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STRICTLY OBSERVABLE LINEAR SYSETEMS

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Abstract

The rudimants of the module theoretic approach to linear system theory are briefly reviewed. Two types of integer invariants of systems are mentioned: the reduced reachability indices, and the latency indices. The reduced reachability indices are related to the problem of reducing a system through the application of causal precompensation. The latency indices are related to the problem of causal factorization of one system over another.

1. INTRODUCTION

In this short note we wish to summerize some of the basic features of the module theoretic approach to linear system theory. Our discussion will be mainly on a des - criptive level, and proofs will be omitted. A more detailed discussion of the topics mentioned below can be found in HAMMER and HEYMANN [1981 a and b].

Formal Laurent series: Consider a discrete time linear time-invariant system Σ. At each instant of time t, the system Σ admits an m-dimensional real vector $u_t \in R^m$ as input, and has a p-dimensional real vector $y_t \in R^p$ as output. Each input sequence $u_{t_o}, u_{t_o+1}, \dots$ to Σ can be formally represented as aLaurent series

$$u = \sum_{t=t_o}^{\infty} u_t z^{-t},$$

where $u_t \in R^m$ for all t, the index t serves as the time marker, and where t_o is the finite time at which the sequence "starts". The set of all such formal Laurent series is denoted by ΛR^m. Each series $u = \Sigma u_t z^{-t}$ in ΛR^m can be naturally divided into three parts : the (strict) past part $u^P := \sum_{t<0} u_t z^{-t}$; the present part $u^o := u_o$; and the (strict) future part $u^F := \sum_{t>0} u_t z^{-1}$.

In the set ΛR^m one can define an operation of addition for every pair of elements $u^i = \sum_{t=t_o^i} u_t^i z^{-t}$, $i = 1,2$, by

$$(1.1) \qquad u^1 + u^2 = \sum_{t=\min(t_o^1, t_o^2)} (u_t^1 + u_t^2) z^{-t}$$

(coefficientwise). Also, given an element $k = \sum_{t=t_o}^{\infty} k_t z^{-t}$ in the set ΛR of scalar Laurent series, one can define an operation of multiplication

$$(1.2) \qquad ku^1 = \sum_{t=t_o+t_o^1}^{\infty} \left[\sum_{j=t_o}^{t-t_o^1} k_t u_{t-j}^1 \right] z^{-t}$$

(convolution). The importance of these operations is that, under them, the set ΛR forms a field, and the set ΛR^m forms an m-dimensional linear space over this field (a ΛR - linear space).

The relevance of ΛR-linearity to our discussion stems from the fact that it is closely related to time invariance. Indeed, the system Σ induces a map $f: \Lambda R^m \to \Lambda R^p$ which assigns to each input sequence $u \in \Lambda R^m$ its corresponding output sequence $y = fu \in \Lambda R^p$. If the map f is ΛR-linear, then, in particular, it commutes with the element $z \in \Lambda R$, that is, $fzu = zfu$ for every $u \in \Lambda R^m$. But, by (1.2), multiplication by z represents a one step time shift of the sequence to the left, so that the last equation implies that f commutes with the time shift operator. Thus, ΛR-linearity implies time invariance (KALMAN, FALB, and ARBIB [1969], WYMAN [1972]). Conversely, under some mild assumptions (see e.g., HAMMER and HEYMANN [1981a, section 2]), if the linear system Σ is time invariant, then the map $f: \Lambda R^m \to \Lambda R^p$ induced byit is ΛR-linear. Summerizing, we have that, in a broad sense, ΛR-linearity is equiva- lent to time invariance of linear systems.

Rings and modules: The set ΛR^m of Laurent series with coefficients in R^m contains, as subsets, the set Ω^+R^m of all (polynomial) elements of the form $\sum_{t=t_o}^{0} u_t z^{-t}$, $t_o \le 0$, and the set Ω^-R^m of all (power series) elements of the form $\sum_{t\geq 0} u_t z^{-t}$. In particular, Ω^+R is the usual set of polynomials with real coefficients, and Ω^-R is the set of all power series in z^{-1} with real coeffi cients. It is known that both of Ω^+R and Ω^-R form principal ideal domains under the operations of addition and multiplication defined in (1.1) and (1.2). The set Ω^+R^m forms a free Ω^+R-module of rank m, and Ω^-R^m forms a free Ω^-R-module of rank m, both under the operations (1.1) and (1.2).

Clearly, the ΛR-linear space ΛR^m is also a free Ω^+R-module, and Ω^+R^m is then an Ω^+R-submodule of it. Thus, we can consider the quotient Ω^+R-module $\Lambda R^m/\Omega^+R^m$. Each element in this quotient module is an equivalence class c of elements of ΛR^m which are equal modulo their polynomial part. Explicitly, two elements $u^i = \Sigma u^i_t z^{-t} \in \Lambda R^m$, $i = 1,2$, belong to the same equivalence class $c \in \Lambda R^m/\Omega^+R^m$ if and only if $u^2_t = u^1_t$ for all $t > 0$ (i.e., the strictly future parts of the sequences are identical). As in any situation involving quotient modules, we can define a canonical projection

$$\pi^+ : \Lambda R^m \to \Lambda R^m/\Omega^+R^m ,$$

which assigns to every element in ΛR^m its equivalence class in $\Lambda R^m/\Omega^+R^m$. By definition, this projection is an Ω^+R-homomorphism.

Analogously, the ΛR-linear space ΛR^m forms an Ω^-R-module as well, and Ω^-R^m is a submodule of it. The quotient Ω^-R-module $\Lambda R^m/\Omega^-R^m$ is then well defined. It consists of equivalence classes each of which contains all those elements in ΛR^m which have the same strictly polynomial part; that is, two elements $u^i = \Sigma u^i_t z^{-t}$, $i = 1,2$, in ΛR^m belong to the same equivalence class in $\Lambda R^m/\Omega^-R^m$ if and only if $u^2_t = u^1_t$ for all $t < 0$. We also obtain an induced canonical projection of Ω^-R-modules

$$\pi^- : \Lambda R^m \to \Lambda R^m/\Omega^-R^m ,$$

which assigns to each element in ΛR^m its equivalence class in $\Lambda R^m/\Omega^-R^m$. The projections π^+ and π^- are repeatedly employed in our discussion below.

Transfer matrices: Let T be an $m\mathrm{x}p$ transfer matrix of a linear time invariant system. Every entry of T is evidently an element in ΛR, and thus T can be regarded as a linear transformation (matrix) $\Lambda R^m \to \Lambda R^p$. Conversely, let $f: \Lambda R^m \to \Lambda R^p$ be a ΛR-linear map. As usual, f can be represented as a matrix relative to specified bases $u_1,\ldots u_m$ in ΛR^m and $y_1,\ldots y_p$ in ΛR^p. Of particular importance is the case when $u_1,\ldots,u_m \in R^m$ and $y_1,\ldots,y_p \in R^p$, where R^m and R^p are regarded as subsets (of "purly present" sequences) of ΛR^m and ΛR^p, respectively. In such case, the matrix representation Z_f of f is called a transfer matrix, and it coincides with the classical concept of transfer matrices. Thus, a ΛR-linear map and a transfer matrix are equivalent quantities, and in our discussionbelow we shall make no distinction among them.

A ΛR-linear map is called polynomial if all the entries of its transfer mantrix are polynomials (in Ω^+R); it is called causal if all the entries of its transfer matrix are in Ω^-R; it is called strictly causal if all the entries of its transfer matrix are in $z^{-1}\Omega^-R$; it is called rational if all the entries of its transfer matrix are fractions of polynomials; it is called a i/o (input/output) map if it is both rational and strictly causal; and, finally, it is called bicausal if it is invertible and if both of it and its inverse are causal.

2. KERNELS AND FACTORIZATION

Let $f: \Lambda R^m \to \Lambda R^p$ be a ΛR-linear map. As we have seen before, such a map represents a linear time invariant system admitting inputs from R^m and having its outputs in R^p. Since f is ΛR-linear, it is evidently also an Ω^+R-homomorphism. Whence, the map π^+f is again an Ω^+R-homomorphism, and $\Delta := \mathrm{Ker}\ \pi^+f$ is an Ω^+R-module. The module Δ consists of all input sequences (to the system represented by f) that lead to output sequences which have zero future parts. It forms an extension of the classical KALMAN [1965] realization module Δ_K which consists of all past input sequences that lead to output sequences having zero future parts. We have that

(2.1) $\Delta_K = \Delta \cap \Omega^+R^m.$

The algebraic significance of the module Δ is that it determines whether polynomial factorization of one map over another is possible, as follows (HAMMER and HEYMANN [1981b]).

(2.2) THEOREM. Let $f_1, f_2: \Lambda R^m \to \Lambda R^p$ be rational ΛR-linear maps.

(i) There exists a polynomial map $P: \Lambda R^p \to \Lambda R^p$ such that $f_2 = Pf_1$ if and only if $\mathrm{Ker}\ \pi^+f_1 \subset \mathrm{Ker}\ \pi^+f_2$.

(ii) There exists a polynomial unimodular map $M: \Lambda R^p \to \Lambda R^p$ such that $f_2 = Mf_1$ if and only if $\mathrm{Ker}\ \pi^+f_1 = \mathrm{Ker}\ \pi^+f_2$.

In general, the module $\mathrm{Ker}\ \pi^+f$ contains both polynomial and non-polynomial elements of ΛR^m, and, when f is noninjective, this module is not finitely generated. (It contains the ΛR-linear space $\mathrm{Ker}\ f$.) Nevertheless, for a particular f, it may happen that $\mathrm{Ker}\ \pi^+f$ consists of polynomial elements only, that is,

(*) $\mathrm{Ker}\ \pi^+f \subset \Omega^+R^m .$

In such case, $\mathrm{Ker}\ \pi^+f$ is equal to the Kalman realization module (1.2). When (*) holds, the map f is called strictly observable (HAMMER and HEYMANN [1981b]). We note that a strictly observable map is necessarily injective. Further, letting I be the identity, we clearly have that $\operatorname{Ker} \pi^+ I = \Omega^+ R^m$. Thus, a strictly obsevable map f satisfies $\operatorname{Ker} \pi^+ f \subset \operatorname{Ker} \pi^+ I$. By Theorem 2.2 this implies that there exists a polynomial map P such that $Pf = I$, i.e., a strictly observable map has a polynomial left inverse . As we show in the next section, the system theoretic significance of strictly observable systems is that they are minimal in the sense that their MacMillan degree cannot be reduced by the application of causal precompensation (see Theorem 3.2(i) below).

In complete analogy, one can also consider the $\Omega^- R$-module $\operatorname{Ker} \pi^- f$. This module consists of all the input sequences that lead to output sequences which are zero inthe past. From the algebraic point of view, this module determines the solution to the problem of causal factorization, as follows (HAMMER and HEYMANN [1981a]).

(2.3) THEOREM. Let $f_1, f_2: \Lambda R^m \to \Lambda R^p$ be ΛR-linear maps.
(i) There exists a causal map $h: \Lambda R^p \to \Lambda R^p$ such that $f_2 = hf_1$ if and only if $\operatorname{Ker} \pi^- f_1 \subset \operatorname{Ker} \pi^- f_2$.
(ii) There exists a bicausal map $\iota: \Lambda R^p \to \Lambda R^p$ such that $f_2 = \iota f_1$ if and only if $\operatorname{Ker} \pi^- f_1 = \operatorname{Ker} \pi^- f_2$.

As we can see, there is a complete analogy between Theorems 2.2 and 2.3 .

3. KERNELS AND INDICES

In the present section we limit our discussion to the case of injective maps (i.e., transfer matrices with ΛR-linearly independent columns). For the more general case, see HAMMER and HEYMANN [1981 a and b].

Let $f: \Lambda R^m \to \Lambda R^p$ be an injective rational ΛR-linear map. We assign next to each one of the modules $\operatorname{Ker} \pi^+ f$ and $\operatorname{Ker} \pi^- f$ a set of integers which turn out to have system theoretic significance. Before doing so, we briefly review the concept of proper bases. Let $d = \sum d_t z^{-t}$ be an element in ΛR^m . The order of d is defined as $\operatorname{ord} d := \min_t \{d_t \neq 0\}$ if $d \neq 0$, and $\operatorname{ord} d := \infty$ if $d = 0$. When d is a polynomial, then the order is just the negative of the degree. The leading coefficient $\hat{d}$ of d is the first nonzero coefficient in the Laurent series expansion, that is $\hat{d} := d_{\operatorname{ord} d}$ if $d \neq 0$, and $\hat{d} := 0$ if $d = 0$. A set of elements $d_1, \dots, d_n \in \Lambda R^m$ is properly independent if the leading coefficients $\hat{d}_1, \dots, \hat{d}_n \in R^m$ are linearly independent over the field of real numbers R. A basis consisting of properly independent elements is called a proper basis. A proper basis $d_1, \dots, d_m \in \Lambda R^n$ is ordered if $\operatorname{ord} d_{i+1} \leq \operatorname{ord} d_i$ for all $i = 1, \dots, m-1$.

Returning now to our modules, we have the following (HAMMER and HEYMANN [1981b]).

(3.1) THEOREM. Let $f: \Lambda R^m \to \Lambda R^p$ be an injective rational ΛR-linear map. Then,
(i) the $\Omega^+ R$-module $\operatorname{Ker} \pi^+ f$ has an ordered proper basis $d_1, \dots, d_m$; and
(ii) if $d'_1, \dots, d'_m$ is any ordered proper basis of the $\Omega^+ R$-module $\operatorname{Ker} \pi^+ f$, then $\operatorname{ord} d'_i = \operatorname{ord} d_i$ for all $i = 1, \dots, m$.

Now, let $f: \Lambda R^m \to \Lambda R^p$ be an injective rational ΛR-linear map, and let $d_1, \dots, d_m$ be an ordered proper basis of $\operatorname{Ker} \pi^+ f$. Then, the reduced reachability indices $\mu_1 \geq \mu_2 \geq \dots \geq \mu_m$ of f are defined as $\mu_i := -\operatorname{ord} d_i$, $i = 1, \dots, m$. In view of Theorem 3.1, these indices are uniquely determined by f. The system theoretic significance of the reduced reachability indices is related to the characterization of the set of all dynamics that can be assigned to a given f by applying causal precompensation. We next discuss this point. Let

$$x_{k+1} = Fx_k + Gu_k$$
$$y_k = Hx_k$$

be a reachable realization of the system represented by f. As is known, the dynamical properties of the system are determined by the pair of matrices (F,G), which we call a semi-realization of f. A semirealization (F,G) of f is canonical if there exists a matrix H such that (F,G,H) represents a canonical realization of f. Finally, the reachability indices (or Kronecker invariants) of a system are discussed in ROSENBROCK [1970], BRUNOVSKY [1970], and KALMAN [1971]. We can now state the following (HAMMER and HEYMANN [1981b])

(3.2) THEOREM. Let $f: \Lambda R^m \to \Lambda R^p$ be an injective linear i/o map with reduced reachability indices $\mu_1 \geq \mu_2 \geq \dots \geq \mu_m$. Then,
(i) For every nonsingular causal precompensator $\iota: \Lambda R^m \to \Lambda R^m$, the reachability indices $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m$ of the system $f\iota$ satisfy $\lambda_i \geq \mu_i$ for all $i = 1, \dots m$. The last condition holds with equality for all $i = 1, \dots, m$ if and only if $f\iota$ is strictly observable.
(ii) Let (F,G) be any reachable pair with m reachability indices $\theta_1 \geq \theta_2 \geq \dots \geq \theta_m$. If $\theta_i \geq \mu_i$ for all $i = 1, \dots, m$, then there exists a nonsingular causal precompensator $\iota: \Lambda R^m \to \Lambda R^m$ such that (F,G) is a canonical semirealization of the system $f\iota$.

In particular, Theorem 3.2 implies that the reduced reachability indices are the minimal reachability indices obtainable through causalprecompensation, and that the dynamical order of a strictly observable system cannot be further reduced through the application of causal precompensation.

Consider now a particular type of causal precompensators - the feedback precompensators, which are defined as follows. Let $r: \Lambda R^p \to \Lambda R^m$ be acausal map, and assume that it is connected as an output feedback around the system $f: \Lambda R^m \to \Lambda R^p$. The resulting system f_r will then be given by

$$f_r = f\iota_r ,$$

where $\iota_r := [I + rf]^{-1}$ is an equivalent (bicausal) precompensator. The following theorem states that a system can be maximally reduced (i.e., transformed into a strictly observable one) also by using causal output feedback alone (HAMMER and HEYMANN [1981b]).

(3.3) THEOREM. *Let* $f: \Lambda R^m \to \Lambda R^p$ *be an injective linear i/o map with reduced reachability indices* $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_m$. *There exists a causal output feedback compensator* $r: \Lambda R^p \to \Lambda R^m$ *such that* f_r *has reachability indices equal to* $\mu_1, \ldots, \mu_m$.

In analogy to the case of $\operatorname{Ker} \pi^+ f$, one can also assign a set of integers to the $\Omega^- R$-module $\operatorname{Ker} \pi^- f$. For this purpose we need the following result (HAMMER and HEYMANN [1981a]).

(3.4) THEOREM. *Let* $f: \Lambda R^m \to \Lambda R^p$ *be an injective* ΛR-*linear map. Then,*

(i) *the* $\Omega^- R$-*module* $\operatorname{Ker} \pi^- f$ *has an ordered proper basis* $d_1, \ldots, d_m$; *and*

(ii) *if* $d_1', \ldots, d_m'$ *is any ordered proper basis of* $\operatorname{Ker} \pi^- f$, *then* $\operatorname{ord} d_i' = \operatorname{ord} d_i$ *for all* $i = 1, \ldots, m$.

Now, let $f: \Lambda R^m \to \Lambda R^p$ be an injective linear i/o map. We define the *latency indices* $\upsilon_1 \geq \upsilon_2 \geq \ldots \geq \upsilon_m$ of f as $\upsilon_i := -\operatorname{ord} d_i - 1$, $i = 1, \ldots, m$. In view of Theorem 3.4, the latency indices are uniquely determined by f. The system theoretic significance of the latency indices is related to the problem of causal factorization with remainders, which is stated as follows.

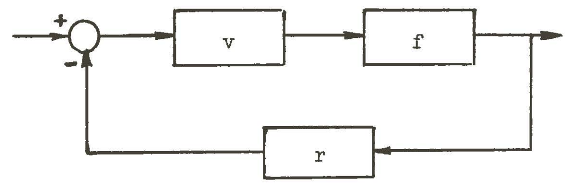
Causal division: Let $f_1, f_2: \Lambda R^m \to \Lambda R^p$ be rational ΛR-linear maps. Find a pair of rational maps $r: \Lambda R^p \to \Lambda R^p$ and $q: \Lambda R^m \to \Lambda R^p$, where r is *causal*, such that

$$f_2 = rf_1 + q, \tag{3.5}$$

and where q has the minimal possible dynamical order.

The problem of causal division appears as an underlying problem in a variety of control theoretic circumstances. One such circumstance is the problem of feedback representation of precompensators, which is stated as follows. Let $f: \Lambda R^m \to \Lambda R^p$ be the transfer of a given system, and suppose that one is required to design around f a classical control configuration of the form

(3.6)



which transforms f into a prescribed transfer matrix f'. In (3.6), $r: \Lambda R^p \to \Lambda R^m$ is a causal output feedback, and $v: \Lambda R^m \to \Lambda R^m$ is a causal precompensator. We add the requirement that v be nonsingular in order to prevent possible loss of degrees of freedom of the control variables. Thus, we have to find causal compensators v and r, where v is nonsingular, such that

$$f' = fv[I + rfv]^{-1}.$$

This problem can be solved in two steps: (i) compute an equivalent precompensator $\iota: \Lambda R^m \to \Lambda R^m$ for which $f' = f\iota$, and (ii) find compensators v and r for which

$$\iota = v[I + rfv]^{-1} .$$

As we see, step (i) can be solved through (the dual of) Theorem 2.3, whereas step (ii) requires the solution of the equation

$$\iota^{-1} = rf + v^{-1},$$

which is of the form (3.5).

Several other circumstances in which equation (3.5) is encountered are indicated in EMRE and HAUTUS [1980].

The connection between the latency indices and the problem of causal division of maps is given by the following result, the proof of which is given in HAMMER and HEYMANN [1981a, proof of Theorem 7.2]. (The reference also includes the required explicit constructions.)

(3.7) THEOREM. *Let* $f: \Lambda R^m \to \Lambda R^p$ *be an injective linear i/o map with latency indices* $\upsilon_1 \geq \upsilon_2 \geq \ldots \geq \upsilon_m$, *and let* $f': \Lambda R^m \to \Lambda R^p$ *be a rational* ΛR-*linear map. There exists a rational causal map* $r: \Lambda R^p \to \Lambda R^p$ *such that* $f' = rf + q$, *where the remainder* $q: \Lambda R^m \to \Lambda R^p$ *has reachability indices* $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m$ *which satisfy* $\lambda_i \leq \upsilon_i$ *for all* $i = 1, \ldots, m$.

The bound on the reachability indices of the remainder q given by Theorem 3.7 is tight in the following sense: There exists a map $f': \Lambda R^m \to \Lambda R^p$ for which, in every equation of the form $f' = rf + q$ with causal r, the reachability indices $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m$ of q satisfy $\lambda_i \geq \upsilon_i$ for all $i = 1, \ldots, m$, where $\upsilon_1 \geq \upsilon_2 \geq \ldots \geq \upsilon_m$ are the latency indices of f (see HAMMER and HEYMANN [1981a, Theorem 7.9]).

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