Proceedings of the 26th International Symposium on Mathematical Theory of Networks and Systems,
University of Cambridge, UK, August 2024.

Nonlinear Sampled-Data Control: Optimal Robust Inter-Sample Tracking

Jacob Hammer *

* *Department of Electrical and Computer Engineering, University of Florida, Gainesville, FL 32611, USA (e-mail: hammer@mst.ufl.edu).*

Abstract: The design and implementation of optimal robust controllers to optimize inter-sample tracking in nonlinear sampled-data control systems is considered. Optimal robust controllers that minimize inter-sample tracking errors are shown to exist for a very broad family of nonlinear systems that includes most systems of practical interest. A relatively simple technique for the design and implementation of such controllers is presented. Results apply to most nonlinear systems of practical interest.

Keywords: nonlinear control, nonlinear state feedback, sampled data control, optimal control.

1. INTRODUCTION

Sampled-data control refers to the control of continuous-time systems by digital controllers through a process of periodic sampling, as seen Figure 1. Sample-data control has become popular in the last few decades due to its low cost and implementation simplicity. In this note, we prove the existence of optimal robust controllers that minimize inter-sample tracking errors in sampled-data systems. We also present a relatively simple design and implementation methodology for such optimal robust controllers. The results are valid for a broad class of nonlinear systems.

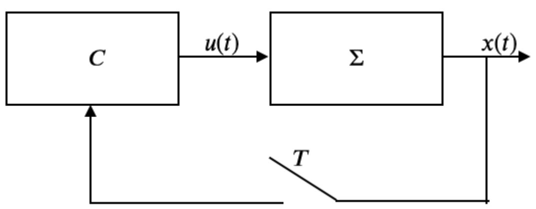


Fig. 1. Sampled Data Control

In Figure 1, the controlled system Σ is a continuous-time nonlinear system with input signal $u(t)$ and state $x(t)$; here, $u(t)$ is produced by the controller C based on samples of $x(t)$ it receives periodically every $T > 0$ seconds. Note that the control system operates in open loop between samples. The system Σ imposes bounds of $K > 0$ and $A > 0$ on its input and state amplitudes, respectively, to accommodate common operating constraints of the system. The controller C must be robust to tolerate uncertainties about the model of Σ.

The task of the controller C is to optimize inter-sample performance to maintain Σ as close as possible to the target state $x_{target} = 0$ between samples, when there is no feedback. The *inter-sample tracking error* is $\ell := \sup_{t\in[0,T]} x^\top(t)x(t)$.

1.1 The controlled system

The system Σ of Figure 1 is described by

$$\Sigma : \begin{array}{l} \dot{x}(t) = f(t, x(t), u(t)), t \geq 0, \\ x(0) = x_0, \end{array} \tag{1}$$

where $x(t) \in R^n$ is the state; $u(t) \in R^m$ is the input; and $f : R^+ \times R^n \times R^m \to R^n$ is the *recursion function*, which is continuously differentiable. Here, R is the set of real numbers, and R^+ represents the non-negative real numbers. The initial state x_0 is in the ball $\rho(\sigma) := \{x : x^\top x \leq \sigma\}$, where $\sigma > 0$ is specified. In functional notation, we represent $x(t)$ by $\Sigma(x_0, u, t)$. The L^2-norm of a vector $x \in R^n$ is $|x|_2 := (x^\top x)^{1/2}$.

Generally, optimal controllers may be difficult to design and implement, since they must generate involved vector-valued multivariable functions of time as input to Σ. One of our tasks is to find controllers that approximate optimal performance and are easier to design and implement. We concentrate on the following topics.

Problem 1. The system Σ of Figure 1 imposes amplitude bounds of $K > 0$ and $A > 0$ on its input and state, respectively, and its model is not precisely known. The state of Σ is sampled at a specified sampling period of $T > 0$.
(i) Obtain conditions for the existence of optimal robust controllers C that minimize inter-sample tracking errors.
(ii) Find easy to design and implement controllers that approximate optimal performance. □

To simplify the mathematical framework, we require input signals of Σ to be continuous functions of time. As discussed in Section 2, practically all signals used in applications are continuous functions of time, when viewed in a sufficiently refined time scale. The restriction to continuous functions of time allows us to obtain a major generalization of the results of Chakraborty and Hammer (2008, 2009); Yu and Hammer (2016); Choi and Hammer (2019), by generalizing from input-affine systems to a family of nonlinear systems that includes most systems of practical interest. Our discussion generalizes classical optimization theory (Kelendzheridze (1961); Pontryagin et al. (1962); Gamkrelidze (1965); Neustadt (1966, 1967); Luenberger (1969); Young (1969); Warga (1972) by proving existence of optimal solutions for broad families of nonlinear systems; by proving robustness of these solutions; and by outlining simple means of implementing approximants of optimal solutions.

This note is organized as follows. Sections 2 and 3 introduce background material, while Section 4 reformulates Problem 1 in formal terms. Existence of optimal robust inter-sample controllers is proved in Section 6; their existence depends on a certain notion of controllability, which is reviewed and refined in Section 5. Section 7 describes relatively simple controllers that approximate optimal performance. Section 8 is an example and Section 9 provides a brief summary of this note.

2. CONTINUOUS INPUT SIGNALS

We assume that the controlled system Σ of Figure 1 receives only continuous input signals. This is not an over-restrictive assumption, since practically all signals in continuous-time applications are continuous functions of time, when viewed in a fine time scale. We introduce our class of continuous signals through their Fourier transform, as follows.

Let $\mathbb{C}$ denote the complex numbers, and let $|c|$ be the absolute value of a complex number c. The L^∞-norm of a matrix $V \in \mathbb{C}^{n\times m}$ is $|V| := \max_{i,j} |V_{ij}|$, while the L^∞-norm of a function of time $V : R^+ \to \mathbb{C}^{n\times m} : t \mapsto V(t)$ is $|V|_\infty := \sup_{t\geq 0} |V(t)|$.

We are interested in the class Ω of Lebesgue measurable complex vector valued functions $\upsilon : R \to \mathbb{C}^m : \omega \mapsto \upsilon(\omega)$, whose inverse Fourier transforms are real vector valued functions of time. Denoting by $\upsilon^i(\omega)$ component i of υ and by $\angle\upsilon^i(\omega)$ its phase, it is known that

$$\Omega := \left\{ \upsilon : R \to \mathbb{C}^m : \begin{array}{l} |\upsilon^i(\omega)| \text{ is even and } \angle\upsilon^i(\omega) \text{ is odd} \\ \text{as a function of } \omega, i = 1, 2, \ldots, m. \end{array} \right\}$$

A member $\upsilon \in \Omega$ induces a function of time $u = \mathscr{F}^{-1}\upsilon$ through the inverse Fourier transform. To assure that u is of finite energy, the square magnitude of υ must be bounded and integrable (Parseval's theorem). This leads us to the following family of exponentially bounded functions

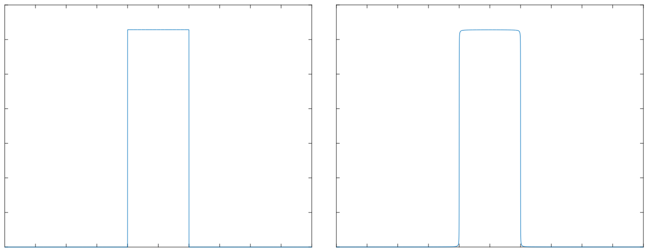
$$\Omega(W,\kappa) := \left\{ \upsilon \in \Omega : |\upsilon(\omega)| \leq W e^{-\kappa|\omega|} \text{ for all } \omega \in R \right\}, \tag{2}$$

where $W, \kappa > 0$. The corresponding time domain family of signals is

$$U(W,\kappa) = \left\{ \mathscr{F}^{-1}\upsilon : \upsilon \in \Omega(W,\kappa) \right\}. \tag{3}$$

We refer to κ as the *smoothing factor*, since, as we show later, it relates to the fastest rise time possible for signals in $U(W,\kappa)$. As an example of members of $U(W,\kappa)$, consider Figure 2, which compares a pulse to its counterpart in $U(W,\kappa)$, using $\kappa = 0.001$. As the figure shows, members of $U(W,\kappa)$ can closely imitate jumps.

Lemma 2. *(i)* All members of $U(W,\kappa)$ are bounded.
(ii) All members of $U(W,\kappa)$ are equally uniformly continuous.



(a) Pulse signal (b) Pseudo bang-bang signal

Fig. 2. 'Smoothed' pulse

Proof. [Proof outline] We outline the proof of (ii). For times $t_1 < t_2$ and a real number $\omega_0 > 0$ we can write

$$|u(t_2) - u(t_1)| \leq \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} W e^{-\kappa|\omega|} \left| e^{j\omega t_2} - e^{j\omega t_1} \right| d\omega + \frac{2W}{\pi\kappa} e^{-\kappa\omega_0} \tag{4}$$

For a fixed ω, the mean value theorem yields $e^{j\omega t_2} - e^{j\omega t_1} = -\omega \sin(\omega t')(t_2 - t_1) + j\omega\cos(\omega t'')(t_2 - t_1)$, for some $t', t'' \in [t_1, t_2]$. Thus, $\left|e^{j\omega t_2} - e^{j\omega t_1}\right| \leq \sqrt{2}|\omega|(t_2 - t_1)$. Substituting into (4), we get

$$|u(t_2) - u(t_1)| \leq \frac{\sqrt{2}W}{\pi\kappa^2}\left[1 - e^{-\kappa\omega_0}(\kappa\omega_0 + 1)\right](t_2 - t_1) + (2W/\pi\kappa)e^{-\kappa\omega_0}. \tag{5}$$

Given $\varepsilon > 0$, choose ω_0 to satisfy $(2W/\pi\kappa)e^{-\kappa\omega_0} < \varepsilon/2$; thereafter, choose $\delta > 0$ to satisfy $\sqrt{2}W/\pi\kappa^2)[1 - e^{-\kappa\omega_0}(\kappa\omega_0 + 1)]\delta < \varepsilon/2$. Then, by (5), we have $|u(t_2) - u(t_1)| < \varepsilon$ for all $|t_2 - t_1| < \delta$ and any $u \in U(W,\kappa)$. □

As input signals to the controlled system Σ must be bounded by $K > 0$, we concentrate on the signal family

$$U(K,W,\kappa) := \{u \in U(W,\kappa) : |u|_\infty \leq K\}. \tag{6}$$

A Hilbert space. Let $\mathcal{H}$ be the Hilbert space of Lebesgue measurable functions $f, g : R \to \mathbb{C}^m$ with inner product $\langle f, g\rangle = \int_{-\infty}^{\infty} \bar{f}^\top(\omega)g(\omega)d\omega$, where $\bar{f}$ is the complex conjugate of f. For members $f, g \in \Omega(W,\kappa)$, a calculation yields $|\langle f, g\rangle| \leq mW^2/\kappa$, so $\Omega(W,\kappa)$ members have bounded inner products. We use the following mathematical notions (e.g., Lusternik and Sobolev (1961)).

Definition 3. A sequence $\{\upsilon_n\}_{n=1}^\infty \subseteq \mathcal{H}$ is *weakly convergent* to $\upsilon \in \mathcal{H}$ if $\lim_{n\to\infty} \langle \upsilon_n, y\rangle = \langle \upsilon, y\rangle$ for all $y \in \mathcal{H}$. A subset $G \subseteq \mathcal{H}$ is *weakly compact* if every sequence in G has a subsequence that converges weakly to a member of G. □

Next, we show that $\Omega(W,\kappa)$ of (2) is weakly compact.

Lemma 4. The set $\Omega(W,\kappa)$ is weakly compact in the topology of the Hilbert space $\mathcal{H}$.

Proof. [Proof outline] The proof is similar to the proof of an analogous statement of Chakraborty and Hammer (2009, 2010). See Hammer (2023) for details. □

We use the following terms.

Definition 5. A family G of functions of time is *pointwise compact* if every sequence $\{g_i\}_{i=1}^\infty \subseteq G$ has a subsequence $\{g_{i_k}\}_{k=1}^\infty$ that converges pointwise to a function $g \in G$, i.e.,

$\lim_{k\to\infty} g_{i_k}(t) = g(t)$ for all t. The family G is *uniformly pointwise compact* if, for any times $t_1 \le t_2$ and any real number $\varepsilon > 0$, there is an integer $N \ge 1$ for which $|g_{i_k}(t) - g(t)| < \varepsilon$ for all $k \ge N$ and all $t \in [t_1, t_2]$. □

The next statement highlights a critical property of $U(K, W, \kappa)$.

Lemma 6. The family $U(K, W, \kappa)$ is uniformly pointwise compact.

Proof. [Proof outline] For a real number $a > 0$, define the function

$$p_a(\omega, r) := \begin{cases} (0,\ldots,0,1,0,\ldots,0)^\top \text{ (1 in row } r), & \omega \in [-a, a], \\ 0, & \omega \notin [-a, a]. \end{cases}$$

Consider a sequence $\{u_k\}_{k=1}^\infty \subseteq U(K, W, \kappa)$. Then, by (3) and (6), $\upsilon_k = \mathscr{F} u_k \in \Omega(W, \kappa)$. As $\Omega(W, \kappa)$ is weakly compact by Lemma 4, there is a subsequence $\{\upsilon_{k_i}\}_{i=1}^\infty$ that converges weakly to a function $\upsilon \in \Omega(W, \kappa)$. Then,

$$\lim_{i\to\infty} \int_{-\infty}^{\infty} \bar{p}_a^\top(\omega, r)[(\upsilon_{k_i}(\omega) - \upsilon(\omega))e^{j\omega t}]d\omega$$
$$= \lim_{i\to\infty} \int_{-\infty}^{\infty} [\overline{p_a^\top(\omega, r)e^{-j\omega t}}][(\upsilon_{k_i}(\omega) - \upsilon(\omega))]d\omega = 0,$$

so that

$$\lim_{i\to\infty} \int_{-a}^{a} (\upsilon_{k_i}^r(\omega) - \upsilon^r(\omega))e^{j\omega t}d\omega = 0, r = 1,\ldots,m. \quad (7)$$

Set $u := \mathscr{F}\upsilon$; then, by (7), it follows that

$$\lim_{i\to\infty} |u_{k_i} - u| = \tfrac{1}{2\pi}|\lim_{i\to\infty} \textstyle\int_{-\infty}^{\infty} (\upsilon_{k_i}^r(\omega) - \upsilon^r(\omega))e^{j\omega t}d\omega$$
$$\lim_{i\to\infty} \textstyle\int_{-a}^{a} (\upsilon_{k_i}^r(\omega) - \upsilon^r(\omega))e^{j\omega t}d\omega| \le \tfrac{W}{\pi\kappa}e^{-\kappa a}$$

As $a > 0$ can be selected arbitrarily large, it follows that the sequence $\{u_{k_i}\}_{i=1}^\infty$ is pointwise convergent to u. A proof that this convergence is uniform is in Hammer (2023). □

3. NONLINEAR SYSTEMS

To incorporate modeling uncertainties and disturbances, we decompose the recursion function f of (1) into a sum

$$f(t, x, u) = f_0(x, u) + f_\gamma(t, x, u) \quad (8)$$

of continuously differentiable functions; here, f_0 is a specified *nominal recursion function*, and f_γ is an unknown *uncertainty function* representing uncertainties and disturbances. The nominal system

$$\Sigma_0 : \dot{x}(t) = f_0(x(t), u(t)), t \ge 0; x(0) = x_0, \quad (9)$$

is time invariant.

The input and state amplitude bounds of Σ restrict f_0 and f_γ to the domain $R^+ \times [-A, A]^n \times [-K, K]^m$. As f_0 and f_γ are continuously differentiable and the domain $[-A, A]^n \times [-K, K]^m$ is compact, there are bounds $B, \gamma(t), \gamma \ge 0$ such that

$$|\partial f_0(c)/\partial \left(\begin{smallmatrix} x \\ u \end{smallmatrix}\right)| \le B; |\partial f_\gamma(t, c'(t))/\partial \left(\begin{smallmatrix} x \\ u \end{smallmatrix}\right)| \le \gamma(t), \gamma(t) \le \gamma, \quad (10)$$

for all $(t, c, c'(t)) \in R^+ \times [-A, A]^n \times [-K, K]^m$, where we assumed that there is a constant $\gamma > 0$ such that $\gamma(t) \le \gamma$ for all t, reflecting boundedness of uncertainty effects. To simplify notation, we use the same bounds for values at the origin:

$$|f_0(0, 0)| \le B \text{ and } |f_\gamma(t, 0, 0)| \le \gamma, t \ge 0. \quad (11)$$

Notation 7. For real numbers $K, A, \sigma > 0$, let $\mathcal{S}_\gamma(\Sigma_0, K, A)$ be the set of all systems consistent with (1), (8), (10), and (11), with input and state amplitudes bounded by K and A, respectively, and initial states in $\rho(\sigma)$; here, Σ_0 is the nominal system. Furthermore,

(i) All input signals belong to $U(K, W, \kappa)$.
(ii) All systems have the same initial state $x_0 \in \rho(\sigma)$.
(iii) All systems have the same controller C. □

Item *(ii)* is valid since the sampler of Figure 1 transmits the actual state $x(0) = x_0$ at $t = 0$. Item *(iii)* represents robustness of the controller C: as it is not known which member of $\mathcal{S}_\gamma(\Sigma_0, K, A)$ the controlled system actually is, C must be suitable for all members.

4. TRACKING IN SAMPLED-DATA CONFIGURATIONS

We develop a periodic framework for operating the controller C. In this framework, C is designed for the first sampling interval $[0, T]$. The operation of C during other sampling intervals $[kT, (k+1)T]$, $k \in \{1, 2, \ldots\}$, is obtained by shifting in time its operation during $[0, T]$. For this to be possible, the states $x(kT)$ at the start of sampling interval $[kT, (k+1)T]$ must be among the initial states $x(0)$. This gives rise to the following (Choi and Hammer (2020)).

Definition 8. A *sampling radius* for a family $\mathcal{S}_\gamma(\Sigma_0, K, A)$ with sampling period $T > 0$ is a real number $\sigma > 0$ such that the following holds for all systems $\Sigma \in \mathcal{S}_\gamma(\Sigma_0, K, A)$ and for all integers $k \ge 0$: for every state $x(kT) \in \rho(\sigma)$, there is an input signal $u_{x(kT)} \in U(K, W, \kappa)$ that steers Σ to achieve $x((k+1)T) \in \rho(\sigma)$ and $|x(t)| \le A$ for all $t \in [kT, (k+1)T]$. □

We investigate the existence of sampling radii in the next section.

4.1 Upholding the state amplitude bound

For a member $\Sigma \in \mathcal{S}_\gamma(\Sigma_0, K, A)$ with initial state x_0, the family of input signals that preserve the sampling radius σ of Definition 8 is

$$U(A, \Sigma, x_0) = \left\{ u \in U(K, W, \kappa) : \begin{array}{c} \sup_{t\in[0,T]} |\Sigma(x_0, u, t)| \le A \\ \text{and } \Sigma(x_0, u, T) \in \rho(\sigma) \end{array} \right\}$$

Input signals must preserve the sampling radius for all members of $\mathcal{S}_\gamma(\Sigma_0, K, A)$. The family of appropriate input signals is

$$U(A, \gamma, x_0) = \bigcap_{\Sigma \in \mathcal{S}_\gamma(\Sigma_0, K, A)} U(A, \Sigma, x_0). \quad (12)$$

4.2 Continuity and compactness

Systems $\Sigma \in \mathcal{S}_\gamma(\Sigma_0, K, A)$ have the following continuity feature (see Hammer (2023) for proof).

Lemma 9. Let $\Sigma \in \mathcal{S}_\gamma(\Sigma_0, K, A)$ have an initial state $x_0 \in \rho(\sigma)$, and let $\{u_i\}_{i=1}^\infty \subseteq U(A, \gamma, x_0)$ be a sequence uniformly pointwise convergent to $u \in U(K, W, \kappa)$. Then, the sequence $\{\Sigma(x_0, u_i, t)\}_{i=1}^\infty$ is uniformly pointwise convergent to $\Sigma(x_0, u, t)$ at all times $t \ge 0$.

The input signals set $U(A, \gamma, x_0)$ of (12) is compact, as follows.

Lemma 10. The set $U(A, \gamma, x_0)$ is uniformly pointwise compact.

Proof. [Proof outline] Let $\{u_i\}_{i=1}^{\infty} \subseteq U(A, \Sigma, x_0)$ be a sequence, where $\Sigma \in \mathcal{S}_\gamma(\Sigma_0, K, A)$ and $x_0 \in \rho(\sigma)$. Since $U(A, \Sigma, x_0) \subseteq U(K, W, \kappa)$, there is, by Lemma 6, a subsequence $\{u_{i_k}\}_{k=1}^{\infty}$ that converges pointwise uniformly to a signal $u \in U(K, W, \kappa)$. By Lemma 9, $\{\Sigma(x_0, u_{i_k}, t)\}_{k=1}^{\infty}$ converges uniformly to $\Sigma(x_0, u, t)$. As $\{u_i\}_{i=1}^{\infty} \subseteq U(A, \Sigma, x_0)$, we have $\Sigma(x_0, u_{i_k}, T) \in \rho(\sigma)$ and $|\Sigma(x_0, u_{i_k}, t)| \le A$ for all $t \in [0, T]$, $k = 1, 2, \ldots$, so that $|\Sigma(x_0, u, t)| \le A$ for all $t \in [0, T]$ and $\Sigma(x_0, u, T) \in \rho(\sigma)$. Thus, $u \in U(A, \Sigma, x_0)$ and $U(A, \Sigma, x_0)$ is uniformly pointwise compact. Since $U(A, \gamma, x_0)$ is then an intersection of compact sets by (12), it is compact. □

4.3 Inter-sample tracking errors

For initial state x_0, signal $u \in U(A, \gamma, x_0)$, and target state $x = 0$, the inter-sample tracking error over the family $\mathcal{S}_\gamma(\Sigma_0, K, A)$ and the sampling interval $[0, T]$ is

$$\ell(\sigma, K, A, \gamma, T, x_0, u) := \sup_{\Sigma \in \mathcal{S}_\gamma(\Sigma_0, K, A),\ t \in [0,T]} |\Sigma(x_0, u, t)|_2^2 . \tag{13}$$

The infimal inter-sample tracking error over all inputs is then

$$\ell^*(\sigma, K, A, \gamma, T, x_0) := \inf_{u \in U(A, \gamma, x_0)} \ell(\sigma, K, A, \gamma, T, x_0, u).$$

We explore the question of whether there is an input signal $u^*(x_0) \in U(A, \gamma, x_0)$ that achieves this infimum by satisfying

$$\ell^*(\sigma, K, A, \gamma, T, x_0) = \ell(\sigma, K, A, \gamma, T, x_0, u^*(x_0)). \tag{14}$$

A signal $u^*(x_0)$, if it exists, is an optimal robust solution to Problem 1 *(i)*, robustly minimizing inter-sample tracking errors. We restate now Problem 1 in formal terms.

Problem 11.
(i) Are there optimal robust solutions $u^*(x_0)$ for every $x_0 \in \rho(\sigma)$?
(ii) Find simple controllers that approximate optimal performance. □

The resolution of Problem 11 depends on the following notion.

5. CONSTRAINED CONTROLLABILITY

Constrained controllability makes it possible to steer the controlled system Σ to the vicinity of the origin, without violating input and state amplitude constraints (compare to Choi and Hammer (2019)).

Definition 12. For a real number $\sigma > 0$, a system $\Sigma \in \mathcal{S}_\gamma(\Sigma_0, K, A)$ is (K, A, σ, T)-*controllable* if there is $\sigma' \in (0, \sigma)$ such that, for every $x_0 \in \rho(\sigma)$, there is an input signal $u_{x_0} \in U(K, W, \kappa)$ for which $\Sigma(x_0, u_{x_0}, T) \in \rho(\sigma')$ and $|\Sigma(x_0, u_{x_0}, t)| \le A$ for all $t \in [0, T]$. □

(K, A, σ, T)-controllability resembles the notion of a sampling radius, except for the contractive requirement $\sigma' < \sigma$ of Definition 12, which helps overcome model uncertainty. Thus, (K, A, σ, T)-controllability is close to necessary for periodic sampling.

Next, a few mathematical facts (Zeidler (1985); Willard (2004)).

Theorem 13.
(i) A continuous functional is lower semi-continuous.
(ii) Let S and A be topological spaces. Assume that, for every member $a \in A$, there is a lower semi-continuous functional $f_a : S \to R$. If $\sup_{a \in A} f_a(s)$ exists at every point $s \in S$, then the functional $f(s) := \sup_{a \in A} f_a(s)$ is lower semi-continuous on S.
(iii) The Generalized Weierstrass Theorem: A lower semi-continuous functional attains a minimum in a compact set. □

Theorem 13 *(ii)* and Lemma 9 imply the following

Corollary 14. The functional $\ell(\sigma, K, A, \gamma, u, T, x_0)$ of (13) is a lower semi-continuous functional of the input signal u over the domain $U(A, \gamma, x_0)$. □

The next statement shows that (K, A, σ, T)-controllability needs to be verified just for one system – the nominal system Σ_0.

Proposition 15. Assume that the nominal system Σ_0 is (K, A_0, σ, T)-controllable. Then, for every $A > A_0$, there is an uncertainty parameter $\gamma > 0$ for which all members of the family $\mathcal{S}_\gamma(\Sigma_0, K, A)$ are (K, A, σ, T)-controllable. Furthermore, the family of input signals $U(A, \gamma, x_0)$ of (12) is not empty for initial states $x_0 \in \rho(\sigma)$.

Proof. [Proof outline] By (K, A_0, σ, T)-controllability of Σ_0, there is, for every $x_0 \in \rho(\sigma)$, an input $u \in U(A_0, \Sigma_0, x_0)$ satisfying $\Sigma_0(x_0, u, T) \in \rho(\sigma')$, where $0 < \sigma' < \sigma$; and $|\Sigma_0(x_0, u, t)| \le A_0$ for all $t \in [0, T]$. Let $\Sigma \in \mathcal{S}_\gamma(\Sigma_0, K, A)$. Denote $x'(t) := \Sigma(x_0, u, t)$, $x(t) := \Sigma_0(x_0, u, t)$, and $\xi(t) = x'(t) - x(t)$; and let $t_1, t_2 \in [0, T]$ $t_1 < t_2$. Then,

$$\xi(t) = \xi(t_1) + \int_{t_1}^{t} [f_0(x'(s), u(s)) + f_\gamma(s, x'(s), u(s))]\, ds - \int_{t_1}^{t} f_0(x(s), u(s)) ds.$$

Using the mean value theorem and (8), (10), and (11), we obtain

$$[1 - nB(t_2 - t_1)] \sup_{s \in [t_1, t_2]} |\xi(s)| \le |\xi(t_1)| + \gamma(nA + 1)(t_2 - t_1).$$

Let $\eta > 0$ satisfy $nB\eta \le 1/2$ with $p := T/\eta$ an integer. Then, setting $t_1 = i\eta$, $i \in \{0, 1, \ldots, p-1\}$, we get

$$\sup_{i\eta \le s \le (i+1)\eta} |\xi(s)| \le 2|\xi(i\eta)| + 2\gamma\eta(nA + 1), \xi(0) = 0,$$

$i = 0, 1, \ldots, p - 1$. This yields

$$\sup_{0 \le s \le T} |\xi(s)| \le \gamma\eta(nA + 1) \sum_{i=1}^{p} 2^i = \gamma\eta(nA + 1)2(2^p - 1).$$

Let $\varepsilon := A - A_0$ and $\varepsilon' := (\sigma - \sigma')/2$. Then, any $\gamma > 0$ satisfying $\gamma < \min\{\varepsilon, \varepsilon'\}/[\eta(nA + 1)2(2^p - 1)]$ fulfills the proposition. □

Proposition 15 substantially simplifies the process of verifying the existence of optimal robust solutions, as we discuss next.

6. EXISTENCE OF OPTIMAL ROBUST CONTROLLERS

Existence of optimal robust controllers that minimize inter-sample tracking errors is a consequence of the generalized Weierstrass theorem (Theorem 13*(iii)*) and the following facts: *(a)* the family of inputs $U(A, \gamma, x_0)$ is compact and not empty by Lemma 10 and Proposition 15; and *(b)* the tracking error $\ell(\sigma, K, A, \gamma, u, T, x_0)$ is a lower semi-continuous functional of u by Corollary 14. This proves

Theorem 16. Under the conditions of Proposition 15, there is, for every initial state $x_0 \in \rho(\sigma)$, an optimal robust input signal $u^*(x_0) \in U(A, \gamma, x_0)$ satisfying (14). □

The signal $u^*(x_0)$ minimizes inter-sample tracking errors; when produced by the controller C, it creates an optimal robust controller for the system Σ during the sampling interval $[0, T]$. It is a robust controller, since optimization was over the family of systems $\mathcal{S}_\gamma(\Sigma_0, K, A)$, which represents uncertainty about the system Σ. The next statement shows that the same signal is optimal for any sampling interval $[kT, (k+1)T]$, after being appropriately shifted in time.

Theorem 17. Refer to the conditions and notation of Theorem 16. The shifted-in-time input signal $u^*(x(kT), t - kT)$, $t \in [kT, (k+1)T]$, is an optimal robust input signal that minimizes inter-sample tracking errors during the sampling interval $[kT, (k+1)T]$, $k \in \{0, 1, \ldots\}$; it achieves the minimal tracking error $\ell^*(\sigma, K, A, \gamma, T, x(kT))$.

Proof. [Proof outline] *(i)* The nominal system Σ_0 is time-invariant. *(ii)* The uncertainty parameter γ of (10) is constant. As a result, the arguments used in the proof of Theorem 16 are valid during any sampling interval (see Hammer (2023) for details). □

By Theorem 17, (K, A_0, σ, T)-controllability of the nominal system Σ_0 is sufficient for the existence of optimal robust controllers, as long as the uncertainty parameter γ is not too large. By the discussion following Definition 12, this implies that (K, A_0, σ, T)-controllability is close to being a necessary and sufficient condition for the existence of optimal robust controllers for the general class of nonlinear systems (1). The controllers operate as follows.

Controller Operation (*outline*). Under conditions of Theorem 17:
• The controller C receives the state $x(kT)$ of the controlled system Σ at the time kT.
• The controller C generates the signal $u^*(x(kT), t - kT)$ as input to Σ during the sampling interval $[kT, (k+1)T]$. □

As optimal input signals may be hard to calculate and implement, we introduce next a family of input signals that approximate optimal performance and are relatively easy to calculate and implement.

7. PSEUDO BANG-BANG SIGNALS

Recall that a bang-bang signal is a signal whose components switch between the values of K and $-K$, where K is the input amplitude bound of the system Σ. *Pseudo bang-bang signals* are continuous function of time that resemble bang-bang signals. Like bang-bang signals, pseudo bang-bang signals are characterized by a string of scalars — their 'switching times' and, as a result, are relatively easy to design and implement. In formal terms:

Definition 18. A *pseudo bang-bang signal* u_{ps} is induced by a bang-bang signal $u_s \in U(K)$ by taking the Fourier transform $\upsilon_s := \mathscr{F} u_s$; multiplying it by $e^{-\kappa|\omega|}$, where $\kappa > 0$ is a desired smoothing factor; and inverting the Fourier transform $u_{ps}(t) = \mathscr{F}^{-1}\left(\upsilon_s(\omega) e^{-\kappa|\omega|}\right)$. A *pseudo bang-bang controller* generates pseudo bang-bang signals as input to the system it controls. □

By Lemma 2, pseudo bang-bang signals are uniformly continuous functions of time. Figure 2 shows that pseudo bang-bang signals resemble their bang-bang counterparts (in the figure, $\kappa = 0.001$). The next statement shows that pseudo bang-bang controllers can approximate the performance of optimal controllers (see Hammer (2023) for proof).

Theorem 19. Assume the conditions and notation of Proposition 15 and Theorem 17, where u^* denotes the optimal input. Let $\Sigma \in \mathcal{S}_\gamma(\Sigma_0, K, A)$ be a system with initial state x_0, state $x(t)$, input signal $u(t)$, and sampling period $T > 0$. Denote by x^* the response of Σ to u^*. Then, for every $\varepsilon > 0$, there are $\kappa, W' > 0$ and a pseudo bang-bang signal $u^\pm(x(kT)) \in U(K + \varepsilon, W', \kappa)$ for which the following is true for all $k = 0, 1, \ldots$: the response $x^\pm(t)$ of Σ to the time-shifted input signal $u^\pm(x(kT), t - kT)$ does not differ by more than ε from the optimal response $x^*(t)$, $t \in [kT, (k+1)T]$; furthermore, $x^\pm((k+1)T) \in \rho(\sigma)$. □

In Theorem 19, increasing accuracy by reducing ε may require more 'switchings' of the pseudo bang-bang signal $u^\pm$ that is generated by a pseudo bang-bang controller approximating optimal performance.

Implementation of pseudo bang-bang controllers is simpler than implementation of other controllers, since pseudo bang-bang signals are characterized just by lists of switching times.

Operation of a pseudo bang-bang controller C.

(refer to Figure 1)

- The feedback sampler supplies the state $x(kT) \in \rho(\sigma)$ of Σ to the controller C at the time $t = kT$.
- The controller C creates the pseudo bang-bang signal $u^\pm(x(kT), t - kT)$ as input to Σ during the time intervals $[kT, (k+1)T]$.
- This input respects the sampling radius σ, so $x^\pm((k+1)T) \in \rho(\sigma)$, and the process continues cyclically over sampling intervals $k = 0, 1, \ldots$ □

8. EXAMPLE

We demonstrate the techniques of this note on a variant of the Michaelis-Menten equation, an equation often used in biological science (Michaelis and Menten (1913); Cao (2011); and others):

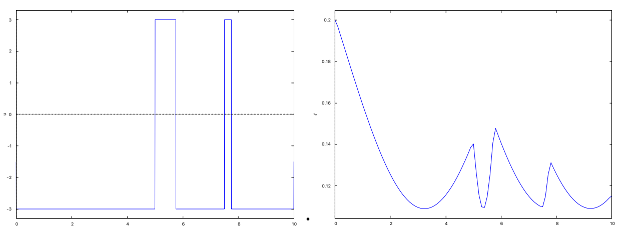
$$\Sigma : \ \dot{x}^1(t) = \frac{(a + u(t))x^2(t)}{(b + x^2(t))}, \ \dot{x}^2(t) = -\frac{(a + u(t))x^1(t)}{(c + x^2(t))}.$$

Here, $x(t) = (x^1(t), x^2(t))^\top$ is the state; $u(t)$ is the input signal, and a, b, and c are unspecified constants with nominal values $a_0 = 2$, $b_0 = 2$, and $c_0 = 5$, and uncertainty domains $1.9 \le a \le 2.1$, $1.9 \le b \le 2.1$, and $4.9 \le c \le 5.1$. The sampling period is $T = 10$; the input and state amplitude bounds are $K = 3$ and $A = 1$, respectively; the sampling radius is $\sigma = 0.2$; and the target state is $x = 0$. The task is to minimize inter-sample tracking errors.

We use pseudo bang-bang signals to simplify calculation and implementation (Section 7) and allow an approximation error of $\varepsilon = 0.01$ (Theorem 19). The design of a pseudo bang-bang controller is briefly described as follows (see Hammer (2023) for details).

Procedure 20. Pseudo bang-bang Controller Derivation (outline).
(i) Use the search process of Choi and Hammer (2019) to calculate a bang-bang input signal $v^{\pm}(x_0)$ that minimizes tracking error.
(ii) Calculate the Fourier transform $V^{\pm}(x_0) = \mathscr{F} v^{\pm}(x_0)$.
(iii) The pseudo bang-bang input signal is then $u^{\pm}(x_0) = \mathscr{F}^{-1}\left(V^{\pm}(x_0,\omega)e^{-\kappa|\omega|}\right)$, where $\kappa > 0$ is a sufficiently small smoothing factor (see Hammer (2023)).
(iv) Repeat *(i)* to *(iii)* for a grid of initial states $x_0 \in \rho(\sigma)$ to complete controller design. □

Figure 3 depicts the outcome of Procedure 20 for $\kappa = 0.001$, $a = 2$, $b = 2$, and $c = 5$, and initial state $x_0 = (0.4, 0.2)^\top$. A single case is depicted to avoid clutter. The maximal tracking error is $\ell = \sup_{t\in[0,T]} |x(t)|_2^2 = 0.2$. As any tracking error must satisfy $\ell \ge |x_0|_2^2 = 0.2$, this outcome is the best possible, so no additional error was introduced by the use of pseudo bang-bang signals. Also, the figure shows that $|x(10)|_2^2 \in \rho(0.2)$, so the sampling radius is preserved, and the process can proceed periodically with $T = 10$.



(a) A pseudo bang-bang input (b) The inter-sample deviation

Fig. 3. An input signal and its response

9. CONCLUSION

This note presents the essentials of a design and implementation method for optimal robust inter-sample tracking of nonlinear sampled-data systems. The method applies to a wide class of nonlinear systems. It is shown that optimal robust inter-sample controllers exist as long as a certain controllability condition is valid. Furthermore, it is shown that pseudo bang-bang controllers, which are relatively easy to design and implement, can approximate optimal performance.

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