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Tolerable Inaccuracies in Linear State Feedback

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***Abstract*—Inaccuracies that a linear state feedback controller can tolerate before losing its ability to stabilize a given system are examined. For high gain feedback, these inaccuracies can be described by a cone in the feedback parameters' space, called the "tolerance cone". This cone describes fractional (percentage) errors the feedback controller can tolerate without jeopardizing the stability of the controlled system. The tolerance cone is determined by the proximity to singularity of a normalized version of the controlled system's controllability matrix.**

I. Introduction

In this note, we investigate the accuracy required of stabilizing linear feedback controllers. Specifically, given a linear system, we examine the parameter uncertainties that a feedback controller can tolerate before losing its ability to stabilize the system. As it turns out, stabilizing feedback controllers can often tolerate rather broad inaccuracies without impairing their ability to stabilize, especially when the feedback gains are high. In fact, in some cases, a stabilizing feedback controller can be obtained simply by assigning any large values to its parameters. This observation conforms with practical experience, where one occasionally encounters systems that seem to work well with almost any choice of large negative feedback.

The inaccuracies that a stabilizing state feedback controller can tolerate depend on a certain normalized version of the controlled system's controllability matrix, called the *normalized controllability matrix* (Section III). This matrix is obtained by dividing the controllability matrix by another matrix derived from the same system. The closer the normalized controllability matrix is to singularity, the more accuracy is required of a stabilizing state feedback controller. This result includes a quantitative estimate of the accuracy required of stabilizing state feedback controllers. It also highlights an important role of the normalized controllability matrix as a critical factor that determines the accuracy requirements of stabilizing state feedback.

Our discussion depends on a certain property of monic polynomials introduced in Section II (a polynomial is *monic* when the highest power of its variable has a coefficient of 1). According to this property, almost every monic polynomial with large positive coefficients has all its roots in the open left half of the complex plane. This fact gives rise to a set of stabilizing high-gain state feedback controllers whose parameters form the unbounded tail of an infinite cone in the feedback parameters' space. The vertex angle of this cone is narrower when the normalized controllability matrix is closer to singularity. As a result, higher accuracy is required when the normalized controllability matrix is close to singularity. Another conclusion is that high gains in the feedback controller relieve some of the burden of accuracy required for stabilization (Section III). Similar results can also be obtained for dynamic output feedback ([6]).

The paper concentrates on linear continuous-time systems. It employs classical control techniques (e.g., [9]) combined with algebraic techniques ([11], [2], [3], and [4]).

II. A Property of Monic Polynomials

We examine now a special family of monic polynomials with large positive coefficients. We show that (i) the members of this family have all their roots in open left half of the complex plane; and (ii) this family includes almost all polynomials with large positive coefficients. These features are used later to investigate properties of stabilizing high-gain linear state feedback. Consider a monic polynomial

$$p_n(s) := s^n + a_1 s^{n-1} + a_2 s^{n-2} + a_3 s^{n-3} + ... + a_i s^{n-i} + ... + a_n, \quad (1)$$

with coefficients given by the special formula

$$a_i := k_i \alpha^{m(n,i)}, \ i = 1, 2, ..., n, \quad (2)$$

where

$$m(n,i) = \left[in - \frac{i(i-1)}{2}\right], \ i = 1, 2, ..., \quad (3)$$

and $\alpha, k_1, k_2, ..., k_n$ are strictly positive real numbers with $\alpha \to \infty$. For example, $m(n,1) = n$; $m(n,2) = 2n-1$; $m(n,3) = 3n - 3$; ...; $m(n,n-1) = (n-1)(n+2)/2$; $m(n,n) = n(n+1)/2$. Normalizing the coefficients by setting $a_0 := 1$ and $k_0 := 1$, we note that consecutive coefficients satisfy

$$\frac{a_i}{a_{i-1}} = \left(\frac{k_i}{k_{i-1}}\right) \alpha^{n-i+1} =: s_i^0, \ i = 1, 2, ..., n. \quad (4)$$

It follows from (3) that $m(n,i) > m(n,i-1)$ for all $i = 1, ..., n$, so that, in $p_n(s)$, lower powers of s have coefficients with higher powers of α.

EXAMPLE 5. $p_2(s) = s^2 + k_1\alpha^2 s + k_2\alpha^3$ and $p_3(s) = s^3 + k_1\alpha^3 s^2 + k_2\alpha^5 s + k_3\alpha^6$, where k_1, k_2, k_3 can be any strictly positive real numbers. ♦

When discussing the case $\alpha \to \infty$, it is convenient to use:

DEFINITION 6. Polynomials $A_1(\alpha)$ and $A_2(\alpha)$ are *α-equivalent* (written $A_1(\alpha) =_{|\alpha} A_2(\alpha)$) if the highest power of α is the same and has the same coefficient in both. ♦

When $A_1(\alpha)$ and $A_2(\alpha)$ are α-equivalent, their ratio $A_1(\alpha)/A_2(\alpha) \to 1$ as $\alpha \to \infty$.

Letting $s_1, s_2, ..., s_n$ be the negatives of the roots of

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$p_n(s)$, we can write

$$p_n(s) = (s + s_1)(s + s_2) \dots (s + s_n).$$

We intend to show that, with some mild exceptions,

$$\frac{s_i}{s_i^0} \to 1 \text{ as } \alpha \to \infty,\ i = 1, 2, \dots, n, \tag{7}$$

where s_i^0 is given by (4). This would indicate that, with some mild exceptions, the polynomials $p_n(s)$ have all their roots inside the open left half of the complex plane for large values of α. To this end, consider the polynomial

$$\begin{aligned} q_n(s) &:= (s + s_1^0)(s^{n-1} + b_1 s^{n-2} + b_2 s^{n-3} + \dots + b_{n-1}) \qquad (8)\\ &= (s + k_1\alpha^n)(s^{n-1} + b_1 s^{n-2} + b_2 s^{n-3} + \dots + b_{n-1}), \end{aligned}$$

where

$$b_i := \left(\frac{k_{i+1}}{k_1}\right)\alpha^{m(n-1,i)},\ i = 1, 2, \dots, n-1,\ b_0 := 0. \tag{9}$$

More succinctly, let β_i be the coefficient of s^{n-i} in $q_n(s)$:

$$q_n(s) = s^n + \beta_1 s^{n-1} + \dots + \beta_n; \tag{10}$$

We show now that $q_n(s)$ is α-equivalent to $p_n(s)$.

LEMMA 11. $\beta_i =_{|\alpha} a_i$ for all $i = 1, 2, \dots, n$.

Proof. A brief calculation shows that

$$\beta_i = b_i + b_{i-1}k_1\alpha^n = \left(\frac{k_{i+1}}{k_1}\right)\alpha^{m(n-1,i)} + \left(\frac{k_i}{k_1}\right)\alpha^{m(n-1,i-1)}k_1\alpha^n,$$

$i = 1, 2, \dots, n$. Using (3), we obtain

$$\beta_i = \frac{k_{i+1}}{k_1}\alpha^{[i(n-1)-\frac{i(i-1)}{2}]} + \frac{k_i}{k_1}\alpha^{[in-\frac{i(i-1)}{2}]},\ i = 1, 2, \dots, n-1,$$

and

$$\beta_n = \alpha^n k_n \alpha^{m(n-1,n-1)}.$$

Employing the fact that $k_1 = 1$ and simplifying, we get

$$\beta_i = \begin{cases} k_i\alpha^{[in-\frac{i(i-1)}{2}]} + k_{i+1}\alpha^{[i(n-1)-\frac{i(i-1)}{2}]}, & i = 1, 2, \dots, n-1, \\ k_n\alpha^{n(n+1)/2}, & i = n. \end{cases} \tag{12}$$

In view of (3), we obtain $\beta_i =_{|\alpha} k_i\alpha^{m(n,i)}$, so that, by (2), $\beta_i =_{|\alpha} a_i$, $i = 1, 2, \dots, n$. ♦

By Lemma 11, the ratios between the coefficients of the polynomials $q_n(s)$ and $p_n(s)$ approach 1 as $\alpha \to \infty$. Qualitatively, this means that, for bounded values of s, we have $q_n(s)/p_n(s) \to 1$ as $\alpha \to \infty$. Referring to (8), define the polynomial $p'_{n-1}(s) := s^{n-1} + b_1 s^{n-2} + b_2 s^{n-3} + \dots + b_{n-1}$; then

$$q_n(s) = (s + s_1^0)p'_{n-1}. \tag{13}$$

Denoting $k'_i := k_{i+1}/k_1$, we obtain from (9) that $b_i = k'_i\alpha^{m(n,i)}$, $i = 1, 2, \dots, n$, so that $p'_{n-1}(s)$ has the same coefficient structure as $p_{n-1}(s)$ of (1). As the coefficients k_i and k'_i represent arbitrary positive numbers, we can identify them with each other. Then, we can rewrite (13) in the form $q_n(s) = (s + s_1^0)p_{n-1}(s)$. By Lemma 11, we conclude that $p_n(s) =_{|\alpha} (s + s_1^0)p_{n-1}(s)$. This gives rise to the relationships

$$p_i(s) =_{|\alpha} (s + s_{n-i+1}^0)p_{i-1}(s),\ i = 2, 3, \dots, n,$$

where s_{n-i+1}^0 is from (4). Qualitatively, these relationships imply that the roots of the polynomial $p_n(s)$ tend toward the values $-s_1^0, -s_2^0, \dots, -s_n^0$ as $\alpha \to \infty$. Stating this fact in accurate form is the main objective of the remaining part of this section. It would imply that when $p_n(s)$ is assigned as the characteristic polynomial of a linear system, the resulting system is asymptotically stable for large values of α.

The family of polynomials $\{p_n(s)\}$ is a rather large family of polynomials. Indeed, every monic polynomial $p(s) = s^n + c_1 s^{n-1} + \dots + c_n$ with coefficients $c_1, c_2, \dots, c_n > 0$ is included in it; simply set $k_1 := 1$, and determine the remaining constants $\alpha, k_2, k_3, \dots, k_n > 0$ from the equations

$$k_i\alpha^{m(n,i)} = c_i,\ i = 1, 2, \dots, n, \tag{14}$$

where $m(n,i)$ is given by (3). Setting $k_1 := 1$ has the effect of normalizing the values of the remaining coefficients.

EXAMPLE 15. For $p(s) = s^3 + c_1 s^2 + c_2 s + c_3$, where $c_1, c_2, c_3 > 0$, equations (14) are $k_1\alpha^3 = c_1$, $k_2\alpha^5 = c_2$, $k_3\alpha^6 = c_3$. Setting $k_1 := 1$, yields $\alpha = \sqrt[3]{c_1}$, $k_2 = c_2/\alpha^5$, $k_3 = c_3/\alpha^6$. ♦

Needless to say, not all monic polynomials with positive coefficients have roots that approach the values (4); a restriction on the coefficients $c_1, c_2, \dots, c_n$ of $p(s)$ is required in order for that to happen as $\alpha \to \infty$. Nevertheless, we show below that this restriction excludes only a small subset of monic polynomials with large positive coefficients.

Let R^{+n} be the set of all vectors in R^n with strictly positive coordinate values; R^{+n} forms a *cone.* In general, a cone is an n-dimensional domain created by the motion of a straight ray whose origin is attached to a fixed point.

For a real number $\gamma > 0$, let $\chi(\gamma)$ be the set of all vectors $w = (w_1, w_2, \dots, w_n) \in R^n$ satisfying

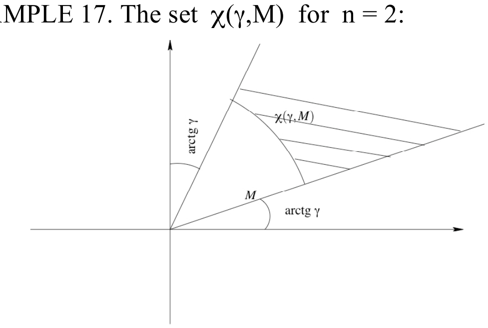
(i) $w_1 > 0$ and (ii) $\gamma \le \frac{w_{i+1}}{w_i} \le \frac{1}{\gamma}$ for all $i = 1, 2, \dots, n-1$.

A slight reflection shows that $\chi(\gamma)$ forms a cone in R^{+n}. The vertex angle of $\chi(\gamma)$ approaches $\pi/2$ as $\gamma \to 0$.

DEFINITION 16. The *horn* of a cone is the infinite open end of the cone; it consists of all vectors of the cone whose length exceeds r, where $r \to \infty$. ♦

Let $\chi(\gamma,M)$ be the set of all vectors $w \in \chi(\gamma)$ satisfying $|w| \ge M$. Then, $\chi(\gamma,M)$ forms the horn of $\chi(\gamma)$ as $M \to \infty$.

EXAMPLE 17. The set $\chi(\gamma,M)$ for $n = 2$:



Consider now a monic polynomial $p(s) = s^n + c_1 s^{n-1} + \dots + c_n$ with positive coefficients. Define by

$$\gamma_i := \frac{c_i}{c_{i-1}},\ i = 1, 2, \dots, n, \text{ where } c_0 := 1,$$

the coefficient ratios, and refer to $\gamma_1, \gamma_2, ..., \gamma_n$ as the *slopes* of $p(s)$. Each set of slopes $\gamma_1, \gamma_2, ..., \gamma_n$ can be considered as a point in R^n. Define the multivariable polynomials

$$\phi_i(\gamma_1,\gamma_2, ...,\gamma_n) := (n-1)(-\gamma_i)^{n-2} + (n-2)\gamma_2(-\gamma_i)^{n-3} + ... + \gamma_2...\gamma_{n-1},$$

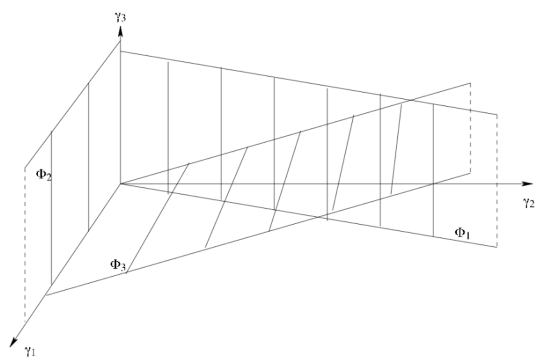
where $i = 1, ..., n$ and $n \geq 3$. Let $\Phi_i(n)$ be the surface of all slopes $(\gamma_1,\gamma_2, ...,\gamma_n) \in R^n$ satisfying

$$\phi_i(\gamma_1,\gamma_2,...,\gamma_n) = 0, i \in \{1, ..., n\}. \qquad (18)$$

The surface $\Phi_i(n)$ is of measure zero with respect to the standard Lebesgue measure in R^n for all $i = 1, 2, ..., n$. Being the union of a finite number of sets of measure zero, the surface $\Phi(n) := \bigcup_{i=1,2,...,n}\Phi_i(n)$ similarly satisfies

LEMMA 19. $\Phi(n)$ is a set of measure zero in R^n (with respect to the standard Lebesgue measure). ♦

EXAMPLE 20. For $n = 3$, we obtain Φ_1: $2(-\gamma_1) + \gamma_2 = 0$, or $\gamma_2 = 2\gamma_1$; Φ_2: $2(-\gamma_2) + \gamma_2 = 0$, or $\gamma_2 = 0$; Φ_3: $2(-\gamma_3) + \gamma_2 = 0$, or $\gamma_2 = 2\gamma_3$. In graphical form:



For a real number $\delta > 0$, build around $\Phi_i(n)$ a domain $S_i(n,\delta)$ given, for each $i = 1, 2, ..., n$, by

$$S_i(n,\delta) := \{\gamma_1, \gamma_2, ..., \gamma_n \in R^n : |\phi_i(\gamma_1, \gamma_2, ..., \gamma_n| \leq \delta\}.$$

Then, define the set

$$S(n,\delta) := \begin{cases} \bigcup_{i=1,2,...,n} S_i(\delta), \text{ for } n \geq 3, \\ \varnothing \text{ for } n = 1, 2, \end{cases} \qquad (21)$$

where $\varnothing$ is the empty set, and consider the difference set

$$V(\delta,M) := \chi(\delta,M) \setminus S(\delta), \qquad (22)$$

where $M, \delta > 0$. By Lemma 19, as $\delta \to 0$, the set $V(\delta,M)$ includes almost all vectors in R^n having positive components and length M or longer; the only vectors excluded are those whose slopes satisfy one of the relations (18). Thus, as $\delta \to 0$, a random choice of an n dimensional vector of length M or longer yields, with probability one, a member of $V(\delta,M)$ (all slopes being equally probable).

The next statement indicates that, for sufficiently large M, a polynomial with coefficients in $V(\delta,M)$ has roots approximately given by (4). Consequently, all roots of such a polynomial are in the open left half of the complex plane. (Re s denotes the real part of a complex number s.)

LEMMA 23. Let $-s_1, -s_2, ..., -s_n$ be the roots of the monic polynomial $p(s) = s^n + c_1s^{n-1} + c_2s^{n-2} + ... + c_n$. Then, for every pair of real numbers $\varepsilon, \delta > 0$, there is a real number $M > 0$ such that, for all coefficients $(c_1, c_2, ..., c_n) \in V(\delta,M)$, the roots of $p(s)$ satisfy

$$\left(\frac{k_i}{k_{i-1}}\right) \alpha^{n-i+1} (1 - \varepsilon) \leq \text{Re } s_i \leq \left(\frac{k_i}{k_{i-1}}\right) \alpha^{n-i+1} (1 + \varepsilon),$$

for all $i = 1, 2, ..., n$, where $k_0 := 1$, $k_1 := 1$, and $k_2, k_3, ..., k_n$ and α are determined by (14). ♦

Lemma 23 indicates that almost all monic polynomials of degree n with sufficiently large positive coefficients have roots approximately equal to those of (4). This implies that all such polynomials have their roots in the open left half of the complex plane. The proof of Lemma 23 is provided below. The following notion is convenient (measures are with respect to the standard Lebesgue measure in R^n).

DEFINITION 24. Let $\Gamma(\delta) \subset R^n$ be a family of subsets depending on $\delta > 0$. Then, $\Gamma(\delta)$ is a *virtual horn* if:

(i) There is a horn $H \subset R^n$ such that the difference set $H \setminus \Gamma(\delta)$ approaches a set of measure zero as $\delta \to 0$.

(ii) $\Gamma(\delta)$ is the union of a finite number m of simply connected sets, where m is independent of δ.

When (i) and (ii) are valid, we say that $\Gamma(\delta)$ is *virtually equal* to the horn H. If G is a subhorn of H, then we say that $\Gamma(\delta)$ *virtually includes* the horn G. ♦

Note that the set $V(\delta,M)$ of (22) forms a virtual horn; it is virtually equal to the horn R^{+n}. In these terms, Lemma 23 can be restated in the following somewhat briefer form.

COROLLARY 25. The set of real coefficients $(c_1, c_2, ..., c_n)$ for which the monic polynomial $p(s) = s^n + c_1s^{n-1} + ... + c_n$ has all its roots in the open left half of the complex plane virtually includes the horn R^{+n}. ♦

Proof (of Lemma 23). Considering (14), we can assume without loss of generality that $p(s)$ has coefficients of the form (2). The roots of $p(s)$ are determined by the equation

$$p(s) = s^n + a_1s^{n-1} + ... + a_n = 0, \qquad (26)$$

where we set $c_i = a_i$, $i = 1, 2, ..., n$, with a_i given by (2). Recall the polynomial $q_n(s) = s^n + \beta_1s^{n-1} + ... + \beta_n$ of (10), whose coefficients are given by (12). The discrepancy between the coefficients of $p(s)$ and $q_n(s)$ is given by

$$\Delta a_j := \beta_j - a_j = k_{j+1}\alpha^{[j(n-1)-\frac{j(j-1)}{2}]}, \; j = 1, 2, ..., n. \qquad (27)$$

Now, by (8), the first root of $q_n(s)$ is $-s_1^0$, where s_1^0 is given by (4). The discrepancies $\Delta a_1, \Delta a_2, ..., \Delta a_n$ between the coefficients of the two polynomials cause the root $-s_1$ of $p(s)$ to deviate from the root $-s_1^0$ of $q_n(s)$. Similarly, these discrepancies also cause each one of the roots $-s_i$ of $p(s)$ to deviate from the value of the root $-s_i^0$ of $q_n(s)$. In the proof, we show that these root deviations satisfy the relations stated in the Lemma. To derive a bound on the root deviations, we use the Taylor series first order error bound.

Write $s_i = s_i^0(1 + \varepsilon_i)$, where ε_i describes the "fractional" impact of the deviations $\Delta a_1, ..., \Delta a_n$ on root i. Note that s_i and ε_i can be complex, but s_i^0 is real. Considering ε_i as a function of the real scalar coefficients $a_1, a_2, ..., a_n$, we can write $\partial s_i/\partial a_j = s_i^0\,[\partial\varepsilon_i/\partial a_j]$, or

$$\frac{\partial \varepsilon_i}{\partial a_j} = \frac{1}{s_i^0}\frac{\partial s_i}{\partial a_j}. \tag{28}$$

To obtain the derivative $\partial s_i/\partial a_j$, we can differentiate (26):

$$\frac{\partial}{\partial a_j}[s_i^n + a_1 s_i^{n-1} + ... + a_n] = n s_i^{n-1}\frac{\partial s_i}{\partial a_j} + a_1(n-1)s_i^{n-2}\frac{\partial s_i}{\partial a_j} + ...$$
$$+ s_i^{n-j} + a_j(n-j)s_i^{n-j-1}\frac{\partial s_i}{\partial a_j} + ... + a_{n-1}\frac{\partial s_i}{\partial a_j} = 0,$$

so that

$$\frac{\partial s_i}{\partial a_j} = \frac{-s^{n-j}}{ns^{n-1} + a_1(n-1)s^{n-2} + a_2(n-2)s^{n-3} + ... + a_{n-1}} =: h(s),$$

where $h(s)$ is the corresponding function. Evaluating the derivative at $s = -s_i^0$ of (7) and assuming that the denominator is not zero, we obtain $\partial s_i/\partial a_j = h(-s_i^0)$.

Let $\Delta\varepsilon_{ij}$ be the contribution of the discrepancy Δa_j to the deviation ε_i. Then, using the first order Taylor series error bound together with (28) and (27), we obtain

$$|\Delta\varepsilon_{ij}| \le \sup\left|\frac{\partial \varepsilon_i}{\partial a_j}\Delta a_j\right| = \sup\left|\frac{1}{s_i^0}\frac{\partial s_i}{\partial a_j}\Delta a_j\right| = \sup\left|\frac{1}{s_i^0}h(-s_i^0)\Delta a_j\right|. \tag{29}$$

To express in terms of the slopes, recall the i-th coefficient slope $\gamma_i := a_i/a_{i-1}$, $i = 1, 2, ..., n$, where $a_0 := 1$. For the nominal coefficients (2), we obtain $\gamma_i = (k_i/k_{i-1})\alpha^{n-i+1}$, so that, by (4), we have $\gamma_i = s_i^0$, $i = 1, 2, ..., n$. Substituting into (29), yields

$$\left|\Delta\varepsilon_{ij}\right| \le \sup\left|\frac{\gamma_i^{n-j-1}\left(k_{j+1}\alpha^{m(n-1,j)}\right)}{n(-\gamma_i)^{n-1}+(n-1)\gamma_1(-\gamma_i)^{n-2}+(n-2)\gamma_1\gamma_2(-\gamma_i)^{n-3}+...+\gamma_1\gamma_2...\gamma_{n-1}}\right| \tag{30}$$

where we assume that the denominator is not zero. Further,

$$\gamma_{j+1}\gamma_j...\gamma_2 = \frac{k_{j+1}}{k_j}\frac{k_j}{k_{j-1}}...\frac{k_2}{k_1}\alpha^{n-j}\alpha^{n-j+1}\alpha^{n-j+2}...\alpha^{n-1} =$$
$$= k_{j+1}\alpha^{\Sigma_{i=0}^{j-1}(n-j+i)} = k_{j+1}\alpha^{[j(n-j)+\Sigma_{i=0}^{j-1} i]} = k_{j+1}\alpha^{[jn-\frac{j(j+1)}{2}]}.$$

Also, since $m(n-1,j)-[jn-(j(j+1)/2)] = j(n-1)-(j(j-1)/2)[jn-(j(j+1)/2)] = 0$, we obtain that $k_{j+1}\alpha^{m(n-1,j)} = \gamma_{j+1}\gamma_j...\gamma_2$. Substituting into (30) yields

$$\left|\Delta\varepsilon_{ij}\right| \le \sup\left|\frac{\gamma_i^{n-j-1}\gamma_{j+1}\gamma_j...\gamma_2}{n(-\gamma_i)^{n-1}+(n-1)\alpha^n(-\gamma_i)^{n-2}+(n-2)\alpha^n\gamma_2(-\gamma_i)^{n-3}+...+\alpha^n\gamma_2...\gamma_{n-1}}\right| \tag{31}$$

for all $n \ge 3$; for $n = 2$, the last denominator term is $\alpha^n\gamma_2$, since $\gamma_1 = \alpha^n$ was already inserted earlier. Thus, for fixed slopes, we have $\left|\Delta\varepsilon_{ij}\right| \to 0$ as $\alpha \to \infty$, as long as none of the slopes $\gamma_1, \gamma_2, ..., \gamma_n$ is a root of the denominator.

By definition of the set $\chi(\delta,M)$, we have $\delta \le \gamma_i \le (1/\delta)$ for all $i = 1, 2, ..., n$, where we have assumed (without loss of generality) that $\delta \le (1/\delta)$. Recalling (21), we obtain from (31) that, for all slopes $\gamma_1, \gamma_2, ..., \gamma_n \notin S(n,\delta)$ and for sufficiently large α, we can write

$$\left|\Delta\varepsilon_{ij}\right| \le \left|\frac{(1/\delta)^{n-1}}{\alpha^n\delta - n\delta^{n-1}}\right| = \left|\frac{1}{\alpha^n\delta^n - n\delta^{2(n-1)}}\right|$$

for all $i, j = 1, 2, ..., n$. Taking into account the contributions of the deviations in all n coefficients, the total relative error ε_i in the root s_i satisfies $|\varepsilon_i| \le \Sigma_{j=1}^n|\Delta\varepsilon_{ij}| \le n/|\alpha^n\delta^n - n\delta^{2(n-1)}|$. The last expression is clearly independent of the index i; setting $\varepsilon := \max_{i=1,...,n}|\varepsilon_i|$, we can rewrite the last inequality in the form $\varepsilon \le n/|\alpha^n\delta^n - n\delta^{2(n-1)}|$. Note that the equality $s_i = s_i^0(1 + \varepsilon_i)$ implies that $\text{Re}\, s_i = s_i^0(1 + \text{Re}\,\varepsilon_i)$, since s_i^0 is real; also, one always has that $|\text{Re}\,\varepsilon_i| \le |\varepsilon_i|$. Finally, taking $\alpha := N$, we conclude that our Lemma is valid for all $M \ge N$, where N satisfies $1/|N^n\delta^n - n\delta^{2(n-1)}| \le \varepsilon/n$. ♦

Furthermore, since all monic polynomials whose roots are in the open left half of the complex plane must have positive coefficients, Corollary 25 yields the following conclusion.

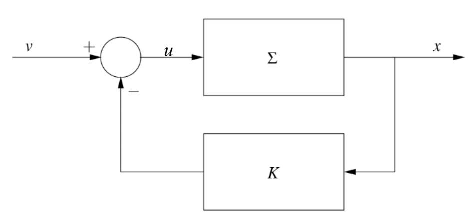
THEOREM 32. The set of real coefficients $(c_1, c_2, ..., c_n)$ for which the monic polynomial $p(s) = s^n + c_1 s^{n-1} + ... + c_n$ has all its roots inside the open left half of the complex plane is virtually equal to the horn R^{+n}.

III. State Feedback And Tolerance Horns

Consider now the implications of Lemma 23 on the accuracy required to implement a static state feedback controller that stabilizes the reachable realization

$$\Sigma : \dot{x}(t) = Ax(t) + Bu(t), \tag{33}$$

where A and B are constant matrices. Assuming Σ has n states and m inputs, a stabilizing constant state feedback matrix K will be $m \times n$:



First, some preliminary notions.

DEFINITION 34. Let $v \in R^n$ be a vector, and let $0 \le \rho \le \pi$ be a number. The *circular cone* $\Xi(v,\rho)$ with *vertex angle* ρ around v is the set of all vectors $w \in R^n$ satisfying

$$\Xi(v,\rho) := \left\{w \ne 0 \in R^n : \cos^{-1}\frac{v\cdot w}{|v||w|} \le \rho\right\}.$$

Let $S \subset R^n$ be a set containing a ray. The *inner span* ρ of S is the vertex angle of the largest circular cone that can fit into S, i.e., $\rho := \sup\{\varphi : \Xi(v,\varphi) \subset S, v \in S\}$. The inner span of a horn is the inner span of the cone generating the horn. ♦

We consider the inner span of a matrix image of R^{+n}.

DEFINITION 35. Let Q be a real matrix with n columns, and let $Q[R^{+n}]$ be the image of R^{+n} through the matrix Q. The *column inner span* of Q is the inner span of the set $Q[R^{+n}]$. ♦

EXAMPLE 36. Consider the matrix

$$Q = \begin{pmatrix} 1-\varepsilon & 1 \\ 1 & 1 \end{pmatrix},$$

where $0 < \varepsilon < 1$. The image set $Q[R^{+2}]$ is then the set of vectors $(a(1-\varepsilon) + b, a + b)^T$, where $a, b > 0$. Using the parallelogram rule, it can be seen that the column inner span θ of Q is $\theta = 0.5\cos^{-1}\left((2-\varepsilon)/\sqrt{2(2-2\varepsilon+\varepsilon^2)}\right)$. Here, $\theta \to 0$ as $\varepsilon \to 0$, i.e., as Q approaches singularity. ♦

Consider the reachable linear input/state system of (33), and let $C' = (B, AB, A^2B, ..., A^{n-1}B)$ be its controllability matrix. As the realization is reachable, we have $\operatorname{rank} C' = n$, where n is the state dimension. The next algorithm extracts from C' an $n\times n$ non-singular matrix called the *reduced controllability matrix* ([1], [10]).

ALGORITHM (37). Deriving the reduced controllability matrix of (33).

Step 1. Let $B_1, B_2, ..., B_m$ be the columns of the input matrix B. Define the $n\times n$ sub-matrices

$C'_i := (B_i, AB_i, ..., A^{n-1}B_i), i = 1, ..., m.$

Step 2. Derive a list of integers by the following recursive process.

a) First, set $n_1 := \max_{i=1,...,m} \operatorname{rank} C'_i$,
and let i_1 be an integer for which $\operatorname{rank} C'_{i_1} = n_1$. If $n_1 = n$, then go to Step 3; otherwise, set $k := 1$ and continue to b).

b) Using recursion, assume that the integers $n_1, n_2, ..., n_k$ and $i_1, i_2, ..., i_k$ have been derived; define

$n'_{k+1} := \max_{i=1,...,m} \operatorname{rank} (C'_{i_1}, C'_{i_2}, ..., C'_{i_k}, C'_i)$,

and let $i_{k+1} \in \{1, ..., m\}$ be an integer for which $\operatorname{rank} (C'_{i_1}, C'_{i_2}, ..., C'_{i_k}, C'_{i_{k+1}}) = n'_{k+1}$. Set

$n_{k+1} := n'_{k+1} - (n_1 + n_2 + ... + n_k)$.

c) If $n_1 + n_2 + ... + n_{k+1} = n$, then go to Step 3; otherwise, repeat Step 2 with the value of k increased by 1.

Step 3. Define the matrices $C_{i_j} := (B_{i_j}, AB_{i_j}, ..., A^{n_j-1}B_{i_j})$, $j = 1, 2, ..., k$. The *reduced controllability matrix* is then
$C := (C_{i_1}, C_{i_2}, ..., C_{i_k})$. ♦

The reduced controllability matrix of a reachable realization is invertible. Consequently, it can be used to induce the similarity transformation

$A' := C^{-1}AC,\ B' := C^{-1}B,$

which yields the so-called *controllability canonical form*

$$\dot{x} = A'x + B'u. \tag{38}$$

By construction, the reduced controllability matrix of the controllability canonical form is the identity matrix.

Now, let T_p be a similarity transformation that takes the realization (38) into the controller form (see [10] for a construction of T_p). Applying this similarity transformation to (38), we obtain the realization

$$\dot{z} = A_c z + B_c u, \tag{39}$$

where the matrices $A_c := T_p^{-1}A'T_p$ and $B_c := T_p^{-1}B'$ are of the form

$$A_c = \begin{pmatrix} (A_1) & \binom{a_{1,1}}{0} & \cdots & \binom{a_{1,k}}{0} \\ (0) & (A_2) & \cdots & \binom{a_{2,k-1}}{0} \\ \cdots & \cdots & \cdots & \cdots \\ (0) & (0) & \cdots & (A_k) \end{pmatrix},\ B_c = \begin{pmatrix} \binom{1}{0} & \binom{0}{0} & \cdots & \binom{0}{0} \\ \binom{0}{0} & \binom{1}{0} & \cdots & \binom{0}{0} \\ \cdots & \cdots & \cdots & \cdots \\ \binom{0}{0} & \binom{0}{0} & \cdots & \binom{1}{0} \end{pmatrix}$$

Here, the blocks $A_1, A_2, ..., A_k$ on the main diagonal are matrices in companion form; the dimension of the matrix A_j is $i_j\times i_j$, and $a_{i,j}$ are scalars. In fact, we have a combination of single input systems in the controller form, where each single input system is given by

$$\Sigma_j : \dot{x}^j = A_j x^j + b_j u^j, \text{ where} \tag{40}$$

$$A_j = \begin{pmatrix} -a_j^1 & -a_j^2 & \cdots & -a_j^{i_j-1} & -a_j^{i_j} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, b_j = \begin{pmatrix} 1 \\ 0 \\ \cdots \\ 0 \\ 0 \end{pmatrix}, j = 1, ..., k. \tag{41}$$

It can be seen that the similarity transformation is $T_p = C_c^{-1}$, where C_c is composed of the controllability matrices of the sub-realizations $\Sigma_1, ..., \Sigma_k$ of (40), as follows. Let C_j be the controllability matrix of the sub-realization (40); it is well known that (* is an unspecified entry)

$$C_j = \begin{pmatrix} 1 & * & * & \cdots & * \\ 0 & 1 & * & \cdots & * \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 1 & * \\ 0 & 0 & \cdots & \cdots & 1 \end{pmatrix}.$$

A direct examination shows that

$$C_c = \begin{pmatrix} C_1 & * & \cdots & * \\ 0 & C_2 & \cdots & * \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & C_m \end{pmatrix},$$

so $\det C_c = 1$, i.e., C_c is invertible. The combined similarity transformation

$$C_N := CC_c^{-1} \tag{42}$$

takes the (33) to the multivariable controller form (38).

DEFINITION 43. The matrix $C_N = CC_c^{-1}$ is the *normalized controllability matrix* of the realization (33). ♦

As seen, the normalized controllability matrix is the similarity transformation taking a realization into controller form. It is also critical in determining accuracy requirements of stabilizing state feedback, as follows. Consider the controller form (39). Apply the static state feedback

$$K_{c,j} = (k_{c,j}^1, k_{c,j}^2, ..., k_{c,j}^{i_j}) \tag{44}$$

to subsystem j of (40), $j = 1, ..., m$. Combining these feedback vectors into one matrix yields the feedback matrix

$$K_c = K_{c,1} \oplus K_{c,2} \oplus ... \oplus K_{c,k} = \begin{pmatrix} K_{c,1} & 0 & 0 \cdots & 0 \\ 0 & K_{c,2} & 0 \cdots & 0 \\ 0 & 0 & 0 \cdots & K_{c,k} \end{pmatrix} \tag{45}$$

Now, let $\mathcal{K}_c$ be the class of static state feedback matrices that stabilize the system (39). Clearly, the class $\mathcal{K}_c$ depends on the values of the entries $a_j^1, ..., a_j^{i_j}$ of (41), $j = 1, 2, ..., k$.

Consider first the case where all these entries are zero, i.e.,

$$a_j^i = 0,\ i = 1, ..., i_j,\ j = 1, ..., k. \qquad (46)$$

Denote by Σ_j^0 the instance of Σ_j satisfying (46). Applying the feedback (44) to Σ_j^0 yields the characteristic polynomial

$$a_j^0(s) := s^{i_j} + k_j^1 s^{i_j-1} + ... + k_j^{i_j}.$$

Let $S^0(i_j)$ be the set of all vectors $K_{c,j}$ for which the polynomial $a_j^0(s)$ has all its roots in the open left half of the complex plane, i.e., the set of all stabilizing feedback compensators for this subsystem. In view of Lemma 23, the set $S^0(i_j)$ includes the virtual horn $V(\delta,M)$ of (22). The set of stabilizing feedback vectors for the system (40) is then

$$S_c(i_j) := S^0(i_j) + (a_j^1, a_j^2, a_j^3, ..., a_j^{i_j}).$$

In view of (45), the set of stabilizing static state feedback compensators for the system (39) is

$$S_c := S_c(i_1) \oplus S_c(i_2) \oplus ... \oplus S_c(i_k).$$

Recalling the similarity transformation C_N of (42), it follows that the static state feedback compensator

$$K := K_c C_N \qquad (47)$$

stabilizes the original realization (33), with K_c being any static state feedback stabilizing the controller form (39). Thus, the set of matrices $S := S_c C_N$ is a set of stabilizing state feedback matrices for the original system (33).

Let $\left(R^{+i_j}\right)^T$ be the set of all row vectors of dimension i_j with positive coefficients. In view of Corollary 25, the set of row vectors $S_c(i_j)$ virtually includes the horn $\left(R^{+i_j}\right)^T$. Denote by C_N^1 the submatrix of C_N that consists of the top-left $i_1 \times i_1$ block; let C_N^2 be the submatrix of C_N consisting of the next $i_2 \times i_2$ block along the main diagonal, and so on. Let $C^+(j) := \left[R^{+i_j}\right]^T C_N^j$ be the horn spanned by the rows of the matrix C_N^i, i = 1, 2, ..., m. Theorem 32 then yields

THEOREM 48. The set of stabilizing static state feedback controllers of the system Σ is virtually equal the sum of horns $C^+ := C^+(1) \oplus C^+(2) \oplus ... \oplus C^+(k)$. ♦

Let ρ_i be the inner row span of the matrix C_N^i, i = 1, 2, ..., k. By (45) and Theorem 48, the non-zero entries of the class of stabilizing static state feedback controllers for the system Σ virtually include a horn with the span $\rho := \min_{i=1, 2, ..., k} \rho_i$. Thus, ρ provides an indication of the tolerance available in the implementation of high-gain stabilizing state feedback for the system Σ. We refer to ρ as the *controllability inner span* of the system Σ. This yields

COROLLARY 49. Let Σ be a reachable input/state system, let ρ be the controllability inner span of Σ, and let S be the family of stabilizing static state feedback controllers of Σ. Then, the non-zero entries of the family S virtually include a horn with row span equal to ρ. ♦

The combination of Theorem 48 and Corollary 49 show that the set of high-gain stabilizing state feedback controllers is almost equal to the horn spanned by the normalized controllability matrix C_N. The closer C_N is to being singular, the less tolerance is available when implementing stabilizing state feedback controllers. This highlights an important connection between implementation tolerance and features of the normalized controllability matrix.

(50) EXAMPLE. Consider a system with the normalized controllability matrix

$$C_N = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix};$$

applying C_N as a similarity transformation yields the controller form

$$\dot{x} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u.$$

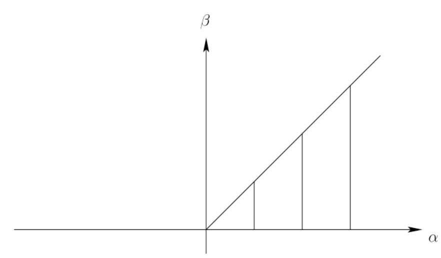
Using the state feedback $k_c = (a,b)$, we obtain the realization

$$\dot{z} = \begin{pmatrix} -a & -b \\ 1 & 0 \end{pmatrix} z + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$$

with the characteristic polynomial $a(s) := s^2 + as + b$. The roots of this polynomial have negative real parts whenever $a, b > 0$ (there is no need to consider horns in this case). Thus, the set of stabilizing feedback vectors $\{K_c\}$ covers the entire first quadrant of our feedback vector space. Transforming this set back to the original coordinate system, we obtain the set of stabilizing feedback controllers

$$S_K = \left\{(a,b)\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} : a, b > 0\right\} = \{((a + b),b) : a, b > 0\}$$

depicted in the following figure.



We see a narrowing of the cone (from the full quadrant). ♦ Similar results are valid for dynamic output feedback ([6]).

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