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Tracking and approximate model matching for non-linear systems

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The design of non-linear tracking systems is investigated in a general setting that guarantees internal stability. It is shown that, by adopting an appropriate design methodology, tracking accuracy can be improved to any desirable level. The proposed methodology is computationally, as well as conceptually, undemanding, and it also leads to a simple solution of a class of non-linear approximate model matching problems.

1. Introduction

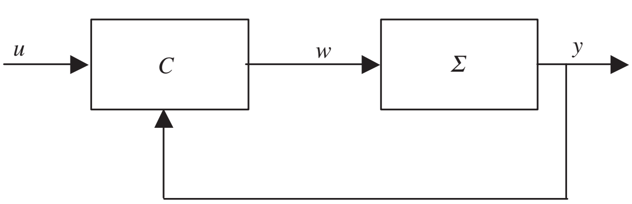
A substantial portion of the theory and practice of control engineering is devoted to the study of tracking systems, namely, of systems whose output is required to follow their input closely. Tracking systems are often characterized in terms of their *tracking error*—the deviation of the output signal from the corresponding input signal. Reduction of the tracking error is a critical design objective in many applications of control systems, including the design of guidance and navigation systems. The present paper addresses the problem of reducing tracking errors in a rather general context that encompasses linear as well as non-linear systems. Needless to say, low tracking errors can be achieved only for signals that are feasible output signals of the system being controlled.

Tracking can be viewed as a special case of approximate model matching. Indeed, consider two systems Ξ and Φ sharing their normed input and output spaces, and let $|\cdot|$ denote a norm over these spaces. Given a real number $\varepsilon > 0$, we say that Φ is a ε-*approximant* of Ξ over the input domain S if

$$|\Phi u - \Xi u| \le \varepsilon \quad \text{for all } u \in S$$

Clearly, the number ε indicates the accuracy by which Φ approximates Ξ.

Consider the following closed loop control configuration.



(1)

Here, Σ is the system being controlled, C is a controller, and Σ_c is the system represented by the closed loop. For notational convenience, we assume that the controller C is constructed so that Σ_c has the same input space as Σ. Noting that the output of Σ_c is the output of Σ, we conclude that the images satisfy

$$\operatorname{Im} \Sigma_c \subset \operatorname{Im} \Sigma \tag{2}$$

Stability of the configuration (1) is, of course, an issue of critical importance. To investigate stability, we limit our attention to bounded input signals u. Thus, we will be mostly interested in the *bounded-input image* $\operatorname{Im}_b \Sigma_c$ formed by the set of output sequences of Σ_c that are generated by bounded input sequences. Similarly, the bounded-input image of Σ, denoted $\operatorname{Im}_b \Sigma$, is formed by all responses of Σ to bounded input sequences. Then, the argument leading to (2) also yields the inclusion

$$\operatorname{Im}_b \Sigma_c \subset \operatorname{Im}_b \Sigma$$

A central topic of our discussion is the existence and the design of a controller C that turns the closed loop system Σ_C into a ε-approximant of a specified system, as follows.

The approximate model matching problem: Given a system Φ, a bounded domain S, and a real number $\varepsilon > 0$, determine whether there is a controller C such that configuration (1) satisfies

$$|\Phi u - \Sigma_c u| \le \varepsilon \quad \text{for all } u \in S \tag{3}$$

If such a controller exists, describe a procedure for its design.

The system Φ of the matching problem is called the *model*. Note that a tracking system is obtained in the special case when the model is the identity system $\Phi = I$.

We introduce some notation. For a real number $\varepsilon > 0$ and an element v of a normed space, let $N_\varepsilon(v)$ designate the neighbourhood of v given by

$$N_\varepsilon(v) = \{w : |w - v| \le \varepsilon\}$$

For a set A, denote by

$$N_\varepsilon(A) = \cup_{v \in A} \quad N_\varepsilon(v)$$

the corresponding neighborhood of the set A.

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Assume now that Σ_c is a ε-approximant of the system Φ. In view of (3), we have $\Phi u \in N_\varepsilon(\Sigma_c u)$ for all $u \in S$, which implies that $\Phi[S] \subset N_\varepsilon(\Sigma_c[S])$. Recalling that S is a bounded domain, we clearly have $N_\varepsilon(\Sigma_c[S]) \subset N_\varepsilon(\mathrm{Im}_b\, \Sigma_c)$, so that

$$\Phi[S] \subset N_\varepsilon(\mathrm{Im}_b \Sigma)$$

When Φ is an invertible system, this leads to the relation

$$S \subset \Phi^{-1}[N_\varepsilon(\mathrm{Im}_b \Sigma)]$$

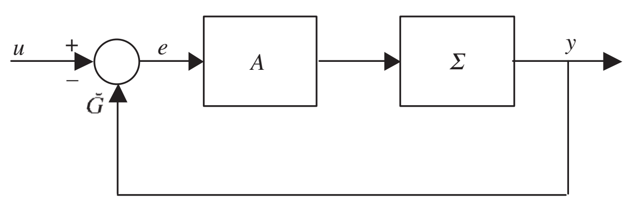
For tracking systems, simply substitute $\Phi = I$, which yields

$$S \subset N_\varepsilon(\mathrm{Im}_b\, \Sigma) \tag{4}$$

Recalling that S forms the set of signals the system must track, the inclusion (4) implies, as one would expect, that only signals that are within ε of the image of Σ can be tracked with an error not exceeding ε. This simple and rather obvious fact forms a fundamental restriction on the operation of tracking systems. It plays an important role in our ensuing discussion. In order to track signals of the class S without error, set $\varepsilon = 0$ in (4), to obtain the following necessary condition for accurate tracking

$$S \subset \mathrm{Im}_b\, \Sigma \tag{5}$$

The methodology developed in this paper builds on the classical control principle according to which high forward gain yields high accuracy in closed loop systems (Black 1934, Newton *et al.* 1957). This principle is based on the use of the following configuration, commonly called the *Black diagram*.



(6)

Here, Σ is the system being controlled, and A is a high gain linear amplifier.

By refining and somewhat expanding this classical principle, we derive a general framework for the design of controllers that achieve tracking and approximate model matching. The resulting framework applies to linear as well as to non-linear systems, and it guarantees internal stability. It can accommodate errors and uncertainties encountered in the modelling and in the operation of engineering systems.

In its simplistic form, the classical principle of using high forward gain applies only to a rather restricted class of systems; for more general systems, it gives rise to instability. The process of extending this principle to achieve internally stable control of linear and non-linear systems leads to controllers with an interesting feature: the high-gain amplifier is combined with a controller that has hysteresis-type properties (§ 5). The resulting closed loop system is then internally stable, and its performance accuracy improves as the amplifier's gain grows. In a broad sense, this reaffirms the advantages of high forward gain, and it results in a rather simple design methodology for a common category of control systems.

The performance limitation (4) brings into focus the need to properly specify tracking signals, and it highlights a distinction between the present approach and the traditional method of designing tracking systems. Traditionally, tracking signals are specified without regard to the restriction (4), and, consequently, cannot always be tracked in their entirety. Instead, one lets the tracking system choose a path that converges asymptotically to the tracking signal. This leaves the designer with incomplete control over the tracking process. In the approach taken in this paper, the tracking signal is selected so that it satisfies the inclusion (4). In this way, the system does not deviate from the tracking signal by more than the permissible error throughout the entire tracking process.

As an example, consider the case of a ground-to-air missile tracking an airplane. In traditional tracking, the missile is given the airplane's flight data, and is left to choose its own approach path to the airplane. This leaves the operator without control over part of the tracking process. In the approach developed in this paper, the entire path of the missile, including takeoff, is specified, subject to the constraint (4). This allows the operator more complete control over the tracking process and the missile performance, and facilitates the selection of a stealthy ascent path.

In many applications, the selection a tracking signal near the image of the controlled system is not an overly taxing process. Often, appropriate signals can be selected based on general characteristics of the system, such as bandwidth restrictions, signal magnitude bounds, bounds on rates-of-change, or other key features.

In addition to the tracking problem, the paper also addresses the problem of approximate model matching for non-linear systems. A simple methodology for approximate model matching is developed in § 4 and 5, based on a special control configuration that employs a high-gain amplifier in its forward path. It is shown that, in this configuration, the accuracy of matching the model improves as the gain grows larger.

In summary, when appropriately extended, the classical principle of using high forward gain becomes an effective methodology for addressing a number of general issues related to the control of non-linear systems. Alternative approaches to the control of non-linear systems can be found in Lasalle and Lefschetz (1961), Lefschetz (1965), Hammer (1984, 1985, 1989, 1994), Desoer and Kabuli (1988), Verma (1988), Sontag (1989), Chen and de Figueiredo (1990), Paice and Moore (1990), Verma and Hunt (1993), Sandberg (1993), Paice and van der Schaft (1994), Baramov and Kimura (1995), Georgiou and Smith (1997), Logemann *et al.* (1999), the references cited in these publications, and others.

The paper is organized as follows. In § 2 we introduce some terminology and provide a qualitative overview of the approach taken in the paper. Section 3 describes a technical concept which underlies much of the discussion of the subsequent sections. The control methodology of the paper is first introduced in § 4, where it is applied to the special case of (linear or non-linear) minimum-phase systems. The results are then generalized in § 5 to linear or non-linear systems that do not have the minimum-phase property.

2. Basic Considerations

2.1. *Preliminaries*

The presentation in the paper is for discrete-time systems, but the same principles can be extended to continuous-time systems as well. Let R be the set of real numbers, let R^m be the set of all m-dimensional real vectors, and let $S(R^m)$ be the set of all sequences $u = \{u_0, u_1, u_2, \ldots\}$ of m-dimensional real vectors, where $u_i \in R^m$, $i = 0, 1, 2, \ldots$ A system Σ with specified initial conditions induces a map $\Sigma : S(R^m) \to S(R^p)$ that transforms input sequences of m-dimensional real vectors into output sequences of p-dimensional real vectors. We write $y = \Sigma u$ to represent the output sequence y generated by Σ from the input sequence u. It will be convenient to assume that the system Σ has an equilibrium point at the zero input sequence, so that $\Sigma 0 = 0$.

As usual, a system Σ is *causal* if its response does not depend on future input values. The system is *strictly causal* if there is a delay of at least one step before input changes are reflected in its response. Finally, the system Σ is *bicausal* if it is invertible, and if Σ and its inverse Σ^{-1} are both causal systems (e.g. Hammer 1984).

The systems we consider are given in terms of a *state space representation*

$$\left.\begin{aligned} x_{k+1} &= f(x_k, u_k) \\ y_k &= h(x_k), \quad k = 0, 1, 2, \ldots \end{aligned}\right\} \tag{7}$$

Here, $x_k \in R^n$ is the *state* of the system at the step k, while u_k and y_k represent the input value and the output value, respectively, at that step. The function $f : R^n \times R^m \to R^n$ is the *recursion function* and $h : R^n \to R^p$ is the *output function*. For convenience, we use the initial condition $x_0 = 0$ for the system, but the discussion can be adapted to more general initial conditions. Note that a system described by (7) is strictly causal, since the output function h does not depend on the input value u_k (e.g. Hammer 1984). We say that the realization (7) is *uniformly continuous* if f and h are uniformly continuous functions. The *input/state part* Σ_s of Σ is the system described by the recursion $x_{k+1} = f(x_k, u_k)$, $k = 0, 1, \ldots$

As our discussion in this paper involves stability issues, we need to introduce some norms over our spaces. First, for a real number a, let $|a|$ be the absolute value of a. For a vector $r = (r^1, r^2, \ldots, r^q) \in R^q$, denote

$$|r| := \max\{|r^i|, i = 1, \ldots, q\}$$

it will be convenient to refer to $|r|$ as the ℓ^∞-*norm* of the vector $\boldsymbol{r}$. Finally, for an element $s \in S(R^q)$, the ℓ^∞-*norm* is given by

$$|s| := \sup_{i \geq 0} |s_i|$$

where $|s| := \infty$ when the supremum does not exist. A subset $S \subset S(R^q)$ is *bounded* if there is a real number $M \geq 0$ such that $|s| \leq M$ for all elements $s \in S$; when the latter holds, we write $|S| \leq M$. To improve clarity, we sometimes refer to such a set as 'ℓ^∞-bounded'. Also, given a real number $\theta \geq 0$, we denote by $S(\theta^q)$ the set of all sequences $s \in S(R^q)$ satisfying $|s| \leq \theta$, i.e. the set of all sequences of q-dimensional vectors bounded by θ.

Another norm that is important to our discussion is the ℓ^1-*norm* given, for a vector $v = (v^1, v^2, \ldots, v^p) \in R^p$, by

$$|v|_1 := |v^1| + |v^2| + \cdots + |v^p|$$

The *weighted* ℓ^1-*norm* $|\cdot|_{1w}$ is defined, for a sequence $y \in S(R^p)$, by

$$|y|_{1w} := \sum_{i=0}^{\infty} 2^{-i} |y_i|^1 \tag{8}$$

It is easy to see that the weighted ℓ^1-norm exists for every bounded sequence $y \in S(R^p)$.

A norm $\langle \cdot \rangle$ over $S(R^p)$ is *compatible* with the weighted ℓ^1-norm if there is a constant a > 0 such that $\langle u \rangle \leq a|u|_w^1$ for all $u \in S(R^p)$. In our ensuing discussion, we will denote by $|| \cdot ||$ a norm that has the following properties: (i) it is compatible with the weighted ℓ^1-norm, and (ii) under it, every closed and ℓ^∞-bounded subset of $S(R^p)$ is compact.

A system $\Sigma : S(R^m) \to S(R^p)$ is *BIBO-stable* (bounded-input bounded-output stable) if, for every real number $M \geq 0$, there is a real number $N \geq 0$ such that $|\Sigma u| \leq N$ whenever $|u| \leq M$. The notion of BIBO-stability is an underlying requirement of every other stability concept. We say that Σ is *stable* if it is BIBO-stable and if it is continuous with respect to the norm $||\cdot||$.

When a system $\Sigma : S(R^p) \to S(R^p)$ is invertible as a set mapping, we denote its inverse by Σ^{-1}, so that $\Sigma^{-1}\Sigma = \Sigma\Sigma^{-1} = I$, the identity system. If Σ^{-1} is BIBO-stable, then Σ is called a *BIBO-minimum phase system*. When Σ is both BIBO-stable and BIBO-minimum phase, then it is called a *BIBO-unimodular system*. Finally, if Σ^{-1} is stable, the we say that Σ is a *minimum phase system* (omitting 'BIBO'), and if Σ is both stable and minimum phase, then it is called a *unimodular system*.

For composite systems, a stronger notion of stability is used. Consider a composite system Ψ that consists of q subsystems. Add an external signal to the output of each one of the subsystems of which the composite system consists. This results in a system with $q+1$ external input signals — the original input signal and the q newly added signals. The composite system Ψ is *internally BIBO-stable* if the following holds for each one of the $(q+1)$ external input signals: the map from the external signal to any signal within the configuration forms a BIBO-stable system. We say that Ψ is *internally stable* if each such map forms a stable system. Internal stability guaranties that a composite system is implementable and well behaved.

2.2. *Ideas from classical control theory*

The classical control solution of the tracking problem is based on the use of high forward gain within the Black diagram (6) (Black 1934). In the diagram, Σ is the system being controlled and A represents an ideal amplifier with gain A. Of course, the use of unity feedback in the diagram requires that the number of outputs of Σ be equal to the number of inputs, say $\Sigma : S(R^m) \to S(R^m)$ ($m=1$ is used in classical control). Let Σ_A denote the input/output relation of the closed loop (6).

In order to concentrate on the tracking problem and ignore other issues, we assume that Σ is a stable system. In case Σ is not stable, the block 'Σ' in (6) must be replaced by a closed loop system that stabilizes Σ. If done appropriately, this process separates the problems of stabilization and tracking without impairing the ability to control the system effectively (Hammer 1994).

Assume now that Σ is a strictly causal and invertible system (it is not necessary to assume that Σ is linear). An examination of (6) yields

$$y = \Sigma A e, \quad e = u - y \tag{9}$$

so that $e = u - \Sigma A e$, or $[I + \Sigma A]e = u$, where I is the identity system. The fact that Σ is strictly causal implies that the system $[I + \Sigma A]$ is invertible (e.g. Hammer 1984), so we can write $e = [I + \Sigma A]^{-1}u$. Using (9) again, we have $y = \Sigma A[I + \Sigma A]^{-1}u$, which is the classical feedback equation

$$\Sigma_A = \Sigma A[I + \Sigma A]^{-1}$$

Taking into account the fact that A represents a constant 'scalar' gain, we can write $y = \Sigma A\{[(1/A)I + \Sigma]A\}^{-1}u = \Sigma A A^{-1}[(1/A)I + \Sigma]^{-1}u$, or

$$y = \Sigma[(1/A)I + \Sigma]^{-1}u \tag{10}$$

Note that (10) is valid even when Σ is a non-linear system. Ignoring for a moment mathematical rigor (§§3 and 4 present a rigorous discussion), one may presume that

$$\lim_{A\to\infty}[(1/A)I + \Sigma] = \Sigma \tag{11}$$

If (11) is accepted as correct and substituted into (10), and if Σ is continuous, one arrives at the conclusion

$$\lim_{A\to\infty}\Sigma[(1/A)I + \Sigma]^{-1} = \Sigma\Sigma^{-1} = I \tag{12}$$

In other words, when the gain A is sufficiently large, one has

$$y \approx u \tag{13}$$

Conclusion (13) seems to indicate that, for large gain A, configuration (6) becomes an accurate tracking system irrespective of the nature of Σ, as long as Σ preserves its invertibility and strict causality.

Great caution has to be exercised when drawing such far reaching conclusions. Indeed, a brief examination of (12) reveals a major difficulty in the case when Σ is not a BIBO-minimum phase system. Indeed, the expression $\Sigma\Sigma^{-1}$ implies that, for large gain A, the input signal of Σ in (6) is (almost) equal to the output signal of Σ^{-1}. When Σ is not a BIBO-minimum phase system, this means that the input of Σ in (6) is unbounded for at least some input signals u, invalidating the internal stability of the configuration. In fact, this qualitative argument indicates that configuration (6) cannot be used with large gain A when Σ is not a BIBO-minimum phase system. Thus, it seems that the favorable tracking properties of the Black diagram are restricted to systems that are BIBO-unimodular (recall that Σ was assumed to be BIBO-stable).

Section 4 provides an accurate analysis of the tracking properties of the Black diagram for non-linear BIBO-stable and minimum phase systems. Generally speaking, the analysis confirms the prowess of the Black diagram to achieve good tracking for such systems, if one allows for a small modification of the diagram. Subsequently, a general investigation of tracking for stable non-linear systems is provided. In the meanwhile, the next section introduces a technical concept that is instrumental to the discussion.

3. Subbounded systems

Let R^+ be the set of all non-negative real numbers. A *bound function* is a strictly increasing continuous function $\alpha : R^+ \to R$, whose image includes R^+. Note that, for a bound function α, the image satisfies $R^+ \subset \operatorname{Im}\alpha$, and the restriction $\alpha : R^+ \to \operatorname{Im}\alpha$ is an isomorphism with an inverse $\alpha^{-1} : \operatorname{Im}\alpha \to R^+$. The notation α^{-1} will always refer to this inverse. Examples of bound functions include the functions $\alpha(\theta) = a\theta$, where a is a positive constant; $\alpha(\theta) = a\theta^2$; or $\alpha(\theta) = a\theta^{1/2}$; or, more generally, all forms of

$$\alpha(\theta) = a\theta^b \tag{14}$$

where a, $b > 0$ are constants. Of course, bound functions may take other forms as well. The following is the basic concept discussed in this section.

Definition 1: A system $\Sigma : S(R^m) \to S(R^m)$ is a *sub-bounded system* if there is a bound function α satisfying $|\Sigma w| \geq \alpha(|w|)$ for all $w \in S(R^m)$. The function α is then called a *lower bound function* of Σ.

The next statement indicates that (invertible) sub-bounded systems are always BIBO-minimum phase systems. Later on, we show that the converse is also true, so that the existence of a lower bound function characterizes a BIBO-minimum phase system.

Proposition 1: *A sub-bounded invertible system $\Sigma : S(R^m) \to S(R^m)$ is a BIBO-minimum phase system.*

Proof: Let $\Sigma : S(R^m) \to S(R^m)$ be a sub-bounded invertible system. There is then a bound function $\alpha : R^+ \to R$ such that $|\Sigma u| \geq \alpha(|u|)$ for all $u \in S(R^m)$. Now, consider a sequence $w \in S(R^m)$, and let $u := \Sigma^{-1}w$. Substituting into the last inequality, we obtain $|\Sigma\Sigma^{-1}w| \geq \alpha(|\Sigma^{-1}w|)$, so that

$$|w| \geq \alpha(|\Sigma^{-1}w|) \tag{15}$$

Now, the fact that α is strictly increasing implies that α^{-1} is also strictly increasing. Consequently, applying α^{-1} to both sides of (15) yields $\alpha^{-1}(|w|) \geq \alpha^{-1}\alpha(|\Sigma^{-1}w|)$, so that $\alpha^{-1}(|w|) \geq |\Sigma^{-1}w|$. This shows that $|\Sigma^{-1}w|$ is bounded whenever $|w|$ is bounded, namely, that Σ is a BIBO-minimum phase system. □

It is easy to show that every linear BIBO-minimum phase system $\Sigma : S(R^m) \to S(R^m)$ has a lower gain function. Indeed, let Σ be such a system. Then, by definition, the inverse Σ^{-1} is an invertible and stable system, so it has a well defined norm $|\Sigma^{-1}| > 0$ satisfying $|\Sigma^{-1}y| \leq |\Sigma^{-1}||y|$ for all $y \in S(R^m)$. Denoting $w := \Sigma^{-1}y$ and substituting into the previous inequality, we get $|w| \leq |\Sigma^{-1}||y| = |\Sigma^{-1}||\Sigma w|$, where we have used the obvious fact $y = \Sigma w$. This yields

$$|\Sigma w| \geq \frac{1}{|\Sigma^{-1}|}|w| \tag{16}$$

so that a linear BIBO-minimum phase system Σ is always sub-bounded, with a lower bound function given by

$$\alpha(\theta) := \frac{1}{|\Sigma^{-1}|}\theta \tag{17}$$

In fact, any function of the form $\beta(\theta) = a\theta$ with $0 < a \leq 1/|\Sigma^{-1}|$ can serve as a lower bound function for Σ.

To generalize the discussion of the last paragraph to non-linear systems, we need the following concept. A preliminary note on notation: let $\theta \geq 0$ be a real number; the set of all sequences $u \in S(R^m)$ satisfying $|u| \geq \theta$ includes all sequences with unbounded norm.

Definition 2: Let $\Sigma : S(R^m) \to S(R^p)$ be a BIBO-stable system. The function $\varphi : R^+ \to R^+$ given for a real number $\theta \geq 0$ by

$$\varphi(\theta) := \inf_{|u| \geq \theta} |\Sigma u| \tag{18}$$

is called the *floor function* of Σ.

The fact that Σ is BIBO-stable means that, for every real number $\theta \geq 0$, there is a real number $B \geq 0$ such that $\Sigma[S(\theta^p)] \subset S(B^p)$. This implies that the function φ of (18) satisfies $\varphi(\theta) \leq B$. Consequently, φ is a bounded function when Σ is BIBO-stable. Another direct consequence of (18) is that φ is a monotone increasing function. By using the function φ, the BIBO-minimum phase property can be characterized as follows.

Proposition 2: *Let $\Sigma : S(R^m) \to S(R^m)$ be an invertible BIBO-stable system, and let φ be its floor function. Then, the following are equivalent:*

(i) *Σ is a BIBO-minimum phase system.*

(ii) *For every unbounded sequence of real numbers $\{\theta_n\}_{n=1}^{\infty}$, the sequence $\{\varphi(\theta_n)\}_{n=1}^{\infty}$ is unbounded.*

Proof: By definition, the system Σ is a BIBO-minimum phase system if and only if Σ^{-1} is BIBO-stable, i.e. if and only if, for every real number $N \geq 0$, there is a real number $B \geq 0$ such that $\Sigma^{-1}[S(N^m)] \subset S(B^m)$. In these terms, our proof will be complete upon showing that the following two statements are equivalent.

(iii) There is a sequence of real numbers $\theta_n \to \infty$ over which φ is bounded.

(iv) There is a real number $N \geq 0$ for which the inverse image $\Sigma^{-1}[S(N^p)]$ is not bounded.

First, assume that (iii) is valid. Then, there is a sequence of real numbers $\theta_n \to \infty$ and a real number $M > 0$ such that $\varphi(\theta_n) \leq M$ for all $n \geq 1$. Choose a real number $\varepsilon > 0$. It follows by (18) that the following is true. For every integer $j \geq 1$, there is an element $u(j) \in S(R^m)$ such that $|u(j)| \geq \theta_j$ while $|\Sigma u(j)| \leq M + \varepsilon$. Then, $u(j) \in \Sigma^{-1}[S((M+\varepsilon)^m)]$ for all $j \geq 1$.

Since $|u(j)| \geq \theta_j \to \infty$, it follows that $\Sigma^{-1}[S((M+\varepsilon)^m)]$ is not a bounded set. Thus, (iii) implies (iv) with $N := M + \varepsilon$.

Conversely, assume that (iv) is valid, i.e. that $\Sigma^{-1}[S(N^p)]$ is not a bounded set. Then, two cases are possible: (a) there is a sequence of elements $u(j) \in \Sigma^{-1}[S(N^p)]$, $j = 1, 2, \ldots$, and an unbounded sequence of real numbers $\theta_j \to \infty$ such that $|u(j)| \geq \theta_j$ for all $j \geq 1$; and (b) there is an unbounded sequence $u \in \Sigma^{-1}[S(N^p)]$. In case (a), since $\Sigma u(j) \in S(N^p)$ for all $j = 1, 2, \ldots$, we have

$$\varphi(\theta_j) = \inf_{|u| \geq \theta_j} |\Sigma u| \leq |\Sigma u(j)| \leq N$$

for all $j = 1, 2, \ldots$, so that φ is bounded by N over all real numbers. This shows that (iv) implies (iii) in case (a). When case (b) is valid, it follows directly by (18) that $\varphi(\theta) \leq N$ for all real numbers $\theta \geq 0$, so that (iv) implies (iii) in case (b) as well. This concludes the proof. □

Consider now a sub-bounded system $\Sigma : S(R^m) \to S(R^m)$. Let φ be the floor function of Σ and let α be its lower bound function. It follows directly from the definitions that

$$\alpha(\theta) \leq \varphi(\theta) \quad \text{for all } \theta \geq 0$$

In fact, a slight reflection shows that the converse direction expressed by the following statement is also true.

Lemma 1: *Let $\Sigma : S(R^m) \to S(R^m)$ be a BIBO-stable and BIBO-minimum phase system, and let φ be its floor function. Then, Σ is a sub-bounded system if and only if there is a bound function $\alpha : R^+ \to R$ satisfying $\alpha(\theta) \leq \varphi(\theta)$ for all $\theta \geq 0$.*

We are now ready to show that the property of being sub-bounded characterizes BIBO-minimum phase systems.

Theorem 1: *Let $\Sigma : S(R^m) \to S(R^m)$ be a BIBO-stable invertible system satisfying $\Sigma 0 = 0$. Then, Σ is a BIBO-minimum phase system if and only if it is sub-bounded.*

Proof: By Proposition 1, every BIBO-stable, invertible, and sub-bounded system is a minimum phase system, so we only need to prove the converse direction. Let $\Sigma : S(R^m) \to S(R^m)$ be a BIBO-stable and BIBO-minimum phase system, and let φ be its floor function. The fact that Σ is BIBO-stable implies that φ is a bounded and monotone increasing function. A lower bound function $\alpha : R^+ \to R$ for Σ can be constructed as follows.

Choose a real number $\Delta > 0$, and define the sequence of numbers

$$\varphi_n := \varphi(n\Delta), \quad n = 0, 1, 2, \ldots .$$

Note that $\varphi_0 = 0$, since $\Sigma 0 = 0$. Also, the fact that Σ is a BIBO-minimum phase system implies, by Proposition 2, that $\varphi_n \to \infty$ as $n \to \infty$. Now, set $n(0) := 0$, and define recursively a strictly increasing sequence of integers $n(1), n(2), n(3), \ldots$, where $n(1)$ is the first integer for which $\varphi_{n(1)} > \varphi_0$. Continuing by recursion, assume that $n(i)$ was defined for an integer $i \geq 1$. Then, $n(i+1)$ is the first integer for which $\varphi_{n(i+1)} > \varphi_{n(i)}$. The sequence $\{\varphi_{n(i)}\}_{i=0}^{\infty}$ is then a strictly increasing sequence of non-negative real numbers.

Now, the function α is defined as follows. Set $\alpha(0) := -1$, and

$$\alpha(\theta) := \alpha(n(i)\Delta) + \frac{\varphi_{n(i)} - \alpha(n(i)\Delta)}{[n(i+1) - n(i)]\Delta}\,\theta,$$

$$n(i)\Delta \leq \theta < n(i+1)\Delta, \quad i = 0, 1, 2, \ldots .$$

Note that the graph of $\alpha(\theta)$ consists of straight line segments with positive slopes, all lying under the graph of φ. Thus, α is a bound function and $\alpha(\theta) \leq \varphi(\theta)$ for all $\theta \geq 0$. Invoking Lemma 1, it follows that α is a lower bound function of Σ, and the proof concludes. □

4. Control of minimum phase systems

4.1. *Tracking*

In this subsection, we utilize the notion of lower bound functions to develop a solution of the tracking problem for BIBO-minimum phase systems. The discussion includes both linear as well as non-linear systems. Let $\Sigma : S(R^m) \to S(R^m)$ be a BIBO-bounded system with a lower bound function α. We define a new function $\gamma : R^+ \to R^+$, called the *lower gain function* of Σ, as follows. Select a real number $\theta_0 > 0$ for which $\alpha(\theta_0) > 0$, and set

$$\gamma(\theta) := \begin{cases} \alpha(\theta)/\theta & \text{for } \theta \geq \theta_0 \\ \alpha(\theta_0)/\theta_0 & \text{for } 0 \leq \theta < \theta_0 \end{cases} \tag{19}$$

Note that the lower gain function is not unique, as it depends on the lower bound function α and on the number θ_0.

For example, in the case of a linear BIBO-minimum phase system, it follows by (17) that a lower gain function is given by the constant function

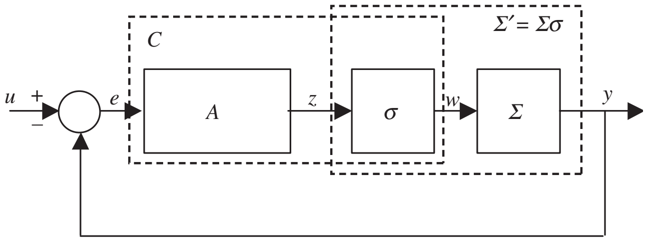
$$\gamma(\theta) = 1/|\Sigma^{-1}| \tag{20}$$

In fact, all constant functions $\gamma(\theta) := c$ with $0 < c \leq (1/|\Sigma^{-1}|)$ are valid lower gain functions of Σ in this case. More generally, when the lower bound function is of the form (14), a lower gain function is given by

$$\gamma(\theta) := \begin{cases} a\,\theta^{b-1} & \text{for } \theta \geq \theta_0 \\ a\,\theta_0^{b-1} & \text{for } 0 \leq \theta < \theta_0 \end{cases}$$

Consider now a strictly causal and sub-bounded system $\Sigma : S(R^m) \to S(R^m)$ having the lower bound function

α and a lower gain function γ. Build the control configuration



(21)

where $A > 0$ represents a constant gain

$$z_k = Ae_k, \quad k = 0, 1, 2, \ldots$$

and σ represents a static controller given by

$$w_k = \sigma(z_k) := \left[\frac{1}{\gamma(\alpha^{-1}(|z_k|)}\right] z_k, \quad k = 0, 1, 2, \ldots. \quad (22)$$

In the special case when Σ is a linear system, it follows by (20) that σ is simply a constant gain controller. When σ is a constant gain controller, it can be combined with the constant gain controller A and eliminated from the diagram. However, when Σ is a non-linear system, the compensator σ may not represent constant gain. We start our investigation of the controller σ by examining its stability properties, showing that it is, in fact, BIBO-unimodular. This will require the following auxiliary technical result. (Note that, by definition, a bound function $\alpha : R^+ \to R$ satisfies $R^+ \subset Im\,\alpha$.)

Lemma 2: *Let $\alpha : R^+ \to R$ be a bound function and let $\gamma(\theta)$ be a lower gain function of the form* (19). *Then, the following are true.*

(i) *For every vector $s \in R^m$, there is a unique vector $t \in R^m$ satisfying the relation $s = t\gamma(|t|)$.*

(ii) $s = t\gamma(|t|)$ if and only if $t = s/\gamma(\alpha^{-1}(|s|))$.

(iii) *For a sequence $w = \{w_0, w_1, w_2, \ldots\} \in S(R^m)$, set $z_k := w_k\gamma(|w_k|)$, $k = 0, 1, 2, \ldots$ Then, the ℓ^∞-norms satisfy $|z| = |w|\gamma(|w|)$.*

Proof: (i) Let $t_1, t_2 \in R^m$ be two vectors satisfying $t_1\gamma(|t_1|) = t_2\gamma(|t_2|)$. Calculating norms on both sides, we get $|t_1|\,\gamma(|t_1|) = |t_2|\,\gamma(|t_2|)$. We consider now several cases. (a) $|t_1|$, $|t_2| < \theta_0$, where θ_0 is from (19); then, $\gamma(|t_1|) = \gamma(|t_2|) = \alpha(\theta_0)/\theta_0$, so the equality $t_1\,\gamma(|t_1|) = t_2\,\gamma(|t_2|)$ clearly implies that $t_1 = t_2$. (b) $|t_1| < \theta_0$ while $t_2 \geq \theta_0$; then, $|t_1|\,\gamma(|t_1|) = |t_1|\alpha(\theta_0)/\theta_0$ while $|t_2|\,\gamma(|t_2|) = \alpha(|t_2|)$. Now, since $|t_1| < \theta_0$, it follows that $|t_1|\,\gamma(|t_1|) = |t_1|\alpha(\theta_0)/\theta_0 < \alpha(\theta_0) \leq \alpha(|t_2|)$, where the last inequality follows from the relation $\theta_0 \leq |t_2|$. Thus, $|t_1|\gamma(|t_1|) \neq |t_2|\gamma(|t_2|)$, so that $t_1\,\gamma(|t_1|) \neq t_2\,\gamma(|t_2|)$, and case (b) is not possible under our assumption. (c) $|t_1|, |t_2| \geq \theta_0$; then, by (19), the equality $|t_1|\,\gamma(|t_1|) = |t_2|\,\gamma(|t_2|)$ implies that $\alpha(|t_1|) = \alpha(|t_2|)$. From the invertibility of α, we conclude that $|t_1| = |t_2|$, so that $\gamma(|t_1|) = \gamma(|t_2|)$. The equality $t_1\,\gamma(|t_1|) = t_2\,\gamma(|t_2|)$ implies then that $t_1 = t_2$. This completes the proof of part (i).

Turning to part (ii), assume that $s = t\,\gamma(|t|)$ and $|t| < \theta_0$. Then, $s = t\,\alpha(\theta_0)/\theta_0$, so that $|s| = |t|\alpha(\theta_0)/\theta_0 < \alpha(\theta_0)$. Using the fact that α is strictly increasing, the last inequality implies that $\alpha^{-1}(|s|) < \theta_0$. Thus, $s/\gamma(\alpha^{-1}(|s|)) = s(\theta_0/\alpha(\theta_0)) = t$, so that (ii) is valid when $|t| < \theta_0$.

Next, assume that $s = t\,\gamma(|t|)$ and $|t| \geq \theta_0$. Then, calculating norms on both sides, we obtain $|s| = |t\,\gamma(|t|)| = |t|\,\gamma(|t|) = \alpha(|t|)$, according to (19). Consequently, $|t| = \alpha^{-1}(|s|)$; substituting into the equation $s = t\,\gamma(|t|)$, and using the fact that $\gamma(\theta) > 0$ for all $\theta > \theta_0$, we can write $t = s/\,\gamma(|t|) = s/\gamma(\alpha^{-1}(|s|))$. This proves one direction of part (ii).

For the converse direction of part (ii), assume that

$$t = s/\gamma(\alpha^{-1}(|s|)) \quad (23)$$

Consider first the case α^{-1} $(|s|) < \theta_0$. Then, since α^{-1} is strictly increasing, it follows that $|s| < \alpha(\theta_0)$. Consequently, in this case, $|t| = (\theta_0/\alpha(\theta_0))|s| < \theta_0$, so that $\gamma(|t|) = (\alpha(\theta_0)/\theta_0)$. The equation $s = (\alpha(\theta_0)/\theta_0)t$, which follows directly from (23), implies then that $s = t\,\gamma(|t|)$. This proves (ii) when $\alpha^{-1}(|s|) < \theta_0$.

In continuation, assume that $\alpha^{-1}(|s|) \geq \theta_0$. Then, $|t| = |s|/\gamma\,(\alpha^{-1}(|s|)) = |s|\alpha^{-1}(|s|)/[\alpha(\alpha^{-1}(|s|))] = \alpha^{-1}(|s|)$, so that $|t| = \alpha^{-1}(|s|)$, or $|s| = \alpha(|t|)$. Substituting into (23), we obtain $t = s/\gamma(\alpha^{-1}(\alpha(|t|))) = s/\gamma(|t|)$, or $s = t\gamma(|t|)$, which completes the proof of (ii).

Finally, regarding part (iii), we again distinguish between two possibilities. First, if $|w_k| < \theta_0$ for all k, then $|w| \leq \theta_0$, and we obtain $z_k = (\alpha(\theta_0)/\theta_0)w_k$ for all k, which implies that $|z| = (\alpha(\theta_0)/\theta_0)|w| = |w|\,\gamma(|w|)$. Next, let K be the set of all integers k for which $|w_k| \geq \theta_0$, and assume that $K \neq \phi$. Then, $|w| \geq \theta_0$, and, for $k \in K$, it follows by (19) that

$$|z_k| = |w_k|\gamma(|w_k|) = \alpha(|w_k|), \quad k \in K$$

Now, fix an integer $k \in K$, and consider the positive real numbers $\alpha(|w_0|), \alpha(|w_1|), \ldots, \alpha(|w_k|)$. Let $\alpha(|w_j|)$ be the largest of these numbers, that is, $\alpha(|w_j|) \geq \alpha(|w_i|)$ for all $i = 0, \ldots, k$. The fact that α is a monotone strictly increasing function implies that $|w_j| \geq |w_i|$ for all $i = 0, \ldots, k$, so that $|w_j| \geq \theta_0$. These facts lead to the following chain of equalities

$$\begin{aligned}|z_0^k| &= \max_{i=0,\ldots k} |z_i| \\ &= \max_{i=0,\ldots,k} \alpha(|w_i|) \\ &= \alpha(|w_j|) = \alpha(\max_{i=0,\ldots,k} |w_i|) \\ &= \alpha|w_0^k|) \quad \text{for all } k \in K\end{aligned}$$

Thus, $|z| = \alpha(|w|)$, and our proof concludes. □

Lemma 3: *The controller σ of* (22) *is BIBO-unimodular.*

Proof: The proof depends on Lemma 2. Combining (22) with part (ii) of Lemma 2 yields $z_k = w_k\gamma(|w_k|)$, $k = 0, 1, 2, \ldots$ Computing norms of both sides, and using part (iii) of Lemma 2, we obtain $|z_k| = \alpha(|w_k|)$, and, using the invertibility of the function α, we conclude that $|w_k| = \alpha^{-1}(|z_k|)$ for all $k \geq 0$. The last two equalities imply that the sequence w is bounded if and only if the sequence z is bounded, and the proof is complete. □

Clearly, configuration 21 is equivalent to configuration (1) with the controller C given by

$$C = \sigma A \tag{24}$$

Defining the system

$$\Sigma' := \Sigma\sigma$$

we have $\Sigma C = \Sigma\sigma A = \Sigma' A$. In other words, instead of controlling the system Σ with the controller $C = \sigma A$, we can control the system Σ' with the constant gain controller A. The interest in this interpretation arises from the fact that the combination $\Sigma' = \Sigma\sigma$ has the following feature, which is critical for accurate tracking.

Definition 3: A BIBO-minimum phase system $\Sigma : S(R^m) \to S(R^m)$ is *linearly sub-bounded* if there are constants $c > 0$ and $d \geq 0$ such that $|\Sigma z| \geq c|z|$ for all bounded input sequences satisfying $|z| \geq d$.

In view of (16), every linear minimum phase system is also linearly sub-bounded. However, in general, a non-linear system can be BIBO-minimum phase without being linearly sub-bounded.

We show below that the Black diagram (6) achieves accurate tracking for all linearly sub-bounded systems, as long as the gain A is sufficiently large. In other words, the Black diagram is effective for tracking with linearly sub-bounded systems. For systems that are sub-bounded, but not linearly sub-bounded, accurate tracking can be achieved by a slight modification of the Black diagram. In this regard, we show that any sub-bounded system can be transformed into a linearly sub-bounded system simply by combining it with the static precompensator σ of (22). The resulting combination $\Sigma' := \Sigma\sigma$ can then be enclosed in a Black diagram, as in (21), and accurate tracking is achieved when the gain A is sufficiently large.

Proposition 3: *Let $\Sigma : S(R^m) \to S(R^m)$ be a BIBO-stable and sub-bounded system with a lower gain function γ, and let $\sigma : S(R^m) \to S(R^m)$ be given by* (22). *Then, the composition $\Sigma' := \Sigma\sigma$ is internally BIBO-stable and linearly sub-bounded.*

Proof: By Lemma 3 and Proposition 1, the systems σ and Σ are both BIBO-unimodular. Consequently, the combination $\Sigma' = \Sigma\sigma$ is BIBO-unimodular as well. Also, the fact that Σ and σ are both stable implies that $\Sigma\sigma$ is internally stable. Thus, it only remains to show that Σ' is a linearly sub-bounded system. Referring to (21), note that z is the input signal and w is the output signal of σ, i.e.,

$$w = \sigma z$$

In addition, w is also the input signal of Σ. Using the definition (22) of σ and Lemma 2, we can write

$$z_k = w_k\gamma(|w_k|), \quad k = 0, 1, 2, \ldots, \quad \text{and} \quad |z| = |w|\gamma(|w|)$$

Letting α be the lower bound function of Σ corresponding to γ, and recalling definition (19) of the gain function, this yields

$$|z| = \alpha(|w|) \quad \text{for all} \quad |w| \geq \theta_0$$

Consequently, $|\Sigma' z| = |\Sigma\sigma z| = |\Sigma w| \geq \alpha(|w|) = |z|$ for all $|w| \geq \theta_0$. We can then write

$$|\Sigma' z| \geq |z| \quad \text{for all} \quad |z| \geq \alpha(\theta_0) \tag{25}$$

Whence, Σ' is linearly sub-bounded, and the proof concludes. □

Much of our ensuing discussion involves internal stability properties of closed loop configurations under conditions of high gain. Formally, we will investigate stability and internal stability properties at the limit, when the forward path gain approaches infinity. This requires the following stronger notions, which refer to systems whose stability properties are not destroyed when a design parameter grows to infinity.

Definition 4: Let $\Psi(A) : S(R^m) \to S(R^p) : u \mapsto \Psi(A)u$ be a system that depends on a real parameter A. The system $\Psi(A)$ is *uniformly BIBO-stable* if there is a real number A_0 such that the following is true for all $A \geq A_0$: for every real number $M \geq 0$, there is a real number $N \geq 0$ such that $|\Psi(A)u| \leq N$ for all input sequences of norm $|u| \leq M$.

The system $\Psi(A)$ is *uniformly BIBO-minimum phase* if there is a real number B_0 such that $\Psi(A)$ is invertible and $\Psi^{-1}(A)$ is uniformly BIBO-stable for all $A \geq B_0$.

Finally, the system $\Psi(A)$ is *uniformly BIBO unimodular* if it is both uniformly BIBO-stable and uniformly BIBO-minimum phase.

We will also need to examine internal stability properties of composite systems with adjustable parameters. The next definition singles out a class of composite systems whose stability properties are preserved as a design parameter approaches infinity.

Definition 5: Let $\Phi(A)$ be a composite system composed of subunits $\Phi^1(A), \ldots, \Phi^q(A)$ that depend on a parameter A. Insert an adder at the output of each subunit, and add an external signal u^i to the output sequence of $\Phi i(A)$, $i = 1, \ldots, q$. Denote by u^0 the input sequence of the composite system. For a given value of the parameter A, let $v^0(A)$ be the output sequence of the composite system, and let $v^i(A)$ be the output sequence of the subunit $\Phi^i(A)$, $i = 1, \ldots, q$. The composite system $\Phi(A)$ is *uniformly internally BIBO-stable* if there is a real number A_0 such that the following is true for all $A \geq A_0$: for every real number $M > 0$, there is a real number $N > 0$ such that $|v^i(A)| \leq N$ for all $i = 0, \ldots, q$ whenever $|u^i| \leq M$ for all $i = 0, \ldots, q$.

The notion of uniform internal stability allows us to examine properties of composite systems at the limit, as the value of a design parameter grows to infinity. Applying this notion to the closed loop configuration (21), we will subsequently show that perfect tracking is achieved at the limit $A \to \infty$, without disturbing internal stability.

Proposition 4: *Let $\Sigma : S(R^m) \to S(R^m)$ be a strictly causal, BIBO-stable, and BIBO-minimum phase system with lower bound function α and lower gain function γ, and let σ be given by* (22). *Then, configuration* (21) *is uniformly internally BIBO-stable.*

Proof: Denote $\Sigma' := \Sigma\sigma$, where σ is given by (22), and note that, since Σ is strictly causal and σ is static, the combination Σ' is strictly causal. Configuration (21) depicts then a closed loop around Σ' with the constant gain controller $A > 0$. Using the notation of (21), we obtain

$$e = u - y \qquad z = Ae \qquad y = \Sigma' z$$

Substitution yields

$$z = Au - Ay = Au - A\Sigma' z$$

or

$$[I + A\Sigma']z = Au$$

Multiplying both sides on the left by $1/A$ results in

$$u = [(1/A)I + \Sigma']z \tag{26}$$

Now, define the system $\Xi := [(1/A)I + \Sigma'] : S(R^m) \to S(R^m)$, so that

$$\Xi z = [(1/A)I + \Sigma']z = u \tag{27}$$

The strict causality of the system Σ' guaranties that Ξ is invertible (e.g. Hammer 1984), so we can write

$$z = \Xi^{-1}u = [1/A)I + \Sigma']^{-1}u \tag{28}$$

The input/output relation induced by configuration (21) can then be expressed in the form

$$y = \Sigma' z = \Sigma'[1/A)I + \Sigma']^{-1}u$$

From (26) we get

$$|u| = |\Xi z| = |(1/A)z + \Sigma' z| \leq |z|/A + |\Sigma' z| \tag{29}$$

Now, fix a real number $\theta > 0$, and consider a sequence $z \in S(\theta^m)$. Since Σ' is BIBO-stable by Proposition 3, there is a real number $D \geq 0$ such that $|\Sigma' z| \leq D$. Then, by (29),

$$|u| \leq \theta/A + D \leq \theta + D$$

for all $A \geq 1$. This shows that the system Ξ is uniformly BIBO-stable.

Next, we show that the system Ξ is uniformly BIBO-minimum phase. Rewriting (27) in the form

$$u = \Xi z = z/A + \Sigma' z$$

we have

$$|u| = |\Xi z| = |z/A + \Sigma' z| \geq \big||\Sigma' z| - |z|/A\big| \tag{30}$$

Now, by (25), we have $|\Sigma' z| \geq |z|$ for all $|z| \geq \alpha(\theta_0)$; consequently, when $A > 1$ and $|z| \geq \alpha(\theta_0)$, we can write $\big||\Sigma' z| - |z|/A\big| \geq |z| - |z|/A = (1 - 1/A)|z|$, or, recalling (30)

$$|u| \geq (1 - 1/A)|z| \quad \text{for all } A > 1 \quad \text{and} \quad |z| \geq \alpha(\theta_0)$$

Thus,

$$|z| \leq 2|u| \quad \text{for all} \quad A \geq 2 \quad \text{and} \quad |z| \geq \alpha(\theta_0)$$

Specifically, when $|u| \leq \theta$, we get $|z| = |\Xi^{-1}u| \leq 2\theta$ for all $A \geq 2$ and $|z| \geq \alpha(\theta_0)$. Thus

$$\text{Either} \quad |z| < \alpha(\theta_0) \quad \text{or} \quad |z| = |\Xi^{-1}u| \leq 2\theta \quad \text{for all} \quad A \geq 2 \tag{31}$$

This proves that Ξ^{-1} is uniformly BIBO-stable, namely, that Ξ is a uniformly BIBO-minimum phase system. Combining with our earlier observation that Ξ is also uniformly BIBO-stable, we conclude that Ξ is uniformly BIBO-unimodular.

We can now prove that configuration (21) is uniformly internally BIBO-stable. To this end, note that additive signals added at the points u, e and y in (21) all have similar effects on the output of the configuration's adder. Also, a signal z' added to z is equivalent to a signal z'/A added to u. Furthermore, since σ is BIBO-unimodular (Lemma 3) and independent of A, it follows that w is uniformly bounded if and only if z is uniformly bounded. The last sentence implies that it suffices to investigate the impact of the signal z and of signals added to z; the effects of the signal w and of signals added to w do not need to be considered separately (see Hammer 1989 for more details on such observations).

Additionally, since $e = z/A$, it is clear that e will be uniformly bounded as $A \to \infty$ if the signal z is uniformly bounded as $A \to \infty$. Finally, since $y = \Sigma\sigma z$ and σ and Σ are both BIBO-stable and independent of A, it follows that the transmission from u to y is uniformly BIBO-stable if the transmission from u to z is such. Thus, in order to prove that (21) is uniformly internally BIBO-stable, we only need to show that the transmission from u to z is uniformly BIBO-stable. However, the latter is a direct consequence of (28) and of our earlier conclusion that Ξ^{-1} is uniformly BIBO-stable. This concludes our proof. □

Proposition 4 allows us to examine the tracking capabilities of (21) under conditions of high gain. In the notation of (21), the tracking error for a gain A is given by

$$t(A) := |u - y| = |e|$$

The next statement shows that (21) offers a general scheme for achieving accurate tracking with sub-bounded systems, be they linear or non-linear systems.

Theorem 2: *Let $\Sigma : S(R^m) \to S(R^m)$ be a strictly causal, BIBO-stable, and BIBO-minimum phase system with the lower bound function α and the lower gain function γ. Enclose Σ in configuration (21) with the controller $C = \sigma A$, where σ is given by (22). Then, for every bounded input sequence u, the tracking error satisfies $\lim_{A\to\infty}\tau(A) = 0$.*

Proof: Let $u \in S(R^m)$ be a bounded input sequence for configuration (21) with norm $|u| = \theta$. A glance at (21) shows that $e = y - u = z/A$, so that $\tau(A) = |e| = |z|/A$ for all $A > 0$. Using (31), this implies that

$$\tau(A) \leq \frac{1}{A}\max\{\alpha(\theta_0), 2\theta\}$$

for all $A \geq 2$, so that $\lim_{A\to\infty}\tau(A) = 0$, and the proof concludes. □

An examination of the proofs of Proposition 4 and Theorem 2 shows that, when the system Σ is linearly sub-bounded, the precompensator σ can be eliminated from configuration (21) without affecting the results of the two statements. In other words, the classical Black diagram 6 allows accurate tracking with any linearly sub-bounded system Σ. Configuration (21) extends the tracking prowess of the Black diagram to all BIBO-minimum phase systems, be they linear or non-linear.

Example 1: Consider the first order system $\Sigma : S(R) \to S(R)$ given by

$$\begin{aligned} x_{k+1} &= x_k/2 + u_k, x_0 = 0 \\ y_k &= \exp(|x_k|) \end{aligned}$$

A simple calculation shows that, in this case, we can use the lower bound function

$$\alpha(\theta) = \exp(2\theta/3) - 1$$

The corresponding lower gain function can then be taken as

$$\gamma(\theta) = [\exp(2\theta/3) - 1]/\theta, \quad \theta \geq 0,$$

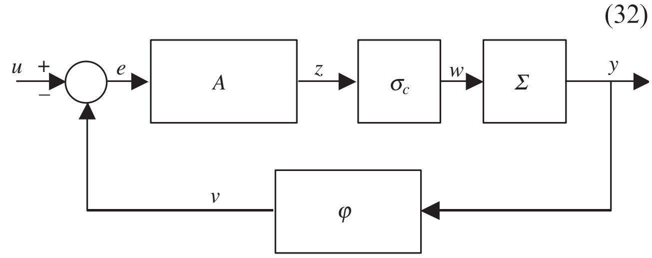
using the continuous extension at $\theta = 0$. The compensator σ, in this case, is given by

$$\sigma(z_k) = \frac{3/2(\log(|z_k| + 1))}{|z_k|} z_k$$

4.2. *Approximate model matching*

The methodology of the previous subsection can be utilized in a number of other applications. An example of such an application is the approximate model matching problem discussed next. Let Σ and φ be two BIBO-stable and BIBO-minimum phase systems, and consider the configuration

(32)



Here, Σ is the system being controlled and φ is used as a feedback compensator; the controller A represents a constant gain amplifier and σ_c is a static compensator to be discussed shortly. Denote by Σ_A the input/output relation of the system represented by the closed loop. To calculate Σ_A, note that

$$y = \Sigma\sigma_c A e, \qquad e = u - \varphi y \tag{33}$$

Substituting, we obtain $e = u - \varphi\Sigma\sigma_c A e$, or $(I + \varphi\Sigma\sigma_c A)e = u$. Assuming that Σ is strictly causal and that all other systems are causal, it follows that the system $(I + \varphi\Sigma\sigma_c A)$ is invertible (e.g. Hammer 1984), and we get

$$\begin{aligned} e &= (I + \varphi\Sigma\sigma_c A)^{-1}u \\ y &= \Sigma\sigma_c A(I + \varphi\Sigma\sigma_c A)^{-1}u \end{aligned}$$

Consequently

$$\Sigma_A = \Sigma\sigma_c A(I + \varphi\Sigma\sigma_c A)^{-1}$$

The fact that φ and Σ are both BIBO-minimum phase systems implies that the combination $\varphi\Sigma$ is BIBO-minimum phase as well. It follows then by Theorem 1 that the combination $\varphi\Sigma$ has a lower bound function α_c. Let γ_c be a lower gain function corresponding to α_c. In analogy with (22), define the compensator σ_c by the relation

$$w_k = \sigma_c(z_k) := \frac{1}{\gamma_c(\alpha_c^{-1}(|z_k|)}z_k, \quad k = 0, 1, 2, \ldots. \tag{34}$$

The next statement shows that, with this compensator, the closed loop system (32) matches the model φ^{-1} as the gain $A \to \infty$.

Theorem 3: *Let $\Sigma : S(R^m) \to S(R^m)$ and $\varphi : S(R^m) \to S(R^m)$ be two BIBO-stable and BIBO-minimum phase systems, where Σ is strictly causal and φ is bicausal. Assume that the inverse system φ^{-1} is continuous with respect to the ℓ^∞-norm. Then, with the compensator σ_c of* (34), *configuration 32 has the following properties:*

(i) *It is uniformly BIBO-internally stable, and*

(ii) $\lim_{A\to\infty} |\Sigma u - \varphi^{-1}u| = 0$ *for every bounded input sequence u; moreover, the limit converges uniformly over all input sequences of norm $|u| \le \theta$, where θ is any positive real number.*

Proof: (i) The proof of this part is similar to the proof of Proposition 4, so we only provide an outline of the proof. Referring to configuration (32) and to (33), we can write $z = Au - A\varphi y = Au - A\varphi\Sigma\sigma_c z$, or $Au = [I + A\varphi\Sigma\sigma_c]z$, which yields $u = [(1/A)I + \varphi\Sigma\sigma_c]z$. Defining the system

$$\Xi_c := [(1/A)I + \varphi\Sigma\sigma_c]\colon\ S(R^m) \to S(R^m) \tag{35}$$

we have

$$u = \Xi_c z \tag{36}$$

Then, Ξ_c is uniformly BIBO-unimodular by an argument similar to the one used to show that Ξ of (27) is uniformly BIBO-unimodular (proof of Proposition 4). We can then rewrite (36) in the form

$$z = \Xi_c^{-1} u \tag{37}$$

Next, since the feedback compensator φ is BIBO-unimodular by assumption, it follows that the signal y is bounded if and only if the signal v is bounded. Combining this fact with the discussion provided in the proof of Proposition 4, we conclude that configuration (32) is uniformly internally BIBO-stable whenever the transmission from u to z is uniformly BIBO-stable. The latter, however, is a direct consequence of (37), since Ξ_c is uniformly BIBO-unimodular. This concludes the proof of part (i) of our Theorem, and we turn to the proof of part (ii).

First note that an argument similar to the one used to derive (25) (proof of Proposition 3) leads to the following conclusion: there is a real number $\theta' \ge 0$ such that

$$|\varphi\Sigma\sigma_c z| \ge |z| \quad \text{for all} \quad |z| \ge \theta' \tag{38}$$

Using (36) and (35), we obtain $|u| \ge \big||\varphi\Sigma\sigma_c z| - |z|/A\big|$; combining with (38), yields

$$|u| = |z| - |z|/A = (1 - 1/A)|z| \ge |z|/2$$

for all $|z| \ge \theta'$ and all $A \ge 2$. In other words, $|z| \le 2|u|$ for all $A \ge 2$, whenever $|z| \ge \theta'$, i.e.

$$|z| \le \max\{\theta', 2|u|\} \tag{39}$$

An examination of configuration (32) shows that $|e| = |u - \varphi y| = |z|/A \le \max\{\theta'/A,\ 2|u|/A\}$ for all $A \ge 2$, where (39) was used. Substituting the relation $y = \Sigma_A u$, we obtain

$$|u - \varphi\Sigma_A u| \le \max\{\theta'/A, 2|u|/A\} \quad \text{for all} \quad A \ge 2.$$

Now, define the quantity

$$s := \varphi\Sigma_A u - u$$

Then, for all $A \ge 2$, we have

$$|s| \le \max\{\theta'/A, 2|u|/A\} \tag{40}$$

and $\varphi\Sigma_A u = u + s$. Using the fact that φ is invertible, this yields $\Sigma_A u = \varphi^{-1}(u + s)$. Subtracting $\varphi^{-1}(u)$ from both sides, we obtain

$$|\Sigma_A u - \varphi^{-1}(u)| = |\varphi^{-1}(u + s) - \varphi^{-1}(u)| \tag{41}$$

The continuity of φ^{-1} with respect to the ℓ^∞-norm implies that, for every real number $\delta > 0$, there is a real number $\varepsilon > 0$ such that $|\varphi^{-1}(u{+}s) - \varphi^{-1}(u)| < \delta$ for all $|s| < \varepsilon$.

Now, let $\theta > 0$ be a real number, and consider all input sequences of norm $|u| \le \theta$. Choose a sequence of real numbers $\delta_1, \delta_2, \ldots$ that converges to zero, and let $\varepsilon_1, \varepsilon_2, \ldots$ be a corresponding sequence of real numbers satisfying the following: $|\varphi^{-1}(u + s) - \varphi^{-1}(u)| < \delta_i$ for all $|s| < \varepsilon_i$, $i = 1, 2, \ldots$ Let $A_1, A_2, \ldots$ be a sequence of positive numbers such that

$$\max\{\theta'/A_i, 2|u|/A_i\} \le \varepsilon_i, \quad i = 1, 2, \ldots.$$

Applying (41) and (40), we conclude that $|\Sigma_A u - \varphi^{-1}(u)| \le \delta_i$ for all gains $A \ge A_i$, which shows that $\lim_{A\to\infty} |\Sigma_\mathrm{A} u - \varphi^{-1}(u)| \le \delta_i$ for all $i = 1, 2, \ldots$ Finally, since $\delta_i \to 0$, it follows that $\lim_{A\to\infty} |\Sigma_A u - \varphi^{-1}(u)| = 0$, and our proof concludes. □

To summarize, we have seen that the Black diagram, with the modification described by configuration 32, facilitates tracking and approximate model matching for non-linear BIBO-minimum phase systems.

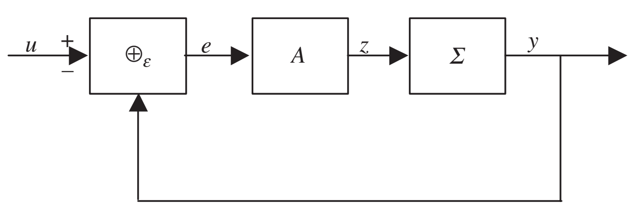
The design procedure is rather simple: all that is required to achieve desirable accuracy of tracking or of model matching is to select the constant gain A sufficiently large. Our next objective is to generalize this approach to systems that are not BIBO-minimum phase systems.

5. Non-minimum phase systems

5.1. *General considerations*

The process of adapting high gain compensators for use with non-minimum-phase systems leads us to a departure from one of the basic tenets of traditional control theory: the requirement that a control system have a unique response. Generally speaking, there is no harm in allowing a system to have a non-unique response, as long as all possible responses do not differ from each other by more than a permissible error bound. In such case, the non-uniqueness has no adverse practical implications. However, from a mathematical standpoint, it leads to a broadening of the class of permissible controllers beyond the family of controllers employed in traditional control or optimization. As we shall see, such broadening of the class of controllers yields improvements in performance.

We use a control configuration with a hysteresis-type response. As shown later, this configuration helps achieve good tracking and approximate model matching for systems that are not necessarily BIBO-minimum phase systems. To be more specific, let $\Sigma : S(R^m) \to S(R^m)$ be the system that needs to be controlled, let A be a constant gain compensator, let $\varepsilon > 0$ be a real number, and consider the following configuration.



(42)

Here, the symbol $\oplus_\varepsilon$ indicates the following operation: given two real numbers a and b

$$a \oplus_\varepsilon b := \begin{cases} 0 & \text{if } |a+b| \leq \varepsilon, \\ a+b-\varepsilon\,\mathrm{sign}(a+b) & \text{if } |a+b| > \varepsilon. \end{cases}$$

In other words, the outcome of the operation is zero if the sum is ε or less; otherwise, the operation reduces the magnitude of the regular sum by ε. A slight reflection shows that this can be restated in the following form.

Lemma 4: *Let a and b be two real numbers. Then, $a \oplus_\varepsilon b$ is the number v of minimal magnitude for which $|a+b-v| \leq \varepsilon$.*

For two vectors $x = (x^1, x^2, \ldots, x^p)$, $z = (z^1, z^2, \ldots, z^p) \in R^p$, we define the operation componentwise:

$$x \oplus_\varepsilon z := (x^1 \oplus_\varepsilon z^1, x^2 \oplus_\varepsilon z^2, \ldots, x^p \oplus_\varepsilon z^p) \tag{43}$$

The next statement is a consequence of Lemma 4.

Lemma 5: *Let x, $z \in R^p$ be two vectors, set $w := x \oplus_\varepsilon z$, and let $A(x, z)$ be the set of all vectors $v \in R^p$ for which $|x+z-v| \leq \varepsilon$. Then, w is the vector of minimal ℓ^1-norm in $A(x, z)$; it also has the minimal ℓ^∞-norm in $A(x, z)$.*

Finally, let v, $w \in S(R^p)$ be two sequences of vectors. The sequence $y := v \oplus_\varepsilon w$ is defined elementwise by

$$y_k := v_k \oplus_\varepsilon w_k, \quad k = 0, 1, 2, \ldots .$$

To examine the sequence $v \oplus_\varepsilon w$, it is convenient to use the weighted ℓ^1-norm defined in (8). A brief examination shows that Lemma 5 leads to the following.

Lemma 6: *For two bounded sequences v, $w \in S(R^p)$, set $s := v \oplus_\varepsilon w$, and let $S(v, w)$ be the set of all sequences $t \in S(R^p)$ satisfying $|v + w - t| \leq \varepsilon$. Then, s is the sequence of minimal weighted ℓ^1-norm in $S(v, w)$; it also has the minimal ℓ^∞-norm in $S(v, w)$.*

Returning to configuration (42), note that the minus sign indicates that

$$e = u \oplus_\varepsilon (-y) \tag{44}$$

We denote the input/output map of the closed loop system (42) by Σ_A^ε to indicate the dependence of the response on the gain A and on the parameter $\varepsilon > 0$.

For a preliminary examination of the control loop (42), assume that all signals are scalar, and that the system Σ represents a scalar constant gain amplifier. Then, the combination ΣA is again a constant gain amplifier, say $\Sigma A = a$, and the entire closed loop system represents a static system. To examine the response, consider first the case when the input signal $u > \varepsilon > 0$. Then, we show that there are two output values possible: $y' > u$ and $y'' < u$. Indeed, in the first case, the loop induces the equation $(u - y' + \varepsilon)a = y'$, which yields

$$y' = a(u + \varepsilon)/(1 + a) \tag{45}$$

In the second case, we have $(u - y'' - \varepsilon)a = y''$, or

$$y'' = a(u - \varepsilon)/(1 + a)$$

When $0 < u < \varepsilon$, the value of y'' is zero, while y' is still given by (45). As we can see, the output value of (42) is not uniquely determined by its input value.

In the above example, the actual output value of the closed loop system depends on the 'initial value' of the output y. In other words, the system exhibits a hysteresis property. As a result, we sometimes refer to ε as the *hysteresis magnitude*. Note that the discrepancy among the different possible output values converges to zero as $\varepsilon \to 0$. Thus, from a practical standpoint, this non-uniqueness of the response causes no adverse effects, as long as ε is selected to comply with the accuracy requirements of the system. Mathematically, however, the non-uniqueness broadens the class of permissible controllers, leading to a potential improvement in performance.

We turn now to a more general examination of configuration (42). First, in view of (43), we can write

$$|u - (y + e)| \le \varepsilon$$

In view of Lemma 6, the signal e can be characterized as the signal $\mu \in S(R^m)$ of minimal weighted ℓ^1-norm for which $(y + \mu)$ is within an ε-neighborhood of the input signal u. Also, since $e = z/A$ with A being a scalar constant gain, we can also say that z is the signal $\varpi \in S(R^m)$ of minimal weighted ℓ^1-norm for which $(y + \varpi/A)$ is within an ε-neighborhood of the input signal u.

Further, since $y = \Sigma A e$ and A represents a scalar constant gain, we have

$$e + y = [I + \Sigma A]e = [(1/A)I + \Sigma]Ae = [(1/A)I + \Sigma]z$$

Thus, we have

Lemma 7: *For a system $\Sigma : S(R^m) \to S(R^m)$ and a sequence $u \in S(R^m)$, let $S(u)$ be the set of all sequences $\mu \in S(R^m)$ for which $|[(1/A)I + \Sigma]\mu - u| \le \varepsilon$. Assume that $S(u)$ is not empty. Then, the signal z of configuration* (42) *is the sequence of minimal weighted ℓ^1-norm in $S(u)$. Also, z has minimal ℓ^∞-norm in $S(u)$.*

In the next subsection we show that, under appropriate conditions, the signal z remains bounded as $A \to \infty$. This implies that $e = z/A \to 0$ as $A \to \infty$, namely, that the discrepancy between the tracking signal u and the output signal y approaches ε as $A \to \infty$. Accordingly, desirable tracking accuracy can be achieved by using a small ε and a high gain A. As discussed later, the error ε can also account for errors in the tracking signal u, in case u is near $\mathrm{Im}_b\, \Sigma$ instead of inside $\mathrm{Im}_b\, \Sigma$.

5.2. *Tracking*

In the present subsection we assume that the controlled system $\Sigma : S(R^m) \to S(R^m)$ is stable. In addition to being BIBO-stable, this requires Σ to be continuous with respect to a topology under which every bounded and closed subset of $S(R^m)$ is compact. A common example of such a topology is the one induced by the weighted ℓ^1-norm (8). For a real number $\varepsilon > 0$ and a sequence $u \in S(R^m)$, set

$$N_\varepsilon(u) := \{v \in S(R^m) : |v - u| \le \varepsilon\} \tag{46}$$

Given a subset $S \subset S(R^m)$, write

$$N_\varepsilon(S) := \{v \in S(R^m) : \ u \in S \quad \text{and} \quad |v - u| \le \varepsilon\}.$$

We are now ready to state the main result of the current subsection. It shows that tracking can be achieved with configuration (42) simply by using a high gain controller.

Theorem 4: *Let $\Sigma : S(R^m) \to S(R^m)$ be a strictly causal and stable system, let $\varepsilon > 0$ be a real number, and let $u \in N_{\varepsilon/2}(\mathrm{Im}_b\, \Sigma)$ be a tracking signal. Then, configuration* (42) *is uniformly BIBO-internally stable, and its response satisfies $\lim_{A\to\infty} |u - \Sigma_A^\varepsilon u| \le \varepsilon$.*

Proof: First, recall that the stability of Σ implies that it is BIBO-stable and continuous with respect to the norm $\|\cdot\|$. By the assumptions of §2.1, this norm is compatible with the weighted ℓ^1-norm, and any closed and bounded subset of $S(R^m)$ is compact under it. Turning to the proof, note that (44) implies that

$$\begin{aligned} N_\varepsilon(u) \supset e + y = e + \Sigma A e = [I + \Sigma A]e \\ = [(1/A)I + \Sigma](Ae) = [(1/A)I + \Sigma]z \end{aligned} \tag{47}$$

Also, since Σ is strictly causal, the system $[I/A + \Sigma]$ has an inverse $[(1/A)I + \Sigma]^{-1}$ (e.g. Hammer 1984). We can then consider the set

$$\Omega := [(1/A)I + \Sigma]^{-1}[N_\varepsilon(u)]$$

Now, the system $[(1/A)I + \Sigma]$ is stable since Σ is stable; consequently, $[(1/A)I + \Sigma]$ represents a continuous map. Combining the latter with the fact that $N_\varepsilon(u)$ is a closed set by its definition (46), we conclude that Ω is a closed subset of $S(R^m)$.

Next, by assumption, $u \in N_{\varepsilon/2}(\mathrm{Im}_b\, \Sigma)$, so there is a bounded sequence $v \in (R^m)$ for which

$$|u - \Sigma v| \le \varepsilon/2; \quad \text{set } \chi := |v| < \infty.$$

Denoting $w := \Sigma v$, we have $w \in N_{\varepsilon/2}(u)$. Now, consider a gain

$$A \ge 2\chi/\varepsilon$$

so that $|v|/A \le \varepsilon/2$. Then, $|[(1/A)I + \Sigma]v - u| = |(1/A)v + (\Sigma v - u)| \le |v|/A + |\Sigma v - u| \le \varepsilon$, so that $[(1/A)I + \Sigma]v \in N_\varepsilon(u)$. This shows that $v \in [(1/A)I + \Sigma]^{-1}[N_\varepsilon(u)] = \Omega$, showing that Ω includes the bounded sequence v. Consequently, the bounded intersection $\Omega \cap (S(\chi^m)$ is not empty; since Ω is closed, so is this intersection. Recalling that, in our topology, every closed and bounded set is compact, we conclude that $\Omega \cap S(\chi^m)$ is a compact set. Together with the fact that the norm $\|\cdot\|$ is compatible with the weighted ℓ^1-norm, this implies that every sequence of elements of $\Omega \cap S(\chi^m)$ with decreasing weighted ℓ^1-norms must have a convergent subsequence with a limit in $\Omega \cap S(\chi^m)$. This implies that $\Omega \cap (S(\chi^m)$ contains an element of minimal weighted

ℓ^1-norm, which we denote by ω^+; since $\omega^+ \in \Omega \cap S(\chi^m)$, it follows that $|\omega^+| \leq \chi$.

We return now to configuration (42). By Lemma 7, the sequence z is of minimal ℓ^1-norm in Ω, so we have, for example, $z = \omega^+ \in \Omega \cap S(\chi^m)$. This directly implies that

$$|z| \leq \chi \tag{48}$$

Noting that χ is independent of the gain A, it follows from (48) that z is bounded, and that its bound is independent of A for all $A \geq 2\chi/\varepsilon$. This shows that the signal z is uniformly bounded as a function of the gain A.

We claim now that the fact that z is uniformly bounded as a function of A entails that configuration (42) is uniformly internally BIBO-stable. Indeed, an examination of the configuration leads to the following conclusions:

(i) The stability of the system Σ gives rise to a real number $M > 0$ such that $\Sigma[S(\chi^m)] \subset S(M^m)$, which, by (48), implies that $|y| \leq M$ for all gains $A \geq 2\chi/\varepsilon$, so that y is uniformly bounded for large A.

(ii) The equality $\varepsilon = z/A$ implies that

$$|e| = |z|/A \leq \chi/A \tag{49}$$

so that $|e| \leq \chi$ for all $A \geq 1$. Consequently, the signal e is uniformly bounded for all $A \geq \max\{2\chi/\varepsilon, 1\}$.

Thus, we conclude that configuration (42) is uniformly internally BIBO-stable. Now, equation (49) directly implies that

$$\lim_{A\to\infty} |e| = 0 \tag{50}$$

since $\lim_{A\to\infty}|e| = \lim_{A\to\infty} |z|/A \leq \lim_{A\to\infty}\chi/A = 0$. Using (47), we have $|u-(e+y)| \leq \varepsilon$, so that $|u-y)| - |e| \leq |u-(e+y)| \leq \varepsilon$, or $|u-y| \leq \varepsilon + |e|$. In view of (50), we obtain

$$\lim_{A\to\infty} |u-y| \leq \varepsilon + \lim_{A\to\infty} |e| = \varepsilon + \lim_{A\to\infty} \chi/A = \varepsilon \tag{51}$$

and our proof concludes. □

Theorem 4 demonstrates the ability of configuration (42) to track a prescribed signal u with an error near ε, as long as the forward gain A is sufficiently large. However, an examination of the proof of Theorem 4 reveals that the necessary gain A may vary from one tracking signal to another. In other words, the tracking accuracy may not be uniform. In the next subsection, we show that uniform tracking accuracy can be achieved for systems possessing a certain reachability property.

5.3. *Uniform tracking error*

We examine now to the problem of controlling the system Σ to track all signals of amplitude not exceeding a specified bound $\theta > 0$. The objective is to achieve a uniform tracking error, so that the tracking errors of all signals of this class are bounded by the same upper bound. In view of the fundamental tracking restriction (5), we restrict our attention to tracking signals of the class

$$S(\theta, \Sigma) := S(\theta^m) \cap \mathrm{Im}_b\Sigma \tag{52}$$

since the system Σ cannot accurately track other signals of amplitude θ or less. The following notion is important when tracking classes of signals.

Definition 6: Let $\Xi(A): S(R^m) \to S(R^m)$ be a system that depends on a real parameter A, and assume that A can take arbitrarily large values. Let $S \subset S(R^m)$ be a class of signals and let $\varepsilon > 0$ be a real number. The system $\Xi(A)$ *uniformly tracks* the class of signals S with an error bounded by ε if the following is valid: for every real number $\delta > 0$, there is a real number $A_0 \geq 0$ such that $|\Xi(A)u - u| \leq \varepsilon + \delta$ for all $A \geq A_0$ and for all signals $u \in S$.

It can be seen that configuration (42) achieves uniform tracking of the class of signals $S(\theta, \Sigma)$ when the system Σ has the following property: all signals of the set $S(\theta, \Sigma)$ can be approximately generated by Σ from input signals of amplitude not exceeding a bound $\chi(\theta)$. Specifically, assume that

$$S(\theta,\Sigma) \subset N_{\varepsilon/2}(\Sigma[S(\chi^m(\theta))]) \tag{53}$$

Then, by replacing the number χ in (51) by the bound $\chi(\theta)$, the proof of Theorem 4 yields the next result. It shows that, in this case, the entire class of signals $S(\theta, \Sigma)$ can be uniformly tracked by configuration (42).

Corollary 1: *Let $\Sigma: S(R^m) \to S(R^m)$ be a strictly causal and stable system, and let θ, $\varepsilon > 0$ be real numbers. Assume that there is a real number $\chi(\theta) > 0$ satisfying* (53), *where $S(\theta, \Sigma)$ is given by* (52). *Then, for the class of signals $S(\theta, \Sigma)$, the system Σ_A^ε is uniformly BIBO-internally stable and achieves uniform tracking with an error bounded by ε.*

When the conditions of Corollary 1 are satisfied, configuration (42) provides an effective, intuitive and simple solution of the tracking problem. Our next objective is to examine the class of systems for which these conditions are valid. Specifically, we show that (53) is valid for a rather broad class of systems that includes all systems possessing an appropriate reachability property.

We start with some notation. Let $\Sigma: S(R^m) \to S(R^p)$ be a strictly causal system, let $j \geq 0$ be an integer, and consider an output sequence $y = \Sigma u \in S(R^p)$. Denote $\Sigma u]_j := y_j$ and $\Sigma u]_0^j := \{y_0, y_1, \ldots, y_j\}$. The strict causality of Σ implies that the string $\Sigma u]_0^{j+1}$ is determined by the input values $u_0^j := \{u_0, \dots, u_j\}$. To emphasize this fact, we will sometimes write $\Sigma u_0^j]_0^{j+1}$ instead of $\Sigma u]_0^{j+1}$. Similarly, when considering the image $\Sigma[S(\theta^m)]_0^{j+1}$, we will sometimes write $\Sigma[\theta^m \times \theta^m \times \cdots \times \theta^m]_0^{j+1}$, where the cross product contains $j+1$ terms, corresponding to input steps 0 to j. Finally, $\mathrm{Im}_b\,\Sigma]_{j+1}$ notes all vectors of R^p that can appear as the $j+1$ step of $\mathrm{Im}_b\,\Sigma$, i.e.

$$\mathrm{Im}_b\Sigma]_{j+1} := \{\Sigma u]_{j+1} : |u| < \infty\}$$

The strict causality of the system Σ implies that the initial output value $\Sigma u]_0$ is determined by the initial condition of the system, and is independent of the input sequence u.

Let $\mu > 0$ be a real number, and consider an element $y \in \mathrm{Im}_b\,\Sigma$. Although bounded, the input sequence that generates y may have a large amplitude. Yet, there might be a sequence $y' \in \mathrm{Im}_b\,\Sigma$ very close to y, say $|y' - y| \le \mu$, which is generated by an input sequence of much lower amplitude. In fact, it is interesting to find the input sequence u of lowest amplitude for which $|\Sigma u - y| \le \mu$. Such a 'lowest amplitude' sequence helps reduce the magnitude of signals within our control configuration, at the cost of a performance error not exceeding μ. The use of such lowest magnitude signals helps guaranty internal stability of control configurations involving the system Σ.

Formally, to examine such lowest magnitude signals, we define the following notion.

Definition 7: Let $\Sigma : S(R^m) \to S(R^m)$ be a causal system satisfying $\Sigma 0 = 0$, and let $\mu \ge 0$ and $\theta > 0$ be two real numbers, where $\mu < \theta$. The *inverse bound function* of Σ is given by

$$\chi(\theta,\mu) := \inf\{\alpha : N_\mu[\Sigma[S(\alpha^m)]] \supset [\mathrm{Im}_b\,\Sigma \cap S(\theta^m)]\}$$

when the infimum exists.

The intuitive significance of the inverse bound function $\chi(\theta, \mu)$ is simple: with an error not exceeding μ, any output signal of Σ of amplitude not exceeding θ can be approximated by the response of Σ to an input signal of amplitude not exceeding $\chi(\theta, \mu)$. When Σ is a minimum phase system, a brief examination shows that $\chi(\theta, 0) \le \alpha^{-1}(\theta)$, where α is a lower bound function of Σ. The inverse bound function is an important tool for guarantying internal stability of composite systems, and it underlies our subsequent discussion of the tracking problem.

Our next objective is to show that the inverse bound function exists under rather general conditions. We start with the following auxiliary result, which shows that, often, the image of a function can be approximated by its image over a bounded set.

Lemma 8: *Let $g : R^n \times R^m \to R^n$ be a uniformly continuous function, and let $B > 0$ be a real number. Assume that there is a non-empty set $S \subset B^n \times R^m$ such that $g[S] \subset B^n$. Then, for every real number $\delta > 0$, there is a real number $D \ge 0$ for which the following is true: for every point $(x, u) \in S$, there is an element $u' \in D^m$ such that $|g(x, u) - g(x, u')| < \delta$.*

Proof: By contradiction, assume that the number D described in the lemma does not exist. Then, there is a sequence of points $(x_i, u_i) \in S$, $i = 1, 2, \dots,$ with the following property:

Property 1: For every real number $E \ge 0$, there is an integer $N(E) \ge 1$ such that $|g(x_i, u_i) - g(x_i, u')| \ge \delta$ for every vector $u' \in E^m$ and for all $i \ge N(E)$.

Now, since $g(x_i, u_i) \in B^n$ for all $i \ge 1$ and since B^n is a compact set, it follows that the sequence $\{g(x_i, u_i)\}_{i=0}^{\infty}$ must have a convergent subsequence $\{g(x_{i(k)}, u_{i(k)}\}_{k=1}^{\infty}$. Furthermore, since $S \subset B^n \times R^m$ and $(x_{i(k)}, u_{i(k)}) \in S$ for all $k \ge 1$, we also have that $\{x_{i(k)}\}_{k=1}^{\infty} \subset B^n$. Using again the fact that B^n is a compact set, we conclude that the sequence $\{x_{i(k)}\}_{k=1}$ has a convergent subsequence $\{x_{i(k(j))}\}_{j=1}$ in B^n. We examine next the sequence $\{(x_{i(k(j))}, u_{i(k(j))})\}_{j=1}^{\infty}$.

The fact that the function g is uniformly continuous implies that there is a real number $\xi > 0$ such that

$$|g(x', u) - g(x, u)| < \delta/2 \tag{54}$$

for all $u \in R^m$ whenever $|x' - \mathrm{x}| < \xi$. By the convergence property of the sequence $\{(x_{i(k(j))}, u_{i(k(j))}\}_{j=1}^{\infty}$, there is an integer $N' \ge 1$ such that $|x_{i(k(j'))} - x_{i(k(j))}| < \xi$ for all integers j', $j \ge N'$. Also, since the sequence $\{g(x_{i(k(j))}, u_{i(k(j))})\}_{j=1}^{\infty}$ is convergent, there is an integer $N'' \ge 1$ such that

$$|g(x_{i(k(q'))}, u_{i(k(q))}) - g(x_{i(k(q'))}, u_{i(k(q))})| < \delta/2 \tag{55}$$

for all integers q', $q \ge N''$. Set $N := \max\,\{N', N''\}$ and define

$$D := \max_{r=1,\dots,N} |u_{i(k(r))}|$$
$$u' := u_{i(k(N))} \in D^m$$

Then, using (54) and (55), we obtain

$$\begin{aligned}
&|g(x_{i(k(r))}, u_{i(k(r))}) - g(x_{i(k(r))}, u')| \\
&\quad = |g(x_{i(k(r))}, u_{i(k(r))}) - g(x_{i(k(r))}, u_{i(k(N))})| \\
&\quad = |[g(x_{i(k(r))}, u_{i(k(r))}) - g(x_{i(k(N))}, u_{i(k(N))})] \\
&\qquad - [g(x_{i(k(r))}, u_{i(k(N))}) - g(x_{i(k(N))}, u_{i(k(N))})]| \\
&\quad \le |[g(x_{i(k(r))}, u_{i(k(r))}) - g(x_{i(k(N))}, u_{i(k(N))})]| \\
&\qquad + |[g(x_{i(k(r))}, u_{i(k(N))}) - g(x_{i(k(N))}, u_{i(k(N))})]| \\
&\quad < \delta/2 + \delta/2 = \delta
\end{aligned}$$

contradicting Property 1. Thus, our initial assumption leads to a contradiction, and the proof is complete. $\square$

For a function $f: R^n \times R^m \to R^n$ and an integer $i \geq 1$, we denote by $f^i(x, u_0, \ldots, u_{i-1})$ the *i-th iteration* of f, given recursively by

$$f^1(x, u_0) := f(x, u_0),$$

$$f^i(x, u_0, \ldots, u_{i-1}) := f(f^{i-1}(x, u_0, \ldots, u_{i-2}), u_{i-1}),$$
$$i = 2, 3, \ldots$$

It will be convenient to use the notation $f|_1^i(x, u_0, \ldots, u_{i-1}) := (f^1(x, u_0), f^2(x, u_0, u_1), \ldots, f^i(x, u_0, \ldots, u_{i-1}))$ for the corresponding string of iterated values. Note that if f is uniformly continuous, so are the functions f^i and $f|_1^i$ for every integer $i \geq 1$. It will be convenient to use the notation $f^i(x, \cdot) : (R^m)^i \to R^n : (u_0, \ldots, u_{i-1}) \to f^i(x, u_0, \ldots, u_{i-1})$ to denote the corresponding partial function. We now adapt the classical notion of reachability to our present needs.

Definition 8: Let $\Sigma : S(R^m) \to S(R^p)$ be a system having a realization of the form (7) with a state space of dimension n and a recursion function f. The system Σ is *locally state-reachable* if the following is true for every real number $B > 0$: for every real number $\zeta > 0$, there is a real number $\varpi > 0$ such that $N_\varpi(f^n(x, u_0, \ldots, u_{n-1}) \subset f^n(x, N_\zeta(u_0, \ldots, u_{n-1}))$ for all $u_0, \ldots, u_{n-1} \in R^p$ and for all $x \in B^n$.

In intuitive terms, a locally state-reachable system is characterized by the following property: any sufficiently small change in the state of the n-th step can be achieved by a small change of the input string. It can be readily confirmed that a reachable linear system is also locally state-reachable.

Next, we review the notion of a detectible system.

Definition 9: Let $\Sigma : S(R^m) \to S(R^p)$ be a system with a realization of the form (7), and let $\Sigma_S : S(R^m) \to S(R^n)$ be the input/state part of Σ. The system Σ is *detectible* if, for every real number $\theta > 0$, there is a real number $B > 0$ such that $|\Sigma_s u| \leq B$ whenever $|\Sigma u| \leq \theta$.

We are now ready to prove the existence of the inverse bound function under rather general conditions.

Proposition 5: *Let $\Sigma : S(R^m) \to S(R^m)$ be a system with a uniformly continuous realization of the form* (7). *Assume that Σ is detectible and locally state-reachable, and that $\Sigma 0 = 0$. Then, Σ has an inverse bound function.*

Proof : Let $x_{k+1} = f(x_k, u_k)$, $y_k = h(x_k)$ be a realization of Σ satisfying the conditions of the Proposition. Here, $f: R^n \times R^m \to R^n$ and $h: R^n \to R^p$ are uniformly continuous functions. Let $\Sigma_s : S(R^m) \to S(R^n)$ be the input/state part of Σ. Further, let $\theta > 0$ be a real number, and consider the set of all output sequences $\mathrm{Im}_b \Sigma \cap S(\theta^m)$. The fact that $\Sigma 0 = 0$ implies that $\mathrm{Im}_b \Sigma \cap S(\theta^m) \neq \phi$ for all $\theta \geq 0$, since $\mathrm{Im}_b \Sigma$ and $S(\theta^m)$ both contain the zero sequence. The detectability of Σ implies that there is a real number $B > 0$ such that $\mathrm{Im}_b \Sigma \cap S(\theta^m) \subset h[\mathrm{Im}_b \Sigma_s \cap S(B^n)]$. Consequently, only states and values of f bounded by B affect $\mathrm{Im}_b \Sigma \cap S(\theta^m)$. Fixing two real numbers θ, $\varepsilon > 0$, we show now that $\chi(\theta, \varepsilon)$ is well defined.

First, by the uniform continuity of the output function h, there is a real number $\eta > 0$ such that $|h(x') - h(x)| < \varepsilon$ whenever $|x' - x| < \eta$. Also, by the uniform continuity of the recursion function f, there is a real number $\zeta > 0$ such that

$$f|_1^n(x, N_\zeta(u_0, \ldots, u_{n-1})) \subset N_{\eta/2}(f|_1^n(x, u_0, \ldots, u_{n-1})) \tag{56}$$

for all x and $u_0, \ldots, u_{n-1}$. As Σ is locally state-reachable by assumption and $|x| \leq B$, there is a real number $\varpi' > 0$ such that $N_{\varpi'}(f^n(x, u, \ldots, u_{n-1})) \subset f_1^n(x, N_\zeta(u_0, \ldots, u_{n-1}))$ for all $x \in B^n$ and all $u_0, \ldots, u_{n-1} \in R^p$. Setting $\varpi := \min\{\varpi', \eta/2\}$, we clearly still have

$$N_\varpi(f^n(x, u_0, \ldots, u_{n-1})) \subset f|_1^n(x, N_\zeta(u_0, \ldots, u_{n-1})) \tag{57}$$

Being interested only in states bounded by B and using the fact that f is uniformly continuous, it follows by Lemma 8 that there is a real number $D > 0$ such that the following is true for all $|x| \leq B$: for every $u := (u_0, \ldots, u_{n-1}) \in R^{nm}$, there is an element $u' := (u'_0, \ldots, u'_{n-1}) \in D^{nm}$, such that

$$|f|_1^n(x, u) - f|_1^n(x, u')| < \varpi \tag{58}$$

Consequently, we can limit our attention to input values bounded by D without incurring errors exceeding ϖ.

Now, consider an output sequence $y \in \mathrm{Im}_b \Sigma \cap S(\theta^m)$. In view of our earlier discussion, there is a sequence $x \in S(B^n)$ such that $y = h(x)$. There is then an input sequence $u \in S(R^m)$ such that $x = \Sigma_s u$. Let x_0 be the initial condition of the system Σ. Then, the state values of Σ at steps that are integer multiples of n are given by

$$x_{kn} = f^n(x_{(k-1)n}, u_{(k-1)n}, u_{(k-1)n+1}, \ldots, u_{(k-1)n+(n-1)}),$$
$$k = 1, 2, \ldots.$$

In view of (56), (57) and (58), there are input vectors bounded by $(D + \zeta)$, i.e. input values $u'_{(k-1)n}, u'_{(k-1)n+1}, \ldots, u'_{(k-1)n+(n-1)} \in (D + \zeta)^m$ such that

$$x_{kn} = f^n(x_{(k-1)n}, u'_{(k-1)n}, u'_{(k-1)n+1}, \ldots, u'_{(k-1)n+(n-1)}), \quad k = 1, 2, \ldots \tag{59}$$

and

$$|f|_1^n(x_{(k-1)n}, u'_{(k-1)n+1}, \dots, u'_{(k-1)n+(n-1)}) - f|_1^n(x_{(k-1)n}, u_{(k-1)n}, u_{(k-1)n+1}, \dots, u_{(k-1)n+(n-1)})| < \eta/2 + \varpi \leq \eta, \quad k = 0, 1, 2, \dots \quad (60)$$

Combining the segments $u'_{(k-1)n}, u'_{(k-1)n+1}, \dots, u'_{(k-1)n+(n-1)}$, $k = 1, 2, \dots$, into a sequence $u' \in S(R^m)$, we clearly obtain that

$$u' \in S((D+\zeta)^m)$$

Furthermore, (59) and (60) imply that the state sequence $x' := \Sigma_s u'$ satisfies

$$|x - x'| < \eta$$

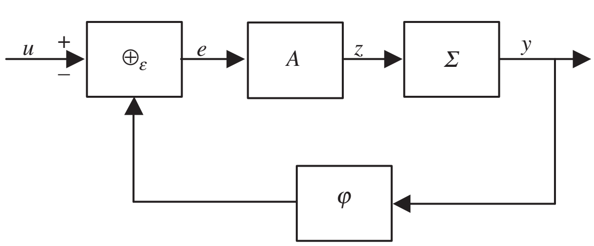
It follows then by the definition of the inverse bound function $\chi(\theta, \varepsilon)$ that $\chi(\theta, \varepsilon) \leq D + \zeta$, which proves that $\chi(\theta, \varepsilon)$ exists. This concludes our proof. □

Combining Proposition 5 and Corollary 1, we obtain the following result, which shows that configuration (42) facilitates tracking for a rather wide class of non-linear systems.

Theorem 5: *Let $\Sigma : S(R^m) \to S(R^m)$ be a stable system with a uniformly continuous realization of the form* (7), *and let ε, $\theta > 0$ be two real numbers. Assume that Σ is detectable and locally state-reachable, and that $\Sigma 0 = 0$. Let $S(\Sigma, \theta)$ be the set of all output signals of Σ that are bounded by θ. Then, for the class of signals $S(\Sigma, \theta)$, the following are true: the closed loop system Σ_A^ε is uniformly BIBO-internally stable, and it uniformly tracks the class of signals $S(\Sigma, \theta)$ with an error bounded by ε.*

5.4. *Approximate model matching*

The general case of the model matching problem can be addressed by using an approach similar to the one used in §4.2 for controlling BIBO-minimum phase systems. Let $\Sigma : S(R^m) \to S(R^m)$ be the system that needs to be controlled and let $\varphi : S(R^m) \to S(R^m)$ be the model that needs to be matched. We say that φ is a *unimodular system* if it is invertible, and if φ and φ^{-1} are both stable systems. Consider now the control configuration.



(61)

Let $\Sigma_{A,\varphi}^\varepsilon$ denote the system represented by the closed loop. We will use the following notion.

Definition 10: Let $\Xi(A)$, $\Psi : S(R^m) \to S(R^m)$ be systems, where $\Xi(A)$ depends on a real parameter A that can take arbitrarily large values. Let $S \subset S(R^m)$ be a class of signals and let $\varepsilon > 0$ be a real number. The system $\Xi(A)$ *uniformly approximates the system Ψ* over the class of signals S with an error bounded by α if the following is valid: for every real number $\beta > 0$, there is a real number $A_0 \geq 0$ such that $|\Xi(A)u - \Psi u| \leq \alpha + \beta$ for all $A \geq A_0$ and for all signals $u \in S$.

The next statement indicates that configuration (61) does indeed provide a solution to the approximate model matching problem under the listed conditions. Its proof can be obtained by combining the arguments used in the proof of Theorem 3 with the arguments leading to Theorem 5, taking into consideration the unimodularity of the model φ.

Theorem 6: *Let $\Sigma : S(R^m) \to S(R^m)$ be a stable system with a uniformly continuous realization of the form* (7), *let $\varphi : S(R^m) \to S(R^m)$ be a unimodular system, and let ε, $\theta > 0$ be two real numbers. Assume that Σ is detectable and locally state-reachable, and that $\Sigma 0 = 0$. Define the class of signals $S := \varphi^{-1}[\mathrm{Im}_b \Sigma \cap S(\theta^m)]$, and let $\delta > 0$ be a real number such that $|\varphi^{-1}(u) - \varphi^{-1}(v)| \leq \delta$ whenever $|u - v| \leq \varepsilon$ and $u \in S$. Then, for the class of signals S, the following are true: the closed loop system $\Sigma_{A,\varphi}^\varepsilon$ is uniformly BIBO-internally stable and it uniformly approximates the system φ^{-1} with an error bounded by δ.*

Note that when the system φ is linear, configuration (61) achieves approximate linearization of the system Σ over the appropriate class of input signals.

6. Conclusions

Starting from the classical control principle that advocates the use high forward gain in feedback control loops, we have developed a general methodology for the design of non-linear tracking systems. For non-minimum phase systems, this methodology involves the use of a control configuration with minor hysteresis properties. An important advantage of the resulting design technique is that it requires a rather small number of variable design parameters—only one variable design parameter (the gain) is required for controlling minimum phase systems, while only two variable design parameters (the gain and the hysteresis magnitude) are required in general. The small number of variable design parameters makes this methodology particularly convenient for design through simulation, as one can easily experiment with the parameter values until a desirable outcome is obtained. The same methodology can be also be used to achieve approximate model matching.

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