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THE CONCEPT OF RATIONALITY AND STABILIZATION OF NONLINEAR SYSTEMS*

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Qualitatively, a system Σ is said to be rational if it can be expressed as a quotient of two stable systems P and Q, as in $\Sigma = PQ^{-1}$. For many years, the concept of rationality has been a cornerstone in the theory of stabilization for linear systems. The purpose of this note is to indicate that the concept of rationality has far reaching implications not only for the theory of linear system stabilization, but also for the more general theory of nonlinear systems stabilization. In fact, the concept of rationality seems to be the key to a stabilization theory for nonlinear systems.

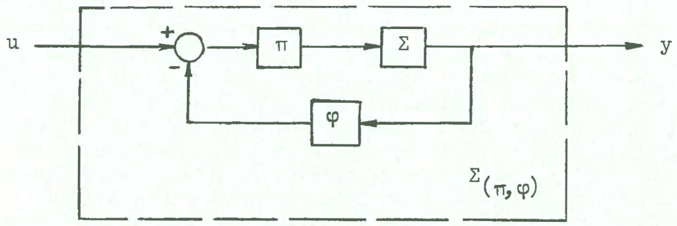
1. INTRODUCTION

The objective of this note is to discuss and to provide an overview of some recent results in the theory of nonlinear systems, included in HAMMER[1984a, 1984b, and 1985]. The pivotal concept in this theory is the concept of rationality. Qualitatively speaking, a nonlinear system Σ is said to be rational if there exists a pair of stable (nonlinear) systems P and Q, with Q invertible, such that $\Sigma = PQ^{-1}$. As is well known, the concept of rationality forms a cornerstone in the theory of stabilization for linear systems. Indeed, one of the first theoretical tools the control Engineer learns is to express the transfer function of a linear time-invariant system as a ratio of two polynomial matrices, which is one way of expressing the system as a ratio of two stable systems. The main point of our present discussion is to show that the concept of rationality is of basic significance to the theory of nonlinear systems stabilization as much as it is of basic significance to the theory of linear systems stabilization. We shall first indicate the fact that only rational nonlinear systems can be (internally) stabilized through the application of additive output feedback. As a consequence of this fact, when aiming at feedback stabilization, one can restrict his attention to rational systems only, namely, if a system Σ is to be stabilized by additive output feedback, then it must be possible to express it as a ratio $\Sigma = PQ^{-1}$ of two stable systems P and Q. We then continue with a preliminary investigation of how the systems P and Q of such a fraction representation can be used to construct nonlinear compensators which will stabilize the given nonlinear system Σ.

We develope the theory explicitly for the case of discrete-time systems. Throughout our discussion, we assume that the given system Σ, which is to be stabilized, is an injective (one to one) system. This assumption is for the purpose of simplifying the technical details of the consideration.

The control configuration that we intend to use for the stabilization of the given system Σ is the following classical one.

(1.1)



Here, π is an in-loop nonlinear precompensator, φ is a nonlinear output feedback compensator, and $\Sigma_{(\pi,\varphi)}$ denotes the resulting closed loop system. We employ an additive feedback configuration even though the involved systems Σ, π, and φ are all nonlinear. The notion of additive feedback seems particularly close to the philosophical origins of the feedback concept, and, throughout its long history of application, has proved to be a powerful stabilization tool in a wide variety of practical situations. Regarding causality, we always assume that the given system Σ is strictly causal (i.e., Σ induces a delay of at least one step between a change in the input and its effect on the output). The compensators π and φ are required to be causal. Finally, we remark that the sum $A + B$ of two nonlinear systems A and B, having the same input and output spaces, is defined pointwise, for every input sequence u, by $(A + B)u = Au + Bu$.

The input/output relationship induced by the composite system $\Sigma_{(\pi,\varphi)}$ of (1.1) can be readily computed (e.g., HAMMER[1984b]), and it is of the form

(1.2) $$\Sigma_{(\pi,\varphi)} = \Sigma\psi_{(\pi,\varphi)},$$

where the equivalent precompensator $\psi_{(\pi,\varphi)}$ is given by

(1.3) $$\psi_{(\pi,\varphi)} = \pi[I + \varphi\Sigma\pi]^{-1}.$$

The inverse in (1.3) always exists when Σ is strictly causal and π and φ are both causal. The main effort is directed toward the following

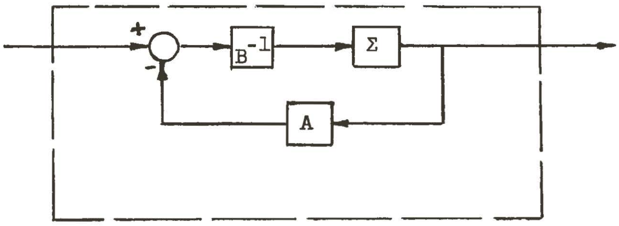
(1.4) DESIGN OBJECTIVE. Find causal compensators π and φ for which the composite system $\Sigma_{(\pi,\varphi)}$ is (internally) stable.

Equation (1.2), though simple and common in the linear theory, is of a rather complicated nature in the nonlinear case. A major simplification arises when one chooses the compensators π and φ in accordance with the following particular form (HAMMER[1985])

(1.5) $\pi = B^{-1}, \quad \varphi = A,$

where B is a stable invertible system (with causal inverse), and where A is a stable (causal) system. This choice of compensators yields the following configuration.

(1.6)



Assume further that the system Σ has a fraction representation $\Sigma = PQ^{-1}$, where P and Q are stable systems and where Q is invertible. As we discuss below, the existence of such a fractionrepresentation is a necessary condition for stabilization. (It is worthwhile to remark here that the existence of such a fraction representation is a rather mild requirement, as can be seen in section 4.) Combining this fraction representation with the particular choice of the compensators (1.5), and inserting into (1.2), we obtain

(1.7)
$$\begin{aligned}\Sigma_{(\pi,\varphi)} &= PQ^{-1}B^{-1}[I + APQ^{-1}B^{-1}]^{-1}\\ &= PQ^{-1}B^{-1}[(BQ + AP)Q^{-1}B^{-1}]^{-1}\\ &= P[AP + BQ]^{-1}.\end{aligned}$$

Letting.

(1.8) $M := AP + BQ,$

we have

$$\Sigma_{(\pi,\varphi)} = PM^{-1}.$$

Thus, since P is stable, the input/output relatioship induced by $\Sigma_{(\pi,\varphi)}$ is stable if the stable map M has a stable inverse M^{-1}. Consequently, in order to achieve a stable input/output relationship, we have to find a (suitable) pair of stable maps A and B for which the combination $(AP + BQ)$ has a stable inverse. This situation is clearly reminiscent of the linear theory. (We remark that, when M^{-1} is stable, the composite system $\Sigma_{(B^{-1},A)}$ will be internally stable, provided the stable systems A and B are 'uniformly' stable (uniformly continuous) in an appropriate sense. We do not elaborate on this point in the present note.) The following question thus becomes of basic importance.

(1.9) QUESTION. Given two stable systems P and Q, when does there exist a pair of stable systems A and B for which the stable map $M := AP + BQ$ has a stable inverse M^{-1}.

Question (1.9) has a well known solution in the linear theory - A and B exist if and only if the transfer matrices representing P and Q are right coprime. In HAMMER[1985] we studied (1.9) for the case where the systems P and Q are nonlinear. We showed there that a close analogy can be drawn between the linear and nonlinear situations, in the sense that a concept of coprimeness is involved in the nonlinear case as well. Qualitatively, we say that two nonlinear systems P and Q are right coprime if, for every unbounded input sequence u, at least one of the output sequences Pu or Qu is unbounded (see exact definition in section 3 below). Then, we show that a pair of stable maps A and B for which $(AP + BQ)$ has a stable inverse exists if and only if P and Q are right coprime. We devote section 3 to a more detailed discussion of this result.

Returning now to our stabilization problem, recall that the maps P and Q of (1.9) originated in the fraction representation $\Sigma = PQ^{-1}$. In view of the last paragraph, we would like P and Q to be right coprime. This leads us to the following

(1.10) QUESTION. When does an injective system Σ possess a right coprime fraction representation, namely, a representation $\Sigma = PQ^{-1}$ where P and Q are stable and right coprime systems.

Of course, in the linear case, every finite dimensional time-invariant system has a coprime fraction representation. In general, the answer to (1.10) depends to a large extent on exactly what definition of stability one adopts. For certain definitions of stability (e.g., Bounded-Input Bounded-Output stability), the existence of right coprime fraction representations is quite general; For some other definitions of stability, right coprime fraction representations may turn out to exist only for limited classes of systems. In our discussion below, we adopt a notion of stability which is a slight modification of the classical Liapunov concept of stability. For this stability notion, coprime fraction representations exist for a rather wide class of systems, which we call the class of 'homogeneous' systems. Qualitatively, a system is homogeneous if it behaves like a continuous map whenever its output is bounded. As we discuss later, the class of homogeneous systems includes the class of all recursive systems having a continuous recursion function, thus including many systems of practical interest. In section 4 below, we shall discuss these questions in more detail, and shall exhibit a construction for coprime fraction representations of nonlinear systems.

2. THE BASIC FRAMEWORK

We introduce now the basic mathematical setup for our discussion. As we have mentioned earlier, the systems we consider are discrete-time systems. The set of input sequences for our systems consists of two-sided infinite sequences with elements in R^m. Formally, we denote by $S(R^m)$ the set of all sequences u of the form $u := \ldots, u_{-1}, u_o, u_1, \ldots$, where $u_i \in R^m$ for all integers i, and where there exists an integer t_u, depending on u, such that $u_j = 0$ for all $j < t_u$. Thus, $S(R^m)$ is the set of all two-sided infinite sequences with elements in R^m which 'start' with zeros. We denote by 0 the zero sequence in $S(R^m)$, consisting of only zero elements. For a sequence $u \in S(R^m)$, we denote by u_k the k-th element of the sequence. By $S_o(R^m)$ we denote the set of all sequences $u \in S(R^m)$ for which $u_j = 0$ for all integers $j < 0$. The space $S_o(R^m)$, which consists of all input sequences remaining zero up to the time zero, will serve as our basic space of input sequences. Given a pair of sequences $u, v \in S(R^m)$, we define their sum $u + v$ elementwise by $(u + v)_i = u_i + v_i$ for all integers i.

A system Σ is a map $\Sigma : S_o(R^m) \to S_o(R^p)$ transforming input sequences in R^m into output sequences in R^p, which satisfies the condition

$$\Sigma 0 = 0.$$

Thus, all systems under consideration have the zero input sequence as a (possibly unstable) equilibrium point. We shall combine systems among themselves in two basic ways - addition and composition (or series combination). The sum of two systems $\Sigma_1, \Sigma_2 : S_o(R^m) \to S_o(R^p)$ is a system $\Sigma := \Sigma_1 + \Sigma_2 : S_o(R^m) \to S_o(R^p)$ defined, for each input sequence $u \in S_o(R^m)$, by $\Sigma u := \Sigma_1 u + \Sigma_2 u$. Clearly, $\Sigma 0 = \Sigma_1 0 + \Sigma_2 0 = 0 + 0 = 0$, and addition preserves the zero equilibrium point. The composition Σ_c of two systems $\Sigma_1 : S_o(R^m) \to S_o(R^p)$ and $\Sigma_2 : S_o(R^p) \to S_o(R^q)$ is the usual composition $\Sigma_c := \Sigma_2\Sigma_1 : S_o(R^m) \to S_o(R^q)$ of the two maps Σ_1 and Σ_2. Again, $\Sigma_c 0 = \Sigma_2\Sigma_1 0 = \Sigma_2 0 = 0$, and the zero equilibrium point is preserved under composition as well.

As an example of a common class of systems, we have the class of recursive systems, defined as follows. A system $\Sigma : S_o(R^m) \to S_o(R^p)$ is <u>recursive</u> if there exists a pair of integers $\eta, \mu \geq 0$ and a function $f : (R^p)^{\eta+1} \times (R^m)^{\mu+1} \to R^p$

such that, for every input sequence $u \in S(R^m)$, the output sequence $y := \Sigma u \in S(R^p)$ can be computed recursively by

$$y_{k+\eta+1} = f(y_k, \ldots, y_{k+\eta} | u_k, \ldots, u_{k+\mu})$$

for all integers k. The vertical line inside the argument of f is used to separate output variables from input variables. The function f is called a recursion function of Σ. Note that the requirement $\Sigma 0 = 0$ implies that $f(0, \ldots, 0 | 0, \ldots, 0) = 0$.

We turn now to a definition of the notion of stability that we adopt for our discussion. As we have indicated in the previous section, the results on coprimeness and on fraction representations of nonlinear systems are sensitive to the particular technical nature of the notion of stability that one employs. For certain types of stability notions, the technical details of the theory may become rather involved, and, in some instances, even the generality may become impaired. The notion of stability has to be chosen with due care so that, on one hand, it would be meaningful for practical applications, and, on the other hand, it would lead to a transparent theory. The notion of stability that we discuss below is a mild variation of the classical Liapunov notion of stability, and it satisfies the above requirements. We need some preliminary terminology before stating the actual definition.

Let $\theta > 0$ be a real number. We denote by $[-\theta, \theta]^m$ the set of all vectors in R^m the entries of which belong to the closed interval $[-\theta, \theta]$. We denote by $S(\theta^m)$ the set of all sequences $u \in S(R^m)$ for which $u_i \in [-\theta, \theta]^m$ for all integers i. Thus, $S(\theta^m)$ is the set of all sequences in $S(R^m)$ bounded by θ. Finally, we let $S_o(\theta^m) := S(\theta^m) \cap S_o(R^m)$ (the intersection) to be the set of all sequences in $S_o(R^m)$ bounded by θ. For a system $\Sigma : S_o(R^m) \to S_o(R^p)$, we denote by $\Sigma[S_o(\theta^m)]$ the image of $S_o(\theta^m)$ through Σ. We say that Σ is BIBO (Bounded-Input Bounded-Output)-stable if, for every real $\theta > 0$, there is a real $N > 0$ such that $\Sigma[S_o(\theta^m)] \subset S_o(N^p)$. As usual, a BIBO-stable system transforms bounded input sequences into bounded output sequences.

Probably, one of the most fundamental contributions of Liapunov was his conception of the close relation between the intuitive notion of stability and the mathematical concept of continuity. Following Liapunov, continuity also is the basis of the notion of stability that we intend to use, so we need to introduce a topology on our spaces of sequences. For the sake of convenience and familiarity, we shall use a topology induced by a metric. Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ be an element of R^m. We denote by ρ the standard norm on R^m given by $\rho(\alpha) := \max\{|\alpha_i|, i = 1, \ldots, m\}$. For an infinite sequence $u \in S_o(R^m)$, we let $\rho(u) := \sup\{2^{-i}\rho(u_i), i \geq 0\}$. For a pair of elements u, v we define $\rho(u,v) := \rho(u - v)$. It is easy to see that ρ is a metric, and so it induces a topology on $S_o(R^m)$ in the standard way. All notions of continuity mentioned below are with respect to this topology. We observe that, under the topology induced by ρ, the space $S_o(\theta^m)$ is compact for every real $\theta > 0$. This fact is of critical importance to our discussion. We are now in a position to define stability.

A system $\Sigma : S_o(R^m) \to S_o(R^p)$ is <u>stable</u> if it is BIBO-stable and if its restriction $\Sigma : S_o(\theta^m) \to S_o(R^p)$ is a continuous map for every real $\theta > 0$. The present definition of stability defers slightly from the classical definition of stability by using the metric ρ instead of the more commonly used norm of L^1. This difference, though being of little practical significance, has a great mathematical advantage, due mainly to the fact that, under the present topology, $S_o(\theta^m)$ is a compact space. Qualitatively speaking, the distinction between the two notions of continuity - the one with respect to L^1 and the one with respect to ρ - comes into effect when regarding the behaviour of the system at 'time infinity'; Over any fixed finite length of time there is no difference between the two notions. Since the boundedness of a stable system at time infinity is separately guarantied in the present definition by the BIBO-stability requirement, the difference between the classical definition of stability and the present definition has no profound practical implications. We thus observe a demonstration of how even slight nuances in the definition of stability may sometimes have substantial effects on the mathematical complexity of the problem. Of course, the topology we employ here is one commonly used when dealing with this type of spaces.

The stability notion discussed in the previous paragraph refers to input/output stability of a system. When dealing with composite systems, one has to use the stronger notion of 'internal stability' which, in addition to input/output stability of the composite system, also requires stability with respect to noises that may affect the ports of the subsystems of which the composite system is composed. We omit here a detailed definition of internal stability.

3. COPRIMENESS OF NONLINEAR SYSTEMS

In the present section we discuss the solution of question (1.9) following HAMMER[1985]. We first restate (1.9) in formal terms. Let $q > 0$ be an integer, let $S \subset S_o(R^q)$ be a subspace, and let $P : S \to S_o(R^p)$ and $Q : S \to S_o(R^m)$ be a pair of stable maps, where Q is invertible. The question that we consider is the following. Under what conditions (on P and on Q) does there exist a pair of stable maps $A : S_o(R^p) \to S_o(R^q)$ and $B : S_o(R^m) \to S_o(R^q)$ for which the map

(3.1) $\quad M := AP + BQ \ : S \to S$

has a stable inverse M^{-1}. In view of the fact that A, P, B, and Q are all stable maps, the map M is stable as well. Thus, M and M^{-1} are both stable. It will be convenient to refer to an invertible map for which it and its inverse are both stable as a unimodular map. We can then rephrase our problem as follows. Find stable maps A and B for which $(AP + BQ)$ is unimodular. In these terms, our question sounds identical to a classical problem in the algebraic theory of polynomial matrices. As is well known, given a pair of polynomial matrices P and Q, one can find a pair of polynomial matrices A and B for which the matrix $(AP + BQ)$ is a polynomial unimodular matrix if and only if P and Q are right coprime. In our ensuing discussion, we show that the situation in the nonlinear case is closely analogous. One can define a concept of coprimeness (directly in terms of P and Q) which guaranties the existence of A and B. The basic idea in this definition of the concept of coprimeness can qualitatively be stated as follows. The maps P and Q are right coprime if, for every unbounded sequence $u \in S$, at least one of the output sequences Pu or Qu is unbounded. In linear single-variable terminology, this amounts to the requirement that P and Q have no unstable zeros in common. The formal definition is as follows (HAMMER[1985]). (Given a map $C : S_1 \to S_2$ and a subset $Z \subset S_2$, we denote by $C^*[Z]$ the inverse image of Z through C, i.e., the set of all elements $u \in S_1$ for which $Cu \in Z$.)

(3.2) DEFINITION. Let $S \subset S_o(R^q)$ be a subspace. Two stable maps $P : S \to S_o(R^p)$ and $Q : S \to S_o(R^m)$ are right coprime if the following conditions hold.

(α) For every real $\tau > 0$ there exists a real $\theta > 0$ such that
$P^*[S_o(\tau^p)] \cap Q^*[S_o(\tau^m)] \subset S_o(\theta^q)$, and

(β) For every real $\tau > 0$, the set $S \cap S_o(\tau^q)$ is a closed subset of $S_o(\tau^q)$. []

As we can see, the notion of coprimeness is defined directly in terms of properties of the maps P and Q and of the space S. That right coprimeness of P and Q is a necessary condition for the existence of stable maps A and B satisfying (3.1) can be seen as follows. Assume that there exist stable maps A and B for which the map $M := AP + BQ$ is unimodular. By contradiction, suppose that condition (α) of (3.2) does not hold. Then, for some real $\tau > 0$, there exists a divergent sequence $\{u^i\}$ of elements of S, and a divergent sequence of real numbers $0 < \theta_1 < \theta_2 < \ldots \to \infty$, so that, for all integers $i \geq 0$, the following holds: $u^i \notin S_o(\theta^q_{i-1})$, but $Q(u^i) \in S_o(\tau^m)$ and $P(u^i) \in S_o(\tau^p)$. Now, since A and B are BIBO-stable, there exists a real $\gamma > 0$ such that $A[S_o(\tau^p)] \subset S_o(\gamma^q)$ and $B[S_o(\tau^m)] \subset S_o(\gamma^q)$. Consequently, $Mu^i = (AP + BQ)u^i = APu^i + BQu^i \in S_o((2\gamma)^q)$ for all integers $i \geq 0$. In other words, $u^i \in M^{-1}[S_o((2\gamma)^q) \cap S]$ for all integers $i \geq 0$. But then, since $\{u^i\}$ is a divergent sequence, it follows that $M^{-1}[S_o((2\gamma)^q) \cap S]$ cannot be bounded, contradicting the fact that M is unimodular. Thus, (α) is a necessary condition for the existence of maps A and B satisfying (3.1) with M a unimodular map.

To see the origin of condition (β) of (3.2), assume that there are stable maps A and B satisfying (3.1) with the particular choice of $M = I$ (the identity map), i.e., $AP + BQ = I$. Assume that for some real $\tau > 0$, the set $S \cap S_o(\tau^q)$ is not a closed subset of $S_o(\tau^q)$. Now, the right hand side of (3.1) has an evident unimodular extension to the closure of S - the identity map. Consider now the terms on the left hand side of (3.1). Let $\{u^i\}$ be a sequence of elements of $S \cap S_o(\tau^q)$ converging to a point $u \in S_o(\tau^q)$ not in S. Clearly, since $AP + BQ = I$, we have $\text{Lim } (AP + BQ)u^i = \text{Lim } u^i = u$. Further, by BIBO-stability, there is a real $\gamma > 0$ such that $AP[S \cap S_o(\tau^q)] \subset S_o(\gamma^q)$ and $BQ[S \cap S_o(\tau^q)] \subset S_o(\gamma^q)$. By compactness of $S_o(\gamma^q)$, this implies that there is a subsequence $\{v^i\}$ of $\{u^i\}$ such that both of the sequences $\{APv^i\}$ and $\{BQv^i\}$ are convergent in $S_o(\gamma^q)$, say, $\text{Lim } APv^i = b \in S_o(\gamma^q)$ and $\text{Lim } BQv^i = b \in S_o(\gamma^q)$. We clearly still have $\text{Lim } v^i = u$, and $a + b = u$. Extend now the two functions AP and BQ to the point u by defining $APu := a$ and $BQu := b$. This extension does, evidently, not violate the BIBO-stability of AP and BQ. In such a way, we can extend AP and BQ to the closure of $S \cap S_o(\tau^q)$, without impairing their BIBO-stability. Consequently, from the solution of (3.1) over $S \cap S_o(\tau^q)$, we obtain a BIBO-stable (possibly not continuous) solution of (3.1) over the closure of $S \cap S_o(\tau^q)$ in $S_o(\tau^q)$. But then, in view of the fact that our argument in the previous paragraph is based on BIBO-stability alone, condition (α) of (3.2) must hold over the closure of $S \cap S_o(\tau^q)$ in $S_o(\tau^q)$; Hence (β).

The main result derived in HAMMER[1985] regarding the concept of coprimeness isthe following.

(3.3) THEOREM. <u>Let</u> $S \subset S_o(R^q)$ <u>be a subspace, and let</u> $P : S \to S_o(R^p)$ <u>and</u> $Q : S \to S_o(R^m)$ <u>be stable maps, where</u> P <u>is injective and</u> Q <u>is a set-isomorphism.</u> <u>If</u> P <u>and</u> Q <u>are right coprime then, for every unimodular map</u> $M : S \to S$, <u>there exist stable maps</u> $A : \text{Im } P \to S_o(R^q)$ <u>and</u> $B : S_o(R^m) \to S_o(R^q)$ <u>satisfying</u> $AP + BQ = M$. (Here, Im P denotes the Image of the map P.)

As we can see from Theorem 3.3, there is a complete formal analogy between the situation here and the situation in the classical theory of polynomial matrices. The methods employed to derive the results here are, of course, of a different nature. For a proof of Theorem 3.3, see HAMMER[1985].

4. FRACTION REPRESENTATIONS OF NONLINEAR SYSTEMS

Let $\Sigma : S_o(R^m) \to S_o(R^p)$ be an injective system. We say that Σ has a <u>right coprime fraction representation</u> if there exists an integer q, a subspace $S \subset S_o(R^q)$, and a pair of stable and right coprime maps $P : S \to S_o(R^p)$ and $Q : S \to S_o(R^m)$, with Q invertible, such that $\Sigma = PQ^{-1}$. Clearly, by the injectivity of

Σ, the map P is injective. As it turns out, the class of systems possessing right coprime fraction representations coincides with the class of so called 'homogeneous systems'. Homogeneous systems are characterized intrinsically, without any direct reference to fraction representations. Qualitatively, a system is homogeneous if it behaves like a continuous function whenever its output is bounded. The formal definition is as follows (HAMMER[1985]).

(4.1) DEFINITION. A system $\Sigma : S_o(R^m) \to S_o(R^p)$ is homogeneous if for every real $\theta > 0$ the following holds: For every subset $S_* \subset S_o(\theta^m)$ for which there exists a real $\tau > 0$ such that $\Sigma[S_*] \subset S_o(\tau^p)$, the restriction of Σ to the closure $\overline{S}_*$ of S_* in $S_o(\theta^m)$ is a continuous map. []

The equivalence between homogeneity and the existence of right coprime fraction representations is stated as the next result (HAMMER[1985]).

(4.2) THEOREM. An injective system $\Sigma : S_o(R^m) \to S_o(R^p)$ has a right coprime fraction representation if and only if it is homogeneous.

We provide next an example of a family of homogeneous systems. Recall that a system $\Sigma : S_o(R^m) \to S_o(R^p)$ is recursive if its output sequence y can be computed from the input sequence u generating it through a recursive relation of the form

$$y_{k+\eta+1} = f(y_k, \ldots, y_{k+\eta} | u_k, \ldots, u_{k+\mu}),$$

for all integers k. The function f is called a recursion function for Σ.

(4.3) PROPOSITION. Let $\Sigma : S_o(R^m) \to S_o(R^p)$ be an injective recursive system. If Σ has a continuous recursion function, then Σ is a homogeneous system.

PROOF. Assume that Σ has a continuous recursion function $f : (R^p)^{\eta+1} \times (R^m)^{\mu+1} \to R^p$. Let $S_* \subset S_o(\theta^m)$ be a subset for which $\Sigma[S_*] \subset S_o(\tau^p)$ for some real $\tau > 0$. Let $\overline{S}_*$ denote the closure of S_* in $S_o(\theta^m)$. We first show that $\Sigma[\overline{S}_*] \subset S_o(\tau^p)$. For every integer $k \geq 0$, denote

$$S_k := \{ ((\Sigma u)_k, \ldots, (\Sigma u)_{k+\eta}, u_k, \ldots, u_{k+\mu}) : u \in S_* \},$$

i.e., S_k is the set of all possible arguments of f at time k for inputs from S_*. Similarly, let

$$S^*_k := \{ ((\Sigma u)_k, \ldots, (\Sigma u)_{k+\eta}, u_k, \ldots, u_{k+\mu}) : u \in \overline{S}_* \}.$$

Clearly, since $S_* \subset S_o(\theta^m)$ and $\Sigma[S_*] \subset S_o(\tau^q)$, we have $S_k \subset ([-\tau,\tau]^p)^{\eta+1} \times ([-\theta,\theta]^m)^{\mu+1}$ for all integers k, and S_k is a bounded set. Using the facts that f is continuous and that here the continuous image of a closed and bounded set is a closed and bounded set, one can readily show by induction that $S^*_k \subset \overline{S}_k$ for all integers k, where $\overline{S}_k$ is the (usual) closure of S_k in $(R^p)^{\eta+1} \times (R^m)^{\mu+1}$. This implies that, for every $u \in \overline{S}_*$, one has $(\Sigma u)_k \in [-\tau,\tau]^p$ for all integers k, or that $\Sigma[\overline{S}_*] \subset S_o(\tau^p)$.

Next, we show that the restriction of Σ to $\overline{S}_*$ is a continuou map. (To simplify notation, we assume here that Σ is causal, i.e., that $\mu \le \eta+1$.) For each integer $n \ge 0$, let $F_n : (R^m)^{n+1} \to (R^p)^{n+1}$ denote the function providing the first output values $(\Sigma u)_o, \ldots, (\Sigma u)_n$ when given the first input values $u_o, \ldots, u_n$, i.e., $F_n(u_o, \ldots, u_n) = ((\Sigma u)_o, \ldots, (\Sigma u)_n)$. (The existence of F_n is a consequence of causality.) Since F_n can be constructed from the continuous function f through a finite number of compositions, F_n is a continuou function for any integer $n \ge 0$. (In the next few sentences we repeatedly employ the definition of the metric ρ for finite dimensional spaces and for infinite sequences.) Choose some real number $\epsilon > 0$. Let $j \ge 0$ be an integer for which $2^{-j}\tau < \epsilon$. By the uniform continuity of F_j on the compact set $([-\theta,\theta])^{j+1}$, there is a real number $\delta > 0$ such that $\rho[F_j(u_o, \ldots, u_j) - F_j(v_o, \ldots, v_j)] < \epsilon$ for all $u_o, \ldots, u_j$ and $v_o, \ldots, v_j$ satisfying $\rho[(u_o, \ldots, u_j) - (v_o, \ldots, v_j)] < \delta$. In view of the fact that $\Sigma u, \Sigma v \in S_o(\tau^p)$ for all $u, v \in \overline{S}_*$, it follows by the choice of j that $\rho(\Sigma u - \Sigma v) < \epsilon$ for all pairs $u, v \in \overline{S}_*$ satisfying $\rho(u - v) < 2^{-j}\delta$. Thus, Σ is continuous over $\overline{S}_*$. []

From Proposition 4.3 we see that many systems of practical interest are indeed homogeneous, and thus possess right coprime fraction representations.

The next question that we wish to address is the question of the uniqueness of right coprime fraction representations. Let $\Sigma : S_o(R^m) \to S_o(R^p)$ be an injective system, and let $\Sigma = PQ^{-1}$ be a right coprime fraction representation, where $P : S \to S_o(R^p)$ and $Q : S \to S_o(R^m)$, and where $S \subset S_o(R^q)$ for some integer $q > 0$. We call S the <u>factorization space</u> of this fraction representation. Now, let $S_1 \subset S_o(R^q)$ be a subspace for which there is a unimodular transformation $M_1 : S_1 \cong S$. Then, it can be readily verified that the two maps $P' := PM_1 : S_1 \to S_o(R^p)$ and $Q' := QM_1 : S_1 \to S_o(R^m)$ are still right coprime (as were P and Q), and that $\Sigma = P'Q'^{-1}$. Thus, from a right coprime fraction representation $\Sigma = PQ^{-1}$ and a unimodular transformation M_1, we can generate a new right coprime fraction representation $\Sigma = P'Q'^{-1}$. The main question that arises in this context is whether <u>all</u> right coprime fraction representations of Σ can be generated in this way from the fixed representation $\Sigma = PQ^{-1}$, by varying the unimodular transformation M. In other words, given a right coprime fraction representation $\Sigma = P_1Q_1^{-1}$, does there always exist a unimodular transformation M such that $P_1 = PM$ and $Q_1 = QM$. As we see from the next theorem, which we reproduce from HAMMER[1985], the answer to this question is in the affirmative.

(4.4) THEOREM. <u>Let</u> $\Sigma : S_o(R^m) \to S_o(R^p)$ <u>be an injective homogeneous system, and let</u> $\Sigma = PQ^{-1}$ <u>and</u> $\Sigma = P_1Q_1^{-1}$ <u>be two right coprime fraction representations of</u> Σ <u>with factorization spaces</u> $S, S_1 \subset S_o(R^q)$, <u>respectively. Then, there exists a unimodular transformation</u> $M : S_1 \to S$ <u>such that</u> $P_1 = PM$ <u>and</u> $Q_1 = QM$.

The final question that we would like to discuss here is the construction of right coprime fraction representations of an injective system. We shall exhibit the construction of one particular such representation. All other right coprime fraction representations of the same system can then be obtained from this one via Theorem 4.4. Let $\Sigma : S_o(R^m) \to S_o(R^p)$ be an injective homogeneous system. As usual, we define the <u>graph</u> $G(\Sigma)$ of Σ as the set of all pairs $(u, \Sigma u)$, where u varies overthe whole space of input sequences $S_o(R^m)$. Clearly, $G(\Sigma)$ is a subset of $S_o(R^m) \times S_o(R^p)$, i.e., a subset of $S_o(R^q)$ with $q := m + p$. Further, let $P_1 : S_o(R^m) \times S_o(R^p) \to S_o(R^m)$ and $P_2 : S_o(R^m) \times S_o(R^p) \to S_o(R^p)$ be the usual projections onto the factors of the product space. Using these projections, we define a pair of maps $P : G(\Sigma) \to S_o(R^p)$ and $Q : G(\Sigma) \to S_o(R^m)$ by setting

$$Px := P_2 x \quad \text{and} \quad Qx := P_1 x$$

for all elements $x \in G(\Sigma)$. By the evident boundedness and continuity of the projections P_1 and P_2, it follows that P and Q are stable systems. Also, a slight reflection shows that Q is a set-isomorphism, and that $\Sigma = PQ^{-1}$. Moreover, using the fact that Σ is an injective homogeneous system, it can be shown that P and Q are actually right coprime, so that $\Sigma = PQ^{-1}$ is a right coprime fraction representation, and its factorization space is $G(\Sigma)$ (see HAMMER[1985] for proof). All other right coprime fraction representations of Σ can be obtained from this one by the application of unimodular transformations.

5. REFERENCES

For a detailed list of references, see the reference lists of the publications listed below.

J. HAMMER

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