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State feedback, confinement and stabilization for non-linear continuous-time systems

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A theory of static state feedback for multivariable continuous-time non-linear systems is formulated. The theory applies to systems described by differential equations of the form $\dot{x}(t) = f(x(t), u(t))$. The basic objective is to design static feedback compensators which achieve the following properties: (*a*) the state-space trajectory of the closed-loop system is confined within a specified subspace, and (*b*) the closed-loop system is internally stable. An explicit method for designing such compensators is developed. The construction of the compensators involves only quantities directly derived from the given function f.

1. Introduction

In general terms, the problem of confinement deals with the design of non-linear compensators which, when connected in a closed loop around a given non-linear system Σ, yield an internally stable system whose output vectors are confined to a specified subdomain of the output space. Let $\mathscr{Y}$ be the output space of the given system Σ and let $\mathscr{V}$ be a suitable subspace of $\mathscr{Y}$. The basic objective is to construct an internally stable closed-loop control configuration around Σ so that all possible output vectors $y(t)$ of the closed loop at the time t satisfy $y(t) \in \mathscr{V}$ for all $t \geqslant 0$. Presently, we assume that there is a coordinate transformation of the output space $\mathscr{Y}$ under which the subspace $\mathscr{V}$ transforms into a rectangular box V. We denote by Σ the system obtained from Σ after this coordinate transformation and, to simplify notation, we regard Σ as the given system. The rectangular confinement problem then reduces to the following.

Let Σ be a given non-linear system, let $y(t)$ denote its output vector at the time t and let n be the dimension of $y(t)$. Also, let $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ be specified real numbers, where $\alpha_i < \beta_i$ for all $i = 1, \ldots, n$. Design an internally stable closed loop configuration around Σ whose output vector $y(t)$ at the time t satisfies $\alpha_i < y_i(t) < \beta_i$ for all $i = 1, \ldots, n$ and all $t \geqslant 0$.

In practical applications of control theory, the confinement problem is invariably encountered; for instance, in almost every design of a control system, it has to be ensured that the amplitudes of the output components do not exceed the specifications of the physical devices on which they appear. Compliance with these specifications is then a rectangular confinement problem. Another practical example of confinement can be presented through the design problem of an autopilot for an aeroplane. As is well known, excessive manoeuvres of an aeroplane may reduce its lift force below the critical level, causing the aeroplane to stall and fall. This danger can be avoided through the use of confinement control to guarantee that the steering surfaces of the aeroplane never tilt beyond the safety range for each level

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of airspeed. Confinement is also of critical importance in the control of biological systems. Consider a biological system consisting of n cells, each of which can be described by k dynamical variables, so that the entire system has nk dynamical state variables. The cells grow and each cell may either divide into two new cells or die. For the sake of demonstration, assume that the division or death of a cell is controlled by one of its dynamical variables, say x_j, so that division of the cell occurs when $x_j > \beta$ and death of the cell occurs when $x_j < \alpha$, where $\alpha < \beta$ are real numbers. Then, in order to prevent the cell from dying or dividing, namely in order to stabilize the number of cells, it is necessary to design a controller guaranteeing that $\alpha < x_j < \beta$ for all appropriate state variables x_j. Thus, the control of biological systems (with reproduction and death) involves the confinement problem in a fundamental way. Note that the problem of controlling such biological systems is of critical importance, since it is related to the restraint of cancerous phenomena, many of which may be regarded as manifestations of instabilities in the reproductive process of cells. We shall discuss the control of biological systems in detail in a separate report.

The present paper presents an implementable solution to the problem of rectangular confinement (with internal stabilization) for non-linear continuous-time systems which have their state as output. The solution is based on the use of static-state feedback. Consider a non-linear system Σ described by a differential equation of the form

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0 \tag{1.1}$$

where $x(t)$ is an n-dimensional real vector describing the output of the system at the time t and $u(t)$ is an m-dimensional real vector describing the input of the system at the time t. The function f is assumed to be continuously differentiable. A representation of the form (1.1) is usually called a state representation of the system Σ and $x(t)$ is identified as the state of the system at the time t. A system that can be represented in the form (1.1) is called an input-state system. Assume for now that (1.1) has a unique solution $x(t)$, $t \geqslant 0$, for all relevant initial conditions x_0 and input functions $u(t)$, $t \geqslant 0$. The system Σ is enclosed in a static state-feedback loop of the form shown in the Figure, where σ is a continuously differentiable function representing the feedback and v represents an external input. The closed-loop system described by the diagram is denoted by Σ_σ and it is still an input–state system, with a state representation given by

$$\dot{x}(t) = f(x(t), \sigma(x(t), v(t))) \tag{1.2}$$

In § 3 we derive necessary and sufficient conditions for the existence of continuously differentiable feedback functions σ which internally stabilize the closed loop (see the Figure) while providing the desired rectangular confinement of the output vector x. Furthermore, whenever such feedback functions exist, a procedure for their computation is outlined.

The results presented in this paper are, in part, a continuous-time version of the results on non-linear discrete-time systems derived by Hammer (1989 b). Technically, however, the details of the present theory are substantially different, owing to the obvious differences between continuous-time and discrete-time systems. Alternative recent investigations on the stabilization of non-linear control systems can be found in the work of Hammer (1984, 1989 a, b, c), Desoer and Kabuli (1988) and Sontag (1989).

2. Basic considerations

Denote by $\mathbb{R}^n$ the set of all n-dimensional real column vectors, and consider an input–state system Σ described by the differential equation

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0 \tag{1.1}$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ for all $t \geqslant 0$, and $f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a function known as the state representation function of the system Σ. Throughout the present discussion it is assumed that the state representation function $f(x, u)$ is continuously differentiable for all relevant x and u. For the sake of simplicity, we assume that the system Σ is time invariant, namely, that the function f does not explicitly depend on the time variable t. The class of non-linear continuous-time systems described by differential equations of the form (1.1) includes most input–state systems encountered in engineering practice.

The system Σ described by (1.1) is to be controlled by inserting it into the closed loop configuration (see the Figure). Here, $\sigma : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m : (x, v) \mapsto \sigma(x, v)$ is again a continuously differentiable function, serving as the state feedback. A few words regarding the dependence of the feedback function σ on the external input variable v are in order. When a function $\sigma(x, v)$ is considered, there is no guarantee in general that it really depends on the variable v. We can then distinguish between two extreme cases;

(*a*) the case where σ does not depend at all on v, in which case $\sigma(x, v) = \sigma(x)$; and

(*b*) the case where $\sigma(x, v)$ is injective in v for every state x.

The first case yields a pure feedback configuration with no external input v, whereas the second case yields a reversible feedback configuration (Hammer 1989 a). Of course, all other intermediate ways of the dependence of $\sigma(x, v)$ on v are also possible, but it seems that in the context of control theory the two extreme cases are the most important ones. The first case corresponds to situations where the closed-loop system operates on its own with no operator, so no external input is present. The second case is of fundamental significance to the theory of fraction representations of non-linear systems (Hammer 1989 a, b, c). The present paper is concerned mainly with case (*a*). Here, the desired characteristics of the closed-loop system are achieved entirely by the feedback $\sigma(x)$, which generates the input of the system Σ within the closed loop through $u(t) = \sigma(x(t))$ and no external input v is provided. In this case, the feedback function σ is simply a continuously differentiable function $\mathbb{R}^n \to \mathbb{R}^m : x \mapsto \sigma(x)$, and the differential equation describing the closed-loop system Σ_σ becomes

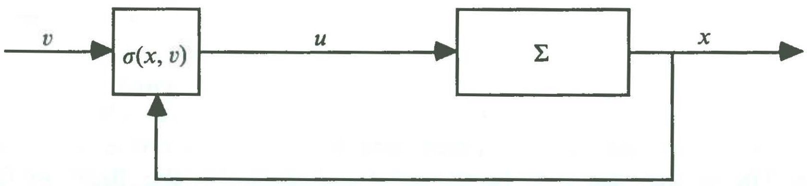
$$\dot{x}(t) = f(x(t), \sigma(x(t))), \quad x(0) = x_0 \tag{2.1}$$

We are now in a position to state the basic technical problem considered in the present paper. First, some notation. The ith component of a vector $x \in \mathbb{R}^n$ is denoted by x_i. Denote by $[-\theta, \theta]^n$, where $\theta > 0$ is a real number, the set of all vectors $x \in \mathbb{R}^n$ for which $|x_i| \leqslant \theta$ for all $i = 1, ..., n$. Further, let $\mathbb{R}^+$ be the set of all non-negative real numbers. This set will serve as our time set, so that the response of a system Σ is simply a function $x : \mathbb{R}^+ \to \mathbb{R}^n$. Denote by $C(\mathbb{R}^n)$ the set of all continuous functions $h : \mathbb{R}^+ \to \mathbb{R}^n$. For a real number $\theta > 0$, let $C(\theta^m)$ be the set of all functions $u \in C(\mathbb{R}^m)$ satisfying $|u_i(t)| \leqslant \theta$ for all $t \geqslant 0$ and all $i = 1, ..., m$, namely, the set of all continuous functions bounded by θ. Next, denote by x^{T} the transpose of a vector $x \in \mathbb{R}^n$. Given two vectors $\alpha, x \in \mathbb{R}^n$, where $\alpha = (\alpha_1 \ldots \alpha_n)^T$ and $x = (x_1 \ldots x_n)^T$, let $x \geqslant \alpha$ (respectively, $x > \alpha$) indicate that $x_i \geqslant \alpha_i$ (respectively, $x_i > \alpha_i$) for all $i = 1, \ldots, n$. For two vectors $\alpha, \beta \in \mathbb{R}^n$, where $\alpha < \beta$, denote by $[\alpha, \beta]$ (respectively, (α, β)) the set of all vectors $x \in \mathbb{R}^n$ satisfying $\alpha_i \leqslant x_i \leqslant \beta_i$ (respectively, $\alpha_i < x_i < \beta_i$) for all $i = 1, \ldots, n$. Also, denote by $C((\alpha, \beta))$ the set of all functions $x \in C(\mathbb{R}^n)$ satisfying $x(t) \in (\alpha, \beta)$ for all $t \geqslant 0$. More generally, for a subset $S \subset \mathbb{R}^n$, let $C(S)$ be the set of all functions $x \in C(\mathbb{R}^n)$ satisfying $x(t) \in S$ for all $t \geqslant 0$. For the sake of clarity, we refer to the problem of rectangular confinement within the domain (α, β) as (α, β)-confinement.

Definition 1: *(α, β)-confinement*

Let Σ be an input–state system described by the differential equation (1.1), where the state representation function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuously differentiable. Let $\alpha, \beta \in \mathbb{R}^n$, where $\alpha < \beta$, be two prescribed fixed vectors, and let $\theta > 0$ be a real number. Find a continuous feedback function $\sigma : [\alpha, \beta] \times [-\theta, \theta]^m \to \mathbb{R}^m, (x, v) \mapsto \sigma(x, v)$, which is continuously differentiable over $(\alpha, \beta) \times [-\theta, \theta]^m$, and for which the following holds. The differential equation (1.2) has a unique solution $x(t)$, $t \geqslant 0$, for any initial condition $x_0 \in (\alpha, \beta)$ and any input function $v \in C(\theta^m)$, and this solution satisfies $\alpha < x(t) < \beta$ for all $t \geqslant 0$.

The problem of (α, β)-confinement by pure state feedback refers to (α, β)-confinement where the feedback function σ does not depend on the variable v, so that $\sigma : [\alpha, \beta] \to \mathbb{R}^m, x \mapsto \sigma(x)$, and no external input is provided for the closed loop system (see the Figure).



A critical ingredient of the problem of rectangular confinement is the fact that the initial condition x_0 of the system Σ is not known in advance and may be any vector within the domain (α, β). For any such initial condition, a unique solution of the differential equation (1.2) describing the closed-loop system is required to exist, and this solution must be confined to the domain (α, β) at all times. Note that the open domain (α, β) is used here in order to permit the incorporation of disturbances; when a disturbance $v(t)$ is added to $x(t)$, the sum $x(t) + v(t)$ is still required to be within the domain $[\alpha, \beta]$ of the feedback function σ, since otherwise the closed-loop system is not well defined. Thus, the function $x(t)$ cannot take values on the boundary of $[\alpha, \beta]$, and must be confined to the interior (α, β). Recalling a comment made in § 1, we emphasize that when rectangular confinement is combined with an appropriate coordinate transformation, it facilitates the confinement of the closed-loop system response to rather general subspaces in state space, and not just to subspaces of the form (α, β).

The present paper presents necessary and sufficient conditions for the existence of a solution σ to the problem of (α, β)-confinement by pure state feedback. When feedback functions σ solving the (α, β)-confinement problem for the system Σ exist, an explicit method for their computation is also described. An important aspect of the solution to the (α, β)-confinement problem presented here is the fact that the necessary and sufficient conditions for the existence of σ, as well as the construction of σ, depend only on quantities directly derived from the given state representation function f of Σ. Thus, the conditions are explicitly verifiable and the construction of σ is implementable. In general, the existence of a solution to the (α, β)-confinement problem for a given system Σ depends, among others, on the specific choice of the bounds α and β; a solution may exist only for some choices of these bounds. The necessary and sufficient conditions derived below can also be used to find the set of bounds α and β for which a solution exists, if there are such bounds.

Furthermore, as we point out later, the solution of the (α, β)-confinement problem also yields internal stabilization of the given system Σ. Consequently, the current results include a theory of internal stabilization by static state feedback, valid for a very general class of non-linear continuous-time systems. The fact that our theory of stabilization is closely linked to the solution of the confinement problem is really a substantial advantage from a practical standpoint, since, in engineering practice, problems of stabilization and confinement are in many cases inseparable. Usually, amplitude bounds at various points of a designed system have to be guaranteed as part of the stabilization process, to avoid physical damage to components.

The basic notion of stability used in the present paper is in the spirit of the Lyapunov notion of stability and is related to continuity in appropriate normed spaces. In order to discuss the notion of stability, we introduce some norms. The usual L^{∞}-norm on $\mathbb{R}^n$ is denoted by $|\cdot|$, and is given by the maximal absolute value of the coordinates $|x| := \max\{|x_1|, \ldots, |x_n|\}$, where $x \in \mathbb{R}^n$ is a vector with the components $x_1, \ldots, x_n$. The L^{∞}-norm on $C(\mathbb{R}^n)$ is also denoted by $|\cdot|$, and is given by

$$|h| := \operatorname*{Sup}_{t \geqslant 0} |h(t)|$$

for a function $h \in C(\mathbb{R}^n)$. Roughly speaking, we shall regard a system Σ as a map that transforms input functions from $C(\mathbb{R}^m)$ into output functions in $C(\mathbb{R}^n)$. The stability of a system Σ is related to its continuity as such a map. The notion of continuity that we use for systems is with respect to a weighted L^{∞}-norm ρ on $C(\mathbb{R}^n)$, which is given by

$$\rho(h) := \operatorname*{Sup}_{t \geqslant 0} 2^{-t}|h(t)| \tag{2.2}$$

for a function $h \in C(\mathbb{R}^n)$.

In order to examine the norm ρ, suppose for a moment that we are interested in the response of our systems only over a finite interval of time, say $[0, T]$, where $T > 0$ is a fixed real number. Let $C_T(\mathbb{R}^n)$ be the set of all continuous functions $h : [0, T] \to \mathbb{R}^n$. Denote by

$$|h| := \operatorname*{Sup}_{t \in [0,T]} |h(t)|$$

the L^{∞}-norm on $C_T(\mathbb{R}^n)$, and by

$$\rho(h) := \operatorname*{Sup}_{t \in [0,T]} 2^{-t}|h(t)|$$

the norm ρ on $C_T(\mathbb{R}^n)$. It is easy to see that on $C_T(\mathbb{R}^n)$ the norm ρ is equivalent to the L^∞-norm. Indeed, let $h \in C_T(\mathbb{R}^n)$ be any function. Then

$$\operatorname*{Sup}_{t\in[0,T]} 2^{-t}|h(t)| \leqslant \operatorname*{Sup}_{t\in[0,T]} |h(t)|$$

and

$$2^T \operatorname*{Sup}_{t\in[0,T]} 2^{-t}|h(t)| = \operatorname*{Sup}_{t\in[0,T]} 2^{T-t}|h(t)| \geqslant \operatorname*{Sup}_{t\in[0,T]} |h(t)|$$

so that $\rho(h) \leqslant |h| \leqslant 2^T\rho(h)$ and $2^{-T}|h| \leqslant \rho(h) \leqslant |h|$ on $C_T(\mathbb{R}^n)$. Using well-known properties of normed spaces, these inequalities imply the equivalence of the two norms $|\cdot|$ and ρ on $C_T(\mathbb{R}^n)$. Since this is true for every finite $T > 0$, we arrive at the following qualitative conclusion: the two norms ρ and $|\cdot|$ on $C(\mathbb{R}^n)$ differ only at the point $t = \infty$, the main difference being that a function which is bounded with respect to the norm ρ is not necessarily bounded with respect to the norm $|\cdot|$ over the infinite time axis $[0, \infty]$. However, for functions that are known to be L^∞-bounded over the entire time axis $[0, \infty]$, the difference between the two norms ρ and $|\cdot|$ is quite minor from a practical standpoint, since they are equivalent over all finite time intervals and since in practical situations only the response over finite time intervals is relevant. Furthermore, although the norm ρ is time-dependent, it is easy to see that continuity with respect to it is not affected by finite time shifts. Thus, with little if any compromise of practical significance, we can replace the standard definition of stability, which requires continuity with respect to the L^∞-norm, with the requirement of continuity with respect to the norm ρ combined with a separate L^∞-boundedness requirement. As we shall see, this replacement yields a substantial simplification of the theory of stabilization for non-linear continuous-time systems. The resulting notion of stability, which is introduced shortly, is analogous to the notion of stability for discrete-time systems used by Hammer (1984, 1989 a, b, c).

Consider a non-linear system Σ described by a differential equation of the form (1.1). Assume the differential equation has a unique solution $x(t)$, $t \geqslant 0$, for any relevant initial condition x_0 and input function u. Formally, we regard Σ as a map $\Sigma : \mathbb{R}^n \times C(\mathbb{R}^m) \to C(\mathbb{R}^n)$ which assigns to each pair $(x_0, u) \in \mathbb{R}^n \times C(\mathbb{R}^m)$ an output function $x \in C(\mathbb{R}^n)$, where $x_0 \in \mathbb{R}^n$ is the initial condition and $u \in C(\mathbb{R}^m)$ is the input function. Then, given a subset $A \subset \mathbb{R}^n \times C(\mathbb{R}^m)$, let $\Sigma\{A\}$ be the image of the set A through Σ, namely, the set of all output functions generated by the system Σ from elements of A.

Definition 2

A system $\Sigma : \mathbb{R}^n \times C(\mathbb{R}^m) \to C(\mathbb{R}^n)$ described by the differential equation (1.1) is bounded input/bounded output stable (BIBO-stable) if the following conditions hold:

(*a*) for every initial condition $x_0 \in \mathbb{R}^n$ and every input function $u \in C(\mathbb{R}^m)$, (1.1) has a unique solution $x(t)$, $t \geqslant 0$;

(*b*) for every pair of real numbers $\omega, \theta > 0$, there exists a real number $M > 0$ such that $\Sigma\{[-\omega, \omega]^n \times C(\theta^m)\} \subset C(M^n)$.

Sometimes, the initial conditions of the system Σ are known to be restricted to a prescribed subset $S \subset \mathbb{R}^n$ and its input functions are known to be restricted to a subset $C \subset C(\mathbb{R}^m)$. We shall indicate such restrictions simply by writing

$\Sigma : S \times C \rightarrow C(\mathbb{R}^n)$. Then, a system $\Sigma : S \times C \rightarrow C(\mathbb{R}^n)$ described by (1.1) is BIBO-stable if (1.1) has a unique solution $x(t)$, $t \geqslant 0$, for all initial conditions $x_0 \in S$ and all input functions $u \in C$; and if for every pair of real numbers $\omega, \theta > 0$ there exists a real number $M > 0$ such that $\Sigma\{\{S \cap [-\omega, \omega]^n\} \times \{C \cap C(\theta^m)\}\} \subset C(M^n)$.

Next, define the norm ρ on the space $\mathbb{R}^n \times C(\mathbb{R}^m)$ by setting

$$\rho(x, u) := |x| + \rho(u) \tag{2.3}$$

for all $x \in \mathbb{R}^n$ and all $u \in C(\mathbb{R}^m)$. The same symbol ρ is used here to simplify notation. The notion of stability employed in the present paper is then defined as follows.

Definition 3

A system $\Sigma : \mathbb{R}^n \times C(\mathbb{R}^m) \rightarrow C(\mathbb{R}^n)$ is stable if it is BIBO-stable and if, for every pair of real numbers $\omega, \theta > 0$, its restriction $\Sigma : [-\omega, \omega]^n \times C(\theta^m) \rightarrow C(\mathbb{R}^n)$ is a continuous function (with respect to the norm ρ).

Similarly, a system $\Sigma : S \times C \rightarrow C(\mathbb{R}^n)$ is stable if it is BIBO-stable and if its restriction $\Sigma : \{S \cap [-\omega, \omega]^n\} \times \{C \cap C(\theta^m)\} \rightarrow C(\mathbb{R}^n)$ is a continuous function for every pair of real numbers $\omega, \theta > 0$.

An important property of the notion of stability given in Definition 3 is the simplicity it lends to the theory of the stabilization of non-linear systems. Indeed, as shown below, a system described by a differential equation of the form (1.1) is stable whenever it is BIBO-stable. In other words, boundedness (with respect to the L^∞-norm) of the output functions $x(t)$, $t \geqslant 0$, implies their continuous dependence on the initial condition x_0 and on the input function u. Considering that boundedness of the output functions is usually quite easy to verify, this leads to a substantial simplification of the mathematical theory of stability and stabilization for continuous-time non-linear systems. In stronger terms, the present definition of stability facilitates the development of simple methods for the synthesis and analysis of stable non-linear control systems, while conforming with the intuitive and practical implications of stability.

We now suggest the following lemma, where $\bar{S}, \bar{A}$ denote the closure of sets $S \subset \mathbb{R}^n$, $A \subset \mathbb{R}^m$, respectively.

Lemma 1

Let $S \subset \mathbb{R}^n$ and $A \subset \mathbb{R}^m$ be subsets, and let $\Sigma : S \times C(A) \rightarrow C(\mathbb{R}^n)$ be a system described by (1.1). Assume that $\Sigma\{S \times C(A)\} \subset C(S)$ and that the state representation function f is continuously differentiable on $\bar{S} \times \bar{A}$. Then, if the system Σ is BIBO-stable, it is also stable.

Remark 1

A simple statement of Lemma 1 is as follows: 'Let $\Sigma : \mathbb{R}^n \times C(\mathbb{R}^m) \rightarrow C(\mathbb{R}^n)$ be a system described by (1.1), where $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a continuously differentiable function. Then, if the system Σ is BIBO-stable, it is also stable.' While true, this statement is somewhat too vague for the present needs.

Proof of Lemma 1

Given a pair $(x_0, u) \in S \times C(A)$, denote by $\Sigma\{(x_0, u)\}(t)$ the output vector at time t generated by the initial condition x_0 and the input function u. Let ω and γ be two positive real numbers, and let $S_\omega := S \cap [-\omega, \omega]^n$ and $A_\gamma := A \cap [-\gamma, \gamma]^m$. By the BIBO-stability of Σ, the response $\Sigma\{(x_0, u)\}(t)$ exists and is unique for all $t \geqslant 0$ whenever $(x_0, u) \in S \times C(A)$, and there is a real number $M > 0$ such that $\Sigma\{S_\omega \times C(A_\gamma)\} \subset C(M^n)$. It only remains to show that the restriction $\Sigma : S_\omega \times C(A_\gamma) \to C(\mathbb{R}^n)$ is a continuous map (with respect to ρ). To this end, choose some real number $\varepsilon > 0$. Let $T > 0$ be a real number satisfying $2^{-T}M < \varepsilon/2$. Now, let $x_1(\cdot), x_2(\cdot) \in \Sigma\{S_\omega \times C(A_\gamma)\}$ be any pair of output functions, and notice that $|x_1(t)| \leqslant M$ and $|x_2(t)| \leqslant M$ for all $t \geqslant 0$. By our choice of T it follows then that

$$\operatorname*{Sup}_{t \geqslant T} 2^{-t}|x_1(t) - x_2(t)| \leqslant 2(2^{-T}M) < \varepsilon$$

so that

$$\rho(x_1 - x_2) < \varepsilon \quad \text{iff} \quad \operatorname*{Sup}_{t \in [0\ T]} 2^{-t}|x_1(t) - x_2(t)| < \varepsilon \tag{2.4}$$

Furthermore, it is clear that the inequality

$$\operatorname*{Sup}_{t \in [0,T]} |x_1(t) - x_2(t)| < \varepsilon$$

implies that

$$\operatorname*{Sup}_{t \in [0,T]} 2^{-t}|x_1(t) - x_2(t)| < \varepsilon$$

Thus, to prove continuity with respect to ρ, it is enough to show that there is a real number $\delta > 0$ such that

$$\operatorname*{Sup}_{t \in [0,T]} |\Sigma(x_{1,0}, u_1)(t) - \Sigma(x_{2,0}, u_2)(t)| < \varepsilon \tag{2.5}$$

for all $(x_{1,0}, u_1), (x_{2,0}, u_2) \in S_\omega \times C(A_\gamma)$ satisfying $\rho((x_{1,0}, u_1) - (x_{2,0}, u_2)) < \delta$. We prove the latter using some standard considerations from the theroy of ordinary differential equations.

First, notice that $M \geqslant \omega$, and that, by its closure and boundedness, the set $\bar{S}_M$ is compact. For similar reasons, $\bar{A}_\gamma$ is also compact. Consider the function $g : \bar{S}_M \times \bar{A}_\gamma \times \bar{S}_M \times \bar{A}_\gamma \to \mathbb{R}$ given by

$$g(x, u, y, v) := \frac{|f(x, u) - f(y, v)|}{|(x, u) - (y, v)|} \tag{2.6}$$

In view of the fact that f is a continuously differentiable function on $\bar{S} \times \bar{A}$, the function g is continuous. Combining this with the fact that the set $\bar{S}_M \times \bar{A}_\gamma \times \bar{S}_M \times \bar{A}_\gamma$ is a compact subset of $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^m$, it follows that there is a real number $K > 0$ such that $g(x, u, y, v) \leqslant K$ for all $(x, u, y, v) \in \bar{S}_M \times \bar{A}_\gamma \times \bar{S}_M \times \bar{A}_\gamma$. This yields the Lipschitz condition

$$|f(x, u) - f(y, v)| \leqslant K|(x, u) - (y, v)| \tag{2.7}$$

for all $(x, u), (y, v) \in S_M \times A_\gamma$.

Next, for any pair $(x_0, u) \in S \times C(A)$, the solution $x(t)$ generated by (x_0, u) satisfies

$$x(t) = x_0 + \int_0^t f(x(\tau), u(\tau))\, d\tau \tag{2.8}$$

Let $x_1(t)$ and $x_2(t)$ be the solutions corresponding to $(x_{1,0}, u_1)$ and $(x_{2,0}, u_2)$, respectively. Recalling that $(x(t), u(t)) \in S_M \times A_\gamma$ for all $t \geqslant 0$ whenever $(x_0, u) \in S_\omega \times C(A_\gamma)$, we obtain

$$\begin{aligned}
|x_1(t) - x_2(t)| &= \left|(x_{1,0} - x_{2,0}) + \int_0^t [f(x_1(\tau), u_1(\tau)) - f(x_2(\tau), u_2(\tau))]\, d\tau\right| \\
&\leqslant |(x_{1,0} - x_{2,0})| + \int_0^t |f(x_1(\tau), u_1(\tau)) - f(x_2(\tau), u_2(\tau))|\, d\tau \\
&\leqslant |(x_{1,0} - x_{2,0})| + K\int_0^t |(x_1(\tau), u_1(\tau)) - (x_2(\tau), u_2(\tau))|\, d\tau \\
&\leqslant |(x_{1,0} - x_{2,0})| + K\int_0^t |u_1(\tau) - u_2(\tau)|\, d\tau + K\int_0^t |x_1(\tau) - x_2(\tau)|\, d\tau
\end{aligned} \tag{2.9}$$

Now, let $u_1, u_2 \in C(A_\gamma)$ be any two input functions satisfying $\rho(u_1 - u_2) < \xi$ for some $\xi > 0$, and let $x_{1,0}, x_{2,0} \in S_\omega$ be any pair of initial conditions satisfying $|x_{1,0} - x_{2,0}| < \zeta$ for some real number $\zeta > 0$. Then, by the definition of ρ, it follows that

$$\operatorname*{Sup}_{t\in[0,T]} |u_1(t) - u_2(t)| < 2^T\xi$$

and (2.9) yields

$$|x_1(t) - x_2(t)| < \zeta + KT2^T\xi + K\int_0^t |x_1(\tau) - x_2(\tau)|\, d\tau \tag{2.10}$$

for all $t \in [0, T]$. Invoking the Bellman–Gronwall inequality results in

$$|x_1(t) - x_2(t)| < (\zeta + KT2^T\xi) \exp(KT) \tag{2.11}$$

for all $t \in [0, T]$. Finally, choose the two real numbers $\xi > 0$ and $\zeta > 0$ to satisfy

$$(\zeta + KT2^T\xi) \exp(KT) < \varepsilon \tag{2.12}$$

and define $\delta := \min\{\zeta, \xi\}$. Then, $|x_1(t) - x_2(t)| < \varepsilon$ for all $t \in [0, T]$ whenever $\rho((x_{1,0}, u_1) - (x_{2,0}, u_2)) < \delta$, and, applying the argument surrounding (2.5), the proof is complete. □

So far, our stability considerations have ignored the possible effects of internal noises and inaccuracies on the performance of the closed-loop system (see the Figure). In order to take these effects into account, the notion of internal stability needs to be reviewed. Consider the system Σ described by (1.1), and assume that various inaccuracies and noises are involved in the implementation of the function $f(x, u)$. To represent these disturbances, we introduce a noise signal $\upsilon_1 \in C(\mathbb{R}^n)$ into the differential equation describing the system Σ in the form

$$\dot{x}(t) = f(x(t), u(t)) + \upsilon_1(t) \tag{2.13}$$

Here, only two restrictions apply to the function υ_1 namely that υ_1 is a continuous function of t, and that it is bounded in the L^∞ sense by a real number $\varepsilon > 0$, such that $\upsilon_1 \in C(\varepsilon^n)$. Next, we also permit the values of the feedback function σ to be corrupted by noise and inaccuracies, so that the input u of the closed-loop system is given by

$$u(t) = \sigma(x(t), v(t)) + \upsilon_2(t) \tag{2.14}$$

where $v \in C(\mathbb{R}^m)$ represents the external input of the closed-loop system (see the Figure), and $\upsilon_2 \in C(\varepsilon^m)$ is a continuous noise function bounded again by ε.

Remark 2

Note that the noise model of (2.14) also includes possible additive noise disturbing the values of the output function $x(t)$. Indeed, suppose that the additive noise $\eta(t)$ disturbs $x(t)$, so that the actual input into the feedback function σ is $x(t) + \eta(t)$, and

$$u(t) = \sigma(x(t) + \eta(t), v(t)) + \upsilon_2(t) \tag{2.15}$$

Now, we deal with stable systems over bounded domains, and there is a real number $N > 0$ such that $(x, v) \in [-N, N]^n \times [-N, N]^m$ over the entire domain of operation of the (stabilized) closed-loop system Σ_σ. Let $\chi > 0$ be a real number. By its continuity, the function σ is uniformly continuous over the compact domain $[-N - \chi, N + \chi]^n \times [-N, N]^m$. Whence, for every real number $\zeta > 0$, there is a real number $\xi > 0, \xi \leqslant \chi$, such that $|\sigma(x + \eta, v) - \sigma(x, v)| < \zeta$ for all $(x, v) \in [-N, N]^n \times [-N, N]^m$ whenever $|\eta| < \xi$. Consequently, for noise signals $\eta(t) \in C(\xi^m)$, we can write $\sigma(x(t) + \eta(t), v(t)) = \sigma(x(t), v(t)) + \upsilon_2'(t)$, where υ_2' is an equivalent noise signal satisfying $|\upsilon_2'(t)| < \zeta$ for all $t \geqslant 0$. Then (2.15) takes the form $u(t) = \sigma(x(t), v(t)) + [\upsilon_2(t) + \upsilon_2'(t)]$, which incorporates the effects of the output noise η, and is of the form (2.14) (with υ_2' included in υ_2).

In addition to the noises υ_1 and υ_2, a noise disturbing the initial condition x_0 is also permitted. However, the effects of this noise have already been included in the notion of input–output stability, since the latter requires the continuous dependence of the output function $x(t)$ on the initial condition x_0. Consequently, no additional consideration of disturbances on the initial condition are necessary.

When the noises υ_1 and υ_2 are incorporated into the configuration in the Figure, they may be regarded as external inputs (over which no control is provided). Then, the closed-loop system Σ_σ can be regarded as the map $\Sigma_\sigma : \mathbb{R}^n \times C(\mathbb{R}^m) \times C(\varepsilon^n) \times C(\varepsilon^m) \to C(\mathbb{R}^n)$, where the terms in the cross product represent the initial condition x_0, the external input function v, the noise υ_1, and the noise υ_2, respectively.

Definition 4

Let $\omega, \theta > 0$ be real numbers, and let $S \subset [-\omega, \omega]^n$ be a subset. The closed-loop system of the Figure is *internally stable* (over the bounded input domain $S \times C(\theta^m)$) if there exists a pair of real numbers $\varepsilon, N > 0$ such that the folllowing hold:

(*a*) $\Sigma_\sigma\{S \times C(\theta^m) \times C(\varepsilon^n) \times C(\varepsilon^m)\} \subset C(N^n)$;

(*b*) The map $\Sigma_\sigma : S \times C(\theta^m) \times C(\varepsilon^n) \times C(\varepsilon^m) \to C(\mathbb{R}^n)$ is continuous (with respect to ρ).

The number ε is referred to as the noise level.

When the noises v_1 and v_2 are present, we refer to the (α, β)-confinement problem as the disturbed (α, β)-confinement problem. In precise terms, the disturbed (α, β)-confinement problem (with external input v) consists of finding a continuous function $\sigma : [\alpha, \beta] \times [-\theta, \theta]^m \to \mathbb{R}^m$ which is continuously differentiable over (α, β), and for which the following holds true: the closed loop system Σ_σ of the Figure satisfies $\Sigma_\sigma\{(\alpha, \beta) \times C(\theta^m) \times C(\varepsilon^n) \times C(\varepsilon^m)\} \subset C((\alpha, \beta))$ for some real number $\varepsilon > 0$. The disturbed (α, β)-confinement problem with pure feedback requires the construction of a continuous function $\sigma : [\alpha, \beta] \to \mathbb{R}^m$ which is continuously differentiable over (α, β), and for which the closed-loop system Σ_σ satisfies that there be a real number $\varepsilon > 0$ such that $\Sigma_\sigma\{(\alpha, \beta) \times C(\varepsilon^n) \times C(\varepsilon^m)\} \subset C((\alpha, \beta))$, where the external input space $C(\theta^m)$ has been deleted from the cross product.

The differential equation describing the closed-loop system Σ_σ with the noises v_1 and v_2 present is given by

$$\dot{x}(t) = f(x(t), \sigma(x(t), v(t)) + v_2(t)) + v_1(t), \quad x(0) = x_0 \tag{2.16}$$

Regarding v_1 and v_2 as (unspecified) input functions, we introduce the augmented input vector $w(t) := (v(t), v_1(t), v_2(t)) \in \mathbb{R}^{m+n+m}$, $t \geq 0$, and define the function

$$g(x, w) := f(x, \sigma(x, v) + v_2) + v_1 \tag{2.17}$$

Then, the differential equation of the closed-loop system becomes

$$\dot{x}(t) = g(x(t), w(t)) \tag{2.18}$$

where the function $g : \mathbb{R}^n \times \mathbb{R}^{m+n+m}$ is continuous (or continuously differentiable) whenever the functions f and σ are continuous (or continuously differentiable). Now, by definition, internal stability of the closed-loop system Σ_σ simply means stability over the augmented input space $S \times C(\theta^m) \times C(\varepsilon^n) \times C(\varepsilon^m)$. By Lemma 1 (in particular, see Remark 1), the latter is equivalent to BIBO-stability over the same input space. Recalling that BIBO-stability means the existence of a unique solution and boundedness, we reach the following conclusion.

Lemma 2

Let Σ be a system described by the differential equation (1.1), where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuously differentiable, and let $\sigma : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ be a continuously differentiable feedback function. Let $\omega, \theta > 0$ be two real numbers, and let $S \subset [-\omega, \omega]^n$ be a subset. Then, the closed-loop system Σ_σ of the Figure is internally stable over the domain S of initial conditions and $C(\theta^m)$ of input functions if and only if there is a pair of real numbers $\varepsilon, N > 0$ such that the following conditions hold:

(*a*) (2.18) has a unique solution $x(t)$, $t \geq 0$, for any initial condition $x_0 \in S$ and any input function $w \in C(\theta^m) \times C(\varepsilon^n) \times C(\varepsilon^m)$;

(*b*) $\Sigma_\sigma\{S \times C(\theta^m) \times C(\varepsilon^n) \times C(\varepsilon^m)\} \subset C(N^n)$.

Lemma 2 provides yet another manifestation of the convenience resulting from the use of the norm ρ. As the lemma shows, even the rather complex idea of internal stability reduces to a simple condition in this framework. In order to verify internal stability for a well-defined system, it is only necessary to verify the boundedness of a certain system over a bounded domain; continuous dependence of the response on the various noise signals is implied by this boundedness. Thus, by using the norm ρ in the definition of stability we gain a substantial simplification of the mathematical difficulty of the stabilization problem with little, if any, compromise of practical significance.

For the case of confinement problems, an even simpler result is obtained. Indeed, let S be a subset of $[-\omega, \omega]^n$ for some $\omega > 0$, and assume that a feedback function σ has been found so that the closed-loop system Σ_σ is well defined and

$$\Sigma_\sigma\{S \times C(\theta^m) \times C(\varepsilon^n) \times C(\varepsilon^m)\} \subset C(S) \tag{2.19}$$

for some $\varepsilon > 0$. Namely, the values of the output function $x(t)$ stay confined within the set S for all $t \geqslant 0$. Since $S \subset [-\omega, \omega]^n$, the inclusion (2.19) directly implies that the closed-loop system is BIBO-stable over the specified input domain (assuming it is well defined there). This means that disturbed confinement alone already guarantees internal stability of the closed-loop system. We summarize this discussion in the subsequent two statements, the first of which is a somewhat technical consequence of Lemma 1 required for later considerations.

Lemma 3

Let Σ be a system described by the differential equation (1.1), where $f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuously differentiable. Let $\omega, \theta > 0$ be two real numbers, let $S \subset [-\omega, \omega]^n$ be a subset and let $\sigma: \bar{S} \times [-\theta, \theta]^m \to \mathbb{R}^m$ be a continuously differentiable feedback function. Assume there is a real number $\varepsilon > 0$ for which the following conditions hold.

(*a*) The differential equation (2.18), with g given by (2.17), has a unique solution for all initial conditions $x_0 \in S$ and all input functions $w \in C(\theta^m) \times C(\varepsilon^n) \times C(\varepsilon^m)$.

(*b*) $\Sigma_\sigma\{S \times C(\theta^m) \times C(\varepsilon^n) \times C(\varepsilon^m)\} \subset C(S)$.

The closed-loop system Σ_σ of the Figure is then internally stable (over the domain $S \times C(\theta^m)$).

Proposition 1

Let Σ be a system described by the differential equation (1.1), where $f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a continuously differentiable function, let $\alpha < \beta$ be two fixed vectors in $\mathbb{R}^n$, and let $\sigma: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ be a continuously differentiable feedback function. If σ is a solution of the disturbed (α, β)-confinement problem for the system Σ, then the closed-loop system Σ_σ is internally stable (over the domain $(\alpha, \beta) \times C(\theta^m)$).

Thus, a solution for the disturbed (α, β)-confinement problem yields internal stabilization of the given system Σ. The derivation of solutions for the disturbed (α, β)-confinement problem is the subject of the next section.

3. Confinement and stabilization

The present section is devoted to the derivation of a solution of the problem of disturbed (α, β)-confinement for a non-linear system Σ described by a differential equation of the form (1.1). By Proposition 1, such a solution also provides internal stabilization of the system Σ. First, we define some notation. Let $\alpha, \beta \in \mathbb{R}^n$ be two fixed vectors satisfying $\alpha < \beta$, and let $\Gamma(\alpha, \beta)$ denote the boundary of the rectangular box $[\alpha, \beta]$. In explicit terms, the boundary consists of $2n$ faces, given by

$$\begin{aligned} \Gamma_i^-(\alpha, \beta) &:= \{(x_1, \ldots, x_n) \in [\alpha, \beta] : x_i = \alpha_i\} \\ \Gamma_i^+(\alpha, \beta) &:= \{(x_1, \ldots, x_n) \in [\alpha, \beta] : x_i = \beta_i\} \end{aligned} \tag{3.1}$$

where $i = 1, \ldots, n$, and

$$\Gamma(\alpha, \beta) = \bigcup_{i=1}^{n} [\Gamma_i^-(\alpha, \beta) \cup \Gamma_i^+(\alpha, \beta)] \tag{3.2}$$

We restrict our attention to the problem of (α, β)-confinement with pure feedback. Let Σ be a system described by the differential equation $\dot{x}(t) = f(x(t), u(t))$, $x(0) = x_0$, where the dimension of x is n, the dimension of u is m, and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a continuously differentiable function. Suppose this system is enclosed in the feedback loop (see the Figure) using a continuously differentiable feedback function $\sigma : \mathbb{R}^n \to \mathbb{R}^m$ with no external input v, so that $u(t) = \sigma(x(t))$. The closed-loop system Σ_σ is then represented by the differential equation $\dot{x}(t) = f(x(t), \sigma(x(t)))$, $x(0) = x_0$. Clearly, the state representation function f has n components $f_1, \ldots, f_n$, each of which represents the derivative of the corresponding coordinate x_i, $i = 1, \ldots, n$, along the system's trajectory. Now, let $\alpha, \beta \in \mathbb{R}^n$ be two fixed vectors with $\alpha < \beta$. Consider the class of feedback functions σ satisfying the following inequalities on the boundary of the box $[\alpha, \beta]$ for some real number $\zeta > 0$.

$$\left.\begin{aligned} f_i(x, \sigma(x)) &\geqslant \zeta \quad \text{for all} \quad x \in \Gamma_i^-(\alpha, \beta), \quad i = 1, \ldots, n \\ f_i(x, \sigma(x)) &\leqslant -\zeta \quad \text{for all} \quad x \in \Gamma_i^+(\alpha, \beta), \quad i = 1, \ldots, n \end{aligned}\right\} \tag{3.3}$$

Notice that the conditions (3.3) refer only to the values of the component functions $f_i(x, \sigma(x))$ on the boundary $\Gamma(\alpha, \beta)$, namely on the faces of the box $[\alpha, \beta]$; the specific values of these functions within the box are not considered. Note also that the feedback function σ needs to be defined only over the domain $[\alpha, \beta]$, since, under (α, β)-confinement, the values of the vector x are confined to this domain during the operation of the closed-loop system. Conditions (3.3) form the basis for our construction of the solution σ to the disturbed (α, β)-confinement problem, and a preliminary indication of their significance is provided by the following result.

Proposition 2

Let $\sigma : [\alpha, \beta] \to \mathbb{R}^m$ be a continuous function solving the disturbed (α, β)-confinement problem. Then, there is a real number $\zeta > 0$ for which conditions (3.3) are satisfied.

Proof

In view of the fact that σ is a solution of the disturbed (α, β)-confinement problem, there is a real number $\varepsilon > 0$ such that the differential equation

$$\dot{x}(t) = f(x(t), \sigma(x(t)) + v_2(t)) + v_1(t), \quad x(0) = x_0 \tag{3.4}$$

has a unique solution $x(t)$, $t \geqslant 0$, for any initial condition $x_0 \in (\alpha, \beta)$ and for any noise signals $v_1 \in C(\varepsilon^n)$ and $v_2 \in C(\varepsilon^m)$; and $x(t) \in (\alpha, \beta)$ for all $t \geqslant 0$. We claim that this implies that (3.3) are satisfied for some $\zeta \geqslant \varepsilon$. Indeed, by contradiction, assume there is a point $x^* \in \Gamma(\alpha, \beta)$ for which (3.3) do not hold for any $\zeta \geqslant \varepsilon$. To be specific, say $x^* \in \Gamma_i^-(\alpha, \beta)$ is a point where $f_j(x^*, \sigma(x^*)) < \varepsilon$. Denote $\delta := \varepsilon - f_j(x^*, \sigma(x^*))$. By the continuity of the functions f and σ, there is a real number $\xi > 0$ such that $|f_j(y, \sigma(y)) - f_j(x^*, \sigma(x^*))| < \delta/2$ for all $y \in [\alpha, \beta]$ satisfying $|y - x^*| < \xi$. Let A be the set of all points $y \in [\alpha, \beta]$ satisfying $|y - x^*| \leqslant \xi$, and, for each $i \neq j$, define

$$m_i := \operatorname*{Sup}_{y \in A} |f_i(y, \sigma(y))|$$

Note that m_i exists and is finite due to the continuity of the functions f and σ and the compactness of A. Let $M := \max_i \{m_i\}$. Consider now the differential equation (3.4) with $\upsilon_2(t) = 0$ for all $t \geqslant 0$ and $\upsilon_1(t) = (0 \ \dots \ 0 \ -\varepsilon \ 0 \ \dots \ 0)^{\mathrm{T}}$ for all $t \geqslant 0$, where the term $-\varepsilon$ appears on the jth coordinate. Then, for all $y \in A$, we have $f_j(y, \sigma(y)) \leqslant -\delta/2$. Let $r > 0$ be a real number satisfying the two inequalities $r < T\delta/2$ and $\xi - r > TM$ for some real number $T > 0$, and let B be the set of all elements $z \in A$ for which $|z - x^*| < r$.

Now, let $x_0 \in (\alpha, \beta)$ be any initial condition belonging to the set B, and let $x(t)$, $t \geqslant 0$, denote the unique solution of (3.4) with this initial condition and the above specified noises υ_1 and υ_2; the solution $x(t)$ exists by virtue of the fact that σ is a solution of the disturbed (α, β)-confinement problem with the noise level ε. Note that, as long as $x(t)$ stays within the set A, we have $\dot{x}_j(t) \leqslant -\delta/2$ and $|\dot{x}_i(t)| \leqslant M$ for all $i \neq j$. Combining this with the inequalities $|x_0 - x^*| < r$, $\xi - r > TM$, and $r < T\delta/2$, and recalling that $x^* \in \Gamma_j^-(\alpha, \beta)$, it follows that there is a time $t' \in [0, T]$ at which $x(t') \in \Gamma(\alpha, \beta) \cap A$; but then $x(t')$ is on the boundary $\Gamma(\alpha, \beta)$, contradicting the fact that $x(t)$ belongs to the interior (α, β) for all $t \geqslant 0$ by the definition of (α, β)-confinement. Consequently, we must have $f_j(x^*, \sigma(x^*)) \geqslant \varepsilon$; since all boundary points $x \in \Gamma(\alpha, \beta)$ can be treated as x^* (with appropriate sign reversals for the cases $x^* \in \Gamma_i^+(\alpha, \beta)$, $i \in \{1, \dots, n\}$), it follows that (3.3) must hold for some $\zeta \geqslant \varepsilon$. □

We conclude then that (3.3) is a necessary condition for the feedback function σ to be a solution of the disturbed (α, β)-confinement problem. Furthermore, it is subsequently shown that (3.3) is also a critical ingredient in a sufficient condition for disturbed (α, β)-confinement. Before deriving sufficient conditions for (α, β)-confinement, some preliminary results are needed.

Let $\alpha, \beta \in \mathbb{R}^n$ be two fixed vectors satisfying $\alpha < \beta$, and let $\theta > 0$ be a real number. Also, let Σ be a system described by the differential equation

$$\dot{x}(t) = g(x(t), w(t)), \quad x(0) = x_0 \tag{3.5}$$

where the dimension of $x(t)$ is n, the dimension of $w(t)$ is p, the input function w is restricted to $C(\theta^p)$, and $g : [\alpha, \beta] \times [-\theta, \theta]^p \to \mathbb{R}^n$ is a continuous function which is continuously differentiable over the domain $(\alpha, \beta) \times [-\theta, \theta]^p$. Consider the case where the function g satisfies the following conditions on the boundary $\Gamma(\alpha, \beta)$ for some real number $\chi > 0$.

$$\left.\begin{aligned} g_i(x, w) &\geqslant \chi \quad \text{for all} \quad w \in [-\theta, \theta]^p \quad \text{and all} \quad x \in \Gamma_i^-(\alpha, \beta), \quad i = 1, \dots, n \\ g_i(x, w) &\leqslant -\chi \quad \text{for all} \quad w \in [-\theta, \theta]^p \quad \text{and all} \quad x \in \Gamma_i^+(\alpha, \beta), \quad i = 1, \dots, n \end{aligned}\right\} \tag{3.6}$$

Proposition 3

Let $\alpha < \beta$ be two fixed vectors in $\mathbb{R}^n$, and let $g : [\alpha, \beta] \times [-\theta, \theta]^p \to \mathbb{R}^n$ be a continuous function which is continuously differentiable over the domain

$(\alpha, \beta) \times [-\theta, \theta]^p$. If (3.6) is satisfied for some real number $\chi > 0$, then, for any initial condition $x_0 \in (\alpha, \beta)$ and any input function $w \in C(\theta^p)$, the differential equation (3.5) has a unique solution $x(t)$, $t \geqslant 0$, and $x(t) \in (\alpha, \beta)$ for all $t \geqslant 0$.

Proof

Assume that (3.6) is satisfied for some real $\chi > 0$. Let $a^* := \frac{1}{3}\min_i \{\beta_i - \alpha_i\}$, and let $\gamma > 0$, $\gamma \leqslant a^*$ be a real number. Denote by $[\alpha + \gamma, \beta - \gamma]$ the set of all vectors $x \in \mathbb{R}^n$ satisfying $\alpha_i + \gamma \leqslant x_i \leqslant \beta_i - \gamma$ for all $i = 1, \ldots, n$. By continuity of the function g over the compact domain $[\alpha, \beta] \times [-\theta, \theta]^p$, there is a real number $\eta' > 0$ such that $|g(y, w) - g(x, w)| < \chi$ for all $x, y \in [\alpha, \beta]$ satisfying $|y - x| < \eta'$ and for all $w \in [-\theta, \theta]^p$. Let $\eta := \min\{\eta', a^*\}$, and let μ be any real number satisfying $0 \leqslant \mu \leqslant \eta$. Then, restricting ourselves to the domain $[\alpha + \mu, \beta - \mu]$, we obtain from (3.6) the conditions

$$\left.\begin{aligned} g_i(x, w) \geqslant 0 \quad \text{for all} \quad u \in [-\theta, \theta]^p \quad \text{and all} \quad x \in \Gamma_i^-(\alpha + \mu, \beta - \mu), \quad i = 1, \ldots, n \\ g_i(x, w) \leqslant 0 \quad \text{for all} \quad u \in [-\theta, \theta]^p \quad \text{and all} \quad x \in \Gamma_i^+(\alpha + \mu, \beta - \mu), \quad i = 1, \ldots, n \end{aligned}\right\} \tag{3.7}$$

Now, let $x_0 \in (\alpha, \beta)$ be any initial condition. There is then a real number $\mu > 0$, $\mu \leqslant \eta$ for which $x_0 \in (\alpha + \mu, \beta - \mu)$. Fix one such μ. We intend to show that, for every initial condition $x_0 \in [\alpha + \mu, \beta - \mu]$, there is a unique solution $x(t)$, $t \geqslant 0$, of the differential equation (3.5), and that $x(t) \in [\alpha + \mu, \beta - \mu]$ for all $t \geqslant 0$. This will clearly prove the proposition.

To this end, let $\omega := \mu/2$, and note that by the continuous differentiability of the function g over the domain $(\alpha, \beta) \times [-\theta, \theta]^p$, it follows that the function

$$h(x, y, w) := \frac{|g(x, w) - g(y, w)|}{|x - y|} \tag{3.8}$$

is continuous over the compact domain $[\alpha + \omega, \beta - \omega] \times [-\theta, \theta]^p$. Consequently, there is a constant $K(\omega) > 0$ such that

$$h(x, y, w) \leqslant K(\omega) \quad \text{for all} \quad (x, y, w) \in [\alpha + \omega, \beta - \omega] \times [-\theta, \theta]^p \tag{3.9}$$

This directly implies that

$$|g(x, w) - g(y, w)| \leqslant K(\omega)|x - y| \tag{3.10}$$

for all $(x, y, w) \in [\alpha + \omega, \beta - \omega] \times [-\theta, \theta]^p$. Furthermore, since the function g is itself continuous over the compact domain $[\alpha + \omega, \beta - \omega] \times [-\theta, \theta]^p$, there is a constant $H(\omega) > 0$ such that $|g(x, w)| \leqslant H(\omega)$ for all $(x, w) \in [\alpha + \omega, \beta - \omega] \times [-\theta, \theta]^p$. Next, fix some input function $w \in C(\theta^p)$, denote $g_w(t, x(t)) := g(x(t), w(t))$, and note that

$$|g_w(t, x) - g_w(t, y)| \leqslant K(\omega)|x - y| \quad \text{and} \quad |g_w(t, x)| \leqslant H(\omega) \tag{3.11}$$

for all $x, y \in [\alpha + \omega, \beta - \omega]$ and all $t \geqslant 0$. Finally, note that the set $[\alpha + \omega, \beta - \omega]$ contains a ball of radius $r(\omega) := \mu - \omega = \mu/2 > 0$ around any initial condition $x_0 \in [\alpha + \mu, \beta - \mu]$. From (3.11) and the last statement, it follows by standard results on the existence and uniqueness of solutions of ordinary differential equations (see, for example Arnold 1973) that the following is true. There is a real number $\delta > 0$ such that the differential equation

$$\dot{x}(t) = g_w(t, x(t)), \quad x(t_0) = x_0 \in [\alpha + \mu, \beta - \mu] \tag{3.12}$$

has a unique solution $x(t)$ over the time interval $[t_0, t_0+\delta]$. Moreover, the number δ can be chosen so that, for fixed α, β and g_w, it depends only on ω (through the quantities $r(\omega)$, $K(\omega)$ and $H(\omega)$). Then, δ is the same for all initial conditions $x_0 \in [\alpha+\mu, \beta-\mu]$ and all initial times $t_0 \geqslant 0$, where the latter follows from the fact that (3.11) holds for all $t \geqslant 0$. Now, take an initial condition $x_0 \in [\alpha+\mu, \beta-\mu]$, and consider the unique solution $x(t)$ of (3.14) over the time interval $[0, \delta]$ with $x(0)=x_0$. We claim that this solution satisfies $x(t) \in [\alpha+\mu, \beta-\mu]$ for all $t \in [0, \delta]$.

Indeed, in view of (3.7) and the definition of g_w, the ith coordinate $x_i(t)$ of the trajectory $x(t)$ cannot leave the domain $[\alpha_i+\mu, \beta_i-\mu]$ owing to the fact that its derivative is non-negative at the lower end of the domain and non-positive at the upper end of the domain. Since this is true for all $i=1, \ldots, n$, it follows that the trajectory $x(t)$, stays within the domain $[\alpha+\mu, \beta-\mu]$ for all $t \in [0, \delta]$. Next, start the differential equation (3.12) at the time $t_0=\delta$ from the initial condition $x_\delta := x(\delta) \in [\alpha+\mu, \beta-\mu]$. By the previous paragraph, there is a unique solution $x(t)$ of (3.12) valid over the time interval $[\delta, 2\delta]$ with $x(\delta)=x_\delta$. Furthermore, the earlier argument of the present paragraph implies that $x(t) \in [\alpha+\mu, \beta-\mu]$ for all $t \in [\delta, 2\delta]$. Combining this with the solution over $[0, \delta]$, we obtain the existence of a unique solution over the interval $[0, 2\delta]$, for which $x(t) \in [\alpha+\mu, \beta-\mu]$ for all $t \in [0, 2\delta]$ and $x(0)=x_0$. Continuing in this manner, we obtain a unique solution $x(t)$, $t \geqslant 0$, satisfying $x(t) \in [\alpha+\mu, \beta-\mu]$ for all $t \geqslant 0$ and $x(0)=x_0$. □

Returning to the investigation of the disturbed (α, β)-confinement problem with pure feedback, we can rewrite (3.4) in the form $\dot{x}(t) = g(x(t), w(t))$, $x(0)=x_0$, where the input function w is generated by the noises v_1 and v_2 through $w(t) = (v_1(t), v_2(t))^{\mathrm{T}} \in \mathbb{R}^{n+m}$ and the function $g : \mathbb{R}^n \times \mathbb{R}^{n+m} \to \mathbb{R}^n$ is given by

$$g(x, w) = f(x, \sigma(x) + v_2) + v_1 \tag{3.13}$$

Clearly, $w \in C(\varepsilon^{n+m})$ whenever $v_1 \in C(\varepsilon^n)$ and $v_2 \in C(\varepsilon^m)$. We can then apply the results of Propositions 2 and 3 to this function g, and obtain necessary and sufficient conditions on the function f for the existence of a feedback function σ solving the disturbed (α, β)-confinement problem. To this end, we first derive a basic technical result. Let $\mathscr{B}_\xi(z)$ denote the closed ball in $\mathbb{R}^m$ having radius $\xi > 0$ and centre at the point z in $\mathbb{R}^m$, namely

$$\mathscr{B}_\xi(z) := \{u \in \mathbb{R}^m : |u - z| \leqslant \xi\} \tag{3.14}$$

Further, given a function $h : \mathbb{R}^n \to \mathbb{R}$, a subset $S \subset \mathbb{R}^n$ and a real number $\zeta > 0$, let $h\{S\} \geqslant \zeta$ indicate the condition $h(y) \geqslant \zeta$ for all $y \in S$. Then, the following is true.

Lemma 4

Let Σ be a non-linear system described by the differential equation (1.1), where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a continuously differentiable function, and let $\alpha, \beta \in \mathbb{R}^n$ be a pair of fixed vectors with $\alpha < \beta$. Then, the disturbed (α, β)-confinement problem by pure feedback has a solution for the system Σ if and only if there is a continuous function $\sigma : [\alpha, \beta] \to \mathbb{R}^m$ continuously differentiable over (α, β) and a pair of real numbers $\xi, \zeta > 0$ such that

$$\left.\begin{aligned} f_i(x, \mathscr{B}_\xi(\sigma(x))) &\geqslant \zeta \quad \text{for all} \quad x \in \Gamma_i^-(\alpha, \beta) \quad \text{and} \quad i = 1, \ldots, n \\ f_i(x, \mathscr{B}_\xi(\sigma(x))) &\leqslant -\zeta \quad \text{for all} \quad x \in \Gamma_i^+(\alpha, \beta) \quad \text{and} \quad i = 1, \ldots, n \end{aligned}\right\} \tag{3.15}$$

Proof

Consider the differential equation $\dot{x}(t) = g(x(t), w(t))$, $x(0) = x_0 \in (\alpha, \beta)$, with the function g of (3.13). It is a direct consequence of our assumptions that g is continuous over $[\alpha, \beta] \times [-\xi, \xi]^{n+m}$ and continuously differentiable over $(\alpha, \beta) \times [-\xi, \xi]^{n+m}$. Now assume that (3.15) holds, set $\varepsilon := \frac{1}{2} \min\{\xi, \zeta\}$, and $\chi := \varepsilon$. Then, slight reflection shows that the function g satisfies

$$\left.\begin{array}{llll} g_i(x, w) \geqslant \chi & \text{for all } w \in [-\varepsilon, \varepsilon]^{n+m} & \text{and all } x \in \Gamma_i^-(\alpha, \beta), & i = 1, \ldots, n \\ g_i(x, w) \leqslant -\chi & \text{for all } w \in [-\varepsilon, \varepsilon]^{n+m} & \text{and all } x \in \Gamma_i^+(\alpha, \beta), & i = 1, \ldots, n \end{array}\right\}$$

and thus the conditions of Proposition 3 hold for the function g with $\theta = \varepsilon > 0$. This implies that the closed-loop system Σ_σ has a unique solution $x(t)$, $t \geqslant 0$, for any initial condition $x_0 \in (\alpha, \beta)$ and for any noise functions $\upsilon_1 \in C(\varepsilon^n)$ and $\upsilon_2 \in C(\varepsilon^m)$; and that $x(t) \in (\alpha, \beta)$ for all $t \geqslant 0$. In other words, the feedback function σ is a solution of the disturbed (α, β)-confinement problem.

Conversely, assume that the function $\sigma : [\alpha, \beta] \to \mathbb{R}^m$ is a solution of the disturbed (α, β)-confinement problem with the noise level $\varepsilon > 0$, and let $\xi := \varepsilon/2$. Let $a \in \mathbb{R}^m$ be a constant vector within the ball $\mathscr{B}_\xi(0)$, and consider the function $\sigma_a : [\alpha, \beta] \to \mathbb{R}^m$ given by $\sigma_a(x) := a + \sigma(x)$. Then σ_a is still continuous over $[\alpha, \beta]$ and continuously differentiable over (α, β), and it is easily seen that it forms a solution of the disturbed (α, β)-confinement problem with the noise level $\xi = \varepsilon/2$. Invoking the argument used in the proof of Proposition 2, we obtain that (3.3) is valid for $\zeta = \varepsilon/2$ with the feedback function σ_a. However, the latter has to hold for any vector $a \in \mathscr{B}_\xi(0)$, which means that (3.15) is valid for $\xi = \varepsilon/2$ and $\zeta = \varepsilon/2$. □

Notice that the conditions of Lemma 4 refer specifically only to the values of the feedback function σ on the boundary $\Gamma(\alpha, \beta)$ of the box $[\alpha, \beta]$; the values of σ within that box are not restricted except for the requirement that σ be continuous over $[\alpha, \beta]$ and continuously differentiable on the interior (α, β). We consider now the question of internal stability for the case of confinement by pure feedback.

Proposition 4

Let Σ be a non-linear system described by the differential equation (1.1), where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a continuously differentiable function, and let $\alpha, \beta \in \mathbb{R}^n$ be a pair of fixed vectors with $\alpha < \beta$. Let $\sigma : [\alpha, \beta] \to \mathbb{R}^m$ be a continuous function which is continuously differentiable over (α, β), and for which (3.15) hold for some real numbers $\xi, \zeta > 0$. The closed-loop system $\Sigma_\sigma : (\alpha, \beta) \to C(\mathbb{R}^n)$ is then internally stable over the domain (α, β) of initial conditions.

Proof

Following the proof of Lemma 4, let g be as in (3.13) and notice that the state representation function of the (disturbed) closed-loop system Σ_σ is given by g. Now, let $x_0 \in (\alpha, \beta)$ be any initial condition, and choose a real number $\mu > 0$ in accordance with the second paragraph of the proof of Proposition 3 (setting $\theta := \varepsilon = \frac{1}{2} \min\{\xi, \zeta\}$, $w := (\upsilon_1, \upsilon_2)^T$ and $p := m + n$ in the latter). Then, $x_0 \in (\alpha + \mu, \beta - \mu)$, and the proof of Proposition 3 implies the following. The closed-loop system Σ_σ has a unique solution $x(t)$, $t \geqslant 0$, for every initial condition $x_0 \in [\alpha + \mu, \beta - \mu]$ and any noise signals $\upsilon_1 \in C(\varepsilon^n)$ and $\upsilon_2 \in C(\varepsilon^m)$, and this solution satisfies $x(t) \in [\alpha+\mu, \beta-\mu]$ for all $t \geq 0$. Consequently, $\Sigma_\sigma\{[\alpha+\mu, \beta-\mu] \times C(\varepsilon^n) \times C(\varepsilon^m)\} \subset C([\alpha+\mu, \beta-\mu])$. Furthermore, since the feedback function σ is continuously differentiable over the domain (α, β), it is continuously differentiable over the closed domain $[\alpha+\mu, \beta-\mu]$. We now invoke Lemma 3 with $\bar{S} = [\alpha+\mu, \beta-\mu]$ and no external input space (i.e., we drop the terms $[-\theta, \theta]^m$ and $C(\theta^m)$ from the statement of the lemma). It follows that the closed-loop system Σ_σ is internally stable over the domain $[\alpha+\mu, \beta-\mu]$ of initial conditions, with noise level $\varepsilon > 0$. Since this is true for all sufficiently small $\mu > 0$ with the same noise level ε, we conclude that Σ_σ is internally stable over the domain (α, β) of initial conditions. □

Of course, the conditions of Lemma 4 cannot be directly used to verify the existence of a solution of the disturbed (α, β)-confinement problem, since they involve the feedback function σ, which is not known in advance. The next objective is to eliminate the function σ from the conditions of the lemma, in order to obtain verifiable conditions involving only the given state representation function f of the system Σ. The new conditions also yield an explicit method for the construction of feedback functions σ that solve the disturbed (α, β)-confinement problem for the system Σ. With this objective in mind, we introduce some basic quantities (see also Hammer 1989 b). As before, as subset $S \subset \mathbb{R}^n \times \mathbb{R}^m$ is the *graph* of a function $g : \mathbb{R}^n \to \mathbb{R}^m$ if $S = \{(x, u) \in \mathbb{R}^n \times \mathbb{R}^m : u = g(x)\}$. Denote by $\Pi_n : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ the standard projection on to the first n-coordinates, so that $\Pi_n(y_1, \ldots, y_{n+m})^T = (y_1, \ldots, y_n)^T$ for every vector $(y_1, \ldots, y_{n+m})^T \in \mathbb{R}^n \times \mathbb{R}^m$. Now let $S \subset \mathbb{R}^n \times \mathbb{R}^m$ and $X \subset \mathbb{R}^n$ be subsets. Then S is a *uniform graph* on X if there is a continuous function $g : X \to \mathbb{R}^m$ and a real number $\xi > 0$ such that $S = \{(x, u) \in X \times \mathbb{R}^m : u \in \mathcal{B}_\xi(g(x))\}$. The function g is then called a *graphing function* on X of the set S, and the number ξ is a *graphing radius*. Intuitively speaking, a uniform graph is simply a 'thickened' graph of a continuous function. It contains the graphs of all continuous functions $g' : X \to \mathbb{R}^m$ satisfying $|g'(x) - g(x)| \leq \xi$ for all $x \in X$. For this reason, the notion of a uniform graph is crucial to the discussion of properties of continuous functions whose values are contaminated by noise.

Returning to the (α, β)-confinement problem of the system Σ described by (1.1), let $\zeta > 0$ be a real number. For each point x of the boundary $\Gamma(\alpha, \beta)$, construct the set of input values

$$U_{f,\zeta}(\alpha, \beta, x) := \left\{ u \in \mathbb{R}^m \;\middle|\; \begin{array}{ll} f_i(x, u) \geq \zeta & \text{for all } i \in \{1, \ldots, n\} \text{ for which } x \in \Gamma_i^-(\alpha, \beta) \\ \text{and} & \\ f_i(x, u) \leq -\zeta & \text{for all } i \in \{1, \ldots, n\} \text{ for which } x \in \Gamma_i^+(\alpha, \beta) \end{array} \right\} \tag{3.16}$$

Note that the set $U_{f,\zeta}(\alpha, \beta, x)$ is obtained simply by solving a set of inequalities determined by the given state representation function f of the system Σ. For boundary points x that are common to several faces of the box $[\alpha, \beta]$, several of the conditions listed on the right-hand side of (3.16) need to be satisfied. Of particular importance to our discussion is the following subset of $\mathbb{R}^n \times \mathbb{R}^m$ directly derived from the subsets $U_{f,\zeta}(\alpha, \beta, x)$.

$$S_f(\alpha, \beta, \zeta) := \{(x, u) \in \mathbb{R}^n \times \mathbb{R}^m : x \in \Gamma(\alpha, \beta), u \in U_{f,\zeta}(\alpha, \beta, x)\} \tag{3.17}$$

Definition 5

Let $f:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^n$ be a continuously differentiable function and let $\alpha<\beta$ be two fixed vectors in $\mathbb{R}^n$. Then, the function f is (α,β)-*uniformly conductive* if there is a real number $\zeta>0$ such that the set $S_f(\alpha,\beta,\zeta)$ contains a uniform graph on the boundary $\Gamma(\alpha,\beta)$.

In intuitive terms, the notion of uniform conductivity is quite simple. It means that a continuous function $g:\Gamma(\alpha,\beta)\to\mathbb{R}^m$ defined on the boundary $\Gamma(\alpha,\beta)$ exists for which the following holds for all $i=1,\ldots,n$: there is a real number $\xi>0$ such that $f_i(x,\mathscr{B}_\xi(g(x)))\geqslant\zeta$ whenever $x\in\Gamma_i^-(\alpha,\beta)$ and $f_i(x,\mathscr{B}_\xi(g(x)))\leqslant-\zeta$ whenever $x\in\Gamma_i^+(\alpha,\beta)$. To discuss the relevancy of uniform conductivity to our present investigation, consider a system Σ described by the differential equation $\dot{x}(t)=f(x(t),u(t))$, and assume there is a continuous feedback function $\sigma:[\alpha,\beta]\to\mathbb{R}^m$ solving the disturbed (α,β)-confinement problem for Σ. Then, by Lemma 4, there are real numbers $\xi,\zeta>0$ for which (3.15) is valid. This then directly implies that $\mathscr{B}_\xi(\sigma(x))\subset U_{f,\zeta}(\alpha,\beta,x)$ for all $x\in\Gamma(\alpha,\beta)$, so that $S_f(\alpha,\beta,\zeta)$ contains a uniform graph on $\Gamma(\alpha,\beta)$. We conclude then that if a solution of the disturbed (α,β)-confinement problem exists, the state representation function f of the system Σ must be (α,β)-uniformly conductive. The main point of the present discussion is that the converse of this statement is also true, namely if the given function f is (α,β)-uniformly conductive, then there is a feedback function σ solving the disturbed (α,β)-confinement problem.

Indeed, assume that the function f is (α,β)-uniformly conductive. Then, on the boundary $\Gamma(\alpha,\beta)$, there is a continuous function $\sigma_\Gamma:\Gamma(\alpha,\beta)\to\mathbb{R}^m$ and real numbers $\xi,\zeta>0$ such that

$$\mathscr{B}_\xi(\sigma_\Gamma(x))\subset U_{f,\zeta}(\alpha,\beta,x)\quad\text{for all}\quad x\in\Gamma(\alpha,\beta)\tag{3.18}$$

Suppose for a moment that the function $\sigma_\Gamma:\Gamma(\alpha,\beta)\to\mathbb{R}^m$ can be extended into a continuous function $\sigma:[\alpha,\beta]\to\mathbb{R}^m$ which is continuously differentiable over the interior (α,β). For this extension σ, we have $\mathscr{B}_\xi[\sigma(x)]\subset U_{f,\zeta}(\alpha,\beta,x)$ for all $x\in\Gamma(\alpha,\beta)$, which means that (3.15) holds for σ with the present values of $\xi,\zeta>0$. By Lemma 4, this implies that σ is a solution of the disturbed (α,β)-confinement problem. Now, whenever f is uniformly conductive, a continuous boundary function σ_Γ satisfying (3.18) is quite easy to derive from the inequalities (3.16), as some later examples will indicate. Thus, the only aspect of the problem that still needs to be considered is the extension of the continuous boundary function σ_Γ into a continuous function $\sigma:[\alpha,\beta]\to\mathbb{R}^m$ which is continuously differentiable over the interior (α,β). Such an extension σ can be derived using standard results from the theory of partial differential equations. For instance, consider the use of the Laplace equation for this purpose. Let $h:\mathbb{R}^n\to\mathbb{R}:(x_1,\ldots,x_n)^{\mathrm{T}}\mapsto h(x_1,\ldots,x_n)$ be a twice continuously differentiable function, and denote by

$$\Delta h:=\frac{\partial^2 h}{\partial x_1^2}+\frac{\partial^2 h}{\partial x_2^2}+\ldots+\frac{\partial^2 h}{\partial x_n^2}\tag{3.19}$$

the Laplace operator. Also, let σ_i be the ith component of the function σ, let $(\sigma_\Gamma)_i$ be the ith component of the function σ_Γ, and let $\sigma_i|_{\Gamma(\alpha,\beta)}$ denote the values of the function σ_i on the boundary $\Gamma(\alpha,\beta)$. Then, by the continuity of σ_Γ and the form of the boundary $\Gamma(\alpha,\beta)$, it follows from the theory of Laplace equations (Petrovsky

1964, ch. 3) that the following is true. For every $i = 1, \ldots, m$, the boundary-value problem

$$\left.\begin{aligned} \Delta\sigma_i &= 0 \\ \sigma_i\big|_{\Gamma(\alpha,\beta)} &= (\sigma_\Gamma)_i \end{aligned}\right\} \tag{3.20}$$

has a unique and continuous solution σ_i over the domain $[\alpha, \beta]$, and this solution is continuously differentiable over the interior (α, β). When the solutions $\sigma_1, \ldots, \sigma_m$ are combined into the vector valued function $\sigma = (\sigma_1, \ldots, \sigma_m)^T$, they create an extension $\sigma : [\alpha, \beta] \to \mathbb{R}^m$ of the boundary function σ_Γ. This extension is continuous over $[\alpha, \beta]$ and continuously differentiable over (α, β); by choice of σ_Γ, it satisfies the conditions (3.15). Thus, invoking Lemma 4, it follows that σ is a solution of the disturbed (α, β)-confinement problem. Of course, partial differential equations other than (3.20) could also be used to obtain suitable extensions σ, or such extensions could be obtained through other methods, without the use of partial differential equations. In any case, when the discussion of the last few paragraphs is combined with Lemma 4 and Proposition 4, we obtain the following theorem, which is the main result of the present paper.

Theorem

Let Σ be a system described by the differential equation $\dot{x}(t) = f(x(t), u(t))$, $x(0) = x_0$, where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuously differentiable and let $\alpha < \beta$ be two fixed vectors in $\mathbb{R}^n$. The following two statements are then equivalent:

(*a*) there exists a feedback function $\sigma : [\alpha, \beta] \to \mathbb{R}^m$ solving the disturbed (α, β)-confinement problem for the system Σ, with the closed-loop system Σ_σ being internally stable for all initial conditions $x_0 \in (\alpha, \beta)$;

(*b*) the given state representation function f of Σ is (α, β)-uniformly conductive.

We can summarize the procedure of solving the disturbed (α, β)-confinement problem for a non-linear system Σ described by a differential equation of the form $\dot{x}(t) = f(x(t), u(t))$ in the following steps.

Step 1

Check whether the given state representation function f is (α, β)-uniformly conductive; if it is, find a continuous boundary function $\sigma_\Gamma : \Gamma(\alpha, \beta) \to \mathbb{R}^m$ which is a graphing function on $\Gamma(\alpha, \beta)$ for the set $S_f(\alpha, \beta, \zeta)$. Computer programs can be developed to check for uniform conductivity and to compute an appropriate function σ_Γ whenever it exists. For low dimensional systems, graphical methods may also be employed.

Step 2

Find a continuous extension $\sigma : [\alpha, \beta] \to \mathbb{R}^m$ of σ_Γ which is continuously differentiable on (α, β). This can be done, for instance, by solving the partial differential equation (3.20). Note that since the closed-loop configuration Σ_σ is internally stable, approximate solutions for the feedback function σ are also adequate.

A few examples on the implementation of Steps 1 and 2 are provided below.

An additional important aspect of the Theorem is the fact that it characterizes all degrees of freedom involved in the solution of the disturbed (α, β)-confinement problem. These consist of the degrees of freedom available in the choice of the boundary function σ_Γ in Step 1 and the degrees of freedom available in the construction of the extension σ in Step 2. In other words, the Theorem presents a complete and constructive solution to the disturbed (α, β)-confinement problem. It also provides a general method for the internal stabilization of non-linear input–state systems through the use of static state feedback. The solution to the problem of confinement with internal stabilization provided here is stated directly in terms of quantities derived from the given state representation function f of the system Σ that needs to be controlled. Note that the solution depends on the amplitude bounds α and β and that a solution may exist only for certain choices of these bounds. The values of α and β for which a solution exists can be derived through the Theorem. We conclude with a few computational examples on the solution of disturbed (α, β)-confinement problems.

Example 1

Consider first a simple system Σ with one input and one state variable, described by the differential equation

$$\dot{x} = 1 + x^2 u \tag{3.21}$$

Assume that it is required to find a feedback function σ with no external inputs, such that the closed-loop system maintains an output amplitude range in the domain $(1, 2)$, whenever started from an initial condition in that domain. The state representation function here is given by

$$f(x, u) = 1 + x^2 u$$

and we have $\alpha = 1$ and $\beta = 2$. The boundary $\Gamma(1, 2)$ consists only of the two points 1, 2, where $\Gamma^-(1, 2) = 1$ and $\Gamma^+(1, 2) = 2$ (we omitted the index i since f has only one component in this case). Choosing $\zeta = 1$ in (3.16), we obtain

$$U_{f,1}(1, 2, 1) = \{u : 1 + 1^2 u \geqslant 1\} = \{u : u \geqslant 0\}$$

$$U_{f,1}(1, 2, 2) = \{u : 1 + 2^2 u \leqslant -1\} = \left\{u : u \leqslant -\frac{1}{2}\right\}$$

Therefore

$$S_f(1, 2, 1) = \begin{cases} (1, u), & \{u \geqslant 0\} \\ (2, u), & \left\{u \leqslant -\frac{1}{2}\right\} \end{cases}$$

Now, in this case, the boundary $\Gamma(1, 2)$ consists of only two discrete points and so every function $\sigma_\Gamma : \Gamma(1, 2) \to \mathbb{R}$ is continuous. For instance, the function $\sigma_\Gamma(1) := 1$ and $\sigma_\Gamma(2) := -1$ is a continuous boundary function, and since $\mathscr{B}_{1/2}(\sigma_\Gamma(1)) \subset U_{f,1}(1, 2, 1)$ and $\mathscr{B}_{1/2}(\sigma_\Gamma(2)) \subset U_{f,1}(1, 2, 2)$, it follows that $S_f(1, 2, 1)$ contains a uniform graph on $\Gamma(1, 2)$. In view of the Theorem this implies that our present confinement problem has a solution. In order to obtain the solution, we only need to extend the boundary function σ_Γ into a continuous function

$\sigma : [1, 2] \to \mathbb{R}$ which is continuously differentiable on $(1, 2)$. One possible choice for σ is clearly given by

$$\sigma(x) = -2x + 3$$

By the Theorem, this function σ yields a closed-loop system Σ_σ which is $(1, 2)$-confined and internally stable for all initial conditions within the inverval $(1, 2)$. The state representation of the internally stable closed-loop system Σ_σ is obtained simply by setting $u := \sigma(x)$ in (3.21), and it is given by

$$\dot{x} = 1 + x^2(-2x + 3)$$

Our construction implies that, for any initial condition $x_0 \in (1, 2)$, this differential equation has a unique solution $x(t)$, $t \geqslant 0$, satisfying $1 < x(t) < 2$ for all $t \geqslant 0$. As we can see, the computation of σ is quite simple.

Example 2

Consider an example of a system Σ with two states and one input, described by the differential equation

$$\left.\begin{aligned} \dot{x}_1 &= 1 + [1 + (x_2)^2]x_1 u \\ \dot{x}_2 &= 1 + [1 + (x_1)^2]x_2 u \end{aligned}\right\} \tag{3.22}$$

For the domain of confinement, we choose $\alpha = (0, 0)^T$ and $\beta = (1, 1)^T$, so that the system has to be confined within the unit square in the first quadrant. The components of the state representation function f here are

$$\left.\begin{aligned} f_1(x_1, x_2, u) &= 1 + [1 + (x_2)^2]x_1 u \\ f_2(x_1, x_2, u) &= 1 + [1 + (x_1)^2]x_2 u \end{aligned}\right\}$$

We need to compute the sets $U_{f,\zeta}(\alpha, \beta, x)$ of (3.16) for some $\zeta > 0$. Choosing $\zeta = 1$, this simply requires solving for u the inequalities

$$\begin{array}{llll} f_1(x_1, x_2, u) \geqslant 1 & \text{for all } x_1, x_2 \text{ satisfying} & x_1 = 0 \text{ and } & 0 \leqslant x_2 \leqslant 1 \\ f_1(x_1, x_2, u) \leqslant -1 & \text{for all } x_1, x_2 \text{ satisfying} & x_1 = 1 \text{ and } & 0 \leqslant x_2 \leqslant 1 \\ f_2(x_1, x_2, u) \geqslant 1 & \text{for all } x_1, x_2 \text{ satisfying} & 0 \leqslant x_1 \leqslant 1 \text{ and } & x_2 = 0 \\ f_2(x_1, x_2, u) \leqslant -1 & \text{for all } x_1, x_2 \text{ satisfying} & 0 \leqslant x_1 \leqslant 1 \text{ and } & x_2 = 1 \end{array}$$

A straightforward calculation yields

$$U_{f,1}(\alpha, \beta, x_1, x_2) = \begin{cases} \{\text{all } u \in \mathbb{R}\} & \text{if } x_1 = 0 \text{ and } 0 \leqslant x_2 < 1 \\ \{\text{all } u \in \mathbb{R} \text{ satisfying } u \leqslant -2/[1 + (x_2)^2]\} & \text{if } x_1 = 1 \text{ and } 0 \leqslant x_2 \leqslant 1 \\ \{\text{all } u \in \mathbb{R}\} & \text{if } 0 < x_1 < 1 \text{ and } x_2 = 0 \\ \{\text{all } u \in \mathbb{R} \text{ satisfying } u \leqslant -2/[1 + (x_1)^2]\} & \text{if } 0 \leqslant x_1 < 1 \text{ and } x_2 = 1 \end{cases}$$

The set $S_f(\alpha, \beta, 1)$ is directly determined by the above expression for $U_{f,1}(\alpha, \beta, x)$ and it is easy to see that it contains a uniform graph on the boundary $\Gamma(\alpha, \beta)$. Indeed, observing that the set $\{\text{all } u \in \mathbb{R} \text{ satisfying } u \leqslant -2\}$ is contained in $U_{f,1}(\alpha, \beta, x)$ for all $x \in \Gamma(\alpha, \beta)$, it follows that the constant function $\sigma_\Gamma : \Gamma(\alpha, \beta) \to \mathbb{R}$ given by $\sigma_\Gamma(x) := -3$ is a graphing function for $S_f(\alpha, \beta, 1)$, with graphing radius $\xi = 1$. Now, one easy way of extending the function σ_Γ into a continuous function

$\sigma : [\alpha, \beta] \to \mathbb{R}$ continuously differentiable over (α, β), is by simply taking the constant function

$$\sigma(x) = -3$$

for all $x \in [\alpha, \beta]$. For this choice of σ, the Theorem implies that the closed-loop system Σ_σ is (α, β)-confined for the present α, β and internally stable for all initial conditions $x_0 \in (\alpha, \beta)$. The state representation of the closed-loop system Σ_σ is obtained by setting $u = \sigma(x) = -3$ in (3.22). It is given by

$$\dot{x}_1 = 1 - 3[1 + (x_2)^2]x_1$$

$$\dot{x}_2 = 1 - 3[1 + (x_1)^2]x_2$$

As we can see from these examples, the computation of the confining and stabilizing feedback function σ is quite simple, and in many cases it is not necessary to resort to the solution of the partial differential equation (3.20) for finding an appropriate extension of the boundary function σ_Γ.

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