \operatorname*{ess\,sup}_{\Sigma \in \mathcal{F}_\gamma(\Sigma_0),\, x_0 \in \rho(\sigma),\, t \in [0, \tau]} \left|\Sigma_\varphi(x_0, t)\right|_2^2. \tag{6.1}$$

As we plan to use infinite time, we utilize only state-feedback functions φ belonging to the family $\Phi(\sigma, K, A, \gamma)$. Then, the infimal deviation over such state-feedback functions is

$$\ell^*(\sigma, K, A, \gamma, \tau) = \inf_{\varphi \in \Phi(\sigma, K, A, \gamma)} \ell(\sigma, K, A, \gamma, \varphi, \tau). \tag{6.2}$$

Theorems 5.5 and 6.1*(ii)* imply the following.

Corollary 6.2. Refer to (3.1) and (6.1). As a function of state-feedback functions $\varphi \in \Phi(\sigma, K, A, \gamma)$, the functionals $\ell(\sigma, K, A, \gamma, \varphi)$ and $\ell(\sigma, K, A, \gamma, \varphi, \tau)$ are weakly lower semi-continuous. □

As the set $\Phi(\sigma, K, A, \gamma)$ is weakly compact by Lemma 5.6, the existence of an optimal solution for Problem 3.1*(i)* follows by

Corollary 6.2 and Theorem 6.1*(iii)*; this leads us to the following (compare to Hammer (2019c)).

Theorem 6.3. Under the conditions of Proposition 4.3, there are optimal robust state-feedback functions $\varphi_\tau^*, \varphi^* \in \Phi(\sigma, K, A', \gamma)$ that implement the minimal deviations $\ell(\sigma, K, A', \gamma, \varphi_\tau^*, \tau) = \ell^*(\sigma, K, A', \gamma, \tau)$ and $\ell(\sigma, K, A', \gamma, \varphi^*) = \ell^*(\sigma, K, A', \gamma)$, respectively. □

Thus, the existence of optimal robust state-feedback functions is assured by (K, A, σ)-controllability of the nominal controlled system Σ_0; the uncertainty parameter γ should not be too large.

7. APPROXIMATION BY BANG-BANG FUNCTIONS

Optimal solutions are often difficult to implement. In this section we use bang-bang feedback functions – piecewise constant functions whose components switch between $-K$ and K – to approximate optimal performance (Hammer (2019b)). Bang-bang functions are easier to calculate and implement than actual optimal functions.

7.1 A disturbance

The errors created by approximating feedback functions must be considered in the context of other disturbances and noise signals that affect the closed-loop system of Figure 1.1. Of particular interest in the present context is the disturbance signal v of Figure 7.1; it affects the input of the state-feedback controller. We let $\Delta > 0$ be the amplitude bound of the signal v, so that v is a member of the family

$$V(\Delta) := \left\{ v \in L_2^{\vartheta,m} : |v|_\infty \le \Delta \right\}.$$

We assume that members of $V(\Delta)$ have a uniform probability distribution. Denoting by $\Delta(x)$ the hyper-square of edge 2Δ centered at a state x, the average signal produced by a state-feedback function φ at a state x in the presence of the disturbance v is

$$\bar{\varphi}(t, x) := \frac{1}{(2\Delta)^n} \int_{\Delta(x)} \varphi(t, z) dz.$$

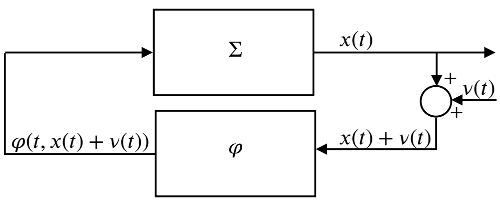

Fig. 7.1. A disturbance signal $v(t)$

7.2 Bang-bang functions approximate optimal performance

A bang-bang state-feedback function for a system $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ is a function : $R^+ \times R^n \to R^m$, whose components switch between $-K$ and K; here, K is the input amplitude bound of Σ. According to Hammer (2019b), the performance of any state-feedback function can be approximated by a bang-bang state-feedback function. Combining this fact with Theorem 6.3, which states that optimal state-feedback functions exist, we obtain the following statement (compare to Hammer (2019c)).

Theorem 7.1. Let the nominal system Σ_0 be (K, A_0, σ)- controllable and let $\varepsilon > 0$ be a real number. Then, there are a bang-bang state-feedback function $\varphi_\pm \in \Phi(\sigma, K, A, \gamma)$ and an uncertainty parameter $\gamma > 0$ satisfying $|\ell(\sigma, K, A, \gamma, \varphi_\pm, \tau) - \ell^*(\sigma, K, A_0, \gamma, \tau)| < \varepsilon$, where $A = A_0 + \varepsilon$. (The feedback signal is averaged over the disturbance signal $v(t)$ of Figure 7.1). □

Thus, bang-bang feedback functions can be used to approximate optimal performance. This simplifies implementation to a significant extent.

7.3 Practical aspects

Bang-bang state-feedback functions that operate over long periods of time may be prone to inducing jitter in the control loop. One way to avoid such jitter is to 'soften' the bang-bang state-feedback function by implementing gradual transitions, instead of the abrupt transitions of a true bang-bang function. As an example, consider a bang-bang state-feedback function φ, whose j-th component φ^j switches from K to $-K$ at the value x_s^i of the i-th state coordinate x^i. Then, this switching can be replace by the gradual transitions

$$\varphi^j := \begin{cases} K & x^i < x_s^i - \delta, \\ K \dfrac{x_s^i - x^i}{\delta} & x^i \in [x_s^i - \delta, x_s^i + \delta], \\ -K & x^i > x_s^i + \delta, \end{cases} \tag{7.1}$$

where $\delta > 0$ is a real number. This modification eliminates jitter. Of course, other remedies can also be used to eliminate jitter. Regardless, 'true' bang-bang functions with abrupt switchings can be used during the process of calculating bang-bang state-feedback functions that approximate optimal performance. Once such functions have been derived, they can be 'softened' prior to implementation.

8. EXAMPLE

Inverted pendulums are common in many engineering applications, including missiles guidance and control, walking robots, and others. The following is a model of a non-linearized inverted pendulum:

$$\Sigma : \begin{array}{l} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = d_1 \sin x_1(t) + d_2 x_2(t) + d_3 \tanh u(t); \end{array} \tag{8.1}$$

here, d_1, d_2, and d_3 are constants with nominal values of $d_1^0 = 12$, $d_2^0 = -0.1$, $d_3^0 = 7$ and uncertainty ranges of $d_1 \in [11.5, 12.5]$, $d_2 \in [-0.105, -0.095]$, and $d_3 \in [6.7, 7.3]$ (approximately 5% uncertainty). The input amplitude bound is $K = 1$, the state amplitude bound is $A = 0.5$, and the initial state domain is $\rho(\sigma)$ with $\sigma = 0.2$.

In some cases, such as the present case, a bang-bang state feedback function $\varphi_\pm$ that approximates optimal performance can be found through insight into the dynamics of the controlled system Σ. Such consideration yields the following bang-bang function (which has been 'softened'):

$$\varphi_\pm = \begin{cases} 1 & \text{if } x_1 \le 0 \text{ and } x_2 < 0.19, \\ -1 & \text{if } x_1 \le 0 \text{ and } x_2 > 0.21, \\ 1 - 100(x_2 - 0.19) & \text{if } x_1 \le 0 \text{ and } x_2 \in [0.19, 0.21], \\ 1 & \text{if } x_1 > 0 \text{ and } x_2 < -0.21, \\ -1 & \text{if } x_1 > 0 \text{ and } x_2 > -0.19, \\ -1 - 100(x_2 + 0.19) & \text{if } x_1 > 0 \text{ and } x_2 \in [-0.21, -0.19]; \end{cases} \tag{8.2}$$

this function is depicted in Figure 8.2 over the time domain $[0, 0.5]$. The response of the closed-loop system with this state-feedback function is shown in Figure 8.1 for the initial state $x_0 = (-0.3, -0.3)^\top \in \rho(0.2)$ and the time interval $[0, 0.5]$. Note that at the end of this time interval, $x(0.5) \in \rho(0.2)$. Therefore, based on the time invariance of Σ, this feedback function can be extended to the entire time axis in 0.5 steps, using the time intervals $[0.5, 1], [1, 1.5], [1.5, 2], \ldots$, as described in the proof of Proposition 4.3. The figure shows that the maximal deviation from the origin for this initial state is $\ell = \max_{t\in[0,0.5]}\{x_1^2(t) + x_2^2(t)\} = 0.18$, equal to the deviation of the initial state x_0. Thus, this is the minimal achievable deviation in this case.

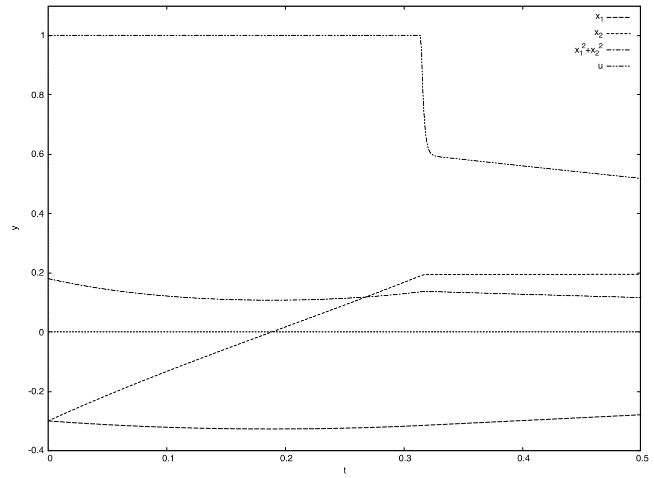

Fig. 8.1. The response

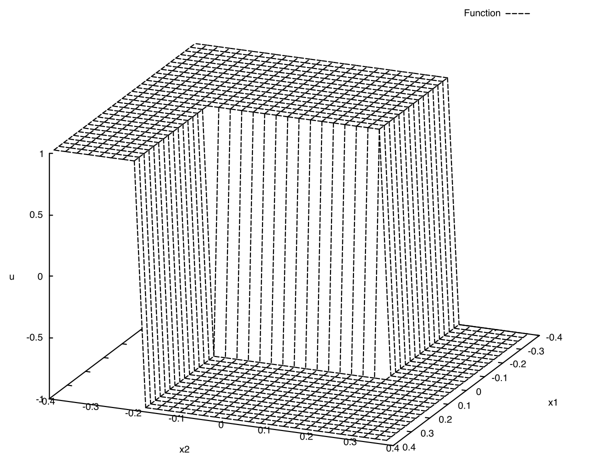

Fig. 8.2. State-feedback function

8.1 A search process

When qualitative considerations do not yield an appropriate bang-bang state-feedback function, the following numerical search process can be used.

Step 1: Use a numerical search process to verify that the controlled system Σ is (K, A, σ)-controllable, as described in Choi and Hammer (2019a). This process also yields a time $\tau > 0$ by which all initial states in $\rho(\sigma)$ can be steered into $\rho(\sigma')$ for some $\sigma' < \sigma$.

Step 2: Choose integers $N, N' \geq 1$ for which τ/N and $2A/N'$ are sufficiently small to satisfy implementation accuracy requirements (recall that states are bounded by A).

Step 3:

(i) Partition the interval $[0, \tau]$ into N sub-intervals of length τ/N: $[0, \tau/N]$, $[\tau/N, 2\tau/N]$, …, $(N-1)\tau/N, \tau]$.

(ii) Partition each state coordinate interval $[-A, A]$ into N' sub-intervals of length $2A/N'$: $[-A, -A + 2A/N']$, $[-A + 2A/N', -A + 4A/N']$, …, $[-A + 2(N'-1)A/N', A]$.

(iii) Denote by C the resulting set of $N(N')^n$ points.

Step 4: Use the following sub-steps to search for bang-bang state-feedback functions φ with an increasing number of switching points in C that satisfy the requirements

$$\sup_{t\in[0,\tau],x_0\in\rho(\sigma)} \left|\Sigma_\varphi(x_0, t)\right| \leq A \quad \text{and} \quad \sup_{x_0\in\rho(\sigma)} \left|\Sigma_\varphi(x_0, \tau)\right|_2^2 \leq \sigma' \tag{8.3}$$

for a $\sigma' < \sigma$.

(a) Starting with $k = 0$ switching points, and increasing the number of switching points in steps $k = 1, 2, \ldots$, search bang-bang state-feedback functions with k switchings in C, testing all possible selections of k switching points in C. Eliminate all bang-bang state-feedback functions that violate (8.3). Among the remaining bang-bang functions, let ℓ_k be the minimal deviation of the closed-loop system's state from $x = 0$ over the time $[0, \tau]$, and let φ_k be a bang-bang state-feedback function that achieves ℓ_k.

(b) The search ends when increasing k above a value k' does not reduce the deviation below $\ell_{k'}$. Then, $\varphi_{k'}$ can serve as a bang-bang state-feedback function. □

9. CONCLUSION

The note proves the existence of optimal robust state-feedback functions that minimize tracking error over the infinite horizon for nonlinear input-affine systems. It also shows that optimal performance can be approximated by bang-bang state-feedback functions – functions that are relatively easy to design and implement.

High accuracy tracking is a design goal in many applications of modern control engineering, ranging from missile guidance to power distribution systems to personal transport systems. Future research will concentrate on the development of fast algorithms for the design of bang-bang state-feedback functions.

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