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State feedback for non-linear control systems

JACOB HAMMER†

A theory of static state feedback for non-linear discrete-time systems is developed. The theory applies to non-linear systems possessing a recursive representation of the form $x_{k+1}=f(x_k, u_k)$, where f is a continuous function, and it deals with the construction of continuous state feedback functions that internally stabilize a given system. The theory yields an explicit method for the computation of stabilizing feedback functions, and several examples of the computation of such functions are provided.

1. Introduction

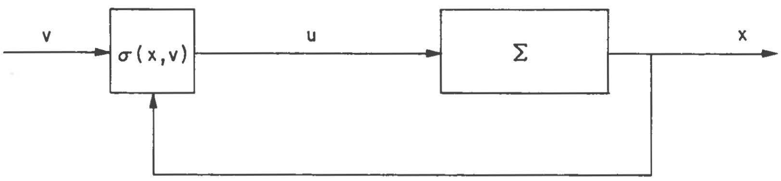
The present paper is devoted to the development of a theory of static feedback for nonlinear discrete-time systems described by recursive representations of the form

$$x_{k+1}=f(x_k, u_k), \quad k=0, 1, 2, \ldots \tag{1.1}$$

Here $\{u_k\}_{k=0}^{\infty}$ is a sequence of m-dimensional real vectors, serving as the input sequence of the system; $\{x_k\}_{k=0}^{\infty}$ is a sequence of p-dimensional real vectors, serving as the output sequence of the system; and f is a continuous function, called the *recursion function.* It is assumed that the initial condition x_0 is specified. For the sake of convenience, we refer to a system described by a recursion of the form (1.1) as an *input/state system.*

The basic control configuration considered is of the form shown in the Figure. The feedback loop is closed through the continuous feedback function $\sigma(x, v)$, and Σ denotes the given system that needs to be controlled. The external input sequence of the closed loop is the sequence $\{v_k\}_{k=0}^{\infty}$ of m-dimensional real vectors, and the overall input/output relation induced by the configuration is denoted by Σ_σ. The configuration depicts *static feedback* in the sense that the current output u of the feedback function σ is determined by the current values of x and v, and is given by the relation

$$u_k=\sigma(x_k, v_k), \quad k=0, 1, 2, \ldots \tag{1.2}$$



When this relation is substituted into the recursive representation (1.1) of the given system Σ, a recursive representation for the closed-loop system Σ_σ results in the form

$$x_{k+1}=f(x_k, \sigma(x_k, v_k)), \quad k=0, 1, 2, \ldots \tag{1.3}$$

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† Center for Mathematical System Theory, Department of Electrical Engineering, University of Florida, Gainesville, Florida 32611, U.S.A.

The main objective of the theory developed in the present paper is to facilitate the computation of feedback functions σ for which the closed-loop system shown in the Figure is internally stable. By 'internally stable' we mean that the closed-loop system is stable, and its stability is preserved when small noises tint the output signals of Σ and σ.

As is well known, the classical theory of feedback mainly deals with *additive feedback* configurations, which are configurations in which the feedback function σ is of the form

$$\sigma(x, v) = v - \phi(x) \tag{1.4}$$

where ϕ is a continuous function. In the present paper we allow a substantial departure from classical feedback theory by not requiring the feedback function σ to be additive. As it turns out, in general, static additive feedback is not adequate for the global stabilization of non-linear input/state systems. When the additivity restriction is removed, a rather general class of non-linear input/state systems can be stabilized by static feedback. In the case that the given system Σ is linear (i.e. when $f(x, u) = Ax + Bu$), the theory developed here includes the results of the standard linear state-feedback theory.

The basic notion of stability that we employ relates to continuity; a system is regarded stable if it induces a continuous map from its space of input sequences to its space of output sequences. Internal stability is then added as an additional requirement, by demanding that the closed-loop system Σ_σ maintain its stability despite small noises that may corrupt the output signals of Σ and of σ. Following Hammer (1986), the present theory deals with stabilization over bounded domains. The basic design objective is to obtain internal stabilization of the given system Σ over a bounded domain of input sequences. The design specifications include the desired maximal amplitudes of the input sequences and of the output sequences of the stabilized closed-loop system. The design procedure then derives a continuous feedback function σ for which the closed-loop system Σ_σ is internally stable and does not exceed the specified output amplitude bound, as long as the input amplitude does not exceed its own specified bound. From a theoretical standpoint, the bounds on the input and output amplitudes can be chosen arbitrarily large; in practice, however, such bounds originate from the physical characteristics of the systems involved. We adopt here the notion of stabilization over a bounded domain for two main reasons. First, it allows the designer to incorporate into the design procedure realistic and imperative considerations relating to the maximal signal amplitudes permitted by the physical setup. And, secondly, it yields a substantial simplification of the mathematical complexity of the stabilization problem.

The basic notion on which our non-linear state-feedback theory rests is the notion of an eigenset of the given recursion function f of the system Σ that needs to be controlled. In order to describe this notion, we need some notation. As usual, $\mathbb{R}^p$ denotes the set of all p-dimensional real vectors, where $p > 0$ is an integer. By $\Pi_p : \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^p$ we denote the standard projection onto the first p coordinates given, for any vector $x = (x_1, \ldots, x_{p+m}) \in \mathbb{R}^p \times \mathbb{R}^m$, by $\Pi_p x := (x_1, \ldots, x_p)$. Now let Σ be a system having a recursive representation $x_{k+1} = f(x_k, u_k)$, where $f: \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^p$ is a continuous function. An *eigenset* of the function f is a subset $E \subset \mathbb{R}^p \times \mathbb{R}^m$ that satisfies the condition $f[E] \subset \Pi_p[E]$, where $f[E]$ simply denotes the image of the set E under the function f. In qualitative terms, an eigenset is simply a set of states and corresponding inputs having the property that when the system is started from a state within the set, any input corresponding to that state generates a new state which is again within the set. The notion of an eigenset is naturally related to the notion of an invariant set commonly used in the theory of autonomous systems.

In the context of feedback theory, the eigensets of interest are non-empty and bounded; namely, they are non-empty eigensets E satisfying $E \subset [-\alpha, \alpha]^p \times [-\alpha, \alpha]^m$ for some real number $\alpha > 0$. Here $[-\alpha, \alpha]^p$ denotes the set of all p-dimensional real vectors whose coordinates belong to the interval $[-\alpha, \alpha]$. As discussed in § 4, the computation of bounded eigensets just involves the solution of certain sets of inequalities derived from the given recursion function f. Once the bounded eigensets of f are known, stabilizing feedback functions σ for the system Σ can be computed, whenever they exist. The necessary and sufficient conditions for the existence of a stabilizing feedback function are rather simple in this framework. Somewhat inaccurately stated, these conditions amount to the requirement that the recursion function f possess a non-empty, open and bounded eigenset. In § 4 we provide a few examples on the computation of stabilizing feedback functions for non-linear input/state systems.

The present paper is written within the framework of Hammer (1984, 1986). Recent studies of the theory of stabilization for non-linear systems can be found in Desoer and Lin (1984), Desoer and Kabuli (1988), Tay and Moore (1989), Sontag (1981, 1989), the references listed in these papers, and others.

2. Notation and basics

The systems that we consider are discrete-time systems, accepting sequences of m-dimensional real vectors as their input and generating sequences of p-dimensional real vectors as their output, where m and p are arbitrary positive integers. For an integer $m > 0$, denote by $S(\mathbb{R}^m)$ the set of all sequences $\{u_0, u_1, \ldots\}$ of m-dimensional real vectors $u_i \in \mathbb{R}^m$, $i = 0, 1, 2, \ldots$. Then a system is simply a map $\Sigma: S(\mathbb{R}^m) \to S(\mathbb{R}^p)$, transforming m-dimensional input sequences into p-dimensional output sequences. Given a subset $S \subset S(\mathbb{R}^m)$, denote by $\Sigma[S]$ the image of the set S under Σ, namely the set of all elements $x \in S(\mathbb{R}^p)$ satisfying $x = \Sigma u$ for some element $u \in S$.

For a vector $u \in \mathbb{R}^m$, denote $|u| := \max\{|u_i|, i = 1, \ldots, m\}$, the maximal absolute value of the coordinates. For a sequence $u \in S(\mathbb{R}^m)$ let $|u| := \sup_{i \geq 0} |u_i|$, so that $|\cdot|$ becomes the usual l^∞ norm. For most of our discussion we shall employ a weighted l^∞ norm, given by $\rho(u) := \sup_{i \geq 0} 2^{-i}|u_i|$ for all $u \in S(\mathbb{R}^m)$. From the norm ρ we construct a metric $\rho(\cdot, \cdot)$ on our spaces of sequences, given by $\rho(u, v) := \rho(u - v)$. For a set $S \subset S(\mathbb{R}^m)$ we denote by $\bar{S}$ the closure of the set with respect to the topology induced by ρ. Unless explicitly stated otherwise, continuity of maps over spaces of sequences is with respect to the topology induced by the metric ρ.

Of particular importance to our discussion are spaces of bounded sequences. For a real number $\theta > 0$ we denote by $S(\theta^m)$ the set of all sequences $u \in S(\mathbb{R}^m)$ satisfying $|u| \leq \theta$, namely, the set of all m-dimensional sequences bounded by θ (with respect to the l^∞ norm). A system $\Sigma: S(\mathbb{R}^m) \to S(\mathbb{R}^p)$ is said to be *BIBO* (*bounded-input bounded-output*) *stable* if for every real number $\theta > 0$ there is a real number $M > 0$ such that $\Sigma[S(\theta^m)] \subset S(M^p)$.

We can now define the basic notion of stability that we employ in our discussion. A system $\Sigma: S(\mathbb{R}^m) \to S(\mathbb{R}^p)$ is *stable* if it satisfies the following two conditions: (i) it is BIBO stable, and (ii) for every real number $\theta > 0$ the restriction $\Sigma: S(\theta^m) \to S(\mathbb{R}^p)$ is a continuous map (with respect to the topology induced by the metric ρ). As shown previously (Hammer 1984, 1986, 1989), this notion of stability is particularly convenient for the solution of the stabilization problem for non-linear discrete-time systems, and it conforms with the qualitative notion of stability that originates from the Lyapunov theory.

It will be convenient to have at our disposition the notion of a homogeneous system, which is defined as follows (Hammer 1987).

Definition 2.1

A system $\Sigma: S(\mathbb{R}^m) \to S(\mathbb{R}^p)$ is a *homogeneous system* if for every real number $\alpha > 0$ and for every subset $S \subset S(\alpha^m)$ the following holds: Whenever there exists a real number $\theta > 0$ such that $\Sigma[S] \subset S(\theta^p)$, the restriction of Σ to the closure $\bar{S}$ of the set S in $S(\alpha^m)$ is a continuous map $\Sigma: \bar{S} \to S(\theta^p)$.

In qualitative terms, a system is homogeneous if it is continuous whenever its outputs are bounded. For our present purposes, we shall need the following class of homogeneous systems (Hammer 1987).

Proposition 2.1

Let $\Sigma: S(\mathbb{R}^m) \to S(\mathbb{R}^p)$ be a system having a recursive representation of the form $x_{k+1} = f(x_k, u_k)$ with the initial condition x_0. If the recursion function $f: \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^p$ is a continuous function then Σ is a homogeneous system.

One of the most important properties of homogeneous systems is the fact that a homogeneous system is stable (i.e. bounded and continuous) whenever it is BIBO stable. This fact is a direct consequence of the definition of a homogeneous system, and it yields a substantial simplification of the theory of stabilization. In view of Proposition 2.1, all of the systems Σ discussed in the present paper are homogeneous systems. Furthermore, in the feedback configuration shown in the Figure let $\Sigma: S(\mathbb{R}^m) \to S(\mathbb{R}^p)$ be an input/state system represented by $x_{k+1} = f(x_k, u_k)$, and let $\sigma: \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^m$ be a continuous feedback function. Then, since f and σ are both continuous functions, it follows by (1.3) and Proposition 2.1 that the closed-loop system Σ_σ is also a homogeneous system.

3. Feedback functions, stabilization and eigensets

The classical theory of feedback deals largely with additive feedback configurations, which require the feedback function σ in the Figure to be of the form (1.4). As it turns out from our ensuing discussion, the restriction to additive feedback functions is too constrictive in our present framework. The class of non-linear input/state systems that can be globally stabilized through the use of static additive feedback is quite small. In generalizing the theory of state feedback to non-additive feedback functions, we should like to maintain two of the most fundamental properties of the feedback operation. These are continuity and reversibility.

The continuity requirement is quite obvious—we require the feedback function σ to be a continuous function. Reversibility is a somewhat more subtle property, requiring the feedback operation to be reversible in the sense that it can be 'undone' by another feedback operation. To explain the reversibility property more fully, consider a classical additive feedback configuration. In this case the feedback function is of the form

$$\sigma = v - \phi(x) =: \sigma^-(x, v) \tag{3.1}$$

In order to apply this feedback function, set $u = \sigma^-(x, v)$ in the Figure and obtain the closed-loop system Σ_{σ^-}, with v serving as the external input. Next, apply the feedback function

$$\sigma^+(x, w) := w + \phi(x) \tag{3.2}$$

to the closed-loop system Σ_{σ^-} by setting $v = \sigma^+(x, w)$. The new closed-loop system is then $(\Sigma_{\sigma^-})_{\sigma^+}$, and, as we show in a moment, it has the same input/output relation as the original system Σ. Thus the feedback operation represented by σ^- is undone by σ^+, with w serving now as the external input. Indeed, in view of (1.3), the recursion function f_{σ^-} of Σ_{σ^-} is given by

$$f_{\sigma^-}(x, v) = f(x, v - \phi(x)) \tag{3.3}$$

where f is the recursion function of the original system Σ. When the feedback function σ^+ is applied to the closed-loop system Σ_{σ^-}, the recursion function $f_{\sigma^-\sigma^+}$ of the resulting system $(\Sigma_{\sigma^-})_{\sigma^+}$ becomes, by (1.3),

$$\begin{aligned} f_{\sigma^-\sigma^+}(x, w) &= f_{\sigma^-}(x, w + \phi(x)) = f(x, (w + \phi(x)) - \phi(x)) \\ &= f(x, w) \end{aligned} \tag{3.4}$$

and the original recursion function is recovered. Thus additive feedback is a reversible feedback operation. On a level of principles, a reversible feedback operation guarantees that no information about the original system Σ is irretrievably lost when the loop is closed. The reversibility of the additive feedback operation has substantial implications in classical feedback theory and practice; for instance, feedback configurations can be used in practice to measure parameters of an unstable system Σ by inserting it into a stabilizing reversible feedback loop. The parameters of Σ can then be recovered from the parameters of the closed loop by reversing the feedback operation. All feedback functions σ considered in the present paper are required to yield reversible feedback operations.

The reversibility requirement lends itself to a fairly straightforward analysis. Let $\sigma: \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^m, (x, v) \mapsto \sigma(x, v)$, be a function, and for every element $x \in \mathbb{R}^p$ denote by $\sigma_x: \mathbb{R}^m \to \mathbb{R}^m$ the partial function given by $\sigma_x(v) := \sigma(x, v)$ for all $v \in \mathbb{R}^m$. The recursion function f_σ of the closed-loop system Σ_σ is then given by

$$f_\sigma(x, v) = f(x, \sigma_x(v)) \tag{3.5}$$

where f is the recursion function of the system Σ. Assume now that the system Σ_σ is itself enclosed in a feedback loop, using the feedback function $\omega: \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^m$, $(x, w) \mapsto \omega(x, w) = v$, so that there are now two feedback loops around Σ, and let $\Sigma_{\sigma\omega}$ denote the final system. Then, as before, the recursion function $f_{\sigma\omega}$ of $\Sigma_{\sigma\omega}$ is given by

$$f_{\sigma\omega}(x, w) = f_\sigma(x, \omega_x(w)) = f(x, \sigma_x \omega_x(w)) \tag{3.6}$$

Now the feedback function ω is required to 'undo' the feedback operation induced by the function σ, so that $f_{\sigma\omega}(x, w) = f(x, w)$ for all x and w. This reversion of the feedback operation has to hold for all input/state systems $\Sigma: S(\mathbb{R}^m) \to S(\mathbb{R}^p)$; namely, it has to hold for all functions $f: \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^p$. Thus we must have $w = \sigma_x \omega_x(w)$, and ω_x must be a right inverse of σ_x for every x. By the classical theory of functions, this means that for all states x the function σ_x must be a surjective (onto) function; the function ω_x, being a right inverse of σ_x, must then be injective (one-to-one). But then, since σ and ω are both required to be feedback functions, σ_x and ω_x must both belong to the same class of functions, and thus must both be injective and surjective, i.e. set isomorphisms. We conclude that, in order to induce a feedback operation that is reversible by feedback, the feedback function σ has to be such that σ_x is a set isomorphism for all states x. Clearly, $\sigma_x : \mathbb{R}^m \to \operatorname{Im} \sigma_x$ is a set isomorphism exactly when $\sigma_x : \mathbb{R}^m \to \mathbb{R}^m$ is injective. This leads to the following definition of the class of permissible feedback functions.

Definition 3.1

Let $\Sigma : S(\mathbb{R}^m) \to S(\mathbb{R}^p)$ be an input/state system, having the recursive representation $x_{k+1} = f(x_k, u_k)$. A *reversible feedback function* for Σ is a continuous function $\sigma : \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^m$, $(x, v) \mapsto \sigma(x, v)$ for which the partial function $\sigma_x : \mathbb{R}^m \to \mathbb{R}^m$, $v \mapsto \sigma_x(v) = \sigma(x, v)$ is an injective function for any possible state x.

Remark 3.1

When restricting ourselves here to the study of feedback functions that satisfy the conditions of Definition 3.1, we do not mean to imply that there is no interest in studying more general classes of feedback functions for which these conditions are not necessarily satisfied. We just single out here a family of 'nice' feedback functions, and we show in the sequel that this family is wide enough to allow global stabilization of a large class of non-linear input/state systems. Dealing with feedback functions that satisfy the conditions of Definition 3.1 is particularly convenient, since such feedback functions preserve many of the classical conceptual properties of feedback configurations.

We turn now to a discussion of stability. Let $\theta > 0$ be a real number. The configuration shown in the Figure is *input/output stable* (for input sequences bounded by θ) if the restriction $\Sigma_\sigma : S(\theta^m) \to S(\mathbb{R}^p)$ is a stable system. One of the advantages of our set-up is the simplicity it yields in the treatment of the notion of input/output stability. Indeed, let $\Sigma : S(\mathbb{R}^m) \to S(\mathbb{R}^p)$ be a system having a recursive representation of the form $x_{k+1} = f(x_k, u_k)$, where $f : \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^p$ is a continuous function, and let $\sigma : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^m$ be a reversible feedback function. Then, using (1.3), the recursive representation of Σ_σ is given by $x_{k+1} = f(x_k, \sigma(x_k, v_k))$, and, since f and σ are both continuous functions, it follows that the system Σ_σ also has a continuous recursion function. Consequently, by Proposition (2.1), the system Σ_σ is homogeneous, and thus, by Definition 2.1, it is stable whenever it is bounded. This proves the following statement.

Proposition 3.1

Let $\Sigma : S(\mathbb{R}^m) \to S(\mathbb{R}^p)$ be a system having a recursive representation of the form $x_{k+1} = f(x_k, u_k)$, where $f : \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^p$ is a continuous function. Let $\sigma : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^m$ be a reversible feedback function and let $\theta > 0$ be a real number. Then the system $\Sigma_\sigma : S(\theta^m) \to S(\mathbb{R}^p)$ is input/output stable if and only if there is a real number $\delta > 0$ such that $\Sigma_\sigma[S(\theta^m)] \subset S(\delta^p)$.

We emphasize that when the system Σ_σ is stable in the context of Proposition 3.1, it is not just BIBO stable but is also continuous with respect to the topology induced by the metric ρ. Thus, in order to guarantee input/output stability over the input domain $S(\theta^m)$, it is enough to make sure that the output sequences are all bounded. This obviously leads to a substantial simplification of the mathematical framework.

As is well known, the notion of input/output stability is too weak a notion of stability for practical applications, since it does not account for the effect of inaccuracies and internal noises. The feedback configurations discussed in the present paper are required to be internally stable in the sense that small noises added to the output of the system Σ or to the output of the feedback function σ do not destroy stability. Formally, denote by $f:\mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^p$ the given recursion function of the system Σ, and let $\sigma:\mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^m$ be the feedback function. To incorporate noise effects, assume that the output of the system Σ is given by the recursion

$$\left.\begin{aligned} x_{k+1} &= f(x_k, u_k) + n_{k+1}, \quad k = 0, 1, 2, \dots \\ x_0 &= x_{00} + n_0 \end{aligned}\right\} \tag{3.7}$$

where $n \in S(\mathbb{R}^p)$ is a noise sequence, and where x_{00} is the specified nominal initial condition. Thus inaccuracies in the initial condition are also permitted. In a similar fashion, the output of the feedback function is given by

$$u_k = \sigma(x_k, v_k) + \nu_k, \quad k = 0, 1, 2, \dots \tag{3.8}$$

where $\nu \in S(\mathbb{R}^m)$ is a noise sequence, and $v \in S(\mathbb{R}^m)$ is the external input sequence of the closed-loop system. We denote by $\Sigma_{\sigma,n,\nu}$ the input/output relation of the closed loop system with the noises n and ν present. We continue to denote by Σ_σ the closed loop without the noises. Clearly, the system $\Sigma_{\sigma,n,\nu}$ can be regarded as a system accepting the three input sequences v, n and ν. To make this fact explicit, we write $\Sigma_{\sigma,n,\nu}: S(\mathbb{R}^m) \times S(\mathbb{R}^p) \times S(\mathbb{R}^m) \to S(\mathbb{R}^p)$, where the first space in the cartesian product represents the space of input sequences v of the closed-loop system, the second space represents the noise n and the third space represents the noise ν. Both noises are assumed to have 'small' amplitudes, not exceeding a bound that is denoted by ε. The notion of internal stability is then defined as follows.

Definition 3.2

The configuration shown in the Figure is *internally stable* (for input sequences bounded by θ) if there is a real number $\varepsilon > 0$ such that $\Sigma_{\sigma,n,\nu}: S(\theta^m) \times S(\varepsilon^p) \times S(\varepsilon^m) \to S(\mathbb{R}^p)$ is a stable system.

In qualitative terms, internal stability means that the output of the closed-loop system is bounded and depends continuously on the input signal v as well as on the noise signals n and ν, where continuity is with respect to the topology induced by the metric ρ. The static state feedback theory developed in the present paper leads to the construction of feedback functions that internally stabilize the given system. In analogy with Proposition 3.1, we can obtain the following simplified condition for internal stability.

Proposition 3.2

Let $\Sigma: S(\mathbb{R}^m) \to S(\mathbb{R}^p)$ be a system having a recursive representation of the form $x_{k+1} = f(x_k, u_k)$, where $f:\mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^p$ is a continuous function. Let

$\sigma: \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^m$ be a reversible feedback function and let $\theta > 0$ be a real number. Then the system Σ_σ is internally stable (for input sequences bounded by θ) if and only if there is a pair of real numbers ε, $\delta > 0$ such that

$$\Sigma_{\sigma,n,v}[S(\theta^m) \times S(\varepsilon^p) \times S(\varepsilon^m)] \subset S(\delta^p)$$

Proof

In view of (1.3), (3.7) and (3.8), the recursion function $f_{\sigma,n,v}$ of the system $\Sigma_{\sigma,n,v}$ is given by

$$f_{\sigma,n,v}(x, v, n, v) = f(x, \sigma(x, v) + v) + n \tag{3.9}$$

By the continuity of the functions f and σ, this implies, through Proposition 2.1, that the system $\Sigma_{\sigma,n,v}: S(\theta^m) \times S(\varepsilon^p) \times S(\varepsilon^m) \to S(\mathbb{R}^p)$ is a homogeneous system, and hence the present assertion is a direct consequence of the definition of a homogeneous system. □

The basic notion on which our state feedback theory rests is the notion of an eigenset of a function, which is defined below. Before stating the definition, we need some notation. Let $\varepsilon > 0$ be a real number. Given a point $x \in \mathbb{R}^n$, denote by $\mathscr{B}_\varepsilon(x)$ the open ball of radius ε around the point x, namely the set of all points $y \in \mathbb{R}^n$ satisfying $|y - x| < \varepsilon$. For a subset $S \subset \mathbb{R}^n$ denote

$$\mathscr{B}_\varepsilon(S) := \bigcup_{x \in S} \mathscr{B}_\varepsilon(x) \tag{3.10}$$

The set $\mathscr{B}_\varepsilon(S)$ is clearly an open neighbourhood of the set S, and it consists of all points $y \in \mathbb{R}^n$ for which there is a point $x \in S$ such that $|y - x| < \varepsilon$. The dimension of the space within which the ball $\mathscr{B}_\varepsilon$ is taken, i.e. n here, is determined by S, and is omitted from the notation. Finally, recall that $\Pi_p: \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^p$ is the standard projection onto the first p coordinates (see § 1).

Definition 3.3

An *eigenset* E of a function $f: \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^p$ is a subset $E \subset \mathbb{R}^p \times \mathbb{R}^m$ satisfying $f[E] \subset \Pi_p[E]$. An *ε-eigenset* $\mathscr{E}$ of the function f is a subset $\mathscr{E} \subset \mathbb{R}^p \times \mathbb{R}^m$ satisfying the condition $f[B_\varepsilon(\mathscr{E})] \subset \Pi_p[\mathscr{E}]$, where $\varepsilon > 0$ is a real number.

In order to point out the significance of eigensets to state feedback theory, consider the closed-loop system $\Sigma_\sigma: S(\theta^m) \to S(\mathbb{R}^p)$ shown in the Figure. Let f be the recursion function of the given system Σ and let σ be the feedback function. Denote by Ω_0 the set of all pairs $(x, u) \in \mathbb{R}^p \times \mathbb{R}^m$ that may appear as arguments of the recursion function f during the operation of the closed-loop system. Letting $\Sigma_\sigma v]_k$ be the kth element of the output sequence $\Sigma_\sigma v$, the set Ω_0 is given by

$$\Omega_0 = \{(x, u) \in \mathbb{R}^p \times \mathbb{R}^m : x = \Sigma_\sigma v]_k,\ u = \sigma(x, v_k) \text{ for some } v \in S(\theta^m) \text{ and } k \in \{0, 1, 2, \ldots\}\} \tag{3.11}$$

We now show that Ω_0 is an eigenset of the recursion function f. Recall that a set $S \subset \mathbb{R}^n$ is *bounded* if there is a real number $\alpha > 0$ such that $S \subset [-\alpha, \alpha]^n$.

Lemma 3.1

If $\Sigma_\sigma: S(\theta^m) \to S(\mathbb{R}^p)$ is input/output stable then the set Ω_0 is a bounded eigenset of the recursion function f of the system Σ.

Proof

Assume that $\Sigma_\sigma : S(\theta^m) \to S(\mathbb{R}^p)$ is input/output stable. Let (x, u) be any point in the set Ω_0. Then there is an input sequence $v \in S(\theta^m)$ and an integer $k \geqslant 0$ such that $x = \Sigma_\sigma v]_k$ and $u = \sigma(x, v_k)$. Denote $y := \Sigma_\sigma v]_{k+1}$ and $w := \sigma(y, v_{k+1})$, and note that, by definition, also $(y, w) \in \Omega_0$, so that $y \in \Pi_p[\Omega_0]$. Furthermore, from the recursive representation of Σ, it follows that $y = f(x, u)$, and hence $f(x, u) \in \Pi_p[\Omega_0]$. Since the latter holds for any point $(x, u) \in \Omega_0$, it follows that $f[\Omega_0] \subset \Pi_p[\Omega_0]$, and Ω_0 is an eigenset of the function f. Next, to show that the set Ω_0 is bounded, recall that, by the input/output stability of Σ_σ, there is a real number $\delta > 0$ such that $\Sigma_\sigma[S(\theta^m)] \subset S(\delta^p)$, which implies that $|x| \leqslant \delta$ for all $x \in \Pi_p[\Omega_0]$. Combining this with the fact that $|v_k| \leqslant \theta$ for all $v \in S(\theta^m)$ and all k, it follows by the continuity of the function $\sigma : \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^p$ that there is a real number $\alpha > 0$ such that $|u| = |\sigma(x, v_k)| \leqslant \alpha$ for all $x \in \Pi_p[\Omega_0]$, all $v \in S(\theta^m)$ and all k. Consequently, by the definition of the norm $|\cdot|$, $|(x, u)| \leqslant \max\{\delta, \alpha\} =: \beta$ for all $(x, u) \in \Omega_0$, and hence the set Ω_0 is bounded by β. □

Assume next that the system $\Sigma_\sigma : S(\theta^m) \to S(\mathbb{R}^p)$ is internally stable, and let $\varepsilon > 0$ be as in Definition 3.3. Let $\Omega := \{(x, u) \in \mathbb{R}^p \times \mathbb{R}^m : x = \Sigma_{\sigma,n,\nu} v]_k,\ u = \sigma(x, v_k) + \nu_k$ for some $v \in S(\theta^m)$, $n \in S(\varepsilon^p)$, $\nu \in S(\varepsilon^m)$ and $k \in \{0, 1, 2, \ldots\}\}$. Define the set

$$\mathscr{E} := \{(f(x, u), \sigma(f(x, u), v)), (x, u) \in \Omega \text{ and } v \in [-\theta, \theta]^m\}$$
$$\cup \{(x_{00}, \sigma(x_{00}, v)), v \in [-\theta, \theta]^m\} \tag{3.12}$$

where f is the recursion function of the system Σ, σ is the feedback function and x_{00} is the nominal initial condition. We now show that $\mathscr{E}$ is a ζ-eigenset of the function f for some real number $\zeta > 0$.

Proposition 3.3

If the system $\Sigma_\sigma : S(\theta^m) \to S(\mathbb{R}^p)$ is internally stable then the set $\mathscr{E}$ of (3.12) is a bounded ζ-eigenset of the recursion function f of the system Σ for some real number $\zeta > 0$.

Proof

Assume that $\Sigma_\sigma : S(\theta^m) \to S(\mathbb{R}^p)$ is internally stable. To consider the boundedness of our sets, let $(x, u) \in \Omega$ be a point. Then, by Proposition 3.2, there is a real number $\delta > 0$ such that $|x| \leqslant \delta$. Combining this with the continuity of the function σ, it follows that there is a real number $\alpha > 0$ such that $|\sigma(x, v)| \leqslant \alpha$ for all $x \in \Pi_p[\Omega]$ and $v \in [-\theta, \theta]^m$. Hence the set Ω is bounded, and, by the continuity of the functions f and σ, it follows that the set $\mathscr{E}$ is bounded as well. Thus it only remains to show that $f[\mathscr{B}_\zeta(\mathscr{E})] \subset \Pi_p[\mathscr{E}]$ for some real number $\zeta > 0$.

It follows directly from (3.12) that $f[\Omega] \subset \Pi_p[\mathscr{E}]$, and so our proof will conclude upon showing that $\mathscr{B}_\zeta(\mathscr{E}) \subset \Omega$. Note that the output x_k of the closed-loop system at time k depends only on the input elements $v_0, \ldots, v_{k-1}$, and the noise elements $n_0, \ldots, n_k$ and $\nu_0, \ldots, \nu_{k-1}$ (where $v_0, \ldots, v_{k-1}$ and $\nu_0, \ldots, \nu_{k-1}$ denote the empty set when $k = 0$). Now let (x, u) be any point in the set Ω. There are then an integer $k \geqslant 0$, input elements $v_0, \ldots, v_{k-1}$, and noise elements $n_0, \ldots, n_k$ and $\nu_0, \ldots, \nu_k$ such that $x = \Sigma_{\sigma,n,\nu} v]_k$ and $u = \sigma(x, v_k) + \nu_k$, where $v \in S(\theta^m)$. From the definition of the set Ω it follows directly that

$$(f(x, u) + n_{k+1}, \sigma(f(x, u) + n_{k+1}, v_{k+1}) + \nu_{k+1}) \in \Omega \tag{3.13}$$

for any $n_{k+1} \in [-\varepsilon, \varepsilon]^p$, $v_{k+1} \in [-\varepsilon, \varepsilon]^m$ and $v_{k+1} \in [-\theta, \theta]^m$, since it is simply the next pair of output and input values of Σ. Now, in view of the fact that σ is a continuous function and its arguments here are bounded, there is a real number $\xi > 0$ such that $|\sigma(f(x, u) + n_{k+1}, v_{k+1}) - \sigma(f(x, u), v_{k+1})| < \frac{1}{2}\varepsilon$ for all $(x, u) \in \Omega$ and $v_{k+1} \in [-\theta, \theta]^m$, whenever $|n_{k+1}| < \xi$. This implies that for any $(x, u) \in \Omega$, $v_{k+1} \in [-\theta, \theta]^m$, $|n_{k+1}| < \xi$ and $z \in \mathscr{B}_{\varepsilon/2}(\sigma(f(x, u), v_{k+1})$ there is an element $v_{k+1} \in (-\varepsilon, \varepsilon)^m$ satisfying $z = \sigma(f(x, u) + n_{k+1}, v_{k+1}) + v_{k+1}$. Thus, letting $\zeta := \min\{\frac{1}{2}\varepsilon, \xi\}$, it follows from (3.13) and the definition of the norm $|\cdot|$ that $\mathscr{B}_\zeta((f(x, u), \sigma(f(x, u), v)) \subset \Omega$ for any $(x, u) \in \Omega$ and $v \in [-\theta, \theta]^m$. Similarly, also $\mathscr{B}_\zeta(x_{00}, \sigma(x_{00}, v)) \subset \Omega$ for all $v \in [-\theta, \theta]^m$. In view of (3.12), the last two facts imply that $\mathscr{B}_\zeta(\mathscr{E}) \subset \Omega$, and our proof concludes. □

Before continuing with our discussion of internal stability, we list some elementary properties of eigensets. Given a subset $S \subset \mathbb{R}^p \times \mathbb{R}^m$, denote by $S(x)$ the set of all elements $u \in \mathbb{R}^m$ for which $(x, u) \in S$.

Proposition 3.4

Let $\mathscr{E}_1$ and $\mathscr{E}_2$ be two ε-eigensets of the function $f: \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^p$. Then the following hold:

(i) the union $\mathscr{E}_1 \cup \mathscr{E}_2$ is an ε-eigenset of the function f;

[redacted]

(iv) let $\mathscr{E} \subset \mathbb{R}^p \times \mathbb{R}^m$ be any subset satisfying the conditions $\Pi_p[\mathscr{E}] = \Pi_p[\mathscr{E}_1]$ and $\mathscr{E}(x) \subset \mathscr{E}_1(x)$ for all $x \in \Pi_p[\mathscr{E}]$; then $\mathscr{E}$ is an ε-eigenset of the function f.

The rather straightforward proof of this proposition is omitted here.

We continue now with our investigation of the implications of the notion of internal stability for non-linear state feedback systems. Recall that the graph of a function $g: \mathbb{R}^p \to \mathbb{R}^m$ is simply a subset of $\mathbb{R}^p \times \mathbb{R}^m$ consisting of all points of the form $(x, g(x))$, $x \in \mathbb{R}^p$. A major role in our theory is played by the notion of a *uniform graph*, which is a subset $S \subset \mathbb{R}^p \times \mathbb{R}^m$ satisfying the following condition: there is a continuous function $g: \mathbb{R}^p \to \mathbb{R}^m$ and a real number $\zeta > 0$ such that $\mathscr{B}_\zeta(g(x)) \subset S(x)$ for all $x \in \Pi_p[S]$. The function g is then called a *graphing function* for the set S. The notion of a uniform graph is quite simple on an intuitive level. First, a uniform graph S contains the graph of the continuous function g. Furthermore, it also contains the graph of any continuous function g' that differs from g by less than ζ, namely any continuous function g' satisfying $|g'(x) - g(x)| < \zeta$ for all $x \in \Pi_p[S]$. The notion of a uniform graph is a natural tool for the description of functions whose values may be corrupted by noise. We are now in a position to introduce the most important notion of our present study.

Definition 3.4

A continuous function $f: \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^p$ is *uniformly conductive* at a point $x_0 \in \mathbb{R}^p$ if it has a bounded ε-eigenset $\mathscr{E}$ for which the set $\mathscr{B}_\varepsilon(\mathscr{E})$ is a uniform graph, and $x_0 \in \Pi_p[\mathscr{E}]$.

In order to point out the significance of a conductive function, we state now the following theorem, which is the main result of the present paper.

Theorem 3.1

Let $\Sigma: S(\mathbb{R}^m) \rightarrow S(\mathbb{R}^p)$ be a system having a recursive representation $x_{k+1} = f(x_k, u_k)$ with the initial condition x_{00}, where $f: \mathbb{R}^p \times \mathbb{R}^m \rightarrow \mathbb{R}^p$ is a continuous function. Then the following two statements are equivalent:

(i) there exists a reversible state feedback function $\sigma: \mathbb{R}^p \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ for which the closed-loop system $\Sigma_\sigma: S(\theta^m) \rightarrow S(\mathbb{R}^p)$ is internally stable, where $\theta > 0$ is a real number;

(ii) the recursion function f is uniformly conductive at the point x_{00}.

Thus we have a complete characterization of internal stabilizability by static state feedback. The significance of this result is twofold. First, from a theoretical point of view, it provides a direct link between properties of the given recursion function f of the system that needs to be stabilized, and the existence of a stabilizing state feedback. From a practical point of view, we shall see later that eigensets of functions can be quite readily computed. Once the eigensets are known, one can check whether f is uniformly conductive, and, if it is, stabilizing feedback functions σ for the system Σ can be directly derived. This yields then an explicit procedure for the computation of stabilizing feedback functions. The proof of Theorem 3.1 consists of Proposition 3.5 and 3.6 below. In the next section we discuss the existence and the computation of eigensets, and we provide a number of examples of the computation of stabilizing reversible feedback functions σ.

Remark 3.2

Note that the nominal initial condition x_{00} of the system Σ in Theorem 3.1 can be chosen as any point in the ε-eigenset $\mathscr{E}$ over which the recursion function f of Σ is conductive. Explicit examples are provided in § 4.

As the first step in the proof of Theorem 3.1, assume that the system Σ_σ is internally stable, and consider the set $\mathscr{E}$ of (3.12). In view of Proposition 3.3, $\mathscr{E}$ is a bounded ζ-eigenset of the recursion function f of the system Σ. Furthermore, it is a direct consequence of our construction of the set $\mathscr{E}$ (see the proof of Proposition 3.3) that

$$(x, \mathscr{B}_\zeta(\sigma(x, v))) \subset \mathscr{B}_\zeta(\mathscr{E}) \text{ for all } x \in \Pi_p[\mathscr{B}_\zeta(\mathscr{E})] \text{ and } v \in S(\theta^m) \tag{3.14}$$

Now define the function $g: \mathbb{R}^p \rightarrow \mathbb{R}^m$ by

$$g(x) := \sigma(x, 0) \tag{3.15}$$

Then, since σ is a continuous function, so also is g. By (3.14), we have $(x, \mathscr{B}_\zeta(g(x))) \subset \mathscr{B}_\zeta(\mathscr{E})$ for all $x \in \Pi_p[\mathscr{B}_\zeta(\mathscr{E})]$. Finally, since $x_{00} \in \Pi_p[\mathscr{E}]$ by (3.12), it follows that the function f is uniformly conductive at x_{00}, with g being the graphing function. This proves the following fact, which simply means that (i) implies (ii) in Theorem 3.1.

Proposition 3.5

Let $\Sigma: S(\mathbb{R}^m) \rightarrow S(\mathbb{R}^p)$ be a system having a recursive representation $x_{k+1} = f(x_k, u_k)$ with the initial condition x_{00}, where $f: \mathbb{R}^p \times \mathbb{R}^m \rightarrow \mathbb{R}^p$ is a continuous function. If there is a reversible feedback function $\sigma:\mathbb{R}^p\times\mathbb{R}^m\to\mathbb{R}^m$ such that $\Sigma_\sigma:S(\theta^m)\to S(\mathbb{R}^p)$ is internally stable then the recursion function f of the system Σ is uniformly conductive at the point x_{00}.

Next we prove the converse direction of Theorem 3.1, namely that condition (ii) implies condition (i).

Proposition 3.6

Let $\Sigma:S(\mathbb{R}^m)\to S(\mathbb{R}^p)$ be a system having a recursive representation $x_{k+1}=f(x_k,u_k)$ with the initial condition x_{00}, where $f:\mathbb{R}^p\times\mathbb{R}^m\to\mathbb{R}^p$ is a continuous function, and let $\theta>0$ be a real number. If the recursion function f is uniformly conductive at the point x_{00} then there exists a reversible state feedback function $\sigma:\mathbb{R}^p\times\mathbb{R}^m\to\mathbb{R}^m$ for which the closed-loop system $\Sigma_\sigma:S(\theta^m)\to S(\mathbb{R}^p)$ is internally stable.

Proof

Assume that f is uniformly conductive at x_{00}. We construct a reversible state feedback function σ for which the closed-loop system $\Sigma_\sigma:S(\theta^m)\to S(\mathbb{R}^p)$ is internally stable. Since f is uniformly conductive, it has a bounded ε-eigenset $\mathscr{E}$ for which $\Pi_p[\mathscr{E}]$ contains the point x_{00} and $\mathscr{B}_\varepsilon(\mathscr{E})$ is a uniform graph. Invoking the definition of a uniform graph, let $g:\mathbb{R}^p\to\mathbb{R}^m$ be a graphing function of $\mathscr{B}_\varepsilon(\mathscr{E})$ and let $\zeta>0$ be a real number for which $(x,\mathscr{B}_\zeta(g(x)))\subset\mathscr{B}_\varepsilon(\mathscr{E})$ for all $x\in\Pi_p[\mathscr{B}_\varepsilon(\mathscr{E})]$. Let $\lambda:=\zeta/3\theta$ and define a function $\sigma:\mathbb{R}^p\times\mathbb{R}^m\to\mathbb{R}^m$, $(x,v)\mapsto\sigma(x,v)$ by

$$\sigma(x,v):=\lambda v+g(x) \tag{3.16}$$

It is then a direct consequence that $(x,\sigma(x,v))\in\mathscr{B}_\varepsilon(\mathscr{E})$ for all $x\in\Pi_p[\mathscr{B}_\varepsilon(\mathscr{E})]$ and for all $v\in[-\theta,\theta]^m$. Furthermore, since $\lambda\neq0$ and g is a continuous function, σ is a reversible feedback function (which is, in fact, additive). Let $\delta>0$ be a bound on the set $\mathscr{E}$, so that $\mathscr{E}\subset[-\delta,\delta]^p\times[-\delta,\delta]^m$. Then, since the recursion function of the closed-loop system Σ_σ is $f(x,\sigma(x,v))$, since $(x,\sigma(x,v))\in\mathscr{B}_\varepsilon(\mathscr{E})$ for all $x\in\Pi_p[\mathscr{B}_\varepsilon(\mathscr{E})]$ and for all $v\in[-\theta,\theta]^m$, since $f[\mathscr{B}_\varepsilon(\mathscr{E})]\subset\mathscr{E}$ and since $x_{00}\in\mathscr{E}$, it follows that for all integers $k\geqslant0$ the kth output vector $\Sigma_\sigma v]_k\in\Pi_p[\mathscr{E}]\subset[-\delta,\delta]^p$. Thus $\Sigma_\sigma[S(\theta^m)]\subset S(\delta^p)$, which, by Proposition 3.1, implies that the closed-loop system Σ_σ is input/output stable.

Furthermore, in order to prove the internal stability of Σ_σ, let $\beta>0$ be a real number such that $|g(x+n)-g(x)|<\frac{1}{3}\zeta$ for all $x\in\bar{\mathscr{B}}_\varepsilon(\Pi_p[\mathscr{E}])$ whenever $n\in\mathbb{R}^p$ satisfies $|n|<\beta$. The existence of β is a direct consequence of the continuity of g over the entire space and the fact that x is restricted to a bounded closed domain. Let $\xi:=\min\{\beta,\frac{1}{3}\zeta\}$. Then, by construction, we have $(x+n,\sigma(x+n,v)+\nu)\in\mathscr{B}_\varepsilon(\mathscr{E})$ for all $x\in\Pi_p[\mathscr{E}]$, $v\in[-\theta,\theta]^m$, $n\in[-\xi,\xi]^p$ and $\nu\in[-\xi,\xi]^m$. The fact that $f[\mathscr{B}_\varepsilon(\mathscr{E})]\subset\mathscr{E}\subset[-\delta,\delta]^p\times[-\delta,\delta]^m$ then implies that $\Sigma_{\sigma,n,\nu}[S(\theta^m)\times S(\xi^p)\times S(\xi^m)]\subset S((\delta+\xi)^p)$, and it follows by Proposition 3.2 that Σ_σ is internally stable. □

The feedback function σ constructed in (3.16), although appropriate for the proof of the proposition, is quite restrictive from a practical point of view. The set of inputs u that it generates for the system Σ at each state x is given by $\mathscr{B}_{\zeta/3}(g(x))$, which, in general, might be only a small subset of the set $\mathscr{B}_\varepsilon(\mathscr{E})(x)$ that describes the permissible inputs u at the state x. This results in an undue restriction of the output vectors that the closed-loop system Σ_σ can reach. In order to increase the set of output vectors generated by the closed-loop system, we can proceed as follows. Let $\mathscr{E}$ be a bounded ε-eigenset over which the recursion function f of the system Σ is conductive, and let $\sigma: \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^m$ be any reversible feedback function satisfying the following condition for some real number $0 < \zeta < \varepsilon$:

$$\sigma_x[-\theta, \theta]^m \subset \mathscr{B}_\zeta(\mathscr{E})(x) \tag{3.17}$$

for all $x \in \Pi_p[\mathscr{B}_\varepsilon(\mathscr{E})]$. Now, since the function σ is continuous over the entire space and since there is a real number $\delta > 0$ such that $\Pi_p[\mathscr{B}_\varepsilon(\mathscr{E})] \subset [-\delta, \delta]^p$, there is a real number $\tau > 0$ such that $|\sigma(x+n, v) - \sigma(x, v)| < \frac{1}{2}(\varepsilon - \zeta)$ for all $x \in \Pi_p[\mathscr{B}_\varepsilon(\mathscr{E})]$ and $v \in [-\theta, \theta]^m$ whenever $n \in [-\tau, \tau]^p$. Let $\xi := \min\{\tau, \frac{1}{2}(\varepsilon - \zeta)\}$. Then one has $(x+n, \sigma(x+n, v) + v) \in \mathscr{B}_\varepsilon(\mathscr{E})$ for all $x \in \Pi_p[\mathscr{E}]$, $v \in [-\theta, \theta]^m$, $n \in [-\xi, \xi]^p$ and $v \in [-\xi, \xi]^m$. Combining this with the fact that $\mathscr{E}$ is a bounded ε-eigenset, i.e. that $f[\mathscr{B}_\varepsilon(\mathscr{E})] \subset \Pi_p[\mathscr{E}] \subset [-\delta, \delta]^p$, it follows that $\Sigma_{\sigma,n,v}[S(\theta^m) \times S(\xi^p) \times S(\xi^m)] \subset S((\delta + \xi)^p)$, and Σ_σ is internally stable by Proposition 3.2. Thus any reversible feedback function σ satisfying (3.17) internally stabilizes the system Σ, and we can state the following.

Theorem 3.2

Let $\Sigma: S(\mathbb{R}^m) \to S(\mathbb{R}^p)$ be a system having a recursive representation $x_{k+1} = f(x_k, u_k)$ with the initial condition x_{00}, where $f: \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^p$ is a continuous function, and let $\theta > 0$ be a real number. Assume that the recursion function f is uniformly conductive at the point x_{00} and let $\mathscr{E}$ be an ε-eigenset of f for which $\mathscr{B}_\varepsilon(\mathscr{E})$ is a uniform graph and $x_{00} \in \Pi_p[\mathscr{E}]$. Then every reversible feedback function $\sigma: \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^m$ satisfying (3.17) yields an internally stable closed-loop system $\Sigma_\sigma: S(\theta^m) \to S(\mathbb{R}^p)$.

It is clear from (3.17) that the function $g(x) := \sigma(x, 0)$ forms a graphing function for the set $\mathscr{B}_\varepsilon(\mathscr{E})$. The feedback function σ constructed in (3.16) obviously belongs to the class of feedback functions satisfying (3.17). Generally speaking, in order to find a feedback function σ that satisfies (3.17), one has to construct a continuous family $\{\sigma_x\}$ of homeomorphisms $\sigma_x: [-\theta, \theta]^m \to \operatorname{Im} \sigma_x$ for which $\operatorname{Im} \sigma_x \subset \mathscr{B}_\zeta(\mathscr{E})(x)$ for all $x \in \Pi_p[\mathscr{B}_\varepsilon(\mathscr{E})]$. In general, the construction of all possible families $\{\sigma_x\}$ is not an easy problem, and, as is well known, it is the subject of homotopy theory. However, some of the families $\{\sigma_x\}$ are quite easy to construct, and one of them is in fact given by (3.16). An additional family $\{\sigma_x\}$, i.e. an additional reversible feedback function σ, that can easily be constructed is described in the following corollary. This feedback function yields, in general, a domain of stabilization that is larger than the one provided by (3.16).

Corollary

In the notation of Theorem 3.2, let g be a graphing function for $\mathscr{B}_\varepsilon(\mathscr{E})$. Let $r: \mathbb{R}^m \to \mathbb{R}$ be a continuous scalar positive valued function satisfying the following conditions: (i) there is a real number $\kappa > 0$ such that $r(x) \geqslant \kappa$ for all $x \in \mathbb{R}^m$; and (ii) there is a real number $0 < \zeta < \varepsilon$ such that for every $x \in \Pi_p[\mathscr{B}_\varepsilon(\mathscr{E})]$ the ball $\mathscr{B}_{r(x)}(g(x)) \subset \mathscr{B}_\zeta[\mathscr{E}](x)$. Define the function

$$\sigma(x, v) := \frac{r(x)}{\theta} v + g(x)$$

Then σ is a reversible feedback function, and the closed-loop system $\Sigma_\sigma : S(\theta^m) \to S(\mathbb{R}^p)$ is internally stable.

Proof

The validity of the corollary is a consequence of the following. First, since $r(x) \geqslant \kappa$, the function σ_x is a homeomorphism for every x. The continuity of the functions $g(x)$ and $r(x)$ implies the continuity of $\sigma(x, v)$. Finally, condition (ii) of the corollary implies that (3.17) is satisfied for the present σ, and the conclusion of the corollary is then valid by Theorem 3.2. □

The feedback function given by (3.16) is a particular case of the feedback functions described by the corollary, with the function $r(x)$ simply being taken as a constant. In practice it is desirable to increase the set Im σ_x as much as possible, so as to provide the system Σ in the Figure with a set of input vectors u that is as large as possible at each instant, since this results in an increased set of reachable output vectors x in the next step. Consequently, in the corollary it is desirable to choose the value of $r(x)$ as large as possible for each x, without violating, of course, the continuity of r and condition (ii) of the corollary. In the next section we discuss the computation of stabilizing feedback functions described by the corollary, and we provide some computational examples. As it turns out, the corollary provides an effective, implementable and relatively simple method for the computation of stabilizing feedback functions for global non-linear control. Of course, in general, a further increase of the domain of stabilization can be obtained through the use of more general feedback functions, as described in Theorem 3.2.

Another point of interest is the connection between our results here and the standard linear state feedback theory. In the next section we show that when the given system Σ is a linear finite-dimensional time-invariant system, the class of feedback functions described by the corollary to Theorem 3.2 includes the classical linear state feedback functions.

4. The construction of eigensets and feedback functions

The present section is devoted to a brief discussion of the computational aspects of the static state feedback theory developed in § 3. A number of detailed examples on the computation of eigensets and feedback functions are provided. It is probably most appropriate to start with an examination of the linear case, to show that our results include standard linear state feedback theory. In addition to providing a connection to previously known material, this will also enable us to exhibit some of the simplest instances of eigensets and feedback functions. Since the purpose of the discussion of the linear case here is mainly didactic, we restrict ourselves to the consideration of a single-input linear system $\Sigma : S(\mathbb{R}) \to S(\mathbb{R}^n)$ with $n > 1$ states. The recursion function is then of the form

$$x_{k+1} = Ax_k + bu_k \tag{4.1}$$

and we assume that the pair (A, b) is reachable and that the initial condition is $x_{00} = 0$. Then, without loss of generality, the pair (A, b) can be taken in controller canonical form, and, denoting by $x_{i,k}$ the ith component of the state vector at time k, the recursion becomes

$$\left.\begin{aligned} x_{i,k+1} &= x_{i+1,k}, \quad i = 1, \ldots, n-1 \\ x_{n,k+1} &= \sum_{i=1}^{n} a_i x_{i,k} + u_k \end{aligned}\right\} \tag{4.2}$$

The components $f_1, \dots, f_n$ of the recursion function f are then given by

$$\left.\begin{aligned} f_i(x, u) &= x_{i+1}, \quad i = 1, \dots, n-1 \\ f_n(x, u) &= \sum_{i=1}^{n} a_i x_i + u \end{aligned}\right\} \tag{4.3}$$

where, somewhat abusing our notation, x_i here denotes the ith component of the vector x. Assume further that the desired bound on the input amplitude is θ and that the desired bound on the output amplitude (including the noise) is δ, so that we need to find a reversible feedback function $\sigma: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ such that Σ_σ is internally stable and $\Sigma_\sigma[S(\theta)] \subset S(\delta^n)$. Now let $\zeta > 0$ be a real number satisfying

$$\sum_{i=1}^{n} |a_i| \zeta < \tfrac{1}{3}\delta, \quad \zeta < \tfrac{1}{3}\delta \tag{4.4}$$

Then it is easy to verify that the set

$$\mathscr{E} = \left\{ (x, u) \in \mathbb{R}^n \times \mathbb{R} : \left| \sum_{i=1}^{n} a_i x_i + u \right| \leqslant \frac{1}{3}\delta, |x_i| \leqslant \frac{1}{3}\delta + \frac{i\zeta}{n}, i = 1, \dots, n \right\} \tag{4.5}$$

is an ε-eigenset of f for $\varepsilon = \zeta/n$. A graphing function for this set is then given by

$$g(x) := -\sum_{i=1}^{n} a_i x_i \tag{4.6}$$

Letting $\lambda := \delta/3\theta$ and defining the function $\sigma: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ by

$$\sigma(x, v) := \lambda v - \sum_{i=1}^{n} a_i x_i \tag{4.7}$$

we obtain a reversible feedback function. By the corollary to Theorem 3.2 it then follows that σ internally stabilizes the system Σ over the input space $S(\theta)$. It is also clear that δ and θ can be chosen arbitrarily large here, and internal stabilization over the entire space can be achieved. Obviously, (4.7) is a standard linear state feedback formula, and thus classical linear state feedback results are included in our framework.

We turn now to a brief discussion of the computational aspects of the non-linear state feedback theory developed in § 3. A more complete study of this topic will be provided in a separate report. As we have seen, the process of computing a static state feedback function that stabilizes a given non-linear input/state system can be divided into three main steps. First, one needs to find an appropriate ε-eigenset of the given recursion function of the system that needs to be stabilized. Then a graphing function for the ε-eigenset needs to be found. Finally, a stabilizing state feedback function σ needs to be computed, using Theorem 3.2 or its corollary. Of course, all of this is under the assumption that the recursion function of the system that needs to be stabilized is uniformly conductive.

Generally speaking, the computation of ε-eigensets of functions involves the solution of certain sets of inequalities. More specifically, let $\Sigma: S(\mathbb{R}^m) \to S(\mathbb{R}^p)$ be an input/state system with the recursive representation $x_{k+1} = f(x_k, u_k)$, where $f: \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^p$ is a continuous function. Assume that the system Σ needs to be stabilized over a range of output amplitudes $|x| \leqslant \delta$. According to our discussion in § 3, we need to find an ε-eigenset $\mathscr{E}$ of the function f for which the projection $\Pi_p[\mathscr{E}]$ onto the state space is bounded by δ. This can be handled in the following way. Find a subset $\mathscr{X} \subset \mathbb{R}^p$ and a real number $\zeta > 0$ for which the following conditions hold: (i) $\mathscr{X} \subset [-\delta, \delta]^p$; (ii) for each element $x \in \mathscr{B}_\zeta(\mathscr{X})$ there is a non-empty bounded subset

$\mathscr{U}(x) \subset \mathbb{R}^m$ such that $f[\mathscr{B}_\zeta(x), \mathscr{B}_\zeta(\mathscr{U}(x))] \subset \mathscr{X}$; and (iii) there is a real number $\alpha > 0$ such that $\mathscr{U}(x) \subset [-\alpha, \alpha]^m$ for all $x \in \mathscr{X}$. Then it follows directly from the definitions that the set

$$\mathscr{E} := \{(x, u) \in \mathbb{R}^p \times \mathbb{R}^m : x \in \mathscr{X} \quad \text{and} \quad u \in \mathscr{U}(x)\} \tag{4.8}$$

is an ε-eigenset of the function f for $\varepsilon = \zeta$. This procedure will in general, yield a class $\mathbb{E}$ of ε-eigensets of the function f, where $\mathbb{E}$ is empty in the case that no such ε-eigensets exist. Now there are two possibilities—either $\mathbb{E}$ contains an ε-eigenset $\mathscr{E}$ for which $\mathscr{B}_\varepsilon(\mathscr{E})$ is a uniform graph, or it does not. In the first case let $\mathscr{E}_g \in \mathbb{E}$ be an ε-eigenset for which $\mathscr{B}_\varepsilon(\mathscr{E}_g)$ is a uniform graph, and let $g(x)$ be a graphing function for $\mathscr{B}_\varepsilon(\mathscr{E}_g)$. Then a stabilizing feedback function σ for the system Σ can be directly computed using Theorem 3.2 or its corollary. Otherwise, if $\mathbb{E}$ does not contain an ε-eigenset $\mathscr{E}$ for which $\mathscr{B}_\varepsilon(\mathscr{E})$ is a uniform graph, it follows by Theorem 3.1 that the system Σ cannot be internally stabilized with output amplitude bounded by the specified bound δ. However, it may still be possible to internally stabilize the system Σ if the output amplitude bound δ is increased.

In qualitative terms, condition (ii) of the previous paragraph is a controllability-type condition. It requires that for every state $x \in \mathscr{X}$ there be a set $\mathscr{U}(x)$ of input values u that steer the state so that it stays within the set $\mathscr{X}$, even if errors (of amplitudes not exceeding ζ) in x or in u are present. Condition (iii) simply requires all relevant input values to have bounded ampitudes; Proposition 3.4 (ii) can be used to help satisfy this condition in the case where the sets of input values turn out to be too large. For low-dimensional systems ε-eigensets can also be found through graphical methods.

We consider now some examples of the computation of ε-eigensets and stabilizing feedback functions. The simplest case is, of course, that of a single-input single-output system, so we consider this first.

Example 4.1

Let $\Sigma : S(\mathbb{R}) \to S(\mathbb{R})$ be the input/state system with the recursive representation

$$x_{k+1} = (x_k)^2 \sin x_k + [1 + (x_k)^2]u_k, \quad k = 0, 1, 2, \ldots \tag{4.9}$$

and the nominal initial condition $x_{00} = 0$. Suppose that it is required to internally stabilize the system Σ. Let $\theta, \delta > 0$ be the desired bounds on the input and output amplitudes of the closed-loop system respectively. From (4.9) the recursion function f of the given system is a continuous function $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by

$$f(x, u) = x^2 \sin x + (1 + x^2)u \tag{4.10}$$

The first step is to construct an appropriate ε-eigenset for the function f. Since the output (including the noise) has to be bounded by δ, we start by considering the set $\mathscr{D}$ of all elements $(x, u) \in [-\delta, \delta] \times \mathbb{R}$ for which $f[\mathscr{D}] \subset [-\frac{2}{3}\delta, \frac{2}{3}\delta]$, namely

$$\mathscr{D} = \{(x, u) \in \mathbb{R} \times \mathbb{R} : |x^2 \sin x + (1 + x^2)u| \leqslant \tfrac{2}{3}\delta, |x| \leqslant \delta\} \tag{4.11}$$

Incidentally, the set $\mathscr{D}$ is a bounded uniform graph, and a continuous graphing function for it is given, for instance, by

$$g(x) = -\frac{x^2 \sin x}{1 + x^2} \tag{4.12}$$

Next, set

$$y := x^2 \sin x + (1 + x^2)u \tag{4.13}$$

Then for any pair of elements x and y the appropriate input value is given by

$$u=\frac{y-x^2\sin x}{1+x^2} \tag{4.14}$$

Thus for each value of x the set $\mathscr{U}'(x)$ of all input values u for which $|y|\leqslant\frac{2}{3}\delta$ consists of all elements $u\in\mathbb{R}$ satisfying

$$\frac{-\frac{2}{3}\delta-x^2\sin x}{1+x^2}\leqslant u\leqslant\frac{\frac{2}{3}\delta-x^2\sin x}{1+x^2} \tag{4.15}$$

Now let $\zeta:=\min\{\frac{1}{3}\delta/(1+x^2), |x|\leqslant\delta\}=\frac{1}{3}\delta/(1+\delta^2)$ and let $\mathscr{U}(x)$ be the set of all elements $u\in\mathbb{R}$ for which

$$\frac{-\frac{2}{3}\delta-x^2\sin x}{1+x^2}+\zeta\leqslant u\leqslant\frac{\frac{2}{3}\delta-x^2\sin x}{1+x^2}-\zeta \tag{4.16}$$

Note that for every pair of elements (x, u), where $|x|\leqslant\delta$ and $u\in\mathscr{U}(x)$, we have $|f(x, u)|\leqslant\frac{2}{3}\delta$. Consequently, setting $\varepsilon:=\zeta$ and recalling that $\zeta<\frac{1}{3}\delta$, it follows that the set

$$\mathscr{E}:=\{(x, u)\in\mathbb{R}\times\mathbb{R}:|x|\leqslant\tfrac{2}{3}\delta, u\in\mathscr{U}(x)\} \tag{4.17}$$

is an ε-eigenset of the function f. Furthermore, it is easy to see that $\mathscr{B}_\varepsilon(\mathscr{E})$ is a uniform graph, and a possible choice of a graphing function for it is

$$g(x)=-\frac{x^2\sin x}{1+x^2}$$

In order to construct a stabilizing feedback function for the system Σ by using the corollary to Theorem 3.2, we define

$$\lambda(x):=\left(\frac{\frac{2}{3}\delta}{1+x^2}-\zeta\right)\Big/\theta \tag{4.18}$$

Noting that $\lambda(x)\geqslant\frac{1}{3}\delta/[(1+\delta^2)\theta]$ for all $|x|\leqslant\delta$, it then follows by our construction and the corollary to Theorem 3.2 that

$$\sigma(x, v)=\lambda(x)v-\frac{x^2\sin x}{1+x^2} \tag{4.19}$$

is a reversible feedback function that internally stabilizes the system Σ for all input sequences belonging to $S(\theta)$. The nominal initial condition of Σ can be any satisfying $|x_{00}|\leqslant\frac{2}{3}\delta$. Since the numbers δ and θ here can be chosen as large as desired, internal stabilization of the system Σ is obtained over an arbitrarily large domain. Note, however, that when δ is increased, the maximal permissible noise amplitude ζ decreases.

As this example demonstrates, the non-linear static state feedback theory developed in § 3 can be used to explicitly compute reversible feedback functions that internally stabilize a given non-linear system. In the example other stabilizing feedback functions can be obtained by changing the graphing function g or the function λ, or by using an entirely different form of the feedback function σ consistent with the conditions of Theorem 3.2. As can be seen from the example, the stabilizing feedback function σ is, in general, non-additive, and non-additive feedback is indeed essential for achieving stabilization over large domains.

We next provide an additional example of the computation of a stabilizing feedback function for a non-linear system. This time the system that needs to be stabilized has two states and one input.

Example 4.2

The term 'nominal' is used here to refer to signals before noise is added. Consider the non-linear input/state system $\Sigma: S(\mathbb{R}) \to S(\mathbb{R}^2)$ described by the recursion

$$\begin{bmatrix} x_{1,k+1} \\ x_{2,k+1} \end{bmatrix} = \begin{bmatrix} (x_{1,k})^2 + x_{2,k} \\ \sin x_{2,k} + [1 + (x_{2,k})^2]u_k \end{bmatrix} \tag{4.20}$$

where x_1 and x_2 are the coordinates of the state vector, and the nominal initial condition is $x_{00} = 0$. The recursion function is clearly given by

$$f(x, u) = \begin{bmatrix} (x_1)^2 + x_2 \\ \sin x_2 + [1 + (x_2)^2]u \end{bmatrix} \tag{4.21}$$

The output sequences of the closed-loop system (including the noise) are required to be bounded by the real number $\delta > 0$; the input sequences are taken from $S(\theta)$, where $\theta > 0$ is a specified real number. In order to construct a stabilizing feedback function, we first need to find an appropriate ε-eigenset for the recursion function f. Assume that the noise level does not exceed $\frac{1}{3}\delta$; namely, take $0 < \varepsilon < \frac{1}{3}\delta$ and consider the set $\mathscr{D}$ of all points $(x_1, x_2, u) \in \mathbb{R}^2 \times \mathbb{R}$ satisfying $|(x_1, x_2)| \leqslant \delta$ and $|f(x, u)| \leqslant \frac{2}{3}\delta$. In explicit form, the set $\mathscr{D}$ consists of all points for which the following conditions hold: (i) $|(x_1, x_2)| \leqslant \delta$; (ii) $|(x_1)^2 + x_2| \leqslant \frac{2}{3}\delta$; and (iii) $|\sin x_2 + [1 + (x_2)^2]u| \leqslant \frac{2}{3}\delta$. Denote

$$y_1 := (x_1)^2 + x_2 \tag{4.22}$$

$$y_2 := \sin x_2 + [1 + (x_2)^2]u \tag{4.23}$$

Then for a given value of y_1

$$x_2 = y_1 - (x_1)^2 \tag{4.24}$$

Now let ζ_1 and ζ_2 be real numbers describing the permissible range for y_1 excluding the noise, so that $-\zeta_1 \leqslant y_1 \leqslant \zeta_2$; as we shall see later, ζ_1 and ζ_2 can be chosen positive here. By (ii), $|y_1| \leqslant \frac{2}{3}\delta$, so that $0 < \zeta_1, \zeta_2 \leqslant \frac{2}{3}\delta$. Since x_1 and y_1 correspond to the same state, also $-\zeta_1 \leqslant x_1 \leqslant \zeta_2$. Let ε_1 denote the maximal permitted noise amplitude on x_1. Then $0 < \varepsilon_1 < \frac{1}{3}\delta$, and the entire range for x_1, including the noise, is $-\zeta_1 - \varepsilon_1 \leqslant x_1 \leqslant \zeta_2 + \varepsilon_1$. For each x_1 it follows from (4.24) that the nominal domain of x_2 is given by

$$-\zeta_1 - (x_1)^2 \leqslant x_2 \leqslant \zeta_2 - (x_1)^2 \tag{4.25}$$

and, since the nominal x_2 is bounded by $\frac{2}{3}\delta$, the two conditions $-\zeta_1 - (x_1)^2 \geqslant -\frac{2}{3}\delta$ and $\zeta_2 - (x_1)^2 \leqslant \frac{2}{3}\delta$ must hold. In order to satisfy the first condition, we need

$$\zeta_1 + (\zeta_1 + \varepsilon_1)^2 \leqslant \tfrac{2}{3}\delta, \quad \zeta_1 + (\zeta_2 + \varepsilon_1)^2 \leqslant \tfrac{2}{3}\delta \tag{4.26}$$

The second condition will then hold since $\zeta_2 \leqslant \frac{2}{3}\delta$.

Next, allowing a noise level of $0 < \varepsilon_2 < \frac{1}{3}\delta$ for x_2, we need, by (4.22), that $-\zeta_1 \leqslant (x_1)^2 + x_2 \leqslant \zeta_2$, or

$$-\zeta_1 \leqslant (y_1 + n_1)^2 + y_2 + n_2 \leqslant \zeta_2 \tag{4.27}$$

where $|n_1| \leqslant \varepsilon_1$ and $|n_2| \leqslant \varepsilon_2$. Now let $\xi_1 < \zeta_1$, $\xi_2 < \zeta_2$ and $\varepsilon' < \varepsilon_1$ be positive real numbers such that (4.27) holds whenever

$$-\xi_1 \leqslant (y_1)^2 + y_2 \leqslant \xi_2, \quad |(y_1, y_2)| \leqslant \tfrac{2}{3}\delta, \quad |n_1| \leqslant \varepsilon', \quad |n_2| \leqslant \varepsilon' \tag{4.28}$$

Defining $z := (y_1)^2 + y_2$, we get from (4.22) and (4.23)

$$z = [(x_1)^2 + x_2]^2 + \sin x_2 + [1 + (x_2)^2]u, \quad -\xi_1 \leqslant z \leqslant \xi_2 \tag{4.29}$$

Thus for every permissible x_1, x_2 and z the input value is determined by

$$u = \frac{z - [(x_1)^2 + x_2]^2 - \sin x_2}{1 + (x_2)^2} \tag{4.30}$$

Further, setting the maximal noise level of the noise v of (3.8) at $|v| \leqslant \min\{\frac{1}{2}\xi_1/(1+\delta^2), \frac{1}{2}\xi_2/(1+\delta^2)\} =: \varepsilon''$, and denoting $\varepsilon := \min\{\varepsilon', \varepsilon''\}$, $\eta_1 := \xi_1 - \varepsilon$ and $\eta_2 := \xi_2 - \varepsilon$, we find that the nominal domain $\mathscr{U}(x)$ for u is

$$\frac{-\eta_1 - [(x_1)^2 + x_2]^2 - \sin x_2}{1 + (x_2)^2} \leqslant u \leqslant \frac{\eta_2 - [(x_1)^2 + x_2]^2 - \sin x_2}{1 + (x_2)^2} \tag{4.31}$$

for any permissible values of x_1 and x_2. It then follows that the set of all points $(x_1, x_2, u) \in \mathbb{R}^2 \times \mathbb{R}$ for which $-\xi_1 \leqslant x_1 \leqslant \xi_2$, $-\xi_1 - (x_1)^2 \leqslant x_2 \leqslant \xi_2 - (x_1)^2$ and $u \in \mathscr{U}(x)$ forms an ε-eigenset for our recursion function f. It is then easy to verify that $\mathscr{B}_\varepsilon(\mathscr{E})$ is a uniform graph, and a graphing function for it can be directly obtained from (4.31). A possible choice for the graphing function is

$$g(x) = \frac{\frac{1}{2}(\eta_2 - \eta_1) - [(x_1)^2 + x_2]^2 - \sin x_2}{1 + (x_2)^2} \tag{4.32}$$

Using this graphing function, a stabilizing feedback function can be derived using the corollary to Theorem 3.2 as follows. Recalling that θ is the desired bound on the input amplitude of the closed-loop system, define $\lambda(x) := \frac{1}{2}(\eta_1 + \eta_2)/\{[1 + (x_2)^2]\theta\}$. Then a reversible feedback function that internally stabilizes the system Σ is given by

$$\sigma(x, v) := \lambda(x)v + g(x) \tag{4.33}$$

where $|v| \leqslant \theta$. Thus we have obtained a stabilizing feedback controller for our system. The domain over which internal stabilization is achieved can be increased by increasing the value of δ. The nominal initial condition of the system Σ can be any (x_{01}, x_{02}) satisfying $-\xi_1 \leqslant x_{01} \leqslant \xi_2$ and $-\xi_1 - (x_{01})^2 \leqslant x_{02} \leqslant \xi_2 - (x_{01})^2$; the noise amplitudes must not exceed ε.

As we have seen throughout our discussion, and in particular in Theorem 3.1, the notion of a uniformly conductive function is the most fundamental notion of the theory of static state feedback for non-linear systems. An input/state system is internally stabilizable if and only if its recursion function is uniformly conductive. We conclude this section with a discussion related to the question of the size of the class of uniformly conductive functions. The result that we provide below in this context is both strong and weak at the same time. It is strong since it shows that the class of uniformly conductive functions is very large, and includes most systems of practical interest. At the same time, it is rather weak, since it is based on linearization techniques, and is thus of a local nature, referring only to stabilization over small amplitudes of the state variables. This contrasts with the scope of the non-linear state feedback theory developed in the present paper, which is a truly global non-linear feedback theory, entirely devoid of linearization techniques or local arguments. Still, the result is relevant to our present discussion, and so we provide it here without proof. It is a direct consequence of Theorem 3.1 and some well-known results on local stabilization of non-linear systems through linearization.

Proposition 4.1

Let $f:\mathbb{R}^p \times \mathbb{R}^m \rightarrow \mathbb{R}^p$ be a function that is continuously differentiable in a neighbourhood of the origin. Let $J := (A, B)$ be the jacobian matrix of the partial derivatives of f at the origin, partitioned into the $p \times p$ matrix A and the $p \times m$ matrix B. If the pair (A, B) is reachable then the function f is uniformly conductive at the origin.

To conclude, a comprehensive static state feedback theory for non-linear systems has been presented in § 3. The theory is of a global nature, and provides necessary and sufficient conditions for stabilization. These conditions are stated in terms of properties of the given recursion function of the system that needs to be stabilized. When stabilization is possible, the theory provides a computational method that yields static reversible feedback controllers that internally stabilize the system. The main step of this method is the derivation of ε-eigensets of the given recursion function, and it involves the solution of certain sets of inequalities. Computer programs can be developed to implement the required computations.

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