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Low Error Operation in Open Loop: Nonlinear Systems with Time Delay ★

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Abstract: Optimal controllers that maintain low operating errors during feedback disruptions are developed for a class of nonlinear systems with time delay. It is shown that such controllers exist and that their performance can be closely approximated by bang-bang controllers that are easy to implement.

Keywords: optimal control, nonlinear control, feedback outage, time delay, bang-bang control.

1. INTRODUCTION

Feedback disruptions are a frequent concern in engineering practice, as they may originate from component failures, from deliberate operating policies (Zhivogyladov and Middleton (2003), Nair et al. (2007), Montestruque and Antsaklis (2004), and others), or from poor operating conditions. We examine here the existence and implementation of automatic controllers that keep operating errors below a specified bound for a class of nonlinear systems with time delays. Applications include transportation, communication, and remotely operated systems (e.g., Sheridan and Ferrell (1963); Bushnell (2001)), digital control systems affected by real-time computing delays (e.g., Ailon and Gil (2000); Imaida et al. (2004), and other applications.

The configuration is depicted in Figure 1, where the controlled system Σ is composed of a dynamical system Σ^- and an input time delay of $\tau > 0$; the input signal $u(t)$ of Σ is generated by the controller C, which loses its feedback link at the time $t = 0$. This feedback disruption may increase operating errors. The goal of C is to generate an input signal $u(t)$ that keeps operating errors below a specified bound for the longest time possible. This would allow maximal time for reinstating feedback.

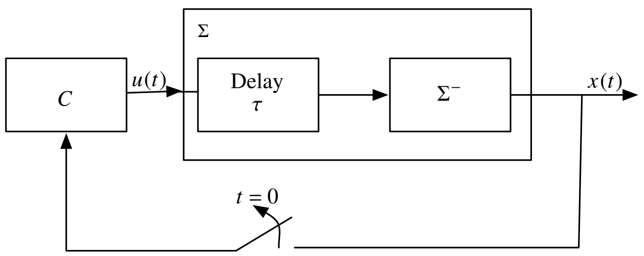

Fig. 1. General configuration

Problem 1. Following a feedback disruption at $t = 0$,

(i) Is there an optimal controller C that keeps operating errors below a specified bound ℓ for the longest time possible?

(ii) Is there a simple-to-implement controller that approximates optimal performance? □

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We show in Section 4 that such optimal controllers exist under rather broad conditions, and in Section 5 we show that optimal performance can be closely approximated by controllers that generate bang-bang input signals for Σ. Bang-bang signals are relatively easy to calculate and implement, since they are determined by a finite string of scalars – their switching times.

1.1 Objectives and preliminaries

After a possible shift of the state-space origin, our goal is to maintain the state $x(t)$ of Σ near the zero state $x = 0$; deviations from the zero state are considered operating errors. Given an *error bound* $\ell > 0$, our goal is to build a controller C that generates an input signal $u(t)$ that keeps the inequality

$$x^T(t)x(t) \le \ell \tag{1}$$

in force for the longest time possible, assuming that the initial state $x(0) = x_0$ of Σ – the last data point provided by the feedback – does satisfy this inequality.

Regarding notation, let R denote the real numbers, and let $|r|$ be the absolute value of $r \in R$. The L^∞–norm of a matrix $A \in R^{n\times m}$ with entries $\{A_{ij}\}$ is $|A| := \max_{i,j} |A_{ij}|$, and the L^∞–norm of a function $v : R^+ \to R^{n\times m} : t \mapsto v(t)$ is $|v|_\infty := \sup_{t\ge 0} |v(t)|$. Here, $|v(t)|$ is the largest absolute value of an entry of $v(t)$ at a time t, while $|v|_\infty$ is the *amplitude* of the signal $v(t)$.

Like most systems encountered in applications, the system Σ permits only bounded input signals. Denoting this bound by $K > 0$, input signals u must satisfy $|u|_\infty \le K$.

1.2 Background

We extend the results of Chakraborty and Hammer (2009, 2010) from the linear case to a class of nonlinear systems with input delays. The discussion depends on optimization theory, including Kelendzheridze (1961), Pontryagin et al. (1962), Gamkrelidze (1965), Neustadt (1966, 1967), Luenberger (1969), Young (1969), Warga (1972), Chakraborty and Hammer (2009, 2010), Chakraborty and Shaikshavali (2009), the references cited in these works, and others. Yet, as best as we know, Problem 1 has not been addressed before for nonlinear systems with input delays.

The present note is organized as follows. Section 2 covers background material and notation, and Section 3 presents auxiliary results. The existence of optimal controllers is proved in

Section 4, while Section 5 shows that bang-bang controllers can achieve close to optimal performance. Finally, an example demonstrating the power of bang-bang controllers in this application is presented in Section 6.

2. BASICS

2.1 The class of input signals

We use the framework of Chakraborty and Hammer (2009, 2010).

Definition 2. Let L^m be the linear space of all Lebesgue measurable functions $f : R \to R^m : t \mapsto f(t)$ that are zero for $t < 0$. For a number $\sigma > 0$, let $L_2^{\sigma,m}$ be the inner product space formed by members of L^m with the inner product

$$\langle f, g\rangle := \int_0^\infty e^{-\sigma t} f^T(t)g(t)dt. \tag{2}$$

Then, the set of input signals bounded by $K > 0$ is

$$U(K) := \left\{u \in L_2^{\sigma,m} : |u|_\infty \le K\right\}, \quad \square \tag{3}$$

The inner product (2) is bounded when f and g are bounded.

2.2 The controlled system

Referring to Figure 1, feedback failure commences at the time $t = 0$. The last information delivered by the feedback is $x(0) = x_0$, the initial state of Σ. A *control input signal* $u(t) \in U(K)$ for Σ starts at $t = 0$. Due to the delay of $\tau > 0$, the impact of $u(t)$ on the state $x(t)$ of Σ commences after $t = \tau$. During the time $[0, \tau]$, the state $x(t)$ is driven by the *residual input signal* $v(t)$, where $|v(t)| \le K$, $t \in [-\tau, 0]$. We set $v(t) := 0$ for $t \notin [-\tau, 0]$, since the values of $v(t)$ outside $[-\tau, 0]$ are inconsequential to our discussion. The signal $v(t)$, being a remnant of earlier operation, is not under our control. The *combined input signal* $w(t)$ satisfies $w(t - \tau) \in U(K)$:

$$w(t) := \begin{cases} v(t) & t \in [-\tau, 0), \\ u(t) & t \ge 0. \end{cases}$$

We concentrate on input affine nonlinear systems with a time delay in the input channel:

$$\Sigma : \ \dot{x}(t) = \begin{cases} a(t, x(t)) + b(t, x(t))v(t - \tau), & t \in [0, \tau), \\ a(t, x(t)) + b(t, x(t))u(t - \tau), & t \ge \tau, \end{cases} \tag{4}$$

where $x(0) = x_0$, and $a : R^+ \times R^n \to R^n$ and $b : R^+ \times R^n \to R^{n\times m}$ are continuous functions. Concisely,

$$\Sigma : \quad \dot{x}(t) = a(t, x(t)) + b(t, x(t))w(t - \tau), t \ge 0, x(0) = x_0. \tag{5}$$

We often use the notation

$$x(t) := \Sigma(x_0, v, u, t) = \begin{cases} \Sigma(x_0, v, t) & t \in [0, \tau), \\ \Sigma(x(\tau), u, t) & t \ge \tau. \end{cases}$$

2.3 Model uncertainties

To represent uncertainties and modeling errors, we set

$$\begin{aligned} a(t, x) &= a_0(t, x) + a_\gamma(t, x), \\ b(t, x) &= b_0(t, x) + b_\gamma(t, x), \end{aligned} \tag{6}$$

where $a_0 : R^+ \times R^n \to R^n$ and $b_0 : R^+ \times R^n \to R^{n\times m}$ are the nominal continuous functions, while $a_\gamma : R^+ \times R^n \to R^n$ and $b_\gamma : R^+ \times R^n \to R^{n\times m}$ are unspecified continuous functions representing uncertainties. All abide by the Lipschitz conditions:

$$\begin{aligned} &|a_0(t, x') - a_0(t, x)| \le \alpha\, |x' - x|, a_0(t, 0) = 0, \\ &\left|a_\gamma(t, x') - a_\gamma(t, x)\right| \le \gamma\, |x' - x|, a_\gamma(t, 0) = 0, \end{aligned} \tag{7}$$

$$\begin{aligned} &|b_0(t, x') - b_0(t, x)| \le \beta\, |x' - x|, |b_0(t, 0)| \le \beta, \\ &\left|b_\gamma(t, x') - b_\gamma(t, x)\right| \le \gamma\, |x' - x|, |b_\gamma(t, 0)| \le \gamma, \end{aligned} \tag{8}$$

for all $x', x \in R^n$ and all $t \ge 0$, where $\alpha \ge 0$, $\beta \ge 0$, and $\gamma \ge 0$ are specified, with γ representing uncertainty.

The residual input signal v is also subject to uncertainty. The nominal residual input signal is $v_0(t)$; the actual signal is known only to the extent that $v \in V(v_0, \gamma)$, where $V(v_0, \gamma)$ is the family of Lebesgue measurable functions $v : [-\tau, 0] \to R^m$, $|v|_\infty \le K$, given by

$$V(v_0, \gamma) := \{v : |v(t) - v_0(t)| \le \gamma \text{ for all } t \in [-\tau, 0]\}. \tag{9}$$

Definition 3. The family $\mathcal{F}_\gamma(x_0, \tau)$: given positive numbers $\alpha, \beta, \gamma, K, \tau$, the family $\mathcal{F}_\gamma(x_0, \tau)$ consists of all systems of the form (5) with the initial state x_0, control input signal $u \in U(K)$, and unspecified residual input signal $v \in V(v_0, \gamma)$, where $a(t, x)$ and $b(t, x)$ are continuous functions satisfying (6), (7), and (8); and $U(K)$ and $V(v_0, \gamma)$ are given by (3) and (9). $\square$

As indicated earlier, due to the time delay, there is no control over the response of Σ during the time $[0, \tau]$. Therefore, in order to satisfy the requirement (1), we must assume that

$$\Sigma^T(x_0, v, u, t)\Sigma(x_0, v, u, t) \le \ell \text{ for all } t \in [0, \tau]. \tag{10}$$

2.4 Problem formulation

For a system $\Sigma \in \mathcal{F}_\gamma(x_0, \tau)$, let $t(x_0, \Sigma, v, u, \ell)$ be the longest time during which (1) is valid for given residual input signal $v \in V(v_0, \gamma)$ and control input signal $u \in U(K)$:

$$t(x_0, \Sigma, v, u, \ell) := \inf\left\{t \ge 0 : \Sigma^T(x_0, v, u, t)\Sigma(x_0, v, u, t) > \ell\right\} \tag{11}$$

Note that $t(x_0, \Sigma, v, u, \ell) \ge \tau$ by (10).

For a fixed control input signal u, let $t(x_0, \gamma, u, \ell)$ be the longest time during which the state of *every* member of $\mathcal{F}_\gamma(x_0, \tau)$ stays below the bound ℓ for *every* residual input signal $v \in V(v_0, \gamma)$:

$$t(x_0, \gamma, u, \ell) = \inf_{\substack{\Sigma\in\mathcal{F}_\gamma(x_0,\tau) \\ v\in V(v_0,\gamma)}} t(x_0, \Sigma, v, u, \ell). \tag{12}$$

Finally, let $t(x_0, \gamma, \ell)$ be the very longest time during which the state of every member of $\mathcal{F}_\gamma(x_0, \tau)$ can be kept below the bound ℓ, for every residual input signal $v \in V(v_0, \gamma)$:

$$t(x_0, \gamma, \ell) = \sup_{u\in U(K)} t(x_0, \gamma, u, \ell). \tag{13}$$

We state now formally the main topic of this note.

Problem 4. Let $K, \gamma, \tau, \ell > 0$ be specified real numbers, let $\mathcal{F}_\gamma(x_0, \tau)$ be as in Definition 3, and denote $x(t) := \Sigma(x_0, v, u, t)$. Using (12) and (13),

(i) Find conditions under which there is an optimal control signal $u(x_0, \gamma, \ell) \in U(K)$ satisfying $t(x_0, \gamma, \ell) = t(x_0, \gamma, u(x_0, \gamma, \ell), \ell)$.

(ii) Find a simple-to-calculate-and-implement control input signal $u^\pm \in U(K)$ for which $t(x_0, \gamma, \ell) \approx t(x_0, \gamma, u^\pm, \ell)$, i.e., a simple control input signal that approximates optimal performance. $\square$

Section 4 below shows that an optimal control input signal $u(x_0, \gamma, \ell)$ of Problem 4*(i)* does exist under rather broad conditions. Section 5 shows that optimal performance can be approximated as closely as desired by a bang-bang control input signal $u^\pm$ that is relatively easy to calculate and implement.

3. PRELIMINARY FACTS

3.1 Bounds

Systems of the family $\mathcal{F}_\gamma(x_0, \tau)$ have no finite escape time, namely, their response does not diverge in finite time. This is a consequence of the continuity of the coefficient functions in (5); of the Lipschitz conditions (7) and (8); and of the fact that all input signals are bounded Lebesgue measurable functions (see Choi and Hammer (2016) for more details):

Proposition 5. For every time $T \geq 0$, there is a real number $M(T) \geq 0$ such that $|\Sigma(x_0, v, u, t| \leq M(T)$ at all times $t \in [0, T]$, for all members $\Sigma \in \mathcal{F}_\gamma(x_0, \tau)$, for all residual input signals $v \in V(v_0, \gamma)$, and for all control input signals $u \in U(K)$. □

Using the fact that continuous functions are bounded over compact domains, Proposition 5 implies the following:

Corollary 6. Let $a(t, x)$ and $b(t, x)$ be as in (5). Then, for every time $T \geq 0$, there is a number $M_{ab}(T) \geq 0$ such that $|a(t, \Sigma(x_0, v, u, t))| \leq M_{ab}(T)$ and $|b(t, \Sigma(x_0, v, u, t))| \leq M_{ab}(T)$ for all $v \in V(v_0, \gamma)$, for all $u \in U(K)$, and for all $\Sigma \in \mathcal{F}_\gamma(x_0, \tau)$. □

3.2 Errors and uncertainties

The uncertainties included in the model of the controlled system Σ and in the residual input signal v, if small, have only a small impact on the response. This is a consequence of Corollary 6, the input-affinity of (5), the Lipschitz conditions (7) and (8), the uncertainty (9) of the residual input signals, and of the fact that all control input signals are bounded by K (see Choi and Hammer (2016) for more details).

Proposition 7. The following holds at all times $t \in [0, \tau]$: for every real number $\delta > 0$, there is a real number $\gamma > 0$ such that $|\Sigma(x_0, v, u, t) - \Sigma(x_0, v_0, u, t)| < \delta$ for all $v \in V(v_0, \gamma)$, for all $u \in U(K)$, and for all $\Sigma \in \mathcal{F}_\gamma(x_0, \tau)$. □

Proposition 7 implies that the scatter that occurs in the response of the controlled system Σ as a result of uncertainties is small, as long as the uncertainties are small. Thus, if the initial state x_0 of Σ is near the origin and the uncertainties are small, operating errors during the period of time $[0, \tau]$ – the period of time during which we have no control over the response – will not violate the operating error bound (10). Beyond the delay time τ, we can mitigate operating errors by using an appropriate control input signal, as discussed in the next section.

4. OPTIMAL SOLUTIONS

4.1 General statement

The following statement asserts the existence of optimal solutions of Problem 4.

Theorem 8. In the notation of Problem 4, (12), and (13):

(i) If $t(x_0, \gamma, \ell) = \infty$, then, for every time $t' \geq 0$, there is a control input signal $u' \in U(K)$ satisfying $t(x_0, \gamma, u', \ell) \geq t'$.

(ii) If $t(x_0, \gamma, \ell) < \infty$, then there is an optimal control input signal $u(x_0, \gamma, \ell) \in U(K)$ satisfying

$$t(x_0, \gamma, \ell) = t(x_0, \gamma, u(x_0, \gamma, \ell), \ell). \quad □$$

The proof of Theorem 8 separates into two cases:

$$\begin{aligned} &\textbf{Case 1: } t(x_0, \gamma, \ell) = \infty; \\ &\textbf{Case 2: } t(x_0, \gamma, \ell) < \infty. \end{aligned} \tag{14}$$

By the definition of supremum, Case 1 directly translates into Theorem 8*(i)*. In Case 2, Theorem 8 follows from the Generalized Weierstrass Theorem, according to which a continuous function attains an extremum in a compact domain. The explicit proof requires the following preliminary notions (e.g., Lusternik and Sobolev (1961), Willard (2004)).

Definition 9. In a Hilbert space H with inner product $\langle \cdot, \cdot \rangle$:

(i) A sequence $\{x_i\}_{i=1}^\infty \subseteq H$ *converges weakly* to an element $x \in H$ if $\lim_{i \to \infty} \langle x_i, y \rangle = \langle x, y \rangle$ for every element $y \in H$.

(ii) A subset $W \subseteq H$ is *weakly compact* if every sequence of elements of W has a subsequence that converges weakly to an element of W. □

The notion of weak compactness will suffice for our application here; the following result is from (Chakraborty and Hammer, 2009, Lemma 3.2).

Lemma 10. The set $U(K)$ of (3) is weakly compact in the topology of the Hilbert space $L_2^{\sigma,m}$. □

The proof of Theorem 8 depends on the following notion of weak continuity (e.g., Willard (2004)).

Definition 11. Let S be a subset of a Hilbert space H, and let z be a point of S.

(i) A functional $F : S \to R$ is *weakly upper semi-continuous* at z if the following is true whenever $F(z)$ is bounded: for every sequence $\{z_i\}_{i=1}^\infty \subseteq S$ that converges weakly to z, and for every $\varepsilon > 0$, there is an integer $N > 0$ such that $F(z_i) - F(z) < \varepsilon$ for all $i \geq N$. If F is weakly upper semi-continuous at every point of S, then F is *weakly upper semi-continuous on S*.

(iii) A function $G : S \to R^n$ is *weakly continuous* at z if the following is true for every sequence $\{z_i\}_{i=1}^\infty \subseteq S$ converging weakly to z: for every $\varepsilon > 0$, there is an integer $N > 0$ such that $|G(z_i) - G(z)| < \varepsilon$ for all $i \geq N$. If G is weakly continuous at every point of S, then G is *weakly continuous on S*. □

The following are taken from Willard (2004).

Theorem 12. (i) A continuous function of a weakly continuous function is weakly continuous.

(ii) A weakly continuous functional is weakly upper semi-continuous.

(iii) A weakly upper semi-continuous functional of a weakly continuous function forms a weakly upper semi-continuous functional.

(iv) Let S and A be topological spaces. Assume that, for every member $a \in A$, there is a weakly upper semi-continuous functional $f_a : S \to R$. If $\inf_{a \in A} f_a(s)$ exists at each point $s \in S$, then the functional $f(s) := \inf_{a \in A} f_a(s)$ is weakly upper semi-continuous on S. □

4.2 Continuity and compactness

Our first goal is to show that the functional $t(x_0, \Sigma, v, u, \ell)$ of (12) is weakly upper semi-continuous in u. For that, we need several convergence features, starting with the following fact that can be verified directly.

Lemma 13. Given $\tau > 0$, let $\{u_i\}_{i=1}^\infty \subseteq U(K)$ be a sequence converging weakly to $u \in U(K)$, and set $u_i^\tau := u_i(t - \tau)$, $i = 1, 2, \ldots$ and $u^\tau(t) := u(t - \tau)$, $t \geq 0$. Then, the sequence $\left\{u_i^\tau\right\}_{i=1}^\infty$ converges weakly to u^τ. □

Next, we show that a weakly convergent sequence of control input signals elicits a convergent sequence of responses.

Lemma 14. Assume that the sequence $\{u_i\}_{i=1}^\infty \subseteq U(K)$ converges weakly to $u \in U(K)$. Then, $\lim_{i \to \infty} \Sigma(x_0, v, u_i, t) = \Sigma(x_0, v, u, t)$ for all $t \geq 0$, for all $\Sigma \in \mathcal{F}_\gamma(x_0, \tau)$, and for all $v \in V(v_0, \gamma)$.

Proof. *(sketch)* Set $x_i(t) := \Sigma(x_0, v, u_i, t)$, $i = 1, 2, \ldots$, $x(t) := \Sigma(x_0, v, u, t)$, and $\xi_i(t) := x_i(t) - x(t)$. We show that $\lim_{i \to \infty} \xi_i(t) = 0$ for every $t \geq 0$. As the initial state x_0 and the residual input signal v are the same for all i, equation (4) implies that $\xi_i(t) = 0$

for all $t \in [0, \tau]$, $i \geq 1$. For $t > \tau$, let $t_1, t_2 \in [\tau, t]$, $t_1 < t_2$, be two times. Then, (4) yields (in the notation of Lemma 13)

$$\xi_i(t_2) = \xi_i(t_1) + \int_{t_1}^{t_2} [a(s, x_i(s)) - a(s, x(s))]ds$$
$$+ \int_{t_1}^{t_2} [b(s, x_i(s))u_i^\tau(s) - b(s, x(s))u^\tau(s)]ds.$$

Invoking (7), (8), and the input bound K, leads to

$$\sup_{t_1 \leq \theta \leq t_2} |\xi_i(\theta)| \leq |\xi_i(t_1)| + (\alpha + \gamma)(t_2 - t_1) \sup_{t_1 \leq \theta \leq t_2} |\xi_i(\theta)|$$
$$+ (\beta + \gamma)K(t_2 - t_1) \sup_{t_1 \leq \theta \leq t_2} |\xi_i(\theta)| \tag{15}$$
$$+ \sup_{t_1 \leq \theta \leq t_2} \left| \int_{t_1}^{\theta} b(s, x(s))[u_i^\tau(s) - u^\tau(s)]ds \right|.$$

Considering the last term, refer to (2) and denote

$$y_\theta(s) := \begin{cases} e^{\sigma s} b(s, x(s)) & 0 \leq s \leq \theta, \\ 0 & \text{else.} \end{cases}$$

Then,

$$\sup_{t_1 \leq \theta \leq t_2} \left| \int_{t_1}^{\theta} b(s, x(s))[u_i^\tau(s) - u^\tau(s)]ds \right| = \sup_{t_1 \leq \theta \leq t_2} \left| \left\langle u_i^\tau - u^\tau, y_\theta \right\rangle \right|. \tag{16}$$

It follows then by Lemma 13 that, for every $\varepsilon > 0$, there is an integer $N_\theta \geq 0$ such that $\left|\left\langle u_i^\tau - u^\tau, y_\theta \right\rangle\right| < \varepsilon$ for all $i \geq N_\theta$.

We show next, by contradiction, that N_θ can be chosen independently of θ. Indeed, assume that there is no integer $N \geq 0$ for which $\left|\left\langle u_i^\tau - u^\tau, y_\theta \right\rangle\right| < \varepsilon$ for all $i \geq N$ and all $\theta \in [t_1, t_2]$. This implies the existence of sequences $\{\theta_j\}_{j=1}^\infty \subseteq [t_1, t_2]$ and $\{i_j\}_{j=1}^\infty \to \infty$ for which

$$\left|\left\langle u_{i_j}^\tau - u^\tau, y_{\theta_j} \right\rangle\right| > \varepsilon \text{ for all } j = 1, 2, \ldots \tag{17}$$

By compactness of the interval $[t_1, t_2]$, there is a convergent subsequence $\{\theta_{j_k}\}_{k=1}^\infty \subseteq \{\theta_j\}_{j=1}^\infty$ converging to a point $\theta' := \lim_{k\to\infty} \theta_{j_k}$. As $\{u_i^\tau\}_{i=1}^\infty$ converges weakly to u^τ, also $\{u_{j_k}^\tau\}_{k=1}^\infty$ converges weakly to u^τ. This implies the existence of an integer $N' \geq 0$ for which

$$\left|\left\langle u_{i_{j_k}}^\tau - u^\tau, y_{\theta'} \right\rangle\right| < \varepsilon/2 \text{ for all } k \geq N'. \tag{18}$$

By (8), this yields

$$\left|\left\langle u_{i_{j_k}}^\tau - u^\tau, y_{\theta_{j_k}} \right\rangle - \left\langle u_{i_{j_k}}^\tau - u^\tau, y_{\theta'} \right\rangle\right|$$
$$= \left| \int_{\theta_{j_k}}^{\theta'} b(s, x(s))[u_{i_{j_k}}(s - \tau) - u(s - \tau)]ds \right| \tag{19}$$
$$\leq (\beta + \gamma)(2K) \left|\theta' - \theta_{j_k}\right|.$$

As $\lim_{k\to\infty} \theta_{j_k} = \theta'$, an integer $N'' \geq N'$ exists for which

$$\left|\theta' - \theta_{j_k}\right| < \frac{\varepsilon}{4(\beta + \gamma)K} \text{ for all } k > N''. \tag{20}$$

Together with (19), (20), and (18) this yields

$$\left|\left\langle u_{i_{j_k}}^\tau - u^\tau, y_{\theta_{j_k}} \right\rangle\right| = \left| \left\langle u_{i_{j_k}}^\tau - u^\tau, y_{\theta_{j_k}} \right\rangle - \left\langle u_{i_{j_k}}^\tau - u^\tau, y_{\theta'} \right\rangle \right.$$
$$\left. + \left\langle u_{i_{j_k}}^\tau - u^\tau, y_{\theta'} \right\rangle \right| < \varepsilon/2 + \varepsilon/2 = \varepsilon \text{ for all } k \geq N'',$$

contradicting (17). Hence, for every $\varepsilon > 0$, there is an integer $N \geq 0$ for which

$$\sup_{t_1 \leq \theta \leq t_2} \left|\left\langle u_i^\tau - u^\tau, y_\theta \right\rangle\right| < \varepsilon \text{ for all } i \geq N. \tag{21}$$

Inserting into (16), we obtain

$$\sup_{t_1 \leq \theta \leq t_2} \left| \int_{t_1}^{\theta} b(s, x(s))[u_i(s - \tau) - u(s - \tau)]ds \right| < \varepsilon \text{ for all } i \geq N.$$

Returning to (15), this yields

$$\{1 - (t_2 - t_1)[(\alpha + \gamma) + (\beta + \gamma)K]\} \sup_{t_1 \leq \theta \leq t_2} |\xi_i(\theta)|$$
$$\leq |\xi_i(t_1)| + \varepsilon \text{ for all } i \geq N. \tag{22}$$

Now, choose $\mu > 0$ for which $\mu[(\alpha + \gamma) + (\beta + \gamma)K] < 1$; then, set $\eta := 1/\{1 - \mu[(\alpha + \gamma) + (\beta + \gamma)K]\}$ and $t_2 := t_1 + \mu$. Inserting into (22), we get $\sup_{t_1 \leq \theta \leq t_1 + \mu} |\xi_i(\theta)| < \eta\, |\xi_i(t_1)| + \varepsilon/\eta$ for all $i \geq N$. As ε can be any positive number, it follows that, for every $\delta > 0$, there is an integer $N > 0$ for which

$$\sup_{t_1 \leq \theta \leq t_1 + \mu} |\xi_i(\theta)| \leq \eta\, |\xi_i(t_1)| + \delta \text{ for all } i \geq N. \tag{23}$$

Next, choose an integer $p \geq (t - \tau)/\mu$ and partition into segments of length μ

$$[\tau, t] \subseteq \{[\tau, \tau + \mu], [\tau + \mu, \tau + 2\mu], \cdots\}.$$

This with (23) yields the recursion

$$\sup_{\tau + (j-1)\mu \leq \theta \leq \tau + j\mu} |\xi_i(\theta)| \leq \eta\, |\xi_i(\tau + (j - 1)\mu)| + \delta, j = 1, 2, \ldots, p$$

for all $i \geq N$. But then, $\sup_{0 \leq \theta \leq t} |\xi_i(\theta)| \leq \left(\sum_{k=0}^{p-1} \eta^k\right) \delta$ for all $i \geq N$. As $\delta > 0$ can be chosen arbitrarily small, we get

$$\lim_{i\to\infty} \sup_{0 \leq \theta \leq t} |x_i(\theta) - x(\theta)| = 0, \tag{24}$$

as required (see Choi and Hammer (2016) for more details). □

Lemma 14 has the following

Corollary 15. As a function of u, the response $\Sigma(x_0, v, u, t)$ is weakly continuous over $U(K)$. □

We examine next the continuity features of the time functional $t(x_0, \Sigma, v, u, \ell)$ of (11).

Lemma 16. The functional $t(x_0, \Sigma, v, u, \ell)$ is weakly upper semi-continuous as a function of u over $U(K)$.

Proof. *(sketch)* Let $\{u_i\}_{i=1}^\infty \subseteq U(K)$ be a sequence converging weakly to $u \in U(K)$. Denote $x_i(t) := \Sigma(x_0, v, u_i, t)$ and $x(t) := \Sigma(x_0, v, u, t)$. Using the error bound ℓ and the class of functions

$$S := \left\{z : R^+ \to R^n : z(t) = \Sigma(x_0, v, g, t) \text{ for some } g \in U(K)\right\},$$

define the functional

$$\Theta(z) := \inf\{t \geq 0 : z^T(t)z(t) > \ell\} : S \to R. \tag{25}$$

By (24), the sequence $x_1(t), x_2(t), \cdots$ converges to $x(t)$ at every $t \geq 0$. We show first that $\Theta(z)$ is upper semi-continuous on S by examining two cases:

Case 1. There is an integer $N' > 0$ for which $\Theta(x_i) \leq \Theta(x)$ for all integers $i \geq N'$.

Case 2. Case 1 is not valid.

Case 1 indicates that $\Theta(x_i) - \Theta(x) \leq 0$, so that $\Theta(x_i) - \Theta(x) < \varepsilon$ for every $\varepsilon > 0$ and all $i \geq N'$, and upper semi-continuity holds.

Case 2 implies there is a subsequence $\{i_k\}_{k=1}^\infty$ and an integer $N'' > 0$ for which $\Theta(x_{i_k}) > \Theta(x)$ for all $k \geq N''$. Then, by (25), there is, for every $\varepsilon > 0$, a time $t' \in [\Theta(x), \Theta(x) + \varepsilon)$ for which

$$x^T(t')x(t') > \ell. \tag{26}$$

Considering that $\lim_{i\to\infty} |x_i(t') - x(t')| = 0$ by (24), it follows that $\lim_{i\to\infty} |x_i^T(t')x_i(t') - x^T(t')x(t')| = 0$, so that, for every $\varepsilon_1 > 0$, there is an integer $N_1 > 0$ satisfying $|x_{i_k}^T(t')x_{i_k}(t') - x^T(t')x(t')| < \varepsilon_1$ for all $k \geq N_1$. By (26), we can take $\varepsilon_1 = [x^T(t')x(t') - \ell]/2$; then, $|x_{i_k}^T(t')x_{i_k}(t') - x^T(t')x(t')| <$

$[x^T(t')x(t')-\ell]/2$ for all $k \geq N_1$, or $x_{i_k}^T(t')x_{i_k}(t') = x^T(t')x(t')$ $+[x_{i_k}^T(t')x_{i_k}(t') - x^T(t')x(t')] > \ell$ for all $k \geq N_1$. By (25), we conclude that $\Theta(x_{i_k}) \leq t'$ for all $k \geq N_1$; as $t' \in [\Theta(x), \Theta(x)+\varepsilon]$, we get $\Theta(x_{i_k}) < \Theta(x)+\varepsilon$ for all $k \geq N_1$. Thus, $\Theta(\cdot)$ is upper semi-continuous on S.

Finally, since $\Sigma(x_0, v, u, t)$ is weakly continuous over $U(K)$ by Corollary 15, since $z^T z : R^n \to R$ is a continuous functional of z, and since $t(x_0, \Sigma, v, u, \ell) = \Theta(\Sigma(x_0, v, u, t))$, the assertion in Case 2 follows by Theorem 12. □

Combining Lemma 16 with Theorem 12*(iv)* yields:

Lemma 17. The functional $t(x_0, \gamma, u, \ell)$ of (12) is weakly upper semi-continuous as a function of u over $U(K)$. □

Now, we can prove Theorem 8.

Proof *(of Theorem 8; sketch).* The proof of Part *(i)* was stated following the theorem; part *(ii)* follows from Lemmas 17 and 10 and the Generalized Weierstrass Theorem, according to which a weakly upper semi-continuous functional attains a maximum in a weakly compact set (e.g., Zeidler (1985)). □

Thus, there is an optimal solution to Problem 4. Optimal solutions, involving vector valued functions of time, are often hard to calculate and implement. The next section shows that optimal performance can be approximated by using signals that are relatively simple to calculate and implement.

5. APPROXIMATING OPTIMAL PERFORMANCE

We show that optimal performance can be approximated by using bang-bang control input signals; this provides a relatively easy to calculate and implement path to optimization. Specifically, let ℓ' be an error bound slightly bigger than the specified error bound ℓ. Then, a bang-bang control input signal $u^{\pm}$ can keep the controlled system below the operating error bound of ℓ' for at least as long as the maximal time $t(x_0, \gamma, \ell)$ achieved by an optimal input signal for the error bound ℓ, as follows.

Theorem 18. The following are true for any error bound $\ell' > \ell$:

(i) If $t(x_0, \gamma, \ell) = \infty$, then, for every time $t' > \tau$, there is a bang-bang control input signal $u'^{\pm} \in U(K)$ (with a finite number of switchings) for which $t(x_0, \gamma, u'^{\pm}, \ell') \geq t'$.

(ii) If $t(x_0, \gamma, \ell) < \infty$, then there is a bang-bang control input signal $u^{\pm} \in U(K)$ (with a finite number of switchings) for which $t(x_0, \gamma, u^{\pm}, \ell') \geq t(x_0, \gamma, \ell)$. □

The proof of Theorem 18 requires an auxiliary result, which is a consequence of Lemmas 10 and 14 and of the continuity of the function $b(t, x)$ of (4).

Lemma 19. Set $x(t) := \Sigma(x_0, v, u, t)$, let $b(t, x)$ be as in (4), and let $t' \geq 0$. Then, for every $\rho > 0$, there is a $\beta(x_0, t', \rho) > 0$ such that $|b(t_1, x(t_1)) - b(t_2, x(t_2))| < \rho$ for all $t_1, t_2 \in [0, t']$ satisfying $|t_1 - t_2| < \beta(x_0, t', \rho)$ for all residual input signals $v \in V(v_0, \gamma)$ and all control input signals $u \in U(K)$. □

Proof. *(of Theorem 18; sketch)* Let $u \in U(K)$ be a control input signal that satisfies either part *(i)* or part *(ii)* of Theorem 8. We show that the performance elicited by u can be approximated by a bang-bang control input signal $u^{\pm} \in U(K)$. Denote $x(t) := \Sigma(x_0, v, u, t)$ and $x^{\pm}(t) := \Sigma(x_0, v, u^{\pm}, t)$. As the initial condition and the residual input signal are the same in both cases,

$$x(t) = x^{\pm}(t) \text{ for all } t \in [0, \tau]. \tag{27}$$

Now, consider a time $t'' > \tau$, where $t'' = t'$ for part *(i)* of Theorem 8 and $t'' = t(x_0, \gamma, \ell)$ for part *(ii)*. Later, we select two times $t_1, t_2 \in [0, t'' - \tau]$, $t_1 < t_2$, and a number $\lambda > 0$ such that $p := (t_2 - t_1)/\lambda$ is an integer, and build the partition

$$[t_1, t_2] = \{[t_1, t_1+\lambda], [t_1+\lambda, t_1+2\lambda], \cdots, [t_1+(p-1)\lambda, t_2]\}.$$

In each segment $[t_1 + q\lambda, t_1 + (q+1)\lambda]$, $q = 1, 2, \ldots, p-1$, we select below m points $\theta_1^q, \theta_2^q, \cdots, \theta_m^q$ to serve as switching points of a bang-bang control input signal $u^{\pm} = (u_1^{\pm}, \cdots, u_m^{\pm})^T \in U(K)$:

$$u_i^{\pm}(t) := \begin{cases} +K, & t \in [t_1 + q\lambda, \theta_i^q), \\ -K, & t \in [\theta_i^q, t_1 + (q+1)\lambda), \text{ if } \theta_i^q < t_1 + (q+1)\lambda, \end{cases}$$

$q = 0, 1, \ldots, p-1$, $i = 1, 2, \ldots, m$.

Here, $\theta_i^q \in [t_1 + q\lambda, t_1 + (q+1)\lambda]$ is calculated from the equality

$$K[2(\theta_i^q - (t_1 + q\lambda)) - \lambda] = \int_{t_1+q\lambda}^{t_1+(q+1)\lambda} u_i(s)ds.$$

This implies for all $i \in \{1, 2, \ldots, m\}$, $q \in \{0, 1, \ldots, p-1\}$ that

$$\int_{t_1+q\lambda}^{t_1+(q+1)\lambda} \left(u_i(s) - u_i^{\pm}(s)\right) ds = 0. \tag{28}$$

To continue, set $\xi(t) := x(t) - x^{\pm}(t)$, $t \in [0, t'']$; by (27),

$$\xi(t) = 0 \text{ for all } t \in [0, \tau]. \tag{29}$$

Using (4) we get

$$\begin{aligned} &\sup_{t\in[\tau+t_1,\tau+t_2]} |\xi(t)| \leq |\xi(\tau+t_1)| \\ &+ \sup_{t\in[\tau+t_1,\tau+t_2]} \left| \int_{\tau+t_1}^{t} \left[a(s, x(s)) - a(s, x^{\pm}(s))\right] ds \right| \\ &+ \sup_{t\in[\tau+t_1,\tau+t_2]} \left| \int_{\tau+t_1}^{t} b(s, x(s))u(s-\tau) - b(s, x^{\pm}(s))u^{\pm}(s-\tau)ds \right|. \end{aligned}$$

Invoking (6), (7), and (8) and rearranging terms leads to

$$\begin{aligned} &[1 - (\alpha+\gamma+(\beta+\gamma)K)(t_2 - t_1)] \sup_{t\in[\tau+t_1,\tau+t_2]} |\xi(t)| \leq |\xi(\tau+t_1)| \\ &+ \sup_{t\in[\tau+t_1,\tau+t_2]} \left| \int_{\tau+t_1}^{t} b(s, x(s)) \left(u(s-\tau) - u^{\pm}(s-\tau)\right) ds \right|. \end{aligned}$$

Choose a number $\eta \in (0, t'' - (\tau + t_1)]$ for which $(\alpha + \gamma + (\beta + \gamma)K)\eta < 1$; set

$$t_2 := t_1 + \eta; \tag{30}$$

and denote $\mu(\eta) := 1/[1 - (\alpha + \gamma + (\beta + \gamma)K)\eta]$. Then,

$$\begin{aligned} &\sup_{t\in[\tau+t_1,\tau+t_2]} |\xi(t)| \leq \mu(\eta)\,|\xi(\tau+t_1)| + \mu(\eta) \sup_{t\in[t_1,t_1+\eta]} \Bigg\{ \\ &\left| \int_{t_1}^{t-\tau} b(s+\tau, x(s+\tau)) \left(u(s) - u^{\pm}(s)\right) ds \right| \Bigg\}. \end{aligned} \tag{31}$$

Regarding the last term, refer to Lemma 19, choose a number $\rho > 0$ and select $\lambda \leq \beta(\rho, x_0, t')$. Let $q(t) \in \{0, 1, 2, \ldots, p-1\}$ be the integer satisfying $t - \tau \in [q(t)\lambda, (q(t)+1)\lambda]$. Using (28), Corollary 6, and noting that $0 \leq q(t)\lambda \leq t_2 - t_1 = \eta$ by (30), gets

$$\begin{aligned} &\sup_{t\in[t_1,t_1+\eta]} \left| \int_{t_1}^{t-\tau} b(s+\tau, x(s+\tau)) \left(u(s) - u^{\pm}(s)\right) ds \right| \\ &= \sup_{t\in[t_1,t_1+\eta]} \left| \sum_{i=0}^{q(t)-1} \int_{t_1+i\lambda}^{t_1+(i+1)\lambda} b(s+\tau, x(s+\tau)) \left(u(s) - u^{\pm}(s)\right) ds \right. \\ &\left. + \int_{t_1+q(t)\lambda}^{t-\tau} b(s+\tau, x(s+\tau)) \left(u(s) - u^{\pm}(s)\right) ds \right| \\ &\leq 2K\rho\eta + 2KM_{ab}(t'')\lambda. \end{aligned}$$

Substituting into (31), we obtain

$$\sup_{t\in[\tau+t_1,\tau+t_2]} |\xi(t)| \le \mu(\eta)\,|\xi(\tau+t_1)| + \mu(\eta)\,[2K\rho\eta + 2KM_{ab}(t'')\lambda]\,.$$

Further, fix $\delta > 0$; then select $\rho > 0$ to satisfy $\mu(\eta)K\rho\eta < \delta/4$, and select $\lambda > 0$ to satisfy $\mu(\eta)KM_{ab}(t'')\lambda < \delta/4$. This yields

$$\sup_{t\in[\tau+t_1,\tau+t_2]} |\xi(t)| \le \mu(\eta)\,|\xi(\tau+t_1)| + \delta. \tag{32}$$

Let $r \ge (t''-\tau)/\eta$ be an integer and build the partition

$$[\tau,t''] \subseteq \{[\tau,\tau+\eta],[\tau+\eta,\tau+2\eta],\ldots,[\tau+(p-1)\eta,\tau+p\eta]\}\,.$$

Then, we get from (32) the recursive relation

$$\sup_{t\in[\tau+i\eta,\tau+(i+1)\eta]} |\xi(t)| \le \mu(\eta)\,|\xi(\tau+i\eta)| + \delta, \xi(\tau) = 0,$$

$i = 0,\ldots,r-1$, so that $\sup_{t\in[\tau,t'']} |\xi(t)| \le \delta\sum_{i=0}^{r}(\mu(\eta))^i$. Finally, the theorem follows by choosing $\delta > 0$ sufficiently small. □

Thus, optimal performance can be closely approximated by bang-bang signals – signals that are relatively easy to calculate and implement.

6. EXAMPLE

Consider the following system Σ:

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} -x_2(t)(0.3+\sin x_1(t)) + (2+\cos t)u(t-0.5) \\ a\dfrac{x_1(t)}{1+x_1^2(t)}\sin t + (2+\sin x_2(t))u(t-0.5) \end{pmatrix},$$

where $-0.5 \le a \le 0.5$ is an unspecified constant; the input time-delay is $\tau = 0.5$ seconds; the initial state is $x_0 = [1,-0.5]^T$; the input signal bound is $K = 1$; the residual input signal $v(t)$ is an unspecified constant signal $v(t) = c$, $-0.1 \le c \le 0.1$, $t \in [-0.5,0]$; and the error bound is $\ell = 4$.

The maximal time $t(x_0)$ during which the inequality $x^T(t)x(t) \le 4$ can be maintained is in this case $t(x_0) \approx 4.1$ seconds (found by a numerical search process; see Choi and Hammer (2016) for details). As shown in Figures 2 and 3, a similar time is achieved by a bang-bang control input signal $u^{\pm}(t)$ with just two switching times (obtained through a numerical search process; see Choi and Hammer (2016) for details). In Figure 3, the parameter values used are:

Set 1: $a = -0.5$, $v(\theta) = -0.1$;
Set 2: $a = 0$, $v(\theta) = 0$;
Set 3: $a = 0.5$, $v(\theta) = 0.1$.

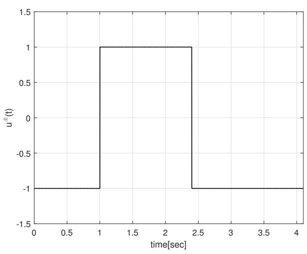

Fig. 2. Bang-bang input

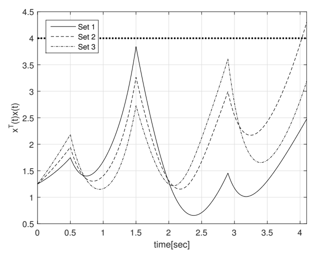

Fig. 3. The response

7. CONCLUSION

We discussed the existence and the implementation of optimal controllers that keep operating errors below a specified bound for the longest time possible during feedback outage. As shown, such optimal controllers do exist and their performance can be approximated as closely as desired by bang-bang signals that are relatively easy to calculate and implement.

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