INTERNATIONAL JOURNAL OF CONTROL
2022, VOL. 95, NO. 4, 1070–1085
https://doi.org/10.1080/00207179.2020.1839790

Taylor & Francis
Taylor & Francis Group

Optimal robust tracking by state feedback: infinite horizon

Jacob Hammer

Department of Electrical and Computer Engineering, University of Florida, Gainesville, FL, USA

ABSTRACT
The existence and implementation of optimal robust state-feedback controllers that achieve optimal tracking over the infinite horizon is considered. The objective is to minimise over all times the deviation of a nonlinear system from a specified target state. It is shown that optimal robust state-feedback controllers that achieve this objective exist for a broad class of nonlinear input-affine systems. It is also shown that optimal performance can be approximated by state-feedback controllers that are relatively easy to design and implement.

ARTICLE HISTORY
Received 29 October 2019
Accepted 17 October 2020

KEYWORDS
Optimal control; tracking; nonlinear control; infinite horizon; optimal feedback

1. Introduction

Tracking over the infinite horizon, namely, keeping a system's response close to a specified target at all times, is a common objective in many applications of modern control engineering. Examples of applications where tracking is a central requirement include airplane autopilot systems, missile guidance systems, control systems for self-driving cars, controllers for personal transport systems, and many others. Tracking systems are also critical in biomedical applications, such as the automatic control of blood glucose concentration for diabetes patients, or the automatic control of heart pacers for heart disease patients.

The present paper develops optimal robust state-feedback controllers that guide a controlled system to track a target state as closely as possible, while facing modelling uncertainties, disturbances, and constraints on the controlled system's input and state amplitudes. The discussion applies to nonlinear input-affine systems. Special attention is devoted to implementation; the paper provides a relatively simple methodology for the design and the implementation of state-feedback controllers that approximate optimal performance.

The controllers considered in this paper are optimal robust state-feedback controllers represented by a state-feedback function φ, as depicted in Figure 1. The controller's task is to maintain the state of the controlled system Σ as close as possible to a specified target state x_{target} at all times $t \geq 0$, while Σ faces modelling uncertainties, disturbances, and constraints on maximal control effort. By appropriately shifting the state coordinates of Σ, we can assume that x_{target} is the zero state $x = 0$, namely, the origin of state space.

In Figure 1, the state of the controlled system Σ at a time t is $x(t)$. The input signal $u(t)$ of Σ is generated by the state-feedback function φ, so that $u(t) = \varphi(t, x(t))$. The closed-loop system is denoted by Σ_φ. The configuration of Figure 1 is a pure state-feedback configuration – no external input signal is necessary. The response $x(t)$ of the closed-loop system is determined by the initial state x_0 of Σ and by the state-feedback function φ. We denote this response by $\Sigma_\varphi(x_0, t)$. Our objective is to find state-feedback functions φ that keep the response $\Sigma_\varphi(x_0, t)$ as close as possible to the target state $x = 0$ at all times $t \geq 0$, while Σ is subject to modelling uncertainties, disturbances, and constraints on its input and state amplitudes.

The controlled system Σ is a nonlinear input-affine system, whose model is not precisely specified. To represent control effort constraints commonly faced by engineers in the field, we impose a bound $K > 0$ on the amplitude of the input signal of Σ and a bound $A > 0$ on the amplitude of the state signal of Σ. Violating these constraints endangers the integrity of the controlled system Σ.

The state-feedback functions φ derived in this paper respect all constraints imposed by the controlled system Σ. In addition, these feedback functions are robust, namely, they achieve the control objective as best as possible, in the face of errors and disturbances that may affect the controlled system Σ.

Uncertainties inherent in the structure, the operation, and the environment of practical control systems make it impossible to maintain the target state $x = 0$ with absolute precision; some deviation from the target state is unavoidable. Our objective is to design optimal robust state-feedback functions that achieve the smallest possible deviation from the target state, despite modelling uncertainties, disturbances, and constraints that affect the controlled system Σ. Specifically, for a state-feedback function φ, let $\ell(\varphi)$ be the maximal deviation of the closed-loop system's state from the target state $x = 0$ over all times $t \geq 0$. Our goal is to find optimal robust state-feedback functions φ^* that achieve the smallest possible deviation $\ell^* = \inf_\varphi \ell(\varphi)$. In other words, we look for optimal state-feedback functions φ^* that satisfy the relation

$$\ell(\varphi^*) = \ell^*. \tag{1.1}$$

An important aspect of our discussion is the consideration of the infinite time horizon: we require the closed-loop system

CONTACT Jacob Hammer hammer@mst.ufl.edu Department of Electrical and Computer Engineering, University of Florida, Gainesville, FL 32611-6130, USA

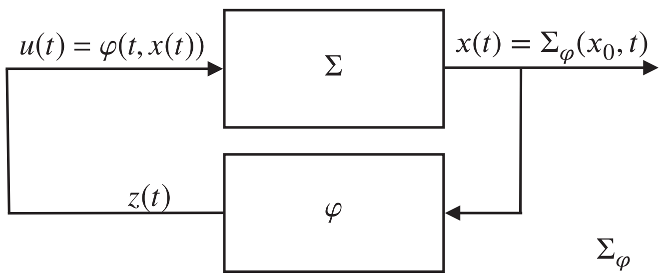

Figure 1. State-feedback configuration.

to achieve the smallest deviation from its target state over all times $t \geq 0$. As a consequence, the state-feedback functions we derive also achieve, in a sense, robust asymptotic stabilisation of the controlled system Σ, as they achieve the minimal deviation from the origin over all times.

One cannot ignore potential difficulties in the design and implementation of optimal state-feedback functions. Generally, optimal state-feedback functions are measurable multi-variable vector-valued functions of time; they can be difficult to calculate and implement. In Section 7, we discuss a methodology that facilitates relatively simple design and implementation of state-feedback functions that produce close-to-optimal performance. We can summarise our objectives as follows.

Problem 1.1: (i) Under what conditions are there optimal robust state-feedback functions φ^* that keep the closed-loop system of Figure 1 as close as possible to its target state, while facing modelling uncertainties, disturbances, and constraints on the input and state amplitudes.

(ii) When such optimal robust state-feedback functions exist, find state-feedback functions that approximate optimal performance and are relatively easy to design and implement.

The existence of optimal robust state-feedback functions that satisfy the requirements of Problem 1.2(i) is discussed in Section 6, where we show that such state-feedback functions exist under rather general conditions. Specifically, we show that such state-feedback functions exist for nonlinear input-affine systems that satisfy a certain controllability condition – the condition of constrained controllability (Choi & Hammer, 2018b, 2019b), which is reviewed and refined in Section 4. In qualitative terms, constrained controllability requires that there be an input signal that drives the controlled system Σ to the vicinity of the target state $x = 0$, without violating input amplitude and state amplitude bounds.

In addition, we point out in Section 4 that constrained controllability is, in fact, also close to being a necessary condition for the existence of solutions of Problem 1.2(i). Thus, constrained controllability is a tight sufficient condition for the existence of optimal state-feedback controllers that fulfill the requirements of Problem 1.2(i).

Part (ii) of Problem 1.2 is addressed in Section 7, where we show that optimal performance can be approximated as closely as desired by bang-bang state-feedback functions; these are piecewise constant state-feedback functions, whose components switch between the two values of $-K$ and K, where K is the input amplitude bound of the controlled system. Bang-bang state-feedback functions are easier to calculate and implement than state-feedback functions in general. Indeed, when the controlled system Σ has m input components, bang-bang functions take values in a finite set with only 2^m points – the set of all combinations of $-K$ and K in m components. This compares favourably to general state-feedback functions, whose components take arbitrary values in the continuum $[-K, K]$. Bang-bang state-feedback functions can approximate optimal performance of other optimisation problems as well, such as minimal time control (Hammer, 2019). Note also that, thanks to the fact that the state amplitude may not exceed the specified error bound A, the domain over which the bang-bang state-feedback function must be constructed is limited.

The present paper expands the framework of Chakraborty and Hammer (2008, 2009, 2010), Chakraborty and Shaikshavali (2009), Yu and Hammer (2016), Choi and Hammer (2018a, 2019a, 2019b) and Hammer (2019) to applications involving state-feedback over an infinite time horizon. In addition to optimal tracking, the methodology derived in this paper also achieves robust asymptotic stabilisation of nonlinear input-affine systems by state feedback.

The considerations in this paper draw on classical studies in the theory of optimal control, including Kelendzheridze (1961), Pontryagin et al. (1962), Gamkrelidze (1965), Neustadt (1966, 1967), Luenberger (1969), Young (1969), and Warga (1972), the references cited in these studies, and many others. Yet, it seems that the topics of this paper – the existence, the design, and the implementation of optimal robust state-feedback controllers for optimal tracking over an infinite time horizon in the presence of modelling uncertainties, disturbances, and operational constraints – have not been previously resolved in the literature.

The paper is organised as follows. Section 2 introduces the class of systems under consideration. A formal statement of Problem 1.2 is provided in Section 3. The notion of constrained controllability, which forms the main requirement for the existence of optimal controllers, is reviewed and refined in Section 4. Section 5 deals with some general properties of state-feedback controllers. The existence of optimal solutions of Problem 1.2(i) is considered in Section 6. The approximation of optimal performance by bang-bang state-feedback functions is discussed in Section 7. Section 8 provides an example, and the paper concludes in Section 9 with a brief summary.

2. Systems and state-feedback functions

2.1 The controlled system's model

Let R be the compactified set of real numbers (i.e. the real numbers augmented by $\pm\infty$); let R^+ be the set of all non-negative real numbers; and let R^n be the set of n-dimensional real vectors. The absolute value of a real number r is $|r|$. The L^∞-norm of an $n \times m$ matrix $V = (V_{ij}) \in R^{n\times m}$ is $|V| := \max_{ij} |V_{ij}|$. For a matrix-valued function $W : R^+ \to R^{n\times m} : t \mapsto W(t)$, the L^∞-norm (also called the *amplitude*) is $|W|_\infty := \sup_{t\geq 0} |W(t)|$. The L^2-norm of a vector $x \in R^n$ is $|x|_2 = (x^\top x)^{1/2}$. For a real number $\sigma > 0$, we use the ball

$$\rho(\sigma) := \left\{x \in R^n : |x|_2^2 \leq \sigma\right\}.$$

Given a real number $K > 0$, the set $[-K, K]^n$ consists of all n-dimensional vectors with components in the interval $[-K, K]$.

The controlled system Σ of Figure 1 is a nonlinear input-affine system described by the differential equation

$$\Sigma : \begin{array}{l} \dot{x}(t) = a(t, x(t)) + b(t, x(t))u(t), \quad t \geq 0, \\ x(0) = x_0, \end{array} \tag{2.1}$$

where t is the time; $u(t) \in R^m$ is the input signal; $x(t) \in R^n$ is the state; and $a : R^+ \times R^n \to R^n$ and $b : R^+ \times R^n \to R^{n\times m}$ are continuously differentiable functions. The initial state x_0 can be any member of the ball $\rho(\sigma)$, i.e.

$$x_0 \in \rho(\sigma), \tag{2.2}$$

where $\sigma > 0$ is specified. We use the notation $\Sigma(x_0, u, t)$ for $x(t)$.

Remark 2.1: As the range of initial states is $\rho(\sigma)$ and the minimal deviation ℓ^* of (1.1) is taken over the entire time axis, it follows that $\ell^* \geq \sigma$.

To represent uncertainty about the model of the controlled system Σ, we decompose the functions a and b of (2.1) into sums of two functions

$$\begin{aligned} a(t, x) &= a_0(x) + a_\gamma(t, x), \\ b(t, x) &= b_0(x) + b_\gamma(t, x), \end{aligned} \tag{2.4}$$

where $a_0 : R^n \to R^n$ and $b_0 : R^n \to R^{n\times m}$ are specified nominal functions; and $a_\gamma : R^+ \times R^n \to R^n$ and $b_\gamma : R^+ \times R^n \to R^{n\times m}$ are unspecified functions that represent modelling uncertainties and disturbances. We assume that all the functions a_0, b_0, a_γ, and b_γ are continuously differentiable. Note that the *nominal system*

$$\Sigma_0 : \begin{array}{l} \dot{x}(t) = a_0(x(t)) + b_0(x(t))u(t), \quad t \geq 0, \\ x(0) = x_0, \end{array} \tag{2.5}$$

is time-invariant. Yet the system Σ may be time-variant, since the functions a_γ and b_γ may depend on the time t. To simplify some formulas in our discussion, we assume that $a_0(0) = 0$.

Practical systems often have restrictions on the maximal control effort they can afford. These restrictions usually appear in the form of bounds on the maximal input and state amplitudes the system can tolerate. Exceeding these amplitude bounds may overload a system's components and cause irreversible harm to the system and its environment. To express such limitations, we impose a constraint of $K > 0$ on the input amplitude of the controlled system Σ, and a constraint of $A > 0$ on the state amplitude of Σ. Thus, we have the constraints

$$|u|_\infty \leq K, \tag{2.6}$$

$$|x|_\infty \leq A. \tag{2.7}$$

To ascertain consistency with the initial state x_0 specification (2.2), we assume that

$$\sqrt{\sigma} < A.$$

Remark 2.2: As the functions a_0, b_0, a_γ, and b_γ are all continuously differentiable, we can utilise the mean value theorem (e.g. Hubbard & Hubbard, 2006), which states that, for two points $x, x' \in [-2A, 2A]^n$, we have

$$a_0(x) - a_0(x') = \frac{\partial a_0(z)}{\partial x}(x - x'), \tag{2.9}$$

where $z \in [-2A, 2A]^n$ is an appropriate point. Note that we utilise here the larger domain $[-2A, 2A]^n$ instead of the domain $[-A, A]^n$, as this will be more convenient for our discussion later on.

Now, since $\partial a_0(z)/\partial x$ is a continuous function and $[-2A, 2A]^n$ is a compact domain, it follows that there is a real number $c \geq 0$ such that $|\partial a_0(z)/\partial x| \leq c$ for all $z \in [-2A, 2A]^n$. Recalling that x is of dimension n, it follows by (2.9) that $|a_0(x) - a_0(x')| \leq nc|x - x'|$. Denoting $\alpha := nc$, we conclude that $|a_0(x) - a_0(x')| \leq \alpha|x - x'|$ for all $x, x' \in [-2A, 2A]^n$. Similar relations apply to the functions b_0, a_γ, and b_γ; recalling that $a_0(0) = 0$, we obtain for all $x, x' \in [-2A, 2A]^n$ the inequalities

$$\begin{array}{ll} |a_0(x) - a_0(x')| \leq \alpha|x - x'|, & a_0(0) = 0, \\ |b_0(x) - b_0(x')| \leq \alpha|x - x'|, & |b_0(0)| \leq \alpha, \\ |a_\gamma(t, x) - a_\gamma(t, x')| \leq \gamma|x - x'|, & |a_\gamma(t, 0)| \leq \gamma, \\ |b_\gamma(t, x) - b_\gamma(t, x')| \leq \gamma|x - x'|, & |b_\gamma(t, 0)| \leq \gamma; \end{array} \tag{2.10}$$

Here, $\alpha, \gamma > 0$ are specified real numbers and we assume that γ is a constant (does not depend on time). Thus, our functions satisfy Lipchitz conditions over the domain of interest.

Finally, since γ relates to uncertainties about the model of the controlled system Σ, it is usually a small number. We refer to γ as the *uncertainty parameter*.

Notation 2.3: Let $K, A, \sigma, \gamma > 0$ be specified real numbers. Denote by $\mathcal{F}_\gamma(\Sigma_0)$ the family of all systems Σ descried by (2.1), (2.4), and (2.10), subject to the following.

(i) Input signal amplitude bounded by K.
(ii) State amplitude bounded by A.
(iii) The initial state $x_0 \in \rho(\sigma)$ is shared by all members of $\mathcal{F}_\gamma(\Sigma_0)$.
(iv) The state-feedback function φ is shared by all members of $\mathcal{F}_\gamma(\Sigma_0)$.

Item (iv) is a result of fact that the family $\mathcal{F}_\gamma(\Sigma_0)$ represents uncertainty about the controlled system, and it is not known which member of $\mathcal{F}_\gamma(\Sigma_0)$ the controlled system Σ is. Therefore, it is not possible to adjust the state-feedback function φ to Σ. Regarding item (iii), the initial state is available by feedback; it can be any member of $\rho(\sigma)$.

2.2 Basic facts

We adopt the mathematical framework of Chakraborty and Hammer (2009, 2010). Let $L_2^{\vartheta,m}$ be the Hilbert space of Lebesgue measurable functions $v, w : R^+ \to R^m$ with the inner product

$$\langle v, w\rangle := \int_0^\infty e^{-\vartheta s} v^\top(s) w(s)\, ds,$$

where $\vartheta > 0$ is a real number. Note that this inner product is bounded when v and w are bounded.

In accord with the constraint (2.6), the class of input signals of the controlled system Σ is

$$U(K) := \left\{u \in L_2^{\vartheta,m} : |u|_\infty \le K\right\}. \tag{2.12}$$

We use the following notions (e.g. Willard, 2004; Zeidler, 1985).

Definition 2.4: Let H be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$.

(i) A sequence $\{v_i\}_{i=1}^\infty \subseteq H$ *converges weakly* to a member $v \in H$ if $\lim_{i\to\infty}\langle v_i, y\rangle = \langle v, y\rangle$ for every $y \in H$.
(ii) A subset W of H is *weakly compact* if every sequence of members of W has a subsequence that converges weakly to a member of W.

The next statement is reproduced from Chakraborty and Hammer (2009, 2010).

Lemma 2.5: *The set $U(K)$ of (2.12) is weakly compact in the topology of the Hilbert space $L_2^{\vartheta,m}$.*

In view of constraint (2.7), the class of permissible responses of the controlled system Σ is

$$X(A) := \left\{x \in L_2^{\vartheta,n} : |x|_\infty \le A\right\}.$$

Let $x_0 \in \rho(\sigma)$ be an initial state and let $\tau > 0$ be a time. Denote by $U(x_0, K, A, \gamma, \tau)$ the class of all input signals in $U(K)$ that generate during the time interval $[0, \tau]$ a response that is bounded by A for all members $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$; namely,

$$U(x_0, K, A, \gamma, \tau) := \{u \in U(K) : |\Sigma(x_0, u, t)| \le A \text{ for all } \Sigma \in \mathcal{F}_\gamma(\Sigma_0) \text{ and all } t \in [0, \tau]\}. \tag{2.15}$$

The following statement shows that this class of input signals is not always empty.

Lemma 2.6: *Let $K, A, \sigma, \gamma > 0$ be real numbers, where $\sqrt{\sigma} < A$, and let $x_0 \in \rho(\sigma)$ be an initial state. Then, there is a time $\tau > 0$ for which $U(K) = U(x_0, K, A, \gamma, \tau)$.*

Proof: Let $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ and let $\tau > 0$ be a time. For an input signal $u \in U(K)$, denote $x(t) := \Sigma(x_0, u, t)$. As $x_0 \in \rho(\sigma)$, we have $|x_0| \le \sqrt{\sigma}$. By (2.1), we get

$$x(t) = x_0 + \int_0^t [a(s, x(s)) + b(s, x(s))u(s)]\, ds.$$

Using the fact that $|u(s)| \le K$ together with (2.10), we get

$$|x(t)| \le |x_0| + [(\alpha+\gamma)A + \gamma + ((\alpha+\gamma)(A+1))K]\, t.$$

Consequently, any time $\tau \in (0, (A - \sqrt{\sigma})/[(\alpha+\gamma)A + \gamma + ((\alpha+\gamma)(A+1))K])$ satisfies the lemma. This concludes our proof. ∎

2.3 State-feedback functions

Let Σ be a system described by (2.1). A state-feedback function φ for Σ, as depicted in Figure 1, is a Lebesgue measurable function $\varphi : R^+ \times R^n \to R^m$. Following Hammer (2019), denote by $L_2^{\vartheta,n,m}$ the Hilbert space formed by measurable functions $v, w : R^+ \times R^n \to R^m$ with the inner product

$$\langle\langle v, w\rangle\rangle := \int_{R^+\times R^n} e^{-\vartheta(s+|z|_2)} v^\top(s,z) w(s,z) d(s,z),$$

where $\vartheta > 0$ is a real number and $d(s, z)$ is the Lebesgue measure element on $R^+ \times R^n$. This inner product is bounded when v and w are bounded.

In the configuration of Figure 1, the input signal of Σ is produced by the state-feedback function φ. As input signals of Σ must be bounded by K, permissible state-feedback functions are restricted to the family

$$\Phi(K) := \left\{\varphi \in L_2^{\vartheta,n,m} : |\varphi(t,x)| \le K \text{forall} (t,x) \in R^+ \times R^n\right\}.$$

The family $\Phi(K)$ has the following property, which is reproduced here from Hammer (2019).

Lemma 2.7: *The set of state-feedback functions $\Phi(K)$ is weakly compact in the topology of the Hilbert space $L_2^{\vartheta,n,m}$.*

Applying a state-feedback function φ to the system Σ of (2.1) yields the closed-loop system Σ_φ of Figure 1 given by

$$\Sigma_\varphi : \begin{array}{l} \dot{x}_\varphi(t) = a(t, x_\varphi(t)) + b(t, x_\varphi(t))\varphi(t, x_\varphi(t)), \\ x_\varphi(0) = x_0. \end{array}$$

We use the notation $\Sigma_\varphi(x_0, t)$ for the response $x_\varphi(t)$.

2.4 Bounded response

For a system $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ with initial state $x_0 \in \rho(\sigma)$, let $\Phi(x_0, K, A, \Sigma, \tau) \subseteq \Phi(K)$ be the class of all state-feedback functions for which the response $\Sigma_\varphi(x_0, t)$ stays bounded by A at all times $t \in [0, \tau]$; explicitly,

$$\Phi(x_0, K, A, \Sigma, \tau) = \{\varphi \in \Phi(K) : |\Sigma_\varphi(x_0, t)| \le A \text{ for all } t \in [0, \tau]\}. \tag{2.8}$$

The set of all state feedback functions $\Phi(x_0, K, A, \Sigma) \subseteq \Phi(K)$ that keep the response $\Sigma_\varphi(x_0, t)$ bounded by A at all times $t \ge 0$ is then given by

$$\Phi(x_0, K, A, \Sigma) = \bigcap_{\tau\ge 0} \Phi(x_0, K, A, \Sigma, \tau),$$

or, equivalently,

$$\Phi(x_0, K, A, \Sigma) = \left\{\varphi \in \Phi(K) : \sup_{t\ge 0} |\Sigma_\varphi(x_0, t)| \le A\right\}.$$

As it is not known which member of $\mathcal{F}_\gamma(\Sigma_0)$ the controlled system Σ is, state-feedback functions cannot be adjusted individually for each member $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$. Therefore, we must concentrate on the class of all state-feedback functions $\Phi(x_0, K, A, \gamma) \subseteq \Phi(K)$ that consists of state-feedback functions which, at all times $t \geq 0$, keep the response of almost every member $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ bounded by A. Formally,

$$\Phi(x_0, K, A, \gamma) = \left\{ \varphi \in \Phi(K) : \operatorname*{ess\,sup}_{\Sigma \in \mathcal{F}_\gamma(\Sigma_0),\, t \geq 0} |\Sigma_\varphi(x_0, t)| \leq A \right\}.$$

As the initial condition x_0 can be any member of $\rho(\sigma)$, we must restrict our attention to state-feedback functions that keep the response bounded by A for all $x_0 \in \rho(\sigma)$. This class of state-feedback functions is given by

$$\Phi(\sigma, K, A, \gamma) = \left\{ \varphi \in \Phi(K) : \operatorname*{ess\,sup}_{\Sigma \in \mathcal{F}_\gamma(\Sigma_0), x_0 \in \rho(\sigma), t \geq 0} |\Sigma_\varphi(x_0, t)| \leq A \right\}. \tag{2.18}$$

We show in Section 4 that this class of state-feedback functions is not empty, as long as the nominal controlled system Σ_0 satisfies a certain controllability condition.

It is sometimes convenient to confine ourselves to a finite interval of time $[0, \tau]$, $\tau > 0$. For this purpose, we define the set of state-feedback functions $\Phi(\sigma, K, A, \gamma, \tau)$ that includes all feedback functions that induce a response bounded by A over the interval $[0, \tau]$, namely,

$$\Phi(\sigma, K, A, \gamma, \tau) = \left\{ \varphi \in \Phi(K) : \operatorname*{ess\,sup}_{\Sigma \in \mathcal{F}_\gamma(\Sigma_0), x_0 \in \rho(\sigma), t \in [0,\tau]} |\Sigma_\varphi(x_0, t)| \leq A \right\}.$$

Now, every state-feedback function in $\Phi(K)$ creates an input signal in $U(K)$; and, vice-versa, every input signal $u \in U(K)$ can be produced by an 'open-loop' feedback function $\varphi(t, x) := u(t)$ belonging to $\Phi(K)$. Consequently, Lemma 2.16 implies that the class of state-feedback functions $\Phi(\sigma, K, A, \gamma, \tau)$ is not always empty:

Lemma 2.9: *Let $K, A, \sigma, \gamma > 0$ be real numbers, where $\sqrt{\sigma} < A$. Then, there is a time $\tau > 0$ for which $\Phi(K) = \Phi(\sigma, K, A, \gamma, \tau)$.*

3. Formal statement of the problem

We proceed now to re-state Problem 1.2 in formal terms. Recall from Notation 2.11 that the system Σ of Figure 1 is an unspecified member of the family $\mathcal{F}_\gamma(\Sigma_0)$ with an initial state that can be any vector $x_0 \in \rho(\sigma)$. Our objective is to find robust state-feedback functions φ which, at all times, keep the closed-loop system Σ_φ of Figure 1 as close as possible to the target state $x = 0$, irrespective of which member $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ the controlled system is, or which initial state $x_0 \in \rho(\sigma)$ is used. This must be accomplished without violating the input signal amplitude bound K and the state amplitude bound A of Σ.

We use the square of the L^2-norm to characterise the deviation of the response from the target state $x = 0$. As it is required to keep the response of Σ bounded by A at all times $t \geq 0$, we must restrict our attention to the family of feedback functions $\Phi(\sigma, K, A, \gamma)$ of (2.18). For a particular state-feedback function $\varphi \in \Phi(\sigma, K, A, \gamma)$, the supremal deviation of the closed-loop system from the origin over all times $t \geq 0$ is given by

$$\ell(\sigma, K, A, \gamma, \varphi) = \operatorname*{ess\,sup}_{\Sigma \in \mathcal{F}_\gamma(\Sigma_0), x_0 \in \rho(\sigma), t \geq 0} \left|\Sigma_\varphi(x_0, t)\right|_2^2. \tag{3.1}$$

The infimal such deviation over all state-feedback functions is then

$$\ell^*(\sigma, K, A, \gamma) = \inf_{\varphi \in \Phi(\sigma, K, A, \gamma)} \ell(\sigma, K, A, \gamma, \varphi), \tag{3.2}$$

where $\ell^*(\sigma, K, A, \gamma) := \infty$ if $\Phi(\sigma, K, A, \gamma) = \varnothing$ (the empty set).

In the forthcoming sections, we answer two critical question in this context: (i) under what conditions is the infimum $\ell^*(\sigma, K, A, \gamma)$ finite; and (ii) if $\ell^*(\sigma, K, A, \gamma)$ is finite, can it be achieved; namely, is there an optimal state-feedback function $\varphi^* \in \Phi(\sigma, K, A, \gamma)$ satisfying

$$\ell^*(\sigma, K, A, \gamma) = \ell(\sigma, K, A, \gamma, \varphi^*). \tag{3.3}$$

If such a state-feedback function exists, then $\ell^*(\sigma, K, A, \gamma)$ is a minimum, not just an infimum. Note that if φ^* exists, it forms a robust state-feedback configuration, since uncertainties and disturbances are taken into consideration by the family $\mathcal{F}_\gamma(\Sigma_0)$. Thus, an optimal state-feedback function φ^*, when it exists, achieves in a robust manner the lowest possible deviation from the target state over the infinite time horizon $t \geq 0$. We can summarise our discussion by restating Problem 1.2 in the following terms.

Problem 3.1: In the notation of (3.2) and (3.3),

(i) Find conditions under which there is an optimal robust state-feedback function φ^*.
(ii) If φ^* exists, find state-feedback functions that approximate optimal performance, while being relatively easy to design and implement. □

Regarding Problem 3.4(i), we show in Section 6 that an optimal state-feedback function φ^* exists, if the nominal controlled system Σ_0 satisfies a certain controllability condition. Our discussion there also indicates that this controllability condition is close to being a necessary condition for the existence of an optimal state-feedback function φ^*.

Problem 3.4(ii) is considered in Section 7, where we show that optimal performance can be approximated as closely as desired by bang-bang state-feedback functions – functions that are relatively easy to design and implement. This provides an additional important application of bang-bang state-feedback functions introduced in Hammer (2019).

4. Constrained controllability

The notion of constrained controllability of Choi and Hammer (2020) (see also Choi & Hammer, 2019b) is a controllability-type feature of the controlled system Σ of Figure 1. It assures that Σ can be driven to the vicinity of the origin without violating specified constraints on the input and state amplitudes of Σ. The following variant of this notion guarantees that every state in the ball $\rho(\sigma)$ can be driven into a ball of a somewhat smaller radius.

Definition 4.1: Let $K, A, \sigma > 0$ be real numbers. A system Σ is (K, A, σ)-*controllable* if there are times $\tau_2 > \tau_1 > 0$ and a real number $\sigma' < \sigma$ such that, for every state $x \in \rho(\sigma)$, there is a time $t_x \in [\tau_1, \tau_2]$ and an input signal $u_x \in U(K)$ satisfying (i) $\Sigma(x, u_x, t_x) \in \rho(\sigma')$, and (ii) $|\Sigma(x, u_x, t)| \leq A$ for all $t \in [0, t_x]$.

We show in Section 6 that (K, A, σ)-controllability is a sufficient condition for the existence of optimal state-feedback functions that satisfy the requirements of Problem 3.4(i). Additionally, this condition is also close to being a necessary condition for the existence of a solution of Problem 3.4(i). Indeed, Problem 3.4(i) requires that there be an input signal in $U(K)$ that keeps the state bounded by A at all times for all initial states. Thus, if we had $\sigma' = A$ in Definition 4.1, the resulting notion would clearly form a necessary condition for the existence of a solution of Problem 3.4(i). In other words, in the definition of (K, A, σ)-controllability, the only additional requirement beyond a necessary condition is the contractive requirement $\sigma' < \sigma$. This contractive requirement helps us accommodate uncertainty and disturbances that afflict the controlled system Σ, while also helping us handle an infinite time horizon.

4.1 Uncertainty, continuity, and convergence

The next statement shows that the entire family $\mathcal{F}_\gamma(\Sigma_0)$ is (K, A, σ)-controllable, if the nominal system Σ_0 is (K, A, σ)-controllable and the uncertainty parameter γ is not too large. This statement is reproduced here (with some minor modifications) from Choi and Hammer (2020).

Proposition 4.2: *Let $K, A_0, \sigma > 0$ be real numbers, where $\sqrt{\sigma} < A_0$, and assume that the nominal system Σ_0 is (K, A_0, σ)-controllable. Then, for every real number $A > A_0$, there is an uncertainty parameter $\gamma > 0$ such that the entire family of systems $\mathcal{F}_\gamma(\Sigma_0)$ is (K, A, σ)-controllable.*

Recall the class of feedback functions $\Phi(\sigma, K, A, \gamma)$ of (2.18), which consists of feedback functions bounded by K that keep the state of the closed-loop system from exceeding the amplitude bound A. The next statement shows that $\Phi(\sigma, K, A, \gamma)$ is not empty, when the nominal system Σ_0 is (K, A, σ)-controllable.

Proposition 4.3: *Let $K, A, \sigma > 0$ be real numbers, where $\sqrt{\sigma} < A$, and assume that the nominal system Σ_0 is (K, A, σ)-controllable. Then, there are an uncertainty parameter $\gamma' > 0$ and an amplitude bound $A' > A$ such that the set of state-feedback functions $\Phi(\sigma, K, A', \gamma)$ is not empty for any uncertainty parameter $\gamma \in (0, \gamma']$.*

Proof: Let $x_0 \in \rho(\sigma)$ be an initial state. According to (2.5), the nominal system Σ_0 is time-invariant. By Definition 4.1 of (K, A, σ)-controllability, there are an input signal $u_{x_0} \in U(K)$, a time $t_1 \in [\tau_1, \tau_2]$, where $\tau_2 > \tau_1 > 0$, and a real number $\sigma' \in (0, \sigma)$ such that (i) $x(t_1) \in \rho(\sigma')$, and (ii) $|\Sigma_0(x_0, u_{x_0}, t)| \leq A$ for all $t \in [0, t_1]$. As $\rho(\sigma') \subset \rho(\sigma)$, it follows again by (K, A, σ)-controllability of Σ_0 that there are an input signal $u_{x(t_1)} \in U(K)$ and a time $t_2 \in [t_1 + \tau_1, t_1 + \tau_2]$, $t_2 \geq t_1 + \tau_1 \geq 2\tau_1$, such that (i) $x(t_2) \in \rho(\sigma')$, and (ii) $|\Sigma_0(x(t_1), u_{x(t_1)}, t)| \leq A$ for all $t \in [t_1, t_2]$. Continuing in this manner, we conclude that, for every integer $k \geq 2$, there is a time $t_k \geq k\tau_1$ and an input signal $u_{x(t_{k-1})} \in U(K)$ such that (i) $x(t_k) \in \rho(\sigma')$, and (ii) $|\Sigma_0(x(t_{k-1}), u_{x(t_{k-1})}, t)| \leq A$ for all $t \in [t_{k-1}, t_k]$. Denoting $t_0 := 0$, define the state-feedback function

$$\varphi(t, x(t)) := u_{x(t_{k-1})}(t), \quad t \in [t_{k-1}, t_k),\ k = 1, 2, \ldots \tag{4.4}$$

As $t_k \geq k\tau_1$ and $\tau_1 > 0$, this defines a state-feedback function φ that is bounded by K at all times and keeps the nominal system's response bounded by A at all times. This state-feedback function can be extended over the entire space $R^+ \times R^n$ to yield a measurable function $\varphi \in \Phi(K)$. When applied to the nominal system Σ_0 as a state feedback, φ keeps the amplitude of the nominal system's state bounded by A at all times. We turn next other members of the family of systems $\mathcal{F}_\gamma(\Sigma_0)$.

For a member $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$, denote $x'(t) := \Sigma_\varphi(x_0, t)$. Now, for an integer $k \geq 1$, consider a time $t \in (t_{k-1}, t_k]$. Using (2.1), (2.4), and (4.4), we can write

$$x'(t) = x'(t_{k-1}) + \int_{t_{k-1}}^{t} [a(s, x'(s)) + b(s, x'(s))u_{x'(t_{k-1})}(s)]\, \mathrm{d}s.$$

Now, assume for a moment that $x'(t_{k-1}) \in \rho(\sigma)$ and $|x'(t)| \leq 2A$ for all $t \in [t_{k-1}, t_k]$. Consider the response $x''(t) = \Sigma_0(x'(t_{k-1}), u_{x'(t_{k-1})})$ of the nominal system Σ_0, starting from the state $x'(t_{k-1})$ at the time t_{k-1}, namely,

$$x''(t) = x'(t_{k-1}) + \int_{t_{k-1}}^{t} [a_0(x''(s)) + b_0(x''(s))u_{x'(t_{k-1})}(s)]\, \mathrm{d}s.$$

Then, since we assumed that $x'(t_{k-1}) \in \rho(\sigma)$, it follows by the definition of the input signal $u_{x'(t_{k-1})}$ that $x''(t_k) \in \rho(\sigma')$ and $|x''(t)| \leq A$ for all $t \in [t_{k-1}, t_k]$.

To continue, denote $\xi(t) := x'(t) - x''(t)$; then,

$$\xi(t_{k-1}) := x'(t_{k-1}) - x''(t_{k-1}) = x'(t_{k-1}) - x'(t_{k-1}) = 0. \tag{4.5}$$

Next, let $t, t' \in [t_{k-1}, t_k]$, $t' > t$, be two times. Then,

$$\xi(t') = \xi(t) + \int_t^{t'} [a_0(x'(s)) + a_\gamma(s, x'(s)) + (b_0(x'(s)) + b_\gamma(s, x'(s)))u_{x'(t_{k-1})}(s)]\, \mathrm{d}s - \int_t^{t'} [a_0(x''(s)) + (b_0(x''(s)))u_{x'(t_{k-1})}(s)]\, \mathrm{d}s.$$

Recalling that $a_0(0) = 0$, we can write

$$|\xi(t)| \leq |\xi(t)| + \int_t^{t'} |a_0(x'(s)) - a_0(x''(s))|\, \mathrm{d}s$$

$$+ \int_t^{t'} |b_0(x'(s)) - (b_0(x''(s))||u_{x'(t_{k-1})}(s)|\, ds$$
$$+ \int_t^{t'} |a_\gamma(s, x'(s)) - a_\gamma(s, 0)|\, ds + \int_t^{t'} |a_\gamma(s, 0)|\, ds$$
$$+ \int_t^{t'} |b_\gamma(s, x'(s)) - b_\gamma(s, 0)||u_{x'(t_{k-1})}(s)|\, ds$$
$$+ \int_t^{t'} |b_\gamma(s, 0)||u_{x'(t_{k-1})}(s)|\, ds.$$

Now, refer to Remark 2.8 and the Lipschitz conditions (2.10) that are valid over the domain $[-2A, 2A]^n$. Applying these Lipschitz conditions, we get

$$\begin{aligned}\sup_{s\in[t,t']} |\xi(s)| &\le |\xi(t)| + \alpha(t'-t) \sup_{s\in[t,t']} |\xi(s)| \\ &+ \alpha(t'-t)K \sup_{s\in[t,t']} |\xi(s)| \\ &+ \gamma(t'-t) \sup_{s\in[t,t']} |\xi(s)| + \gamma(t'-t) \\ &+ \gamma(t'-t)K \sup_{s\in[t,t']} |\xi(s)| + \gamma K(t'-t).\end{aligned}$$

Rearranging terms, we get

$$\begin{aligned}&[1-(\alpha+\gamma)(1+K)(t'-t)] \sup_{s\in[t,t']} |\xi(s)| \\ &\le |\xi(t)| + \gamma(1+K)(t'-t).\end{aligned}$$

Now, let $\mu > 0$ be a real number such that $(\alpha+\gamma)(1+K)\mu \le 1/2$ and the ratio $r := (t_k - t_{k-1})/\mu$ is an integer. Then, using $t' = t + \mu$, we get

$$\sup_{s\in[t,t+\mu]} |\xi(s)| \le 2|\xi(t)| + 2\gamma(1+K)\mu. \qquad (4.6)$$

Next, build the partition of the interval $[t_{k-1}, t_k]$ given by

$$\{[t_{k-1}, t_{k-1}+\mu], [t_{k-1}+\mu, t_{k-1}+2\mu], \ldots, [t_{k-1}+(r-1)\mu, t_k]\},$$

and define the quantity

$$\zeta_j := \sup_{s\in[t_{k-1}+(j-1)\mu, t_{k-1}+j\mu]} |\xi(s)|, \quad j = 1, 2, \ldots, r. \qquad (4.7)$$

Using the fact that $\xi(t_{k-1}) = 0$ by (4.5) and setting $\zeta_0 := \xi(t_{k-1})$, we obtain from (4.6) the relations

$$\zeta_j \le 2\zeta_{j-1} + 2\gamma(1+K)\mu, \quad j = 1, 2, \ldots, r,$$
$$\zeta_0 = 0.$$

Invoking properties of linear relations, this yields $\zeta_j \le (2^j - 1)2\gamma(1+K)\mu$, $j = 1, 2, \ldots, r$. In view of (4.7), this leads to

$$\sup_{s\in[t_{k-1},t_k]} |\xi(s)| \le (2^{r+1} - 2)\gamma(1+K)\mu, \quad k = 1, 2, \ldots \qquad (4.8)$$

Now, referring to the statement of the proposition, let $\varepsilon > 0$ be a real number satisfying $\varepsilon \le \min\{A, (\sigma - \sigma')/2\}$. Then, by (4.8), an uncertainty parameter $\gamma' = \varepsilon/[(2^{r+1} - 2)(1+K)\mu]$ satisfies the statement of the proposition with $A' = A + \varepsilon$. This concludes our proof. ■

Since the set of feedback functions $\Phi(\sigma, K, A', \gamma)$ is not empty by Proposition 4.3, the infimal deviation from the target state $x = 0$ must be bounded, as follows.

Theorem 4.4: *Under the notation and conditions of Proposition 4.3, the infimal deviation $\ell^*(\sigma, K, A', \gamma)$ of (3.2) is finite.*

The next statement refers to the class of input signals $U(x_0, K, A, \gamma, \tau)$ of (2.15). The first part of the statement shows that the response of the controlled system Σ depends continuously on its input signal, while the second part of the statement shows that the class $U(x_0, K, A, \gamma, \tau)$ of input signals is weakly compact.

Theorem 4.5: *Let $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ be a system with initial state $x_0 \in \rho(\sigma)$, let $\tau > 0$ be a time, and let $\{u_i\}_{i=1}^\infty \subseteq U(x_0, K, A, \gamma, \tau)$ be a sequence of input signals that converges weakly to the input signal u. Then,*

- (i) *The sequence of output values $\{\Sigma(x_0, u_i, t)\}_{i=1}^\infty$ converges to $\Sigma(x_0, u, t)$ uniformly over the time interval $[0, \tau]$; and*
- (ii) *The set $U(x_0, K, A, \gamma, \tau)$ is weakly compact.*

Proof: Part (i). In view of Remark 2.8 and the Lipschitz condition of (2.10), the proof of a similar statement in Yu and Hammer (2016) applies here as well, proving Part (i).

Part (ii). Let $\{v_i\}_{i=1}^\infty \subseteq U(x_0, K, A, \gamma, \tau)$ be a sequence of input signals. Then, since $U(x_0, K, A, \gamma, \tau) \subseteq U(K)$, we have that $\{v_i\}_{i=1}^\infty \subseteq U(K)$, and it follows by Lemma 2.14 that there is a subsequence $\{v_{i_j}\}_{j=1}^\infty$ that converges to a signal $v \in U(K)$. Now, let $t \in [0, \tau]$ be a time. In view of Part (i), we have that

$$\lim_{j\to\infty} \Sigma(x_0, v_{i_j}, t) = \Sigma(x_0, v, t) \qquad (4.11)$$

for all $t \in [0, \tau]$. But then, since $v_{i_j} \in U(x_0, K, A, \gamma, \tau)$ for all $j = 1, 2, \ldots$, it follows that $\Sigma(x_0, v_{i_j}, t) \in [-A, A]^n$ for all $j = 1, 2, \ldots$ Considering that $[-A, A]^n$ is a compact set in R^n, we have that $\lim_{j\to\infty} \Sigma(x_0, v_{i_j}, t) \in [-A, A]^n$. Combining this with (4.11), we obtain that $\Sigma(x_0, v, t) \in [-A, A]^n$. As this is true for all $t \in [0, \tau]$, it follows that $v \in U(x_0, K, A, \gamma, \tau)$, and our proof concludes. ■

5. State-feedback and precompensation

It is widely recognised that there is an intimate connection between feedback and precompensation (e.g. Hammer, 1986). Of course, feedback has many advantages over precompensators: feedback tends to better accommodate modelling uncertainties and disturbances, and feedback is the only means of robustly stabilising unstable systems. Yet, there are mathematical arguments that become simpler when feedback is represented by precompensation. In the present section, we explore the formal connection between feedback and precompensation to prepare tools that help us address Problem 3.4. Specifically, the tools developed in this section help to show that the response of the closed-loop system Σ_φ of Figure 1 depends in a continuous manner on the state-feedback function φ. A similar result was derived in Hammer (2019) by a different approach.

5.1 Equivalent precompensation

Consider a sequence $\{\varphi_i\}_{i=1}^{\infty} \subseteq \Phi(K)$ of state-feedback functions, whose members are inserted (individually) into the state-feedback loop of Figure 2 around the controlled system $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$. The figure depicts an additive state-feedback configuration with an external input signal $w(t)$. We denote the closed-loop system by $\Sigma_{(+)\varphi_i}$ to distinguish it from the configuration Σ_φ of Figure 1, which has no external input signal.

With the exception of the external input signal w, all signals in the configuration of Figure 2 depend on the feedback function φ_i. The output signal is x_i; the signal generated by the state-feedback function φ_i is z_i; and the input signal of the controlled system Σ is u_i. As can be seen in the figure, the signal z_i is generated from the signal u_i by a composition of the controlled system Σ and the state-feedback function φ_i. Using the symbol $\circ$ to denote such a composition, we obtain the relations

$$\begin{aligned} z_i &= \varphi_i \circ \Sigma u_i, \\ x_i &= \Sigma u_i, \\ u_i &= w + z_i, \end{aligned} \tag{5.1}$$

$i = 1, 2, \ldots$ Using I to denote the identity system, we obtain from (5.1) the relation

$$(I - \varphi_i \circ \Sigma)u_i = w, \quad i = 1, 2, \ldots \tag{5.2}$$

The following statement shows that $(I - \varphi_i \circ \Sigma)$ is a set isomorphism.

Proposition 5.1: *In the configuration of Figure 2, let $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ be the controlled system and let φ_i be the state-feedback function. Let $\tau > 0$ be a time, let $B > 0$ be a real number, and let S be a class of input signals for which all signals in the configuration of Figure 2 are bounded in amplitude by B during the time interval $[0, \tau]$. Then, over S, the system $(I - \varphi_i \circ \Sigma)$ is an isomorphism (in measure) during the time interval $[0, \tau]$.*

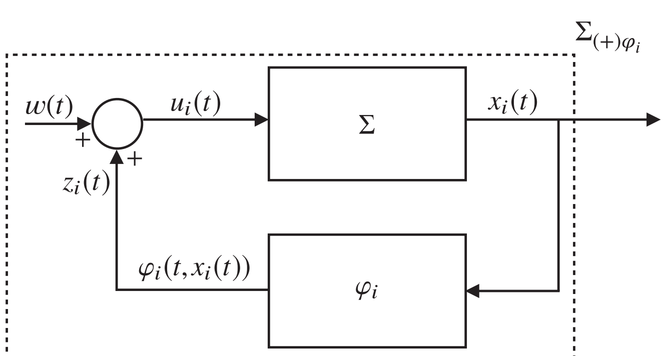

Figure 2. Feedback with external input

Proof: We use the notation of the proposition and assume that $B \le 2A$. To simplify notation, denote $f(t) := a(t, x(t)) + b(t, x(t))u(t)$, so that (2.1) takes the form

$$\dot{x}(t) = f(t). \tag{5.4}$$

We discretise the controlled system Σ of (2.1). Let $\delta > 0$ be a real number. Using the mean value theorem (e.g. Hubbard & Hubbard, 2006), we can write

$$x((k+1)\delta) = x(k\delta) + f(\xi)\delta, \tag{5.5}$$

where ξ is an intermediate point in the interval $[k\delta, (k+1)\delta]$. Setting $\varepsilon := f(\xi) - f(k\delta)$, we have

$$f(\xi) = f(k\delta) + (f(\xi) - f(k\delta)) = f(k\delta) + \varepsilon. \tag{5.6}$$

Using the Lipchitz conditions (2.10), we can write

$$\begin{aligned} |\varepsilon| &\le |a(\xi, x(\xi)) - a(k\delta, x(k\delta))| + K|b(\xi, x(\xi)) - b(k\delta, x(k\delta))| \\ &\le (1 + K)(\alpha + \gamma)|x(\xi) - x(k\delta)|. \end{aligned} \tag{5.7}$$

Similarly, we have the bound

$$\begin{aligned} \sup |f(t)| &= \sup |a(t, x(t)) + b(t, x(t))u(t)| \\ &\le (\alpha + \gamma)(A + AK + K) =: M. \end{aligned}$$

Combining this with (5.4) and the fact that $\xi \in [k\delta, (k+1)\delta]$ yields

$$|x(\xi) - x(k\delta)| \le \sup_{\xi \in [k\delta, (k+1)\delta]} |x(\xi) - x(k\delta)| \le M\delta. \tag{5.8}$$

Substituting into (5.7), we get

$$|\varepsilon| \le (1 + K)(\alpha + \gamma)M\delta. \tag{5.9}$$

In view of (5.5), (5.6), and (5.9), we can write

$$x((k+1)\delta) = x(k\delta) + f(k\delta)\delta + \varepsilon\delta. \tag{5.10}$$

Now, select $\delta > 0$ so that the ratio $N := \tau/\delta$ is an integer; set $k = t/\delta$; and introduce the discretised system

$$\Sigma^D : \begin{aligned} z_{k+1} &= z_k + f(k\delta)\delta, \\ z_0 &= x_0. \end{aligned} \tag{5.11}$$

Comparing (5.10) to (5.11), applying properties of linear recursions, and using (5.9) together with the fact that $N\delta = \tau$, gives rise to the inequality

$$\max_{k=0,1,\ldots,N} |x(k\delta) - z_k| \le N|\varepsilon|\delta \le \tau(1 + K)(\alpha + \gamma)M\delta.$$

Consequently,

$$\lim_{\delta \to 0} \left\{ \max_{k=0,1,\ldots,N} |x(k\delta) - z_k| \right\} = 0.$$

Next, denote

$$\varepsilon' := \tau(1 + K)(\alpha + \gamma)M\delta, \tag{5.12}$$

so that

$$\max_{k=0,1,\ldots,N} |x(k\delta) - z_k| \le \varepsilon'.$$

Then,

$$(I - \varphi \circ \Sigma)u = u(t) - \varphi(t, x(t)) = u(t) - \varphi(k\delta + \zeta, z_k + \varepsilon'), \tag{5.13}$$

where

$$|\zeta| \le \delta. \tag{5.14}$$

In (5.13), the pair (ζ, ε') induces a shift on the domain of the feedback function φ. When integrating the expression $\varphi(k\delta + \zeta, z_k + \varepsilon')$ using the Lebesgue integral, this shift will induce a difference not exceeding $2K(\zeta + n\varepsilon'(2A)^{n-1})$ in the value of the integral, since φ is bounded by K and the state is bounded by $2A$. Consequently,

$$\left|(I - \varphi \circ \Sigma) - (I - \varphi\Sigma^D)\right| \le 2K(\zeta + n\varepsilon'(2A)^{n-1}) \tag{5.15}$$

in measure. Combining this with (5.12) and (5.14) implies that the discretised system $u(k\delta) - \varphi(k\delta, z_k)$ converges in measure to the continuous-time system $u(t) - \varphi(t, x(t))$ as $\delta \to 0$. This means that $I - \varphi \circ \Sigma^D$ converges in measure to $I - \varphi \circ \Sigma$ as $\delta \to 0$.

Now, it is well established that the discrete-time system $I - \varphi \circ \Sigma^D$ forms a set isomorphism (e.g. Hammer, 1986). Assume then that two input signals u, v yield the same response in continuous-time, namely, that $(I - \varphi \circ \Sigma)u = (I - \varphi \circ \Sigma)v$ for all times $t \in [0, \tau]$. Applying (5.15), we obtain

$$\begin{aligned}&\left|(I - \varphi \circ \Sigma^D)u - (I - \varphi \circ \Sigma^D)v\right| \\ &\quad \le \left|(I - \varphi \circ \Sigma)u - (I - \varphi\Sigma^D)u\right| \\ &\qquad + \left|(I - \varphi \circ \Sigma)v - (I - \varphi\Sigma^D)v\right| \\ &\quad \le 4K(\zeta + n\varepsilon'(2A)^{n-1})\end{aligned}$$

for all $t \in [0, \tau]$. Therefore, letting $\delta \to 0$, it follows by (5.12) and (5.14) that $(I - \varphi \circ \Sigma^D)u$ converges in measure to $(I - \varphi \circ \Sigma^D)v$ over the time interval $[0, \tau]$. The latter implies that, as $\delta \to 0$, the value $u(k\delta)$ convenes in measure to $v(k\delta)$ for all $k = 0, 1, \ldots, N$. This, in turn, implies that u and v converge to each other in measure over the time interval $[0, \tau]$. Thus, $(I - \varphi \circ \Sigma)$ is injective (in measure). Surjectivity of $(I - \varphi \circ \Sigma)$ is shown similarly, based on the surjectivity of the associated discretised system $(I - \varphi \circ \Sigma^D)$. This concludes our proof. ■

Continuing with our discussion of the composite system $(I - \varphi \circ \Sigma)$, let $\{\varphi_i\}_{i=1}^{\infty} \subseteq \Phi(K)$ be a sequence of state-feedback functions that converges weakly to a state-feedback function φ. Assume that all members of this sequence, as well as φ, satisfy the requirements of Proposition 5.3 over a time interval $[0, \tau]$, $\tau > 0$. Then, the composite system $(I - \varphi_i \circ \Sigma)$ is invertible for all integers $i \ge 1$ at all times $t \in [0, \tau]$; namely, the inverse $(I - \varphi_i \circ \Sigma)^{-1}$ exists for all $t \in [0, \tau]$. Note that, according to the proof of Proposition 5.3, the time interval $[0, \tau]$ can be any time interval during which all signals in the configuration of Figure 2 are bounded. As systems in practice must operate with bounded signals, this time interval includes all times of practical interest.

Now, substituting the inverse system into (5.2), we can write

$$u_i = (I - \varphi_i \circ \Sigma)^{-1} w, \quad i = 1, 2, \ldots, \tag{5.16}$$

during the time interval $[0, \tau]$. Inserting this relation into (5.1) yields

$$x_i = \Sigma u_i = \Sigma \circ (I - \varphi_i \circ \Sigma)^{-1} w, \quad i = 1, 2, \ldots$$

Seeing that $x_i = \Sigma_{(+)\varphi_i} w$ in Figure 2, we obtain

$$\Sigma_{(+)\varphi_i} = \Sigma \circ (I - \varphi_i \circ \Sigma)^{-1}, \quad i = 1, 2, \ldots, \tag{5.17}$$

during the time interval $[0, \tau]$. Thus, a state-feedback function φ_i has the same input/output effect as the precompensator $(I - \varphi_i \circ \Sigma)^{-1}$.

Certainly, the use of feedback has substantial advantages over the use of precompensators, as feedback is the only means to stabilise an unstable system. Feedback also has the potential to attenuate effects of uncertainties and disturbances that may afflict the controlled system Σ. These advantages of feedback motivate our present discussion. Notwithstanding, for certain formal mathematical calculations, the relationship (5.17) is helpful, and it has been utilised in this manner throughout the history of control theory.

5.2 Continuity

We examine now the dependence of the closed loop system Σ_φ of Figure 1 on the state-feedback function φ. The objective is to show that Σ_φ is a continuous function of φ. We start with a review of some notions from mathematical analysis (e.g. Willard, 2004; Zeidler, 1985).

Definition 5.2: Let S be a subset of a Hilbert space H, and let z be a point of S. A functional $F : S \to R$ is *weakly lower semi-continuous* at z if the following is true for every sequence $\{z_i\}_{i=1}^{\infty} \subseteq S$ that converges weakly to z: whenever $F(z)$ is bounded, there is, for every real number $\varepsilon > 0$, an integer $N > 0$ such that $F(z) - F(z_i) < \varepsilon$ for all $i \ge N$.

A function $G : S \to R^n$ is *weakly continuous* at z if, for every real number $\varepsilon > 0$, there is an integer $N > 0$ such that $|G(z) - G(z_i)| < \varepsilon$ for all $i \ge N$. □

The next statement shows that the composite system $[I - \varphi \circ \Sigma]$ depends continuously on the state-feedback function φ. Needless to say, we are interested only in cases where all signals are bounded, as these are the only cases of practical value.

Lemma 5.3: *Let $\{\varphi_i\}_{i=1}^{\infty} \subseteq \Phi(K)$ be a sequence of state-feedback functions that converges weakly to a state-feedback function φ. Let $B > 0$ be a real number, and let $S \subseteq U(B)$ be a set of external input signals for which all signals in the configuration of Figure 2 are bounded by B at all times for all state-feedback functions φ_i, $i = 1, 2, \ldots$ Then, the sequence $\{[I - \varphi_i \circ \Sigma]\}_{i=1}^{\infty}$ converges weakly to $[I - \varphi \circ \Sigma]$ for almost every system $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$.*

Proof: Consider a function $g \in L_2^{\vartheta, n, m}$, and let $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ be a system. Note that, by the definition of composition, $(\varphi_i - \varphi) \circ$

$\Sigma u = \varphi_i \circ \Sigma u - \varphi \circ \Sigma u$ for an input signal u. Then, the three inner products $\langle g \circ \Sigma u, \varphi_i \circ \Sigma u\rangle$, $\langle g \circ \Sigma u, \varphi \circ \Sigma u\rangle$, and $\langle g \circ \Sigma u, (\varphi_i - \varphi) \circ \Sigma u\rangle$ are, respectively, restrictions of the inner products $\langle\langle g, \varphi_i\rangle\rangle$, $\langle\langle g, \varphi\rangle\rangle$, and $\langle\langle g, (\varphi_i - \varphi)\rangle\rangle$ to the output signal Σu. Also, since the sequence $\{\varphi_i\}_{i=1}^{\infty}$ converges weakly to φ, the sequence $\langle\langle g, (\varphi_i - \varphi)\rangle\rangle$ converges to 0 as $i \to \infty$.

Now, regard the inner product $\langle g \circ \Sigma u, (\varphi_i - \varphi) \circ \Sigma u\rangle$ as a function of Σ. Then, since the sequence $\langle\langle g, (\varphi_i - \varphi)\rangle\rangle$ converges to 0 as $i \to \infty$, the sequence $\langle g \circ \Sigma u, (\varphi_i - \varphi) \circ \Sigma u\rangle$ converges to 0 as $i \to \infty$ for almost every system $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$. As this is true for every $g \in L_2^{\vartheta,n,m}$, it follows that the sequence $\{\varphi_i \circ \Sigma u\}_{i=1}^{\infty}$ converges weakly to $\varphi \circ \Sigma u$ for almost every $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$. Finally, as addition of the identity system I does not affect convergence, the previous sentence implies that the sequence $\{[I - \varphi_i \circ \Sigma]\}_{i=1}^{\infty}$ converges weakly to $[I - \varphi \circ \Sigma]$ for almost every $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$. This concludes our proof. ■

The next statement shows that the signal u of Figure 1 depends continuously on the state-feedback function φ. A similar statement was derived in Hammer (2019) via a different approach.

Lemma 5.4: *Assume that all signals in the configuration of Figure 1 are bounded. Then, the signal u is a weakly continuous function of the state-feedback function $\varphi \in \Phi(K)$ for almost every controlled system $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$.*

Proof: Note first that, by setting $w = 0$ in the configuration of Figure 2, we obtain the the configuration of Figure 1; we consider $w = 0$ below. Now, let $\{\varphi_i\}_{i=1}^{\infty} \subseteq \Phi(K)$ be a sequence of state-feedback functions that converges weakly to a state-feedback function φ, and assume that all signals of the configuration of Figure 2 are bounded by a bound $B > 0$ for all state-feedback functions φ_i, $i \geq 1$. By (5.16), we have $u_i = [I - \varphi_i \circ \Sigma]^{-1}w$, where, by our assumption, $u_i \in U(B)$. By Lemma 2.14, the sequence $\{u_i\}_{i=1}^{\infty}$ has a subsequence $\{u_{i_k}\}_{k=1}^{\infty}$ that converges weakly to a signal $u \in U(B)$. Now, for each integer $j \in \{1, 2, \ldots\}$, define the set

$$S_j := \bigcup_{k \geq j} u_{i_k} = \bigcup_{k \geq j} [I - \varphi_{i_k} \circ \Sigma]^{-1} w.$$

Denote by $\bar{S}_j$ the weak closure of the set S_j. As u is the weak limit of the sequence $\{u_{i_k}\}_{k=1}^{\infty}$, it follows that

$$\bigcap_{j \geq p} \bar{S}_j = \{u\}, \quad p = 1, 2, \ldots \tag{5.21}$$

Further, for an integer $q \geq j$, consider the set $V_{q,j} := [I - \varphi_{i_q} \circ \Sigma]\bar{S}_j$. By Lemma 5.19, as $q \to \infty$, the set $V_{q,j}$ is weakly convergent to the set $[I - \varphi \circ \Sigma]\bar{S}_j$ for almost every system $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$. Also, since $w = [I - \varphi_{i_k} \circ \Sigma]u_{i_k}$ for all $k = 1, 2, \ldots$, we have that $w \in [I - \varphi_{i_q} \circ \Sigma]\bar{S}_j$ for all $q \geq j$. The last two sentences imply that

$$w \in [I - \varphi \circ \Sigma]\bar{S}_j. \tag{5.22}$$

Next, using the fact that $[I - \varphi \circ \Sigma]$ is an isomorphism by Proposition 5.3, we obtain from (5.22) and (5.21) the relation

$$w \in \bigcap_{j \geq 1} [I - \varphi \circ \Sigma]\,\bar{S}_j = [I - \varphi \circ \Sigma]\left[\bigcap_{j \geq 1} \bar{S}_j\right] = [I - \varphi \circ \Sigma]\,\{u\}.$$

Therefore, $w = [I - \varphi \circ \Sigma]u$ for almost every $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$. By Proposition 5.3, we can write $u = [I - \varphi \circ \Sigma]^{-1}w$ for almost every $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$. This shows that u is the weak limit of the sequence $[I - \varphi_{i_k} \circ \Sigma]^{-1}w$. Thus, the signal u is a weakly continuous function of the state-feedback function φ for almost every system $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$. As the configuration of Figure 1 is obtained by taking $w = 0$ in Figure 2, our proof concludes. ■

The next statement shows that the response of the closed-loop system Σ_φ of Figure 1 depends in a continuous manner on the feedback function φ. This is one of the main results of this section.

Theorem 5.5: *Let $K, A, \sigma > 0$ be real numbers, where $\sqrt{\sigma} < A$, and assume that the nominal system Σ_0 is (K, A, σ)-controllable. Then, there are an uncertainty parameter $\gamma > 0$ and an amplitude bound $A' > A$ such that the closed-loop system Σ_φ of Figure 1 is a weakly continuous function of the state-feedback function $\varphi \in \Phi(\sigma, K, A', \gamma)$ for almost every system $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$.*

Proof: Referring to the configuration of Figure 2, we take $w = 0$; this makes this configuration identical to the configuration of Figure 1. Recall Proposition 4.3, according to which the class of feedback functions $\Phi(\sigma, K, A', \gamma)$ is not empty; we restrict our attention to feedback functions $\varphi \in \Phi(\sigma, K, A', \gamma)$. By Lemma 5.20, the response of the system $(I - \varphi \circ \Sigma)^{-1}$ is a weakly continuous function of the feedback function φ. In addition, Theorem 4.10(i) states that Σ is a weakly continuous function of its own input signal. As $\Sigma_{(+)\varphi} = \Sigma \circ (I - \varphi \circ \Sigma)^{-1}$ by (5.17), these facts imply that the response of $\Sigma_{(+)\varphi}$ is a weakly continuous function of the state-feedback function φ. Finally, since Σ_φ is $\Sigma_{(+)\varphi}$ when $w = 0$, our proof concludes. ■

We show next that the class of state-feedback functions $\Phi(\sigma, K, A, \gamma)$ of (2.18) is weakly compact.

Lemma 5.6: *The class of state-feedback functions $\Phi(\sigma, K, A, \gamma)$ is weakly compact in the topology of the Hilbert space $L_2^{\vartheta,n,m}$.*

Proof: Let $\{\varphi_k\}_{k=1}^{\infty} \subseteq \Phi(\sigma, K, A, \gamma)$ be a sequence of state-feedback functions. As $\Phi(\sigma, K, A, \gamma) \subseteq \Phi(K)$ and $\Phi(K)$ is weakly compact by Lemma 2.17, there is a subsequence $\{\varphi_{k_i}\}_{i=1}^{\infty}$ that converges weakly to a member $\varphi \in \Phi(K)$. By Theorem 5.23, this implies that, for every $t \geq 0$ and for every initial state $x_0 \in \rho(\sigma)$, the sequence of vectors $\{\Sigma_{\varphi_{k_i}}(x_0, t)\}_{i=1}^{\infty}$ converges, and

$$\lim_{i \to \infty} \Sigma_{\varphi_{k_i}}(x_0, t) = \Sigma_\varphi(x_0, t) \tag{5.25}$$

for all $t \geq 0$. But then, since $\{\varphi_k\}_{k=1}^{\infty} \subseteq \Phi(\sigma, K, A, \gamma)$, we have that $\Sigma_{\varphi_{k_i}}(x_0, t) \in [-A, A]^n$ for all $i = 1, 2, \ldots$ and all $t \geq 0$.

As $[-A, A]^n$ is a compact set in R^n, the latter implies that $\lim_{i\to\infty} \Sigma_{\varphi_{k_i}}(x_0, t) \in [-A, A]^n$ for all $x_0 \in \rho(\sigma)$ and all $t \geq 0$. In view of (5.25), we get that $\Sigma_\varphi(x_0, t) \in [-A, A]^n$ for all $t \geq 0$, so that $\varphi \in \Phi(\sigma, K, A, \gamma)$. This concludes our proof. ■

In the next sections, we employ the tools developed in the present section to prove the existence of optimal robust state-feedback functions.

6. Existence of optimal robust state-feedback functions

We turn now to an examination of the existence of solutions of Problem 3.4(i). We show in this section that optimal robust state-feedback functions exist whenever the nominal controlled system Σ_0 is (K, A, σ)-controllable and the uncertainty parameter γ is not too large. Our discussion depends on a few facts from mathematical analysis quoted in the next statement (e.g. Willard, 2004; Zeidler, 1985); part (iii) of the next statement is the Generalised Weierstrass Theorem.

Theorem 6.1: (i) *A weakly continuous functional is weakly lower semi-continuous.*
(ii) *Let S and A be topological spaces. Assume that, for every member $a \in A$, there is a weakly lower semi-continuous functional $f_a : S \to R$. If $\sup_{a\in A} f_a(s)$ exists at every point $s \in S$, then the functional $f(s) := \sup_{a\in A} f_a(s)$ is weakly lower semi-continuous on S.*
(iii) *A weakly lower semi-continuous functional attains a minimum in a weakly compact set.*

Consider now a finite interval of time $[0, \tau]$, $\tau > 0$. Let $\ell(\sigma, K, A, \gamma, \varphi, \tau)$ be the supremal deviation of the closed-loop system Σ_φ from the target state $x = 0$ over the time interval $[0, \tau]$, where the supremum is taken over all members $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ and over all initial conditions $x_0 \in \rho(\sigma)$; namely,

$$\ell(\sigma, K, A, \gamma, \varphi, \tau) := \operatorname*{ess\,sup}_{\Sigma\in\mathcal{F}_\gamma(\Sigma_0), x_0\in\rho(\sigma), t\in[0,\tau]} \left|\Sigma_\varphi(x_0, t)\right|_2^2. \tag{6.2}$$

As our ultimate aim is to take τ to infinity, we must restrict our attention to state-feedback functions that keep the response of the closed-loop system bounded by A at all times, namely, to state-feedback functions that are members of the family $\Phi(\sigma, K, A, \gamma)$ of (2.18). The infimal deviation over the time interval $[0, \tau]$ that can be achieved by such a state-feedback function is

$$\ell^*(\sigma, K, A, \gamma, \tau) = \inf_{\varphi\in\Phi(\sigma,K,A,\gamma)} \ell(\sigma, K, A, \gamma, \varphi, \tau). \tag{6.3}$$

Then, an optimal robust state-feedback functions φ^*_τ, if it exists, is one that satisfies the equality

$$\ell(\sigma, K, A, \gamma, \varphi^*_\tau, \tau) = \ell^*(\sigma, K, A, \gamma, \tau).$$

Such feedback functions optimise performance over the time interval $[0, \tau]$. Later, we prove the existence of optimal robust state-feedback functions φ^* that optimise performance over all times $t \geq 0$, as characterised by (3.3). By (3.1) and (3.2), we can write $\ell^*(\sigma, K, A, \gamma) = \sup_{\tau\to\infty} \ell^*(\sigma, K, A, \gamma, \tau)$; this supremum is bounded since feedback functions are in the set $\Phi(\sigma, K, A, \gamma)$.

Recall that, by Theorem 5.23, the closed-loop system Σ_φ is a weakly continuous function of the state-feedback function φ. Therefore, so is the functional $|\Sigma_\varphi(x_0, t)|_2^2$. Combining this fact with Equations (3.1) and (6.2), it follows by Theorem 6.1(ii) that the following is true.

Corollary 6.2: *The functionals $\ell(\sigma, K, A, \gamma, \varphi)$ and $\ell(\sigma, K, A, \gamma, \varphi, \tau)$ of (3.1) and (6.2), respectively, are weakly lower semi-continuous functionals of the state-feedback function φ.*

Now, according to Lemma 5.24, the set of state-feedback functions $\Phi(\sigma, K, A, \gamma)$ is weakly compact, and, according to Corollary 6.4, the functionals $\ell(\sigma, K, A, \gamma, \varphi)$ and $\ell(\sigma, K, A, \gamma, \varphi, \tau)$ are weakly lower semi-continuous functionals of the state-feedback function φ. Therefore, the Generalised Weierstrass Theorem (quoted above as Theorem 6.1(iii)) implies the existence of state-feedback functions φ^* and φ^*_τ, which achieve minima of $\ell(\sigma, K, A, \gamma, \varphi)$ and $\ell(\sigma, K, A, \gamma, \varphi, \tau)$, respectively. This proves the following statement, which forms the main result of the current section. It shows that there are optimal robust state-feedback functions that solve Problem 3.4(i).

Theorem 6.3: *Let $K, A_0, \sigma > 0$ be real numbers, where $\sqrt{\sigma} < A_0$; let $\tau > 0$ be a time, and refer to (6.3) and (3.2). Assume that the nominal system Σ_0 is (K, A_0, σ)-controllable, and let $\gamma > 0$ be an uncertainty parameter that satisfies the requirements of Proposition 4.2 with the state amplitude bound $A > A_0$. Then, there are optimal robust state-feedback functions $\varphi^*_\tau, \varphi^* \in \Phi(\sigma, K, A, \gamma)$ that achieve the (finite) minimal values $\ell(\sigma, K, A, \gamma, \varphi^*_\tau, \tau) = \ell^*(\sigma, K, A, \gamma, \tau)$ and $\ell(\sigma, K, A, \gamma, \varphi^*) = \ell^*(\sigma, K, A, \gamma)$, respectively.*

In summary, we have seen in this section that, under a broad controllability condition, there are optimal robust state-feedback functions that keep almost every member of the family $\mathcal{F}_\gamma(\Sigma_0)$ as close as possible to the target state at all times. The existence of such state-feedback functions is guaranteed by (K, A, σ)-controllability of the nominal controlled system Σ_0, as long as the uncertainty parameter γ is not too large.

In the paragraphs following Definition 4.1, we have seen that (K, A, σ)-controllability of the nominal controlled system is also close to being a necessary condition for the existence of optimal state-feedback functions. Thus, (K, A, σ)-controllability of the nominal controlled system is a tight sufficient condition for the existence of solutions of Problem 1.2(i). In the next section, we turn to Problem 1.2(ii) and show that optimal performance can be approximated by state-feedback functions that are relatively easy to design and implement.

7. Approximating optimal performance

We address now the issue raised in Problem 1.2(ii), namely, the development of state-feedback functions that approximate optimal performance, while being relatively easy to design and implement. Generally, state-feedback functions are multivariable vector-valued Lebesgue measurable functions of time: they have $n+1$ variables – the n state components of the controlled system Σ plus the time; and they have m components as their values, corresponding to the m components of the input vector of Σ. Designing and implementing such general functions may raise substantial challenges. In the present section, we apply the methodology of Hammer (2019), where it was shown that performance achieved by any state-feedback function can be approximated as closely as desired by the performance achieved by bang-bang state-feedback functions. Recall that bang-bang state-feedback functions are piecewise-constant functions, whose components switch between the two values of $-K$ and K, where K is the input amplitude bound of the controlled system Σ. Thus, bang-bang functions take their values in a finite set of only 2^m vectors – the set of all m-dimensional vectors with components of $-K$ or K. This makes bang-bang state-feedback functions easier to design and implement. In addition, as the state amplitude may not exceed the specified bound A, the state-space domain over which the bang-bang feedback function must be constructed is limited.

7.1 An additive disturbance

Needless to say, replacing an optimal state-feedback function by a bang-bang state-feedback function introduces errors into the feedback loop of Figure 1. These errors must be considered in conjunction with other errors, disturbances, and noises that may affect the closed-loop system. In line with this observation and following Hammer (2019), we take into account the effect of an external noise or disturbance signal $\nu(t) \in R^n$, $t \geq 0$, shown in Figure 3. As can be seen in the figure, the signal ν corrupts the input signal of the state-feedback function φ. We assume that ν is bounded by a specified amplitude bound $\Delta > 0$, and that it is a random signal with a uniform probability distribution. The class of such signals is

$$V(\Delta) := \left\{ \nu \in L_2^{\vartheta,m} : |\nu|_\infty \leq \Delta \right\}.$$

Let $\Delta(x)$ denote the hyper-square of edge 2Δ centred at a point x in the state set of the controlled system Σ. Then, with the disturbance ν active, the average signal produced by the state-feedback function φ at a time $t \geq 0$ is given by

$$\bar{\varphi}(t,x) := \frac{1}{(2\Delta)^n} \int_{\Delta(x)} \varphi(t,z)\,\mathrm{d}z.$$

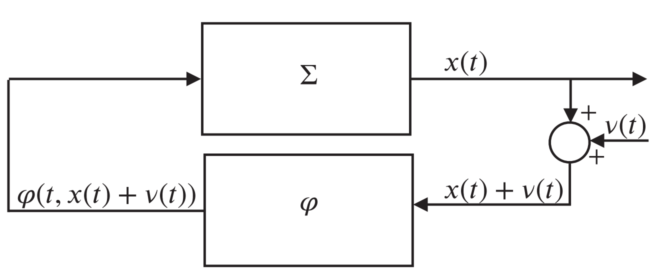

Figure 3. A common disturbance signal $\nu(t)$.

7.2 Approximating optimal performance

Denote by $\mathbb{K}^m$ the set of m-dimensional vectors with components of $-K$ or K, where $K > 0$ is the input amplitude bound of the controlled system Σ. The set $\mathbb{K}^m$ has 2^m members. At each point in time, a bang-bang state-feedback function take its values in the set $\mathbb{K}^m$. The following formal definition of bang-bang state-feedback functions is reproduced from Hammer (2019).

Definition 7.1: Let Σ be a system with m inputs, n states, an input amplitude bound K, and a state amplitude bound A. Let $\tau > 0$ be a time. A *bang-bang state-feedback function* for Σ over the time interval $[0,\tau]$ is a function $\varphi_\pm : [0,\tau] \times [-A,A]^n \to \mathbb{K}^m$ characterised as follows. There is a partition of the domain $[0,\tau] \times [-A,A]^n$ into a finite number $p \geq 1$ of hyper-rectangles $\sigma_1, \sigma_2, \ldots, \sigma_p$, where each component of $\varphi_\pm$ takes a constant value of $-K$ or K in the interior of each hyper-rectangle σ_j, $j \in \{1,2,\ldots,p\}$.

The next statement shows that the performance achieved by bang-bang state-feedback functions can be as close as desired to optimal performance. This is the main result of the current section.

Theorem 7.2: *Let $K, A_0, A, \sigma > 0$ be real numbers, where $\sqrt{\sigma} < A_0 < A$, and let $\tau > 0$ be a time. Assume that the nominal system Σ_0 is (K, A_0, σ)-controllable, and refer to* (6.2) *and* (6.3). *Then, for every real number $\varepsilon > 0$, there are a bang-bang state-feedback function $\varphi_\pm \in \Phi(\sigma,K,A,\gamma)$ and an uncertainty parameter $\gamma > 0$ such that $|\ell(\sigma,K,A,\gamma,\varphi_\pm,\tau) - \ell^*(\sigma,K,A_0,\gamma,\tau)| < \varepsilon$, when the feedback signal is averaged over the disturbance signal $\nu(t)$ of Figure 3.*

Remark 7.3: In Theorem 7.2, the response of the closed-loop system $\Sigma_{\varphi_\pm}$ may exceed the amplitude bound A_0 by ε. Note that $\varepsilon > 0$ can be selected as small as desired.

The proof of Theorem 7.2 depends on the next statement, which is reproduced here from Hammer (2019).

Theorem 7.4: *Let $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ be a system with initial state $x_0 \in \rho(\sigma)$. Let $\tau > 0$ be a finite time, and let $\varphi \in \Phi(\sigma,K,A,\gamma)$ be a state-feedback function. Then, for every real number $\varepsilon > 0$, there are a bang-bang state-feedback function $\varphi_\pm \in \Phi(K)$ and an uncertainty parameter $\gamma > 0$ such that the difference between the responses satisfies $|\Sigma_\varphi(x_0,t) - \Sigma_{\varphi_\pm}(x_0,t)| < \varepsilon$ at all times $t \in [0,\tau]$ and for almost all systems $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$; here, feedback is averaged over the disturbance signal $\nu(t)$ of Figure 3.*

We can prove now Theorem 7.2.

Proof: By Theorem 6.5, there is an optimal state-feedback function $\varphi^*_\tau \in \Phi(K)$. Let $\delta > 0$ be a real number. By Theorem 7.4, there is a bang-bang state-feedback function $\varphi_\pm \in \Phi(K)$ such that $|\Sigma_{\varphi^*_\tau}(x_0,t) - \Sigma_{\varphi_\pm}(x_0,t)| < \delta$ for all $t \in [0,\tau]$. Now, for any vectors $x, y \in R^n$, we can write $(x+y)(x-y)^\top = xx^\top - yy^\top + yx^\top - xy^\top$. As $yx^\top$ is a scalar, we have $yx^\top = (yx^\top)^\top = xy^\top$; substituting into the previous equality yields the relation $(x+y)(x-y)^\top = xx^\top - yy^\top$. Thus, $|xx^\top - yy^\top| \leq |(x+y)||$

$(x-y)|$. Applying this relation to our quantities and recalling that the response of Σ is bounded by A, we obtain

$$\begin{aligned}&\left|\ell(\sigma,K,A,\gamma,\varphi_{\pm},\tau)-\ell^{*}(\sigma,K,A_0,\gamma,\tau)\right|\\&\leq \sup_{t\in[0,\tau]}\left|\Sigma_{\varphi_\tau^*}(x_0,t)+\Sigma_{\varphi_\pm}(x_0,t)\right|\left|\Sigma_{\varphi_\tau^*}(x_0,t)-\Sigma_{\varphi_\pm}(x_0,t)\right|\\&\leq 2A\delta.\end{aligned}$$

Consequently, choosing $\delta \leq \varepsilon/(2A)$ satisfies the requirements of the theorem. This concludes our proof. ■

As we can see from Theorem 7.2, implementation can be simplified by using bang-bang state-feedback functions instead of optimal feedback functions, without causing significant performance degradation. As it turns out, in many cases of practical interest, relatively simple bang-bang state-feedback functions – namely, bang-bang state-feedback functions with sparse switching points – provide performance that is almost indistinguishable from optimal performance. One such case is demonstrated by the example of Section 8.

In some cases, bang-bang state-feedback functions that approximate optimal performance can be obtained by qualitative considerations based on the dynamical features of the controlled system. This is the case in the example of Section 8 below. In cases where qualitative considerations are not effective, bang-bang state-feedback functions that approximate optimal performance can be derived through a numerical search process (see Section 8 below and Hammer, 2019).

7.3 Implementation considerations

When implementing bang-bang state-feedback functions that must operate over extended periods of time, one must give consideration to the potential appearance of jitter – the needless to-and-fro switching around a switching surface. To prevent jitter, one may replace abrupt switching surfaces of bang-bang state-feedback functions by gradual transitions. For instance, suppose that a switch from K to $-K$ of the j-the coordinate φ^j of a bang-bang state-feedback function φ is indicated at the point x_s^i of the ith state coordinate. One may replace this abrupt transition by a gradual transition as follows. Select a real number $\delta > 0$ and perform the transition from K to $-K$ gradually over the interval $[x_s^i-\delta, x_s^i+\delta]$, by setting

$$\varphi^j := \begin{cases} K & x^i < x_s^i - \delta, \\ K\dfrac{x_s^i - x^i}{\delta} & x^i \in [x_s^i-\delta, x_s^i+\delta], \\ -K & x^i > x_s^i + \delta. \end{cases} \tag{7.5}$$

This would prevent potential jitter around the switching surface. A construction along these lines is demonstrated in the example of Section 8.

Alternatively, one may introduce hysteresis into a bang-bang state-feedback function to prevent jitter. This is accomplished by including values of the state-feedback function among the variables that determine switching points. Referring to the state-feedback function φ of the previous paragraph, hysteresis in the component φ^j near the switching point x_s^i can be accomplished by setting

$$\varphi^j := \begin{cases} -K & \text{if } \varphi^j = K \text{ and } x^i > x_s^i + \delta, \\ K & \text{if } \varphi^j = -K \text{ and } x^i < x_s^i - \delta. \end{cases} \tag{7.6}$$

Notwithstanding, abrupt switching surfaces can be used during the process of deriving bang-bang state-feedback functions. Once the derivation is complete, the resulting bang-bang state-feedback function can be 'softened' prior to implementation by a process such as the one described in (7.5) or in (7.6).

8. Illustration and discussion

8.1 Example

The model of an inverted pendulum is encountered in many control engineering applications, including missile control systems, dynamic stabilisation systems of civil engineering structures, walking robots, personal transport vehicles, and others. We consider the control of the following inverted pendulum (Bian et al., 2014):

$$\Sigma : \begin{aligned}&\dot{x}_1(t) = x_2(t),\\&\dot{x}_2(t) = d_1 \sin x_1(t) + d_2 x_2(t) + d_3 \tanh u(t).\end{aligned} \tag{8.1}$$

Here, d_1, d_2, and d_3 are constant parameters; their nominal values are $d_1^0 = 12$, $d_2^0 = -0.1$, $d_3^0 = 7$, and they are subject to uncertainty ranges of about 5%, so that $d_1 \in [11.5, 12.5]$, $d_2 \in [-0.105, -0.095]$, and $d_3 \in [6.7, 7.3]$. The input amplitude bound of Σ is $K = 1$, and the state amplitude bound of Σ is $A = 0.5$. The initial state can be any member of $\rho(0.2)$, i.e. $\sigma = 0.2$.

A numerical search process, such as the one described in Choi and Hammer (2019a), verifies that the nominal system is (K, A, σ)-controllable for the current K, A, and σ. In the following paragraphs, we describe a bang-bang state-feedback functions $\varphi_\pm$, which, when connected to Σ, achieves (almost) minimal deviation from the target state $x = 0$ over the entire time axis $t \geq 0$. The state-feedback function $\varphi_\pm$ is given by (8.2); it is a softened version of a bang-bang function, as discussed in Subsection 7.3. With this state-feedback function, the response of the closed-loop system is plotted in Figure 4. The plot uses a 'close to worst case' initial condition $x_0 = (-0.3, -0.3)^\top \in \rho(0.2)$ (Note that $0.3^2 + 0.3^2 = 0.18 < 0.2$). To avoid clutter, the figure is drawn only for the nominal system; similar response is obtained for other values of the parameters d_1, d_2, and d_3.

The plot of Figure 4 shows the response of the closed-loop system with the state-feedback function $\varphi_\pm$ during the time interval $[0, 0.5]$. As we can see, the state amplitude does not exceed the bound $A = 0.5$ during this time interval, and, at the end of this time interval, the system's state is within the domain $\rho(0.2)$. Based on time invariance of Σ, the feedback function $\varphi_\pm$ can be shifted in time and applied again at the time $t = 0.5$, to guide Σ during the time interval $[0.5, 1]$. During this new time interval, the resulting closed-loop system's state amplitude does not exceed the bound $A = 0.5$, and its state is within $\rho(0.2)$ at the end. Continuing in this manner, shift the state-feedback function $\varphi_\pm$ of (8.2) in time repeatedly to time intervals that

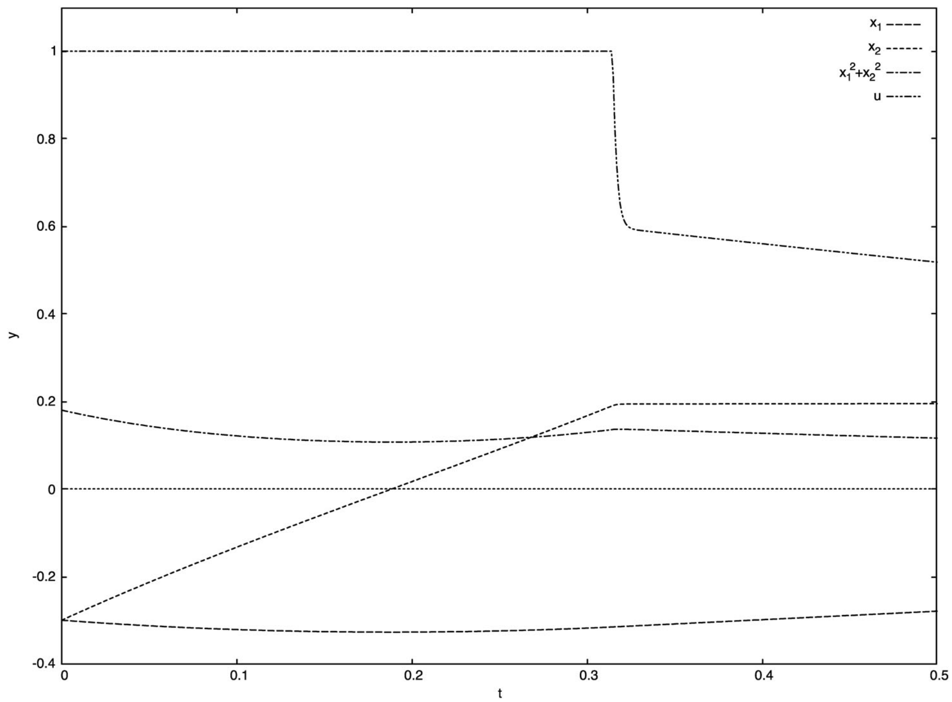

Figure 4. The response.

start at multiples of 0.5 and last 0.5 seconds; namely, the time intervals $[0.5, 1], [1, 1.5], [1.5, 2], \ldots$ This results in a feedback function that keeps the state of Σ within its specified amplitude bound at all times $t \geq 0$; it also achieves the minimal deviation from the target state $x = 0$ over the entire time axis $t \geq 0$, as we discuss next.

Figure 4 shows that the maximal state amplitude of the closed-loop system's response is $\max_{t \in [0,0.5]}\{|x_1(t)|, |x_2(t)|\} = 0.327$; this complies with the amplitude bound $A = 0.5$ imposed by the controlled system's characteristics. The plot also shows that the maximal deviation from the target state is $\ell = \max_{t \in [0,0.5]}\{x_1^2(t) + x_2^2(t)\} = 0.18$ for this state-feedback function with the current initial condition $x(0) = (-0.3, -0.3)^\top$. An examination of Figure 4 together with features of the state-feedback function $\varphi_\pm$ of (8.2) shows that this feedback functions yields a deviation of $\ell^* = 0.2$ over the entire time axis $t \geq 0$. Recalling that the minimal deviation always satisfies $\ell^* \geq \sigma$ and that $\sigma = 0.2$ in this case, it follows that $\varphi_\pm$ actually achieves the minimal deviation.

$$\varphi_\pm = \begin{cases} 1 & \text{if} x_1 \leq 0 \text{ and } x_2 < 0.19, \\ -1 & \text{if } x_1 \leq 0 \text{ and } x_2 > 0.21, \\ 1 - 100(x_2 - 0.19) & \text{if } x_1 \leq 0 \text{ and } x_2 \in [0.19, 0.21], \\ 1 & \text{if } x_1 > 0 \text{ and } x_2 < -0.21, \\ -1 & \text{if } x_1 > 0 \text{ and } x_2 > -0.19, \\ -1 - 100(x_2 + 0.19) & \text{if } x_1 > 0 \text{ and} \\ & x_2 \in [-0.21, -0.19]. \end{cases} \tag{8.2}$$

The function $\varphi_\pm$ is depicted in the plot of Figure 5. Methods for deriving bang-bang state-feedback functions are discussed in the next subsection.

8.2 Deriving bang-bang state-feedback functions

For certain controlled systems, such as the inverted pendulum (8.1), it is possible to derive bang-bang state-feedback functions that approximates optimal performance simply through insight into the dynamics of the controlled system. For systems where such insight is unavailable, a generic search process like the one described below can be used to derive bang-bang state feedback functions that approximate optimal performance.

As a preliminary step, it is necessary to verify that the controlled system Σ is (K, A, σ)-controllable. This can be accomplished by a numerical search process like the one described by Choi and Hammer (2019a). Such a search process also yields a time $\tau > 0$ by which every state in $\rho(\sigma)$ can be driven into the interior of $\rho(\sigma)$ by an appropriate input signal.

The time τ can be used in conjunction with Theorem 7.4 and the search process of Subsection 8.2.1 below to derive an appropriate bang-bang state feedback function over the time interval $[0, \tau]$. Then, following a process similar to the one described earlier in Subsection 8.1, this state-feedback function can be extended to the entire time axis through a time shifting process based on time-invariance of the nominal controlled system Σ_0 (see also Proof of Proposition 4.3). Larger values of τ may also be tested.

8.2.1 Deriving bang-bang state-feedback functions through a search process

Let τ be the time mentioned in the previous paragraph, and let A be the bound of the controlled system's state amplitude; then state coordinates are restricted to the interval $[-A, A]$. Select two integers $N, N' \geq 1$ such that the ratios τ/N and $2A/N'$ are sufficiently small for required implementation accuracy. The values of N and N' depend on the functions that appear in the

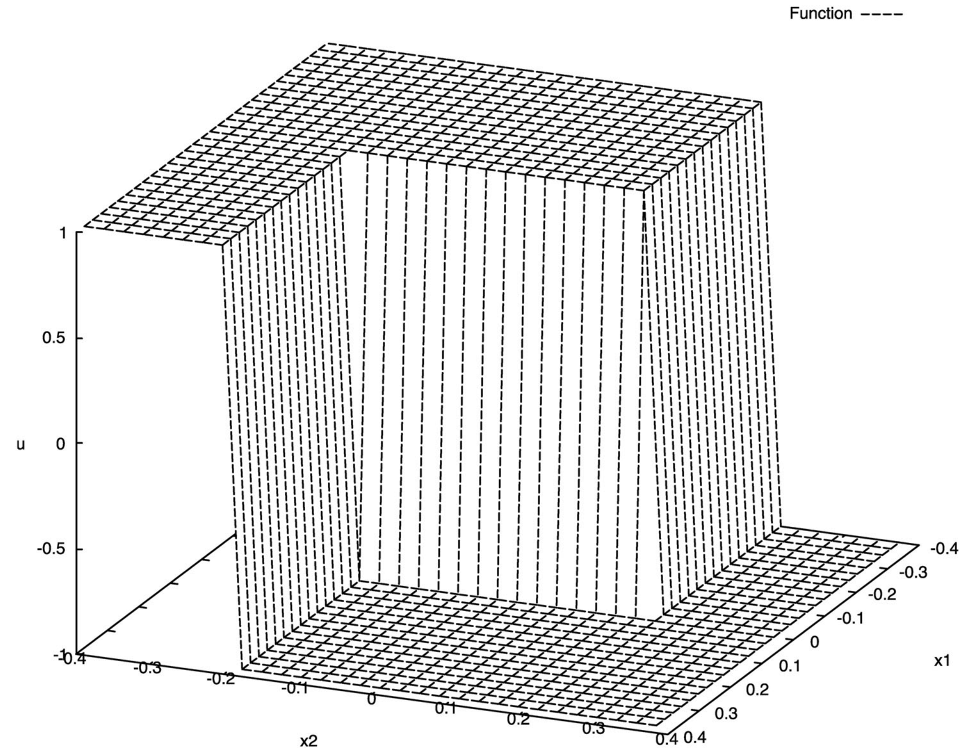

Figure 5. State-feedback function.

differential equation of the controlled system Σ. After selecting N and N', proceed as follows.

Preliminary Step 1: Partition the interval $[0, \tau]$ into N sub-intervals of length τ/N: $[0, \tau/N]$, $[\tau/N, 2\tau/N]$, $[2\tau/N, 3\tau/N], \ldots, (N-1)\tau/N, \tau]$.

Preliminary Step 2: Partition each state coordinate interval $[-A, A]$ into N' sub-intervals of length $2A/N'$: $[-A, -A + 2A/N']$, $[-A + 2A/N', -A + 4A/N']$, $[-A + 4A/N', -A + 6A/N'], \ldots, [-A + 2(N'-1)A/N', A]$.

The result is a set C of $N(N')^n$ points in the domain $[0, \tau] \times [-A, A]^n$. Using points of the set C as potential switching points, test the performance of bang-bang state-feedback functions with an increasing number of switchings, as described qualitatively by the following.

Algorithm 8.3 (qualitative): *Let $\rho(\sigma)$ be the domain of initial states of the controlled system and refer to the notation of Subsection 8.2.1. The interest is in bang-bang state-feedback functions $\varphi : [0, \tau] \times [-A, A]^n \to \mathbb{K}^m$ that satisfy the condition*

$$\sup_{t \in [0,\tau], x_0 \in \rho(\sigma)} \left| \Sigma_\varphi(x_0, t) \right| \leq A \quad \textit{and}$$

$$\sup_{x_0 \in \rho(\sigma)} \left| \Sigma_\varphi(x_0, \tau) \right|_2^2 \leq \sigma' \quad \textit{for some } \sigma' < \sigma. \tag{8.4}$$

The search is over bang-bang state-feedback functions with an increasing number of switching points from the set C, as follows. For an integers $k = 0, 1, 2, \ldots$, search over bang-bang state-feedback functions with k switchings in C; move the k switching points through all points of C, testing each resulting bang-bang state-feedback function and its negative. Among all bang-bang state-feedback functions with k switchings in C that satisfy condition (8.4)*, denote by ℓ_k the lowest deviation of the closed-loop system's state from the target state $x = 0$ over the time interval $[0, \tau]$. Denote by φ_k a bang-bang state-feedback function with k switchings that achieves the deviation ℓ_k. Terminate the search when further increase of k beyond a value of k' does not yield a significant reduction of ℓ_k below the value of $\ell_{k'}$. Then, use $\varphi_{k'}$ as a bang-bang state-feedback function approximating optimal performance.*

In many cases, as in the case of the example of Subsection 8.1, the resulting bang-bang state-feedback functions have relatively sparse switching points and are easy to implement.

9. Conclusion

The paper concentrates on the existence and the design of optimal robust state-feedback controllers that provide optimal tracking for nonlinear input-affine systems over the infinite time horizon. The effects of uncertainty about the controlled system's model were taken into account, as were the effects of constraints on the input amplitude and the state amplitude of the controlled system. It was shown that optimal robust state-feedback controllers exist, as long as the controlled system satisfies the condition of constraint controllability. It was also shown that this condition is close to being a necessary condition for the existence of optimal robust state-feedback controllers.

Once the existence of optimal robust state-feedback controllers was established, the paper describes a relatively simple methodology for the design and implementation of state-feedback controllers that approximate optimal performance.

This implementation is based on the use of bang-bang state-feedback functions – piecewise constant functions whose components switch between two specified values. Bang-bang state-feedback functions are simpler to design and implement than general state-feedback functions. An example of a non-linearised inverted pendulum was used to demonstrate the results.

Future research efforts will concentrate on the development of fast numerical algorithms for the design of bang-bang state-feedback functions that approximate optimal performance.

Disclosure statement

No potential conflict of interest was reported by the author(s).

References

Bian, T., Jiang, Y., & Jiang, Z.-P. (2014). Adaptive dynamic programming and optimal control of nonlinear nonaffine systems. *Automatica*, *50*(10), 2624–2632. https://doi.org/10.1016/j.automatica.2014.08.023

Chakraborty, D., & Hammer, J. (2008). Bang-bang functions: Universal approximants for the solution of min-max optimal control problems. In *Proceedings of the international symposium on dynamic games and applications*.

Chakraborty, D., & Hammer, J. (2009). Optimal control during feedback failure. *International Journal of Control*, *82*(8), 1448–1468. https://doi.org/10.1080/00207170802510265

Chakraborty, D., & Hammer, J. (2010). Robust optimal control: Low-error operation for the longest time. *International Journal of Control*, *83*(4), 731–740. https://doi.org/10.1080/00207170903406249

Chakraborty, D., & Shaikshavali, C. (2009). An approximate solution to the norm optimal control problem. In *Proceedings of the IEEE international conference on systems, man, and cybernetics* (pp. 4490–4495).

Choi, H.-L., & Hammer, J. (2018a). Optimal robust control of nonlinear time-delay systems: Maintaining low operating errors during feedback outages. *International Journal of Control*, *91*(2), 297–319. https://doi.org/10.1080/00207179.2016.1278273

Choi, H.-L., & Hammer, J. (2018b). Recovering in minimal time from feedback loss: Bounded overshoots. In *Proceeding of the American control conference* (pp. 4670–4675).

Choi, H.-L., & Hammer, J. (2019a). Fastest recovery after feedback disruption: Nonlinear delay-differential systems. *International Journal of Control*, *92*(4), 717–733. https://doi.org/10.1080/00207179.2017.1367103

Choi, H.-L., & Hammer, J. (2019b). Fastest recovery from feedback loss: Bounded overshoot. *International Journal of Control*, *92*(9), 2077–2090. https://doi.org/10.1080/00207179.2018.1427891

Choi, H.-L., & Hammer, J. (2020). Periodic sampling: Maximizing the sampling period. *International Journal of Control*, *93*(6), 1303–1316. https://doi.org/10.1080/00207179.2018.1504234

Gamkrelidze, R. (1965). On some extremal problems in the theory of differential equations with applications to the theory of optimal control. *SIAM Journal on Control*, *3*, 106–128.

Hammer, J. (1986). Fraction representations of nonlinear systems: A simplified approach. *International Journal of Control*, *46*(2), 455–472. https://doi.org/10.1080/00207178708933910

Hammer, J. (2019). Optimal robust state-feedback control of nonlinear systems: Minimal time to target. *International Journal of Control*. To appear. https://doi.org/10.1080/00207179.2019.1597279

Hubbard, J., & Hubbard, B. (2006). *Vector calculus, linear algebra, and diferential forms: A unified aproach*. Prentice Hall.

Kelendzheridze, D. (1961). On the theory of optimal pursuit. *Soviet Mathematics Doklady*, *2*, 654–656.

Luenberger, D. G. (1969). *Optimization by vector space methods*. Wiley.

Neustadt, L. (1966). An abstract variational theory with applications to a broad class of optimization problems I, general theory. *SIAM Journal on Control*, *4*, 505–527. https://doi.org/10.1137/0304040

Neustadt, L. (1967). An abstract variational theory with applications to a broad class of optimization problems II, applications. *SIAM Journal on Control*, *5*, 90–137. https://doi.org/10.1137/0305007

Pontryagin, L., Boltyansky, V., Gamkrelidze, R., & Mishchenko, E. (1962). *The mathematical theory of optimal processes*. Wiley.

Warga, J. (1972). *Optimal control of differential and functional equations*. Academic Press.

Willard, S. (2004). *General topology*. Dover Publications.

Young, L. (1969). *Lectures on the calculus of variations and optimal control theory*. W. B. Saunders.

Yu, Z., & Hammer, J. (2016). Fastest recovery after feedback disruption. *International Journal of Control*, *89*(10), 2121–2138. https://doi.org/10.1080/00207179.2016.1149885

Zeidler, E. (1985). *Nonlinear functional analysis and its applications III*. Springer-Verlag.