SIAM J. CONTROL AND OPTIMIZATION
Vol. 21, No. 1, January 1983

0363-0129/83/2101-0001 $01.25/0

STRICTLY OBSERVABLE LINEAR SYSTEMS*

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Abstract. A theory of strictly observable linear systems is developed in a module theoretic framework which is consistent with the classical algebraic theory of linear time invariant realization. The theory incorporates in a unified framework the reduction of linear systems through precompensation, through state feedback, and through dynamic output feedback.

1. Introduction. In Hautus and Heymann [1978] and in Hammer and Heymann [1981], the foundations for an algebraic theory of linear systems were formulated, using the linear realization theory of Kalman [1965] (see also Kalman, Falb, and Arbib [1969], Chapt. 10) as the starting point. In Hautus and Heymann [1978] emphasis has been placed on the input/state behavior and on static state feedback using the theory of $K[z]$-modules ($K[z]$ being the ring of polynomials in z over a field K). In Hammer and Heymann [1981] the theory has been extended to investigate the structure of dynamic as well as static output feedback. It has been shown there that an important role in the theory of output feedback is played by the *latency* structure and the *latency kernel* of the system. The latency structure is characterized by the class of system inputs whose corresponding outputs are identically zero prior to the time $t=0$. This structure is algebraically expressed by modules over the ring $K[[z^{-1}]]$ of power series (in z^{-1} over the field K) and led to a rich structure theory as evidenced in Hammer and Heymann [1981].

In the present paper we focus on a "dual" class of inputs, namely, those that generate outputs terminating at $t=0$. This leads to a $K[z]$-module structure and in particular to the concept of *strict observability* which is the main theme of the paper. The basic definition of strict observability in our framework is that in the above mentioned class of inputs all elements are polynomial (i.e., terminating at or before $t=0$).

The concept of strict observability is closely related to various concepts that have been studied (from various different points of view) in the literature. Probably the first time the concept appeared was in Basile and Marro [1969] and in the paper by Nikolskii [1970] who defined a linear system to be *ideally observable* if its state can be observed from knowledge of the output alone (without knowledge of the corresponding input). Nikolskii showed that ideal observability holds if and only if the observability is maintained under every static state feedback law. The same concept was introduced independently in Rappaport and Silverman [1971], who called it *perfect observability* (see also Payne and Silverman [1973]). In Heymann [1972] the concept of *feedback irreducibility* was introduced and a system was called *feedback irreducible* if its observability is invariant (i.e., indestructible) under state feedback. Irreducibility was also studied in Morse [1975], where a system was defined to be *irreducible* if the subspace v^*, i.e., the largest (A, B)-invariant subspace in the kernel of C, is zero

* Received by the editor February 24, 1981, and in revised form December 18, 1981.

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(see also Morse [1973]). Morse showed that his definition of irreducibility is equivalent to feedback irreducibility and obtained various other results on irreducible systems. The equivalence of irreducibility and strong observability was shown in Molinary [1976], and more recently the equivalence between the various concepts mentioned above was discussed in Hautus [1979], where also an important characterizing rank condition was given. Other recent related papers are Fuhrmann and Willems [1979], [1981] and Khargonekar and Emre [1980].

In this paper we study the effects of bicausal precompensation (i.e., cascade control) and of state as well as output feedback on the structural properties of linear systems, with strict observability playing a central role in the theory.

In § 2 we give some mathematical preliminaries reviewing the algebraic setup. In § 3 strict observability is formally defined and some basic consequences that follow immediately from the definition are discussed. In particular, the structure of injective precompensation orbits is investigated. Theorem 3.2 states that every injective input-output (i/o) map can be made strictly observable by bicausal precompensation, and Theorem 3.3 states the fact that an injective i/o map can be rendered strictly observable, also, by static state feedback in *every possible* realization. Theorem 3.4 summarizes the properties of strictly observable i/o maps. In § 4 the structure of bounded Ω^+K-modules is discussed. In § 5 the structure of precompensation orbits of injective i/o maps is investigated in detail. Reduced reachability indices are defined. In Theorem 5.1 a Wiener–Hopf type factorization is proved for injective i/o maps. A characterization of injective precompensation orbits based on the reduced reachability indices is given in Theorem 5.3, and a "dynamics assignment" theorem (by precompensators) is given in Theorem 5.5. Section 6 is devoted to an investigation of the effect of dynamic output feedback. Theorem 6.5 states that an injective i/o map can be made strictly observable, also, by output feedback. Theorem 6.6 gives an "index assignment" result, i.e., it states that by output feedback any admissible set of reachability indices can be attained for systems defined over infinite fields. In § 7 contact is made between the present theory and the geometric control theory of Wonham and Morse and supremal (A, B)-invariant subspaces in ker C are characterized. Finally, in § 8 generalization to noninjective i/o maps is discussed.

2. Preliminaries on the mathematical setup. The reader is assumed to be familiar with the mathematical setup and terminology of Hautus and Heymann [1978] as well as Hammer and Heymann [1981], which we review briefly.

For a field K and a K-linear space S, we denote by ΛS the set of all formal Laurent series in z^{-1} of the form

$$s = \sum_{t=t_0}^{\infty} s_t z^{-t}, \qquad s_t \in S. \tag{2.1}$$

It can then be seen that, with coefficientwise addition and convolution multiplication, the set ΛK forms a field, and under similar operations the set ΛS becomes a ΛK-linear space. When S is finite dimensional, then so is also ΛS (as a ΛK-linear space) and $\dim_{\Lambda K} \Lambda S = \dim_K S$.

The set ΛS contains as subsets the set $\Omega^+ S$ of (polynomial) elements of the form $\sum_{t\leqq 0} s_t z^{-t}$, and the set $\Omega^- S$ of (power series) elements of the form $\sum_{t\geqq 0} s_t z^{-t}$. In particular, $\Omega^+ K$ and $\Omega^- K$ form principal ideal domains under the operations defined in ΛK. Furthermore, $\Omega^+ S$ and $\Omega^- S$ are free modules over $\Omega^+ K$ and $\Omega^- K$, respectively, and in case the K-linear space S is finite dimensional, both of these modules are of rank equal to $\dim_K S$.

Let U and Y be K-linear spaces. A ΛK-linear map $\bar{f}:\Lambda U \to \Lambda Y$ represents a linear time invariant system with *input value space* U and *output value space* Y (Wyman [1972]). The *order* of the ΛK-linear map $\bar{f}$ is defined as $\operatorname{ord}\bar{f} := \inf\{\operatorname{ord}\bar{f}u - \operatorname{ord} u \mid 0 \neq u \in \Lambda U\}$, and, in case U and Y are finite dimensional, $\operatorname{ord}\bar{f} > -\infty$. Below we shall always assume that U and Y are finite dimensional and we denote

$$m := \dim_K U \quad \text{and} \quad p := \dim_K Y.$$

Further, let L denote the K-linear space of K-linear maps $U \to Y$. With every ΛK-linear map $\bar{f}:\Lambda U \to \Lambda Y$ one associates an element $\mathcal{T}_f = \sum T_t z^{-t}$ in ΛL, called the *transfer function* of $\bar{f}$. The coefficients T_k of the transfer function are given by $T_k := p_k \cdot \bar{f} \cdot i_u$, where the K-linear maps p_k and i_u are defined as

$$i_u : U \to \Lambda U : u \mapsto u \qquad \text{(injection)}$$

$$p_k : \Lambda Y \to Y : \sum y_t z^{-t} \mapsto y_k.$$

It can then be readily seen that the action of $\bar{f}$ on an element $u = \sum u_t z^{-t} \in \Lambda U$ is given by the convolution formula

$$\bar{f}u = \sum_t \left(\sum_k T_k U_{t-k} \right) z^{-t}. \tag{2.2}$$

For conciseness, we shall frequently identify ΛK-linear maps with their transfer functions.

Next, we define some terminology. Let s be an element in ΛS. Then, s is called (i) *polynomial* if $s \in \Omega^+ S$, (ii) *strictly polynomial* if $s \in z\Omega^+ S$, (iii) *causal* if $s \in \Omega^- S$, (iv) *strictly causal* if $s \in z^{-1}\Omega^- S$, (v) *static* if $s \in S$, and (vi) *rational* if there exists a nonzero polynomial $\psi \in \Omega^+ K$ such that ψs is polynomial. We denote by $\Lambda_r S$ the set of all rational elements in ΛS, so that $\Lambda_r K$ is the field of polynomial quotients, and $\Lambda_r S$ is a $\Lambda_r K$-linear space.

The above terminology also applies to ΛK-linear maps $\bar{f}:\Lambda U \to \Lambda Y$ through the respective properties of their transfer functions as elements of ΛL. Upon applying the convolution formula (2.2), it is easy to verify that $\bar{f}$ is: (i) polynomial if and only if $\bar{f}[\Omega^+ U] \subset \Omega^+ Y$, (ii) strictly polynomial if and only if $\bar{f}[\Omega^+ U] \subset z\Omega^+ Y$, (iii) causal if and only if $\bar{f}[\Omega^- U] \subset \Omega^- Y$, (iv) strictly causal if and only if $\bar{f}[\Omega^- U] \subset z^{-1}\Omega^- Y$, (v) static if and only if $\bar{f}[U] \subset Y$, and (vi) rational if and only if $\bar{f}[\Lambda_r U] \subset \Lambda_r Y$. A strictly causal and rational ΛK-linear map $\bar{f}:\Lambda U \to \Lambda Y$ is called a *linear i/o (input-output) map*. A ΛK-linear map $\bar{l}:\Lambda U \to \Lambda U$ is called *bicausal* if it is causal and has a causal inverse.

We associate with a linear i/o map $\bar{f}$ a number of related constructs. First, we define the two $\Omega^+ K$-homomorphisms

$$j^+ : \Omega^+ U \to \Lambda U \qquad \text{(natural injection)},$$

$$\pi^+ : \Lambda Y \to \Lambda Y / \Omega^+ Y (=: \Gamma^+ Y) \qquad \text{(canonical projection)}.$$

Then we associate with every linear i/o map $\bar{f}:\Lambda U \to \Lambda Y$ the $\Omega^+ K$-homomorphism

$$\tilde{f} := \pi^+ \cdot \bar{f} \cdot j^+$$

called the *restricted linear i/o map*. We also associate with $\bar{f}$ its *output value map* defined as

$$f := p_1 \cdot \bar{f} \cdot j^+ : \Omega^+ U \to Y.$$

The output value map gives the output value at (time) $t = 1$, and is, in general, (only) a K-linear map. In certain cases, there exists an Ω^+K-module structure on Y, compatible with its K-linear structure, such that the output value map f is an Ω^+K-homomorphism as well. If this is the case then $\bar{f}$ is called a *linear i/s (input-state) map.*

By a realization of a restricted linear i/o map $\tilde{f}:\Omega^+U \to \Gamma^+Y$, we refer to a triple (X, g, h), where X is an Ω^+K-module, and where $g:\Omega^+U \to X$ and $h:X \to \Gamma^+Y$ are Ω^+K-homomorphisms satisfying $\tilde{f} = h \cdot g$. The module X is called the *state space*. The realization (X, g, h) is called *reachable* if g is surjective and *observable* if h is injective. Clearly, the condition $\tilde{f} = h \cdot g$ implies that $\ker g \subset \ker \tilde{f}$. Conversely, if $\ker g \subset \ker \tilde{f}$, there exists an Ω^+K-homomorphism $h:X \to \Gamma^+Y$ such that (X, g, h) is a realization of $\tilde{f}$.

Given a realization (X, g, h), the map $g:\Omega^+U \to X$ can be viewed as the output response map of a linear i/o map $\bar{g}:\Lambda U \to \Lambda X$, which, in fact, is a linear i/s map. We say that $\bar{g}$ is *reachable* if g is surjective. Finally, if (X, g, h) is a realization of $\bar{f}$, there exists a (static) map $H:X \to Y$ such that $\bar{f} = H \cdot \bar{g}$. The last formula is called a *state representation* of $\bar{f}$.

In the present paper we shall be particularly interested in the following type of Ω^+K-modules. An Ω^+K-submodule $\Delta \subset \Lambda S$ is called *bounded* if there exists an integer $k < \infty$ such that $\operatorname{ord} \delta \leqq k$ for every nonzero element $\delta \in \Delta$. If Δ is a nonzero and bounded module, then the least integer k satisfying this order inequality is called the (*order*) *bound* of Δ. Clearly, Ω^+S itself and all its Ω^+K-submodules are examples of bounded modules. A more detailed examination of the structure of bounded modules is given in § 4 below.

3. Strict observability: Basic properties. Let $\bar{f}:\Lambda U \to \Lambda Y$ be a linear i/o map and, as before, let $\pi^+:\Lambda Y \to \Gamma^+Y$ denote the canonical projection. We introduce the following:

DEFINITION 3.1. A linear i/o map $\bar{f}:\Lambda U \to \Lambda Y$ is called *strictly observable* if $\ker \pi^+\bar{f} \subset \Omega^+U$.

It follows immediately from the definition that if $\bar{f}$ is strictly observable then $\ker \pi^+\bar{f}$ is bounded and the only ΛK-linear space contained in it is the null space. Since, obviously, $\ker \bar{f} \subset \ker \pi^+\bar{f}$, it follows that if $\bar{f}$ is strictly observable then $\bar{f}$ is *injective*, (i.e., $\ker \bar{f} = 0$). In Hammer and Heymann [1981, Lemma 5.11] it was shown that *every injective linear i/s map is strictly observable.*

Let ΛU be a fixed ΛK-linear space and consider the class of all rational bicausal ΛK-linear maps $\Lambda U \to \Lambda U$. Clearly, this class forms a (noncommutative) group under the operation of composition. Under the action of this group (with elements acting as bicausal precompensators), the class of linear i/o maps $\Lambda U \to \Lambda Y$ is partitioned into (mutually exclusive) equivalence classes called (*bicausal*) *precompensation orbits.* We next investigate these orbits.

First observe that if a linear i/o map is injective, then so is every element in its precompensation orbit. Thus, an orbit is either *injective* or *noninjective*. Since, as we have seen, strict observability implies injectivity, it follows that if a precompensation orbit contains strictly observable elements it is injective. The theorem below, the proof of which is postponed to § 5 (see Proof 5.2), states that the converse of the above statement is also true, namely that every injective orbit contains strictly observable elements.

THEOREM 3.2. *Let $\bar{f}:\Lambda U \to \Lambda Y$ be an injective linear i/o map. Then, there exists a bicausal precompensator $\bar{l}:\Lambda U \to \Lambda U$ such that $\bar{f}\bar{l}$ is strictly observable.*

Consider a reachable realization (X, g, h) of a linear i/o map $\bar{f}:\Lambda U \to \Lambda Y$; let $\bar{g}:\Lambda U \to \Lambda X$ be the i/s map associated with g, and let $\bar{f} = H \cdot \bar{g}$ be the corresponding state representation for $\bar{f}$. Also let $\bar{l}:\Lambda U\to\Lambda U$ be a bicausal precompensator for $\bar{f}$. We say that $\bar{l}$ has a *static state feedback representation* in the realization (X, g, h) if there exists a pair of static ΛK-linear maps $F:\Lambda X\to\Lambda U$ and $G:\Lambda U\to\Lambda U$, with G invertible, such that $\bar{l}=(I+F\bar{g})^{-1}G$.

In Hautus and Heymann [1978 Thm. 5.7], it was shown that $\bar{l}$ has a static state feedback representation in a reachable realization (X, g, h) if and only if $\bar{l}^{-1}[\ker\tilde{g}]\subset\Omega^+U$.

Suppose now that $\bar{f}:\Lambda U\to\Lambda Y$ is an injective linear i/o map and that $\bar{l}:\Lambda U\to\Lambda U$ is a bicausal precompensator for $\bar{f}$ such that $\bar{f}\bar{l}$ is strictly observable, that is, $\ker\pi^+\bar{f}\bar{l}\subset\Omega^+U$. Let (X, g, h) be any reachable realization of $\bar{f}$ and let $\tilde{g}$ be the restricted i/s map associated with g. Then $\ker\tilde{g}=\ker g\subset\ker\tilde{f}$ (the equality following from the i/s property), and it follows that

$$\bar{l}^{-1}[\ker\tilde{g}]\subset\bar{l}^{-1}[\ker\tilde{f}]\subset\bar{l}^{-1}[\ker\pi^+\bar{f}]=\ker\pi^+\bar{f}\bar{l}\subset\Omega^+U.$$

By the previous paragraph, we conclude that $\bar{l}$ has a static state feedback representation over $\bar{g}$ and we just proved:

THEOREM 3.3. *Let $\bar{f}:\Lambda U\to\Lambda Y$ be an injective linear i/o map and let $\bar{l}:\Lambda U\to\Lambda U$ be a bicausal precompensator such that $\bar{f}\bar{l}$ is strictly observable. Then $\bar{l}$ has a static state feedback representation in every reachable realization of $\bar{f}$.*

In Heymann [1972], a transfer matrix was called *feedback irreducible* if under the application of static state feedback in a canonical realization, the resultant closed loop system is necessarily also canonical, that is, the observability property is preserved. We shall see that strict observability is equivalent to feedback irreducibility so that Theorem 3.2 combined with Theorem 3.3 is equivalent to Theorem 6.64 in Heymann [1972].

Let $\bar{f}:\Lambda U\to\Lambda Y$ be a strictly observable linear i/o map and let $\delta(\bar{f})$ denote its McMillan degree. If $\bar{f}'$ is any other i/o map in the bicausal precompensation orbit of $\bar{f}$, then by Theorem 3.3, $\bar{f}$ can be obtained from $\bar{f}'$ by static state feedback in any reachable realization of $\bar{f}'$. It follows, therefore, that $\delta(\bar{f})\leqq\delta(\bar{f}')$, where $\delta(\bar{f}')$ is the McMillan degree of $\bar{f}'$. Thus, all strictly observable linear i/o maps in a given (injective) bicausal precomposation orbit have the same McMillan degree δ, which is the minimal degree among all McMillan degrees of elements in the orbit. Furthermore, strict observability implies feedback irreducibility. Conversely, suppose that $\bar{f}$ is a feedback irreducible linear i/o map and let $\bar{f}=H\cdot\bar{g}$ be a canonical state representation of $\bar{f}$. By Theorem 3.2, there exists a bicausal precompensator $\bar{l}$ such that $\bar{f}':=\bar{f}\bar{l}$ is strictly observable, and by Theorem 3.3, $\bar{l}$ has a static state feedback representation over $\bar{g}$. It then follows (see Hautus and Heymann [1978], Cor. 5.9) that $\bar{g}':=\bar{g}\bar{l}$ is also a reachable linear i/s map and $\ker\tilde{g}=\bar{l}^{-1}[\ker\tilde{g}]$. From the feedback irreducibility of $\bar{f}$ it follows that the state representation $\bar{f}'=H\bar{g}'$ is also canonical, whence $\ker\tilde{f}'=\ker\tilde{g}'$. Consequently,

$$\begin{aligned}\ker\pi^+\bar{f}&=\bar{l}[\ker\pi^+\bar{f}\bar{l}]=\bar{l}[\ker\tilde{f}']=\bar{l}[\ker\tilde{g}']\\&=\bar{l}\bar{l}^{-1}[\ker\tilde{g}]=\ker\tilde{g}\subset\Omega^+U,\end{aligned}$$

so that $\bar{f}$ is strictly observable. Our preceding discussion is summarized in the following:

THEOREM 3.4. *Consider the class of linear i/o maps in a fixed injective precomposition orbit. Let δ be the minimal McMillan degree of elements in the orbit. Then the following statements are equivalent*:

(i) *$\bar{f}$ is strictly observable.*

(ii) *$\bar{f}$ is feedback irreducible.*

(iii) *$\bar{f}$ has McMillan degree δ.*

(iv) *Every i/o map $\bar{f}'$ in the orbit can be transformed into $\bar{f}$ by static state feedback in any reachable realization.*

Consider now two linear i/o maps $\bar{f}_1:\Lambda U\to\Lambda Y$ and $\bar{f}_2:\Lambda U\to\Lambda W$, and assume that there exists a polynomial map $P:\Lambda Y\to\Lambda W$ such that $\bar{f}_2=P\cdot\bar{f}_1$. Then if $u\in\ker\pi^+\bar{f}_1$ (i.e., if $\bar{f}_1(u)\in\Omega^+Y$) it follows also that $\bar{f}_2(u)=P\cdot\bar{f}_1(u)\in\Omega^+W$, that is, $u\in\ker\pi^+\bar{f}_2$. We conclude then, that the existence of a polynomial map P such that $\bar{f}_2=P\cdot\bar{f}_1$ implies that $\ker\pi^+\bar{f}_1\subset\ker\pi^+\bar{f}_2$. That the converse of the above statement is also true will be shown in the ensuing discussion. First, we need the following auxiliary result (proof omitted), which is a consequence of the Smith–McMillan canonical form theorem:

LEMMA 3.5. *Let $\bar{f}:\Lambda U\to\Lambda Y$ be a rational ΛK-linear map, let $r:=\dim_{\Lambda K}\operatorname{Im}\bar{f}$ and let $Y_0\subset Y$ be any r-dimensional subspace. Then there exists a polynomial unimodular map $M:\Lambda Y\to\Lambda Y$ such that* $\operatorname{Im}M\cdot\bar{f}=\Lambda Y_0$.

We also require the following result (compare Hammer and Heymann [1981, Lem. 5.1]).

LEMMA 3.6. *Let $\bar{f}:\Lambda U\to\Lambda Y$ be a ΛK-linear map. If $\mathcal{R}\subset\ker\pi^+\bar{f}$ is a ΛK-linear subspace, then $\mathcal{R}\subset\ker\bar{f}$.*

With the above lemmas we can now state and prove the polynomial factorization theorem:

THEOREM 3.7. *Let $\bar{f}_1:\Lambda U\to\Lambda Y$ and $\bar{f}_2:\Lambda U\to\Lambda W$ be rational ΛK-linear maps. There exists a polynomial ΛK-linear map $P:\Lambda Y\to\Lambda W$ such that $\bar{f}_2=P\cdot\bar{f}_1$ if and only if* $\ker\pi^+\bar{f}_1\subset\ker\pi^+\bar{f}_2$.

Proof. That the condition of the theorem is necessary was seen in the discussion preceding Lemma 3.5. To prove sufficiency, assume that $\ker\pi^+\bar{f}_1\subset\ker\pi^+\bar{f}_2$. Let $r:=\dim_{\Lambda K}\operatorname{Im}\bar{f}_1$ and let Y_0 be any r-dimensional subspace of Y. By Lemma 3.5 there exists a unimodular polynomial map $M:\Lambda Y\to\Lambda Y$ such that $\operatorname{Im}M\cdot\bar{f}_1=\Lambda Y_0$. If we denote $\bar{f}_0:=M\bar{f}_1$, it follows immediately from the necessity condition above combined with the fact that both M and M^{-1} are polynomial maps, that $\ker\pi^+\bar{f}_0=\ker\pi^+\bar{f}_1$, whence $\ker\pi^+\bar{f}_0\subset\ker\pi^+\bar{f}_2$. Lemma 3.6 then implies that $\ker\bar{f}_0\subset\ker\bar{f}_2$ so that there exists a ΛK-linear map $P_0:\Lambda Y\to\Lambda W$ such that $P_0\cdot\bar{f}_0=\bar{f}_2$. Let $Y_1\subset Y$ be a direct summand of Y_0 in Y, that is, $Y=Y_0\oplus Y_1$. Also, let $\bar{q}:\Lambda Y\to\Lambda Y$ denote the projection onto ΛY_0 along ΛY_1, i.e., if $y=y_0+y_1\in\Lambda Y$ is the decomposition of y into its components $y_0\in\Lambda Y_0$ and $y_1\in\Lambda Y_1$, then $\bar{q}(y)=y_0$. We now define the map $P:=P_0\cdot\bar{q}\cdot M$ and for each $u\in\Lambda U$ we have

$$P\cdot\bar{f}_1(u)=P_0\cdot\bar{q}\cdot M\bar{f}_1(u)=P_0\bar{q}\bar{f}_0(u)=P_0\bar{f}_0(u)=\bar{f}_2(u),$$

whence $P\cdot\bar{f}_1=\bar{f}_2$. To conclude the proof we need to show that P is polynomial, and since by definition M is polynomial, it suffices to prove that so also is $P_0\cdot\bar{q}$. To this end, we first note that every element $y\in\Omega^+Y$ decomposes uniquely as $y=y_0+y_1$ with $y_0\in\Omega^+Y_0$ and $y_1\in\Omega^+Y_1$. Thus

$$P_0\bar{q}(y)=P_0(y_0)=P_0\bar{q}(y_0)=P_0\bar{q}M\bar{f}_1(u)=P\cdot\bar{f}_1(u)=\bar{f}_2(u)$$

for some $u\in\ker\pi^+\bar{f}_1$. Since by hypothesis $\ker\pi^+\bar{f}_1\subset\ker\pi^+\bar{f}_2$, it follows that $\bar{f}_2(u)=P_0\bar{q}(y)\in\Omega^+W$, and the proof is complete. □

COROLLARY 3.8. *Let $\bar{f}_1,\bar{f}_2:\Lambda U\to\Lambda Y$ be two rational ΛK-linear maps. There exists a unimodular polynomial map $M:\Lambda Y\to\Lambda Y$ such that $\bar{f}_2=M\bar{f}_1$ if and only if* $\ker\pi^+\bar{f}_1=\ker\pi^+\bar{f}_2$.

Proof. Necessity follows immediately from Theorem 3.7. To prove sufficiency, assume that $\ker\pi^+\bar{f}_1=\ker\pi^+\bar{f}_2$. Then by Lemma 3.6, $\ker\bar{f}_1=\ker\bar{f}_2$ so that

$\dim \operatorname{Im} \bar{f}_1 = \dim \operatorname{Im} \bar{f}_2 := r$. Let $Y_0 \subset Y$ be an r-dimensional K-linear subspace and let $M_1, M_2 : \Lambda Y \to \Lambda Y$ be unimodular polynomial maps such that $\operatorname{Im} M_1 \bar{f}_1 = \operatorname{Im} M_2 \bar{f}_2 = \Lambda Y_0$ (see Lemma 3.5). Denoting $\bar{f}_{10} := M_1 \bar{f}$ and $\bar{f}_{20} := M_2 \bar{f}_2$, we obviously also have $\ker \pi^+ \bar{f}_{10} = \ker \pi^+ \bar{f}_{20}$. By Theorem 3.7, there exist then polynomial maps $P_{10}, P_{20} : \Lambda Y \to \Lambda Y$ such that $\bar{f}_{20} = P_{10} \bar{f}_{10}$ and $\bar{f}_{10} = P_{20} \bar{f}_{20}$. Let $Y_1 \subset Y$ be a direct summand of Y_0 in Y and let $\bar{q} : \Lambda Y \to \Lambda Y$ be the projection defined in the proof of Theorem 3.7. Now define the polynomial maps $P_1 = \bar{q}(P_{10} - I)\bar{q} + I$ and $P_2 = \bar{q}(P_{20} - I)\bar{q} + I$ where I is the identity map in ΛY. Clearly then also $\bar{f}_{20} = P_1 \cdot \bar{f}_{10}$ and $\bar{f}_{10} = P_2 \cdot \bar{f}_{20}$, and also $P_2 \cdot P_1 = P_1 \cdot P_2 = I$. (The reader can verify these facts by direct computation.) It follows that P_1 is unimodular and the unimodular map $M := M_2^{-1} P_1 M_1$ satisfies the condition of the corollary. $\square$

We conclude the section with an additional characterization of strict observability.

COROLLARY 3.9. *Let $\bar{f} : \Lambda U \to \Lambda Y$ be a linear i/o map. Then $\bar{f}$ is strictly observable if and only if it has a polynomial left inverse.*

Proof. First observe that if $I : \Lambda U \to \Lambda U$ is the identity map, then $\ker \pi^+ I = \Omega^+ U$. Thus, be definition, $\bar{f}$ is strictly observable if and only if $\ker \pi^+ \bar{f} \subset \ker \pi^+ I$. By Theorem 3.7 this kernel inclusion holds if and only if there is a polynomial map $P : \Lambda Y \to \Lambda U$ such that $\mathrm{P} \cdot \bar{f} = I$, concluding the proof. $\square$

4. Bounded Ω^+K-modules. Let $f : \Lambda U \to \Lambda Y$ be an injective linear i/o map, say of order k. It is then readily seen that $\ker \pi^+ \bar{f}$ is a bounded Ω^+K-submodule of ΛU and its order bound is less than or equal to $(-)k$. Indeed, if $u \neq 0$ has order greater than $(-)k$, then $\operatorname{ord} \bar{f}(u) \geqq \operatorname{ord} \bar{f} + \operatorname{ord} u > 0$, whence $\bar{f}(u) \in z^{-1}\Omega^- Y$, and since $\bar{f}(u) \neq 0$ it follows that $u \notin \ker \pi^+ \bar{f}$.

In the present section we shall study the structure of bounded Ω^+K-submodules of ΛU. We emphasize again that U is a finite dimensional K-linear space. The structure of bounded Ω^+K-submodules of ΛU is essentially the same as that of submodules of $\Omega^+ U$, which was discussed in some detail in Hautus and Heymann [1978, § 6] and also in Forney [1975].

Let $\Delta \subset \Lambda U$ be a bounded Ω^+K-submodule with order bound k^Δ, and for each integer j, let $S_j \subset U$ be the K-linear space spanned by the leading coefficients $\hat{u} \in U$ of all elements $u \in \Delta$ which satisfy $\operatorname{ord} u \geqq j$. In this way, we obtain a chain of K-linear spaces

$$U \supset \cdots \supset S_{j-1} \supset S_j \supset \cdots \supset S_{k^\Delta - 1} \supset S_{k^\Delta} \supset S_{k^\Delta + 1} = 0 \tag{4.1}$$

Now, by the finite dimensionality of U, there exists an integer $k_\Delta (\leqq k^\Delta)$ such that $S_{k_\Delta} \neq S_{k_\Delta + 1}$ and $S_{k_\Delta - j} = S_{k_\Delta}$ for all $j > 0$. We call the chain $\{S_j\}$ the *order chain* of Δ and the nonincreasing sequence of integers $\{\mu_j\}$, $\mu_j := \dim S_j$, we call the *order list* of Δ. In the special case when $\Delta = \ker \pi^+ \bar{f}$, where $\bar{f}$ is a linear i/o map, we refer to the order chain and the order list of Δ, respectively, as the *reduced reachability chain* and the *reduced reachability list of $\bar{f}$.*

PROPOSITION 4.2. *Let $\Delta, \Delta' \subset \Lambda U$ be bounded Ω^+K-submodules with order chains $\{S_j\}$ and $\{S'_j\}$ and order lists $\{\mu_j\}$ and $\{\mu'_j\}$, respectively.*

(i) *If $\Delta' \subset \Delta$ then for each integer j, $S'_j \subset S_j$ and $\mu'_j \leqq \mu_j$.*

(ii) *If $\Delta' \subset \Delta$ and for each integer j, $\mu'_j = \mu_j$, then $\Delta' = \Delta$.*

Proof. (i) This is an immediate consequence of the preceding discussion.

(ii) If $\Delta' \subset \Delta$, then the equalities $\mu'_j = \mu_j$ imply that $S'_j = S_j$ for all j, and if $u \in \Delta$ is any element, there exists an element $u' \in \Delta'$ such that $\operatorname{ord}(u - u') > \operatorname{ord} u$. Further, $u - u' \in \Delta$ so that by the same argument, there is an element $u'' \in \Delta'$ such that $\operatorname{ord}(u - u' - u'') > \operatorname{ord}(u - u')$. Proceeding stepwise, we finally find elements $u', u'', \cdots, u^r \in \Delta'$

such that $\operatorname{ord}(u-u'-u''-\cdots-u^r)>k^\Delta$, where k^Δ is the order bound of Δ. Since $u-u'-\cdots-u^r\in\Delta$, we conclude that $u-u'-\cdots-u^r=0$, whence $u=u'+u''+\cdots+u^r\in\Delta$, so that also $\Delta\subset\Delta'$, and we conclude that $\Delta=\Delta'$, as claimed. □

We turn now to a brief review of some results on proper bases for ΛK-linear spaces and Ω^+K-modules. A set of elements $u_1,\cdots,u_k\in\Lambda U$ is called *properly independent* if and only if their leading coefficients $\hat{u}_1,\cdots,\hat{u}_k\in U$ are K-linearly independent. A basis for a subspace $\mathcal{R}\subset\Lambda U$ is called a *proper basis* if it consists of properly independent elements. If $u_1,\cdots,u_k\in\Lambda U$ is a properly independent set of vectors, then it is also ΛK-linearly independent, as was shown e.g. in Hammer and Heymann [1981, Lem. 4.2], where also the following characterization of proper independence was proved. (See also Forney [1975].)

LEMMA 4.3. *A set of nonzero elements $u_1,\cdots,u_k\in\Lambda U$ is properly independent if and only if for every set of scalars $\alpha_1,\cdots,\alpha_k\in\Lambda K$, or alternatively, if and only if for every set of scalars $\alpha_1,\cdots,\alpha_k\in\Omega^+K$, the following holds*:

$$\operatorname{ord}\sum_{i=1}^{k}\alpha_iu_i=\min\{\operatorname{ord}\alpha_iu_i\,|\,i=1,\cdots,k\}.$$

Proper bases play a role in the theory of causal ΛK-linear maps analogous to the role of bases in general in the theory of linear maps. In particular, let $\bar{f}:\Lambda U\to\Lambda Y$ be a ΛK-linear map and let $u_1,\cdots,u_m$ be a basis for ΛU. If $\bar{f}$ acts causally on every element u_i, that is $\operatorname{ord}f(u_i)\geqq\operatorname{ord}u_i$, it is not necessarily implied that $\bar{f}$ is a causal map. Yet, if $u_1,\cdots,u_m$ is a proper basis, the causality of $\bar{f}$ is assured. This is shown in the following proposition (see also Wolovich [1974]):

PROPOSITION 4.4. *Let $u_1,\cdots,u_m$ be a proper basis for the ΛK-linear space ΛU and let $\bar{f}:\Lambda U\to\Lambda Y$ be a ΛK-linear map. Then $\bar{f}$ is causal if and only if* $\operatorname{ord}\bar{f}(u_i)\geqq\operatorname{ord}u_i$ *for all $i=1,\cdots,m$.*

Proof. The "only if" part is true by definiton. To prove the "if" part, assume $\operatorname{ord}\bar{f}(u_i)\geqq\operatorname{ord}u_i$, $i=1,\cdots,m$, let $0\neq u\in\Lambda U$ be any element and write $u=\sum_{i=1}^m\alpha_iu_i$ for appropriate scalars $\alpha_i\in\Lambda K$, $i=1,\cdots,m$. Then,

$$\begin{aligned}\operatorname{ord}\bar{f}(u)=\operatorname{ord}\sum_{i=1}^{m}\alpha_i\bar{f}(u_i)&\geqq\min\{\operatorname{ord}\alpha_i\bar{f}(u_i)|i=1,\cdots,m\}\\&\geqq\min\{\operatorname{ord}\alpha_iu_i|i=1,\cdots,m\}=\operatorname{ord}u,\end{aligned}$$

where the last step is by Lemma 4.3. Thus $\bar{f}$ is causal. □

Through a similar application of Lemma 4.3, we also have the following:

COROLLARY 4.5. *Let $u_1,\cdots,u_m$ be a proper basis for ΛU and let $l:\Lambda U\to\Lambda U$ be a ΛK-linear map. Then $\bar{l}$ is bicausal if and only if the following conditions both hold*:

(i) $\operatorname{ord}\bar{l}(u_i)=\operatorname{ord}u_i$, $i=1,\cdots,m$, *and*

(ii) *The set $\bar{l}(u_1),\cdots,\bar{l}(u_m)$ is a proper basis for ΛU.*

A basis $u_1,\cdots,u_m$ of an Ω^+K-module $\Delta\subset\Lambda U$ is called *proper* if $u_1,\cdots,u_m$ are properly independent, and it will be called *ordered* if $\operatorname{ord}u_i\geqq\operatorname{ord}u_{i+1}$ for all $i=1,2,\cdots,m-1$.

THEOREM 4.6. *Let $\Delta\subset\Lambda U$ be a bounded Ω^+K-submodule with order chain $\{S_j\}$ and order list $\{\mu_j\}$. Then* (i) *there exists an ordered proper basis for Δ.* (ii) *If $u_1,\cdots,u_m$ is an order proper basis for Δ, then the following hold*:

$$\operatorname{ord}u_j=i\text{ for }\mu_{i+1}<j\leqq\mu_i\text{ and }i\leqq k^\Delta;\tag{4.7}$$

(4.8) *For each integer $i\leqq k^\Delta$, the set of leading coefficients $\hat{u}_1,\cdots,\hat{u}_{\mu_i}$ forms a basis for S_i.*

Proof. The proof is essentially the same as that of Theorem 6.11 in Hammer and Heymann [1981], which deals with proper bases for $\Omega^- K$-submodules of ΛU. We shall therefore give only an outline. In view of the chain property of the S_i, there exists a set $u_1^0, \cdots, u_m^0$ of vectors in U (where $m = \mu_{k_\Delta} = \operatorname{rank} \Delta$) such that for each $k_\Delta \leqq i \leqq k^\Delta$, $u_1^0, \cdots, u_{\mu_i}^0$ is a basis for S_i. Then, for each $k_\Delta \leqq i \leqq k^\Delta$ and each $\mu_{i+1} < j \leqq \mu_i$, there is an element $u_j \in \Delta$ having order i and leading coefficient $\hat{u}_j = u_j^0$. Obviously, the set $u_1, \cdots, u_m$ is properly independent and the $\Omega^+ K$-module Δ' generated by $u_1, \cdots, u_m$ satisfies $\Delta' \subset \Delta$. That actually $\Delta' = \Delta$ follows upon application of Proposition 4.2 (ii). Hence $u_1, \cdots, u_m$ is an ordered proper basis for Δ and satisfies conditions (4.7) and (4.8) by construction . Finally, that each ordered proper basis has these properties follows from the fact that for each integer j, every ordered proper basis $u_1, \cdots, u_m$ of Δ has precisely μ_j elements whose order is greater than or equal to j and $\operatorname{span}_K \{\hat{u}_1, \cdots, \hat{u}_k\} = S_j$. □

Let $\Delta, \Delta' \subset \Lambda U$ be $\Omega^+ K$-submodules. An $\Omega^+ K$-homomorphism $q : \Delta \to \Delta'$ is called *order preserving* if $\operatorname{ord} q(u) = \operatorname{ord} u$ for each $0 \neq u \in \Delta$. If an order preserving q is surjective it is obviously an isomorphism, and we call it in this case an *order (preserving) isomorphism*. The submodules Δ and Δ' are then called *order isomorphic* (compare with the polynomial case in Hautus and Heymann [1978]).

PROPOSITION 4.9. *Let $\Delta, \Delta' \subset \Lambda U$ be bounded $\Omega^+ K$-submodules. Then Δ and Δ' are order isomorphic if and only if they have the same order lists.*

Proof. If Δ and Δ' are order isomorphic, then it follows directly from Theorem 4.6 and Corollary 4.5 that they have the same order lists. Conversely, assume that the bounded modules Δ and Δ' are nonzero and have the same order lists. Then, by Theorem 4.6, the following hold: (i) Δ and Δ' have ordered proper bases $u_1, \cdots, u_m$ and $u'_1, \cdots, u'_{m'}$, respectively, (ii) $m' = m$, and (iii) $\operatorname{ord} u'_i = \operatorname{ord} u_i$ for all $i = 1, \cdots, m$. By Hammer and Heymann [1981, Thm. 4.4], there exist then elements $u_{m+1}, \cdots, u_n$ and $u'_{m+1}, \cdots, u'_n$ such that both of the sets $u_1, \cdots, u_n$ and $u'_1, \cdots, u'_n$ form proper bases of ΛU, and $\operatorname{ord} u'_i = \operatorname{ord} u_i$ for all $i = 1, \cdots, n$. But then, the ΛK-linear map $\bar{l} : \Lambda U \to \Lambda U$ defined through its values as $\bar{l} u_i = u'_i$, $i = 1, \cdots, n$, is bicausal by Corollary 4.5, and, since evidently $\bar{l}[\Delta] = \Delta$, our proof is complete. □

It will be convenient in the sequel to define for a bounded $\Omega^+ K$-module $\Delta \subset \Lambda U$ of rank m, a set of integers $\{\nu_1, \cdots, \nu_m\}$ called the *degree indices* of Δ, as follows. Let $u_1, \cdots, u_m$ be an ordered proper basis of Δ and for each $i = 1, \cdots, m$ define $\nu_i = -\operatorname{ord} u_i$. The relationship between the degree indices and the order list of Δ is established by Theorem 4.6 through (4.7) and (4.8), as follows:

$$\nu_j = -i \quad \text{for } \mu_{i+1} < j \leqq \mu_i, \quad i \leqq k^\Delta. \tag{4.10}$$

An $\Omega^+ K$-submodule $\Delta \subset \Lambda U$ is called *full* if it contains a basis for ΛU. In case Δ is a bounded module, then, clearly, Δ is full if and only if $\operatorname{rank}_{\Omega^+ K} \Delta = \dim U$.

5. The precompensation orbit of injective i/o maps. In the present section we shall study the structure of the $\Omega^+ K$-module $\ker \pi^+ \bar{f}$ for injective linear i/o maps. We shall also investigate the structural invariants of bicausal precompensation orbits.

It is well known from linear realization theory (see e.g. Fuhrmann [1976]) that in view of the rationality of $\bar{f}$, $\ker \pi^+ \bar{f} j^+$ is a full submodule of ΛU. It follows then immediately, since $\ker \pi^+ \bar{f} j^+ \subset \ker \pi^+ \bar{f}$, that $\ker \pi^+ \bar{f}$ is also full.

Let $\bar{f} : \Lambda U \to \Lambda Y$ be an injective linear i/o map. We define the *reduced reachability indices* $\{\nu_1, \cdots, \nu_m\}$ of $\bar{f}$ as the degree indices of $\ker \pi^+ \bar{f}$. We observe that in view of the strict causality of $\bar{f}$, the ν_i are all positive integers. Indeed, if $0 \neq u \in \ker \pi^+ \bar{f}$, then $0 \neq \bar{f}(u) \in \Omega^+ Y$ and $\operatorname{ord} u < \operatorname{ord} \bar{f}(u) \leqq 0$.

Consider now an injective linear i/o map $\bar{f}:\Lambda U \to \Lambda Y$ and let $\bar{l}:\Lambda U \to \Lambda U$ be a bicausal precompensator for $\bar{f}$. Clearly $\bar{l}$ is also an order preserving Ω^+K-isomorphism on ΛU, and since $\bar{l}[\ker \pi^+\bar{f}\bar{l}] = \ker \pi^+\bar{f}$ we see (in view of Proposition 4.9) that the reduced reachability list or, equivalently, the set of reduced reachability indices, is an orbital invariant of bicausal precompensation. Combining this fact with Corollary 3.8, we obtain the following central factorization result:

THEOREM 5.1. *Let $f, f':\Lambda U \to \Lambda U$ be injective linear i/o maps with reduced reachability indices $\{\nu_1, \cdots, \nu_m\}$ and $\{\nu'_1, \cdots, \nu'_m\}$, respectively. Then $\nu_i = \nu'_i$, $i = 1, \cdots, m$ if and only if there exists a polynomial unimodular map $M:\Lambda Y \to \Lambda Y$ and a bicausal precompensator $\bar{l}:\Lambda U \to \Lambda U$ such that $\bar{f}' = M\bar{f}\bar{l}$.*

Proof. If $\bar{f}'M\bar{f}\bar{l}$ with M polynomial unimodular and $\bar{l}$ bicausal, then, by Corollary 3.8, $\ker \pi^+\bar{f}' = \ker \pi^+\bar{f}\bar{l}$, whence $\bar{l}[\ker \pi^+\bar{f}'] = \ker \pi^+\bar{f}$, and by Proposition 4.9, $\ker \pi^+\bar{f}'$ and $\ker \pi^+\bar{f}$ have the same order lists (or equivalently the same reduced reachability indices). Conversely, if $\nu_i = \nu'_i$ for $i = 1, \cdots, m$, then $\ker \pi^+\bar{f}$ and $\ker \pi^+\bar{f}'$ have the same order lists, and by Proposition 4.9 are order isomorphic. Thus, there exists a bicausal isomorphism on ΛU, say $\bar{l}^{-1}$, such that $\ker \pi^+\bar{f}' = \bar{l}^{-1}\ker \pi^+\bar{f} = \ker \pi^+\bar{f}\bar{l}$. By Corollary 3.8 there exists then a unimodular polynomial map $M:\Lambda Y \to \Lambda Y$ such that $\bar{f}' = M\bar{f}\bar{l}$, concluding the proof. □

A factorization of the type obtained in Theorem 5.1 is sometimes called in the literature a Wiener–Hopf factorization (compare Fuhrmann and Willems [1979]).

Before proceeding with our discussion, we turn to the proof of Theorem 3.2, which is an immediate consequence of Theorem 5.1.

Proof 5.2. *Proof of Theorem* 3.2. Assume that $\bar{f}$ has reduced reachability indices $\{\nu_1, \cdots, \nu_m\}$. The injectivity of $\bar{f}$ implies that $r := \dim Y \geqq m (= \dim U)$. Let $\bar{f}':\Lambda U \to \Lambda Y$ be the ΛK-linear map whose transfer matrix is given by

$$\mathcal{T}_{\bar{f}'} = \begin{bmatrix} z^{-\nu_1} & & 0 \\ & \ddots & \\ 0 & & z^{-\nu_m} \\ \hline & 0 & \end{bmatrix}.$$

Clearly $\bar{f}'$ is strictly observable and has the same reduced reachability indices $\{\nu_1, \cdots, \nu_m\}$ as $\bar{f}$. Theorem 5.1 then implies that $\bar{f} = M\bar{f}'\bar{l}$ for some polynomial unimodular map M and a bicausal ΛK-linear map $\bar{l}$. Then the map $\bar{l}^{-1}$ is a bicausal precompensator for $\bar{f}$ and the map $\bar{f}'' = \bar{f}\bar{l}^{-1}$ is strictly observable since $\bar{f}'' = M\bar{f}'$, and by Corollary 3.8 $\ker \pi^+\bar{f}'' = \ker \pi^+\bar{f}' (\subset \Omega^+U)$. □

We conclude this section with a discussion of the problem of "dynamics assignment" through bicausal precompensation. That is, we ask to what extent it is possible to modify a system's essential dynamic characteristics through the application of bicausal precompensator.

We first recall some classical concepts. Let $\bar{f}:\Lambda U \to \Lambda Y$ be a linear i/o map and let $\tilde{f} := \pi^+ \cdot \bar{f} \cdot j^+$ be the restricted i/o map associated with $\bar{f}$. The Ω^+K-submodule $\Delta := \ker \tilde{f} = \ker \pi^+\bar{f} \cap \Omega^+U$, called the *realization kernel* (*or realization module*) of $\bar{f}$, uniquely defines the class of all canonical realizations of $\bar{f}$ (see e.g. Hautus and Heymann [1978]). In particular, let $X_\Delta := \Omega^+U/\Delta$, let $g_\Delta := \Omega^+U \to X_\Delta$ be the canonical projection and let $h_\Delta : X_\Delta \to \Lambda Y/\Omega^+Y$ denote the (unique) Ω^+K-homomorphism such that $\tilde{f} = g_\Delta \cdot h_\Delta$. Then $(X_\Delta, g_\Delta, h_\Delta)$ is a canonical realization of $\tilde{f}$. Thus, the realization kernel Δ characterizes the essential dynamical properties of $\bar{f}$ and its *reachability indices* are the degree indices of the realization kernel Δ. (The reachability indices are, of course, the well known *Kronecker invariants* of canonical realizations of $\bar{f}$—see also Hautus and Heymann [1978], Kalman [1971] and Kailath [1980]).

Let $\bar{f}: \Lambda U \to \Lambda Y$ be an injective i/o map with reachability indices $\{\sigma_1, \cdots, \sigma_m\}$ and reduced reachability indices $\{\nu_1, \cdots, \nu_m\}$. Since, clearly, $\ker \tilde{\bar{f}} = \ker \pi^+ \bar{f} j^+ \subset \ker \pi^+ \bar{f}$, it follows upon application of Proposition 4.2 (i) and formula (4.10), that $\sigma_i \geqq \nu_i$ for $i = 1, \cdots, m$. We have seen previously that the reduced reachability indices are orbital invariants for injective orbits of precompensation and they are shared by all i/o maps in the orbit. This, in particular, holds also for the strictly observable i/o maps. If $\bar{f}$ is strictly observable, then $\ker \pi^+ \bar{f} \subset \Omega^+ U$, whence $\ker \pi^+ \bar{f} = \ker \pi^+ \bar{f} j^+$, implying that the reduced reachability indices of $\bar{f}$ coincide with its reachability indices, that is, $\sigma_i = \nu_i$, $i = 1, \cdots, m$. Conversely, suppose an i/o map $\bar{f}$ in the precompensation orbit satisfies $\sigma_i = \nu_i$, $i = 1, \cdots, m$. Then $\ker \pi^+ \bar{f} = \ker \pi^+ \bar{f} j^+$ (see Proposition 4.2 (ii)), and it follows that $\ker \pi^+ \bar{f} \subset \Omega^+ U$, implying that $\bar{f}$ is strictly observable. We just proved the following:

THEOREM 5.3. *Consider a fixed injective bicausal precompensation orbit O and let $\{\nu_1, \cdots, \nu_m\}$ denote the reduced reachability indices of elements in O. Consider an i/o map $\bar{f} \in O$ with reachability indices $\{\sigma_1, \cdots, \sigma_m\}$. Then* (i) $\sigma_i \geqq \nu_i$, $i = 1, \cdots, m$; (ii) $\sigma_i = \nu_i$, $i = 1, \cdots, m$ *if and only if $\bar{f}$ is strictly observable.*

In § 3 we saw that the McMillan degrees of i/o maps in an injective bicausal precompensation orbit are bounded below by the McMillan degree of the strictly observable i/o maps in the orbit. Since the McMillan degree of an i/o map is equal to the sum of its reachability indices, this result is of course contained in Theorem 5.3, which gives a much stronger minimality result.

Before we proceed further, we wish to make a few remarks on the explicit construction of $\ker \pi^+ \bar{f}$ and the computation of the reduced reachability indices. Suppose $\bar{f}$ is an injective linear i/o map with transfer matrix $\mathcal{T} = \mathcal{T}(z^{-1})$. Then $\operatorname{rank} \mathcal{T} = m$ and we let $\psi = \psi(z)$ denote the least common denominator of the entries of $\mathcal{T}$. Then $\psi \cdot \mathcal{T}$ is a polynomial matrix and there exists a unimodular polynomial matrix M, such that $M(\psi \cdot \mathcal{T}) = \left[\begin{smallmatrix} D \\ 0 \end{smallmatrix}\right]$, where D is a nonsingular polynomial matrix. Hence $M \cdot \mathcal{T} = \left[\begin{smallmatrix} \psi^{-1} D \\ 0 \end{smallmatrix}\right]$, and we claim that $\ker \pi^+ \bar{f} = \psi \cdot D^{-1} \Omega^+ U$. Indeed, $u \in \ker \pi^+ \bar{f}$ if and only if $\bar{f}(u) = \mathcal{T} \cdot u \in \Omega^+ Y$ (where we do not distinguish sharply between the map and its associated transfer matrix). But, since M is a unimodular polynomial matrix, $\mathcal{T} \cdot u \in \Omega^+ Y$ if and only if $M \cdot \mathcal{T} \cdot u \in \Omega^+ Y$, which in turn holds if and only if $\psi^{-1} D u \in \Omega^+ Y$. Now, $\ker \pi^+ \bar{f}$ is a full bounded submodule of ΛU and hence has an ordered proper basis $\{d_1, \cdots, d_m\}$. The reduced reachability indices of $\bar{f}$ are then $\{\nu_1, \cdots, \nu_m\}$ where $\nu_i = -\operatorname{ord} d_i$. Finally, we note that upon defining the matrix $D_+ := [d_1, \cdots, d_m]$, we can also write $\ker \pi^+ \bar{f} = D^+ \Omega^+ U$, whence there exists a unimodular polynomial matrix N such that $(\psi^{-1} D) N = D_+$.

LEMMA 5.4. *Let K be an infinite field and let $\Delta \subset \Omega^+ U$ be a full $\Omega^+ K$-submodule with order indices $\{\sigma_1, \cdots, \sigma_m\}$, $(\sigma_i \leqq \sigma_{i+1})$. Further, let $\{\nu_1, \cdots, \nu_m\}$, $(\nu_i \leqq \nu_{i+1})$, be a set of positive integers such that $\nu_i \leqq \sigma_i$, $i = 1, \cdots, m$. Let Y be a K-linear space such that* $\dim Y = r \geqq m$. *Then there exists an injective linear i/o map $\bar{f}: \Lambda U \to \Lambda Y$ with reduced reachability indices $\{\nu_1, \cdots, \nu_m\}$ and* $\ker \tilde{\bar{f}} = \Delta$.

Proof. Let $d_1, \cdots, d_m$ be an ordered proper basis for Δ and define the matrix $D := [d_1, \cdots, d_m]$. Let $\alpha \in K$ be any element which is not a zero of $\det D$. (Such an α exists since K is infinite.) For each $i = 1, \cdots, m$ let $\delta_i := \sigma_i - \nu_i$ and define the $(m \times m)$-matrix $D_0 := \operatorname{diag}((z-\alpha)^{\delta_1}, \cdots, (z-\alpha)^{\delta_m})$. We now let $\bar{f}: \Lambda U \to \Lambda Y$ be the K-linear map whose transfer matrix is defined by

$$\mathcal{T} = \begin{bmatrix} D_0 & D^{-1} \\ & 0 \end{bmatrix},$$

where the zero submatrix is $(m-r) \times m$ and may be empty. To see that $\bar{f}$ has the desired properties, note first that D_0 and D are right coprime and $\ker \tilde{f} = D\Omega^+U$ (see also Hautus and Heymann [1978]). Furthermore, $\ker \pi^+\bar{f} = DD_0^{-1}\Omega^+U$, whence it follows that the set $(z-\alpha)^{-\delta_1}\cdot d_1, \cdots, (z-\alpha)^{-\delta_m}\cdot d_m$ forms a proper basis for $\ker \pi^+\bar{f}$, and since $\operatorname{ord}[(z-\alpha)^{-\delta_i}d_i] = -\nu_i$, the proof is complete. □

THEOREM 5.5. *Let K be an infinite field and consider an injective bicausal precompensation orbit O with reduced reachability indices $\{\nu_1, \cdots, \nu_m\}$. Let $\Delta \subset \Omega^+U$ be a full Ω^+K-submodule with order indices $\{\sigma_1, \cdots, \sigma_m\}$. There exists an i/o map $\bar{f} \in O$ such that* $\ker \tilde{f} = \Delta$ *if and only if $\sigma_i \geqq \nu_i$, $i = 1, \cdots, m$.*

Proof. Necessity follows from Theorem 5.3 (i). To see the sufficiency, let $\sigma_i \geqq \nu_i$, $i = 1, \cdots, m$. By Lemma 5.4 there exists an injective linear i/o map f_0, (not necessarily in O), which has $\{\nu_1, \cdots, \nu_m\}$ as reduced reachability indices and $\ker \tilde{f}_0 = \Delta$. Let $\bar{f}_1$ be any i/o map in O. By Theorem 5.1 there exist then a unimodular polynomial map M and a bicausal ΛK-linear map $\bar{l}$ such that $f_1 = M\bar{f}_0\bar{l}$. Thus, $\bar{f} := \bar{f}_1\bar{l}^{-1} = M\bar{f}_0$, where $\bar{f} \in O$, and by Corollary 3.8 $\ker \pi^+\bar{f} = \ker \pi^+\bar{f}_0$, concluding the proof. □

It is noteworthy that the requirement of infinite fields in Theorem 5.5 and Lemma 5.4 is an essential one. To demonstrate this fact, consider the following elementary example. Let K be the field of integers modulo 2. Let $\dim Y = \dim U = 1$ and consider as realization kernel the module $\Delta = z(z+1)\Omega^+U$. The degree index of Δ is $\sigma = 2$, but Δ cannot be (canonically) associated with precompensation orbits whose reduced reachability index is $\nu = 1$.

6. Strict observability and output feedback. In § 3 we have seen that every injective i/o map can be rendered strictly observable by static state feedback. The main result of the present section is that every injective i/o map can be rendered strictly observable also by application of (dynamic) causal output feedback.

We begin with some preliminaries. Let $\bar{f}: \Lambda U \to \Lambda Y$ be a linear i/o map and let $\bar{l}: \Lambda U \to \Lambda U$ be a bicausal precompensator for $\bar{f}$. We shall say that $\bar{l}$ is *$\bar{f}$-causal* if there exist a causal ΛK-linear (ouput feedback) map $\bar{g}: \Lambda Y \to \Lambda U$, and an invertible static map $V: \Lambda U \to \Lambda U$ such that $\bar{l} = (I + \bar{g}\bar{f})^{-1}V$. Similarly, we shall say that $\bar{l}$ is *$\bar{f}$-polynomial* if there exist a polynomial ΛK-linear map $\bar{g}: \Lambda Y \to \Lambda U$ and an invertible static map $V: \Lambda U \to \Lambda U$ such that $\bar{g}\cdot\bar{f}$ is strictly causal and $\bar{l} = (I + \bar{g}\bar{f})^{-1}V$. Denoting $f' := \bar{f}\bar{l}$, we obtain through a simple calculation that if $\bar{l}$ is $\bar{f}$-polynomial, then $\bar{l}^{-1}$ is $\bar{f}'$ polynomial, and if $\bar{l}$ is $\bar{f}$-causal, then $\bar{l}^{-1}$ is $\bar{f}'$-causal.

While it is always true that $\ker \pi^+\bar{f} = \bar{l}[\ker \pi^+\bar{f}']$, it is in general not true that a similar formula relates $\ker \tilde{f}'$ $(\subset \ker \pi^+\bar{f}')$ with $\ker \tilde{f}$ $(\subset \ker \pi^+\bar{f})$. An exception to this general situation is given in the following:

LEMMA 6.1. *Let $\bar{f}: \Lambda U \to \Lambda Y$ be a linear i/o map, let $\bar{l}: \Lambda U \to \Lambda U$ be a bicausal precompensator and let $\bar{f}' := \bar{f}\bar{l}$. If $\bar{l}$ is $\bar{f}$-polynomial, then* $\ker \tilde{f} = \bar{l}[\ker \tilde{f}']$.

Proof. If $u \in \ker \tilde{f}$ then $u \in \Omega^+U$ and $\bar{f}(u) \in \Omega^+Y$. Since $\bar{g}$ is a polynomial map, it then also follows that $\bar{g}\bar{f}(u) \in \Omega^+U$. Thus, $\bar{l}^{-1}(u) = V^{-1}(I + \bar{g}\bar{f})u \in \Omega^+U$. Moreover $\bar{f}'\bar{l}^{-1}(u) = \bar{f}\bar{l}\bar{l}^{-1}(u) = \bar{f}(u) \in \Omega^+Y$. Hence $\bar{l}^{-1}(u) \in \ker \tilde{f}'$ (or $u \in \bar{l}[\ker \tilde{f}']$) and consequently $\ker \tilde{f} \subset \bar{l}[\ker \tilde{f}']$. The inverse inclusion follows similarly from the fact that $\bar{l}^{-1}$ is $\bar{f}'$-polynomial, and the lemma follows. □

Combining Lemma 6.1 with Proposition 4.9, we obtain

THEOREM 6.2. *Let $\bar{f}: \Lambda U \to \Lambda Y$ be a linear i/o map, let $\bar{l}: \Lambda U \to \Lambda U$ be a bicausal precompensator and write $\bar{f}' = \bar{f}\cdot\bar{l}$. If $\bar{l}$ is $\bar{f}$-polynomial then $\bar{f}$ and $\bar{f}'$ have the same sets of reachability indices.*

Consider now an injective linear i/o map $\bar{f}: \Lambda U \to \Lambda Y$, and let $\bar{l}: \Lambda U \to \Lambda U$ be a bicausal precompensator for $\bar{f}$. Clearly, in view of the injectivity of $\bar{f}$, there exist a ΛK-linear map $\bar{g}: \Lambda Y \to \Lambda U$ and an invertible static map $V: \Lambda U \to \Lambda U$ such that

$\bar{l}=(I+\bar{g}\bar{f})^{-1}V$ and $\bar{g}\bar{f}$ is strictly causal. Next, let $\bar{g}=\bar{g}^-+\bar{g}^+$ where $\bar{g}^-$ is causal and $\bar{g}^+$ is (strictly) polynomial. Then $\bar{g}^-\cdot\bar{f}$ is obviously strictly causal, and so also is $\bar{g}^+\cdot\bar{f}$, being the difference of two strictly causal maps. Thus, we have the following:

$$\begin{aligned}\bar{l}&=(I+\bar{g}\bar{f})^{-1}V=(I+\bar{g}^-\bar{f}+\bar{g}^+\bar{f})^{-1}V\\&=(I+\bar{g}^-\bar{f})^{-1}[I+\bar{g}^+\bar{f}(I+\bar{g}^-\bar{f})^{-1}]^{-1}V=\bar{l}^-\cdot\bar{l}^+,\end{aligned}\tag{6.3}$$

where $\bar{l}^-:=(I+\bar{g}^-\bar{f})^{-1}$ is a bicausal precompensator for $\bar{f}$ and is $\bar{f}$-causal, and where $\bar{l}^+:=[I+\bar{g}^+\bar{f}(I+\bar{g}^-\bar{f})^{-1}]^{-1}V=(I+\bar{g}^+(\bar{f}\bar{l}^-))^{-1}V$ is a bicausal precompensator for $\bar{f}\bar{l}^-$ and is $(\bar{f}\bar{l}^-)$-polynomial. If we now apply Theorem 6.2, we conclude that the maps $\bar{f}\bar{l}$ and $\bar{f}\bar{l}^-$ have the same sets of reachability indices, the important fact being that $\bar{l}^-$ represents a (dynamic) causal output feedback around $\bar{f}$. This proves the following:

THEOREM 6.4. *Let $\bar{f}:\Lambda U\rightarrow\Lambda Y$ be an injective linear i/o map and assume that $\bar{l}:\Lambda U\rightarrow\Lambda U$ is a bicausal precompensator for $\bar{f}$ such that $\bar{f}\bar{l}$ is rational and has reachability indices $\sigma_1,\cdots,\sigma_m$. Then there exists a causal ΛK-linear map $\bar{g}:\Lambda Y\rightarrow\Lambda U$ such that $\bar{f}(I+\bar{g}\bar{f})^{-1}$ also has reachability indices $\sigma_1,\cdots,\sigma_m$.*

As an immediate consequence of Theorems 3.2 and 6.4, we have the following result:

THEOREM 6.5. *Let $\bar{f}:\Lambda U\rightarrow\Lambda Y$ be an injective linear i/o map. Then $\bar{f}$ can be transformed into a strictly observable map by application of causal (dynamic) output feedback.*

Finally, upon application of Theorem 6.4 to Theorem 5.5, we obtain

THEOREM 6.6. *Let K be an infinite field and let $\bar{f}:\Lambda U\rightarrow\Lambda Y$ be an injective i/o map with reduced reachability indices $\nu_1,\cdots,\nu_m$. For every set of integers $\sigma_1,\cdots,\sigma_m$ satisfying $\sigma_i\geqq\nu_i$, $i=1,\cdots,m$, there exists a causal ΛK-linear map $\bar{g}:\Lambda Y\rightarrow\Lambda U$ such that $\bar{f}(I+\bar{g}\bar{f})^{-1}$ has $\sigma_1,\cdots,\sigma_m$ as reachability indices.*

7. Some further properties of ker $\pi^+\bar{f}$. In the present section, we wish to make formal contact between the present theory and some concepts that appeared in the linear system theory literature. In particular, we wish to make contact with concepts from the geometric theory as expounded by Wonham and Morse (see e.g. Wonham [1979]). It will be assumed that the reader is familiar with the basic concepts of that theory, and with the basic algebraic framework of linear realization theory (as presented, e.g., in Hautus and Heymann [1978]).

Let $\bar{f}:\Lambda U\rightarrow\Lambda Y$ be a linear i/o map and let (X,g,h) be a reachable realization of $\bar{f}$ (i.e., $\pi^+\bar{f}j^+=h\cdot g$, and $g:\Omega^+U\rightarrow X$ is surjective). The *unobservable subspace* (submodule) of (X,g,h) is defined as $\ker h\subset X$ and we say that (X,g,h) is *observable* if $\ker h=0$, i.e., if h is injective. Let $\bar{g}:\Lambda U\rightarrow\Lambda X$ denote the (extended) i/s map associated with g and let $\bar{f}=H\bar{g}$ be the corresponding state representation (i.e., $H=p_1\cdot h$).

We recall that a subspace $S\subset X$ is called *weakly invariant* if the controlled trajectory for every $x\in S$ can be maintained in S by choice of control action. Weakly invariant subspaces coincide with the well known (A,B)-invariant spaces of geometric linear system theory (see in particular Hautus [1979] for comparison of the various concepts). Of particular interest is the maximal weakly invariant subspace contained in $\ker H$, which is frequently denoted in the literature by v^*. We shall show below that v^* is related to $\ker\pi^+\bar{f}$ and, in particular, that $v^*=p_1\bar{g}[\ker\pi^+\bar{f}]$.

The Ω^+K-module $\ker\pi^+\bar{f}$ consists of the class of all inputs for which the corresponding output is identically zero for all $t\geqq 1$. Let $u\in\ker\pi^+\bar{f}$ be any control and write $u=u^++u^-$, where $u^+\in\Omega^+U$ and $u^-\in z^{-1}\Omega^-U$. Then $0=p_1\bar{f}u=p_1H\bar{g}u=$

$Hp_1\bar{g}u = Hp_1\bar{g}u^+ + Hp_1\bar{g}u^-$ and, in view of the strict causality of $\bar{g}$, $p_1\bar{g}u^- = 0$ and we have $p_1\bar{g}u = p_1\bar{g}u^+ \in \ker H$. The state $p_1\bar{g}u^+ = gu^+ \in X$ is the state at time $t = 1$ generated by the control u^+. This state is maintained in $\ker H$ by application (after $t = 0$) of the control u^-, and hence it is clear that $p_1\bar{g}u \in v^*$, so that $p_1\bar{g}[\ker \pi^+\bar{f}] \subset v^*$. To see that the inverse inclusion also holds, let $x \in v^*$ be any state. In view of the reachability of (X, g, h), there exists $u^+ \in \Omega^+U$ such that $x = gu^+ = p_1\bar{g}u^+$. Further, there exists $u^- \in z^{-1}\Omega^-U$ such that the corresponding state trajectory (starting at x) remains in $\ker H$, i.e., the output trajectory is identically zero. Thus, $p_k\bar{f}(u^+ + u^-) = 0$ for all $k \geqq 1$, whence $u = u^+ + u^- \in \ker \pi^+\bar{f}$. We summarize the above discussion in the following:

THEOREM 7.1. *Ler (X, g, h) be a reachable realization of a liner i/o map $\bar{f} : \Lambda U \to \Lambda Y$ and let $\bar{f} = H \cdot \bar{g}$ be the corresponding (reachable) state representation. Then the maximal weakly invariant subspace in* $\ker H$ *is given by*

$$v^* = p_1\bar{g}[\ker \pi^+\bar{f}]. \tag{7.2}$$

We shall next investigate several properties of v^* and its relation to unobservability and feedback. First, the following can be readily verified.

LEMMA 7.3. *Let $\bar{f} : \Lambda U \to \Lambda Y$ be a linear i/o map and let $\bar{f} = H\bar{g}$ be a state representation for $\bar{f}$. Then,*

(i) $p_1\bar{g}[\ker \pi^+\bar{f}] \subset \ker H$.

(ii) *If $\Delta \subset \Lambda U$ is an Ω^+K-module satisfying $p_1\bar{g}[\Delta] \subset \ker H$, then $\Delta \subset \ker \pi^+\bar{f}$.*

Consider now the reachable realization (X, g, h) and let $\bar{f} = H \cdot \bar{g}$ be the associated state representation. Clearly, the unobservable subspace $S = \ker h$ satisfies $S \subset \ker H$, and it is easily seen that, in fact, S is the *maximal* Ω^+K-module contained in $\ker H$. Let us apply static state feedback $F : X \to U$ in the reachable realization (X, g, h) (see Hautus and Heymann [1978] for details). Then the reachable extended linear i/s map $\bar{g} : \Lambda U \to \Lambda X$ is transformed into the reachable i/s map $\bar{g}_F := \bar{g}(I + F\bar{g})^{-1}$, and the i/o map $\bar{f}$ is transformed into $\bar{f}_F := \bar{f}(I + F\bar{g})^{-1}$ (so that $\bar{f}_F = H\bar{g}_F$). Let $g_F := p_1 \cdot \bar{g}_F \cdot j^+$ be the output response map associated with $\bar{g}_F$. Then there is a reachable realization (X_F, g_F, h_F) of $\bar{f}_F$, and we denote the unobservable subspace of this realization by $S_F := \ker h_F$. We then have the following theorem, which gives a sharp insight into the nature of the subspace $v^*(= \bar{p}_1\bar{g}[\ker \pi^+\bar{f}])$.

THEOREM 7.4. *Let $\bar{f} : \Lambda U \to \Lambda Y$ be an injective linear i/o map, let (X, g, h) be a reachable realization and let $\bar{f} = H \cdot \bar{g}$ be the associated state representation. Then the following hold:*

(i) *For every static state feedback $F : X \to U$, $S_F \subset p_1\bar{g}[\ker \pi^+\bar{f}]$.*

(ii) *There exists a static state feedback $F_0 : X \to U$ (for which $\bar{f}_{F_0}$ is strictly observable) such that $S_{F_0} = p_1\bar{g}[\ker \pi^+\bar{f}]$.*

Proof. (i) The reachability of (X, g, h) implies that, for each feedback F, the realization (X_F, g_F, h_F) is also reachable (see e.g. Hautus and Heymann [1978]). Hence g_F is surjective and there is an Ω^+K-module $\Delta \subset \Omega^+U$ such that $S_F = g_F[\Delta] = p_1\bar{g}_F[\Delta]$ and, since $S_F \subset \ker H$, it follows by Lemma 7.3 that $\Delta \subset \ker \pi^+\bar{f}_F$. Thus, denoting $\bar{l} := (I + F\bar{g})^{-1}$, we obtain

$$S_F = p_1\bar{g}_F[\Delta] \subset p_1\bar{g}_F[\ker \pi^+\bar{f}_F] = p_1\bar{g}\bar{l}[\ker \pi^+\bar{f}\bar{l}] = p_1\bar{g}\bar{l}\bar{l}^{-1}[\ker \pi^+\bar{f}] = p_1\bar{g}[\ker \pi^+\bar{f}],$$

as claimed.

(ii) By Theorem 3.3, there exists an F_0 such that $\bar{f}_{F_0}$ is strictly observable (i.e., $\ker \pi^+\bar{f}_{F_0} \subset \Omega^+U$), so that $g_{F_0}[\ker \pi^+\bar{f}_{F_0}] \subset X$ is an Ω^+K-module. Then since $g_F[\ker \pi^+\bar{f}_{F_0}] = p_1\bar{g}_{F_0}[\ker \pi^+\bar{f}_{F_0}] = p_1\bar{g}[\ker \pi^+\bar{f}]$, it follows that $p_1\bar{g}[\ker \pi^+\bar{f}]$ $(= p_1\bar{g}_{F_0}[\ker \pi^+\bar{f}_{F_0}])$ is an Ω^+K-module in $\ker H$, so that $p_1\bar{g}[\ker \pi^+\bar{f}] \subset S_{F_0}$. Combining this with (i) above, we have that $p_1\bar{g}[\ker \pi^+\bar{f}] = S_{F_0}$. □

In the special case when $\bar{f}$ is a strictly observable i/o map, we have that $p_1\bar{g}[\ker \pi^+\bar{f}]=g[\ker g]=0$, implying that for every static state feedback $F:X\to U$, $S_F=0$. Thus, the observability is preserved under state feedback, in agreement with Theorem 3.4.

8. Remarks on noninjective i/o maps. We turn now to some observations and comments on noninjective i/o maps. If $\bar{f}:\Lambda U\to\Lambda Y$ is a linear i/o map, we say that $\bar{f}$ has a *static kernel* if there exists a K-linear subspace $U_0\subset U$ such that $\ker\bar{f}=\Lambda U_0$. If $\ker\bar{f}$ is static, $\bar{f}$ can be made injective by simple restriction of the input value space. The noninjectivity of $\bar{f}$ then stems from the fact that its input value space was chosen to be too large. We proceed now to extend the framework of our theory to noninjective i/o maps.

In Hammer and Heymann [1981, Prop. 5.6], it was shown that a linear i/s map always has a static kernel. Consider now a linear i/s map $\bar{f}:\Lambda U\to\Lambda Y$ and assume that $\ker\bar{f}=\Lambda U_0$ for a subspace $U_0\subset U$. Choose a direct sum complement $U_1\subset U$ for U_0 such that $U=U_0\oplus U_1$ and let $P_1:U\to U_1$ denote the projection of U onto U_1 along U_0. There evidently exists then an injective i/s map $\bar{f}_1:\Lambda U_1\to\Lambda Y$ such that

$$\bar{f}=\bar{f}_1P_1. \tag{8.1}$$

The above restriction procedure, and the fact that injective i/s maps are always strictly observable, motivate us in extending the concept of strict observability to noninjective i/o maps as follows:

DEFINITION 8.2. A linear i/o map $\bar{f}:\Lambda U\to\Lambda Y$ is called *extended strictly observable* if the following conditions hold:

(i) $\bar{f}$ has a static kernel $\Lambda U_0\subset\Lambda U$.

(ii) There exists a subspace $U_1\subset U$ such that $U_1\oplus U_0=U$ and a strictly observable i/o map $f_1:\Lambda U_1\to\Lambda Y$ such that $\bar{f}=\bar{f}_1\cdot P_1$, where $P_1:U\to U_1$ is the projection onto U_1 along U_0.

The following theorem generalizes Theorem 3.2 to noninjective linear i/o maps.

THEOREM 8.3. *Let $\bar{f}:\Lambda U\to\Lambda Y$ be a linear i/o map. There exists a bicausal precompensator $\bar{l}:\Lambda U\to\Lambda U$ such that $\bar{f}\bar{l}$ is extended strictly observable.*

The proof of Theorem 8.3 depends on the following:

LEMMA 8.4. *Let $\bar{f}:\Lambda U\to\Lambda Y$ be a linear i/o map. There exists a bicausal precompensator $\bar{l}:\Lambda U\to\Lambda U$ such that $\bar{f}\bar{l}$ has a static kernel.*

The proof of Lemma 8.4 depends on (and is an easy consequence of) the existence of proper bases for ΛK-linear spaces as discussed in Hammer and Heymann [1981]. The details of the proof are omitted.

Proof 8.5. *Outline of proof of Theorem* 8.3. By Lemma 8.4, there exists a bicausal precompensator $\bar{l}:\Lambda U\to\Lambda U$ such that the map $\bar{f}':=\bar{f}\bar{l}$ has a static kernel ΛU_0. There exists then a direct sum complement ΛU_1 to ΛU_0 and an injective i/o map $\bar{f}'':\Lambda U_1\to\Lambda Y$ such that $\bar{f}'=\bar{f}''\cdot P_1$, where $P_1:U\to U_1$ is the projection onto U_1 along U_0. By Theorem 3.2 there exists a bicausal precompensator $\bar{l}_2:\Lambda U_1\to\Lambda U_1$ such that $\bar{f}_1:=\bar{f}''\cdot\bar{l}_2$ is strictly observable. Finally, it can be shown that $\bar{l}_2$ can be extended to a bicausal ΛK-linear map $\bar{l}_3:\Lambda U\to\Lambda U$ such that $\bar{l}_2P_1=P_1\bar{l}_3$, and we have $\bar{f}_1\cdot P_1=\bar{f}''\bar{l}_2P_1=\bar{f}''P_1\bar{l}_3=\bar{f}'\bar{l}_3=\bar{f}(\bar{l}_1\bar{l}_3)$, concluding the proof. □

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