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STATE FEEDBACK FOR NONLINEAR CONTINUOUS-TIME SYSTEMS: STABILIZATION AND THE CREATION OF INVARIANT SUBSPACES

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ABSTRACT

A theory of static state feedback for multivariable continuous-time nonlinear systems is formulated. The theory applies to systems described by differential equations of the form $\dot{x}(t) = f(x(t),u(t))$. The basic objective is to design static state feedback compensators which achieve the following properties: (i) the state space trajectory of the closed loop system is confined within a specified subspace, and (ii) the closed loop system is internally stable. An explicit method for designing such compensators is developed. The construction of the compensators involves only quantities directly derived from the given function f.

1. INTRODUCTION

In many practical applications it is necessary to stabilize a nonlinear system, while guarantying that its trajectories stay confined within a specified subspace of the output space. For example, in most design applications, it is required that the amplitudes of the components of the output vector do not exceed a prespecified level, usually determined by the physical specifications of the devices on which they appear. Thus, the output vectors are required to stay confined within the subspace determined by the maximal amplitude levels. Other examples abound.

Consider for instance the design problem of an autopilot for an airplane. As well known, excessive maneuvers may reduce the airplane's lift force below the critical level, causing the airplane to stall and fall. In order to avoid this danger, the autopilot controller can be designed so that the steering surfaces of the airplane never tilt beyond the safety range for each level of airspeed. Here again it is necessary to confine the output of the system to within a certain subspace of the output space.

Another class of examples is related to the control of biological systems. Consider a biological system consisting of n cells, each of which can be described by k dynamical variables, so that the entire system has nk dynamical state variables. The cells may grow or recede, and each cell may divide into two new cells or die. For the sake of demonstration, assume that the division or death of a cell is controlled by one of its dynamical variables, say x_j, so that division of the cell occurs when $x_j > \beta$ and death of the cell occurs when $x_j < \alpha$, where $\alpha < \beta$ are fixed real numbers. Then, in order to prevent the cell from dying or dividing, namely, in order to stabilize the number of cells, it is necessary to design a controller guarantying that $\alpha < x_j < \beta$ for all appropriate state variables x_j. Thus, it is again necessary to confine the system to within a proper subspace of its output space. Note that the problem of controlling biological systems is of critical importance, since it is related to the restrain of cancerous phenomena, many of which may be regarded as manifestations of instabilities in the reproductive process of cells.

For systems whose output is their state, the problem of confining the output to a specified subspace becomes the problem of creating an invariant subspace within the state space of the system. Whenever the system is started from an initial condition in that subspace, it is required to stay within the subspace at all times. In addition, internal stabilization of the entire control configuration is also necessary. The objectives are to be achieved by nonlinear static state feedback.

Technically, let $\boldsymbol{\Sigma}$ be the given system, let $\mathcal{Y}$ be its output space, and let $\mathcal{V}$ be a suitable subspace of $\mathcal{Y}$. The basic objective is to construct an internally stable closed loop control configuration around $\boldsymbol{\Sigma}$ so that all possible output vectors $y(t)$ of the closed loop system at the time t satisfy $y(t) \in \mathcal{V}$ for all $t \geq 0$, whenever $y(0) \in \mathcal{V}$. Presently, we assume that there is a coordinate transformation of the output space $\mathcal{Y}$ under which the subspace $\mathcal{V}$ transforms into a rectangular box V. We denote by Σ the system obtained from $\boldsymbol{\Sigma}$ after this coordinate transformation, and, to simplify notation, we regard Σ as the given system. Then, our basic control problem reduces to the following

(1.1) RECTANGULAR CONFINEMENT PROBLEM. Let Σ be a given nonlinear system, let $y(t)$ denote its output vector at the time t, and let n be the dimension of $y(t)$. Also, let $\alpha_1, ..., \alpha_n, \beta_1, ..., \beta_n$ be specified real numbers, where $\alpha_i < \beta_i$ for all $i = 1, ..., n$. Design an internally stable closed loop configuration around Σ, whose output vector $y(t)$ at the time t satisfies $\alpha_i < y_i(t) < \beta_i$ for all $i = 1, ..., n$ and all $t \geq 0$.

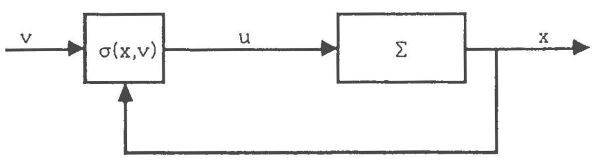
In HAMMER [1989d], an implementable solution was presented to the problem of rectangular confinement (with internal stabilization) for nonlinear continuous-time systems which have their state as output. The solution is based on the use of static state feedback. The purpose of the present talk is to review and reinterpret these results, and present them in concise form.

Consider a nonlinear system Σ described by a differential equation of the form

$$\dot{x}(t) = f(x(t),u(t)), \quad x(0) = x_0, \tag{1.2}$$

where $x(t)$ is an n-dimensional real vector describing the output of the system at the time t, and $u(t)$ is an m-dimensional real vector describing the input of the system at the time t. The function f is assumed to be continuously differentiable. A representation of the form (1.2) is usually called a *state representation* of the system Σ, and $x(t)$ is identified as the *state* of the system at the time t. A system that can be represented in the form (1.2) is called an *input/state system*. Assume for now that (1.2) has a unique solution $x(t)$, $t \geq 0$, for all relevant initial conditions x_0 and input functions $u(t)$, $t \geq 0$. The system Σ is enclosed in a static state-feedback loop of the form

(1.3)



where σ is a continuously differentiable function representing the feedback, and v represents an external input. The closed loop system described by the diagram is denoted by Σ_σ. As can be easily seen, Σ_σ is still an input/state system, with a state representation given by

(1.4) $\dot{x}(t) = f(x(t),\sigma(x(t),v(t)))$.

The present note presents necessary and sufficient conditions for the existence of continuously differentiable feedback functions σ which internally stabilize the closed loop (1.3) while providing desired rectangular confinement of the output vector x. Furthermore, whenever such feedback functions exist, a procedure for their computation is outlined. Finally, a computational example is provided.

The technical background for this presentation is taken from HAMMER [1989d and b]. Alternative recent investigations on the stabilization of nonlinear control systems can be found in HAMMER [1984, 1989a, b, c], DESOER and KABULI [1988], SONTAG [1989], VERMA [1988], TAY and MOORE [1989], the references cited in these papers, and others.

2. NOTATION AND BASICS.

The systems considered here are described by differential equations of the form

(2.1) $\dot{x}(t) = f(x(t),u(t)),\ x(0) = x_0$,

where $t \geq 0$; $x(t) \in R^n$ is the state of the system; and $u(t) \in R^m$ is the input of the system. The function $f : R^n \times R^m \to R^n : (x,u) \mapsto f(x,u)$ is called the *state representation function* of the system, and is assumed to be continuously differentiable for all relevant x and u. The system represented by (2.1) is denoted by Σ. For the sake of simplicity, we assume that the system Σ is time invariant, namely, that the function f does not explicitly depend on the time variable t.

In order to control the system Σ, we insert it into the closed loop configuration (1.3). Here, $\sigma : R^n \times R^m \to R^m : (x,v) \mapsto \sigma(x,v)$ is again a continuously differentiable function, serving as the state-feedback, and v is the external input variable of the closed loop. The way the feedback function σ depends on the external input variable v is of interest, and the following two extreme cases arise: (i) σ does not depend at all on v, in which case $\sigma(x,v) = \sigma(x)$; and (ii) $\sigma(x,v)$ is injective in v for every state x. The first case yields a pure feedback configuration with no external input v, whereas the second case yields a reversible feedback configuration (HAMMER [1989a]), which plays an important role in the theory of fraction representations for nonlinear systems. It seems that in the context of control theory, these two extreme cases are the most important ones. The present note deals with case (i), where the external influence on the closed loop is only through the initial condition x_0. The feedback function σ is then simply a continuously differentiable function $R^n \to R^m : x \mapsto \sigma(x)$, and the closed loop system Σ_σ becomes

(2.2) $\dot{x}(t) = f(x(t),\sigma(x(t))),\quad x(0) = x_0$.

Before stating our basic technical problem, we need some notation. Let x_i be the i-th component of a vector $x \in R^n$. Denote by $[-\theta,\theta]^n$, where $\theta > 0$ is a real number, the set of all vectors $x \in R^n$ for which $|x_i| \leq \theta$ for all $i = 1, ..., n$. As usual, our time set is the set R^+ of all non-negative real numbers, so that the response of the system Σ is simply a function $x : R^+ \to R^n$. Let $C(R^n)$ be the set of all continuous functions $h : R^+ \to R^n$. For a real number $\theta > 0$, let $C(\theta^m)$ be the set of all functions $u \in C(R^m)$ satisfying $|u_i(t)| \leq \theta$ for all $t \geq 0$ and all $i = 1, ..., m$, namely, the set of all continuous functions bounded by θ. Denote by x^T the transpose of a vector $x \in R^n$. Given two vectors $\alpha, x \in R^n$, where $\alpha = (\alpha_1, ..., \alpha_n)^T$ and $x = (x_1, ..., x_n)^T$, let $x \geq \alpha$ (respectively, $x > \alpha$) indicate that $x_i \geq \alpha_i$ (respectively, $x_i > \alpha_i$) for all $i = 1, ..., n$. For two vectors $\alpha, \beta \in R^n$, where $\alpha < \beta$, denote by $[\alpha,\beta]$ (respectively, (α,β)) the set of all vectors $x \in R^n$ satisfying $\alpha_i \leq x_i \leq \beta_i$ (respectively, $\alpha_i < x_i < \beta_i$) for all $i = 1, ..., n$. Let $C((\alpha,\beta))$ be the set of all functions $x \in C(R^n)$ satisfying $x(t) \in (\alpha,\beta)$ for all $t \geq 0$. More generally, for a subset $S \subset R^n$, let $C(S)$ be the set of all functions $x \in C(R^n)$ satisfying $x(t) \in S$ for all $t \geq 0$. We consider rectangular confinement within the domain (α,β), and refer to it as *(α,β)-confinement*.

(2.3) (α,β)-CONFINEMENT BY PURE STATE FEEDBACK. Let Σ be an input/state system described by the differential equation (2.1), where the state representation function $f : R^n \times R^m \to R^n$ is continuously differentiable. Let $\alpha, \beta \in R^n$, where $\alpha < \beta$, be two fixed vectors. Find a continuous feedback function $\sigma : [\alpha,\beta] \to R^m : x \mapsto \sigma(x)$, which is continuously differentiable over (α,β), and for which the following holds. The differential equation (2.2) has a unique solution $x(t)$, $t \geq 0$, for any initial condition $x_0 \in (\alpha,\beta)$, and this solution satisfies $\alpha < x(t) < \beta$ for all $t \geq 0$. ♦

The definition of confinement uses the open domain (α,β) order to permit the incorporation of disturbances; when a disturbance $\nu(t)$ is added to $x(t)$, the sum $x(t) + \nu(t)$ is still required to be within the domain $[\alpha,\beta]$ of the feedback function σ. Consequently, the trajectory $x(t)$ itself has to be confined to the interior.

Note that when rectangular confinement is combined with an appropriate coordinate transformation of the state space, it facilitates the confinement of the closed-loop system response to rather general subspaces, and not just to subspaces of the form (α,β).

The solution of the (α,β)-confinement problem reviewed here (from HAMMER [1989d]) also yields an explicit and implementable method for internal stabilization of the given system Σ by state feedback. It is valid for a very general class nonlinear continuous-time input/state systems. The fact that the resulting theory of stabilization is closely linked to the solution of the confinement problem is really a substantial advantage from a practical standpoint, since in engineering practice problems of stabilization and confinement are in many cases inseparable. Usually, amplitude bounds at various points of a designed system have to be guaranteed as part of the stabilization process, to avoid physical damage to components. The necessary and sufficient conditions for the existence of a stabilizing state feedback function σ, as well as the construction of σ, depend only on quantities directly derived from the given state representation function f of Σ. The conditions are explicitly verifiable and the construction of σ is implementable.

In preparation for a discussion of the notion of stability, we introduce some norms. The usual L^∞-norm on R^n is denoted by $|\cdot|$, and is given by the maximal absolute value of the coordinates $|x| := \max\{|x_1|, ..., |x_n|\}$. The L^∞-norm on $C(R^n)$ is also denoted by $|\cdot|$, and is given by $|h| := \mathrm{Sup}_{t \geq 0} |h(t)|$ for a function $h \in C(R^n)$. The notion of continuity that we use for systems is with respect to a weighted L^∞-norm ρ on $C(R^n)$, which is given by

(2.4) $\rho(h) := \mathrm{Sup}_{t \geq 0}\, 2^{-t}|h(t)|,\ h \in C(R^n)$.

We shall combine the norm ρ with a separate boundedness requirement, and use it in our definition of stability. The resulting stability notion conforms with the intuitive conception of stability in control theory (see HAMMER [1989d] for a detailed discussion).

Let Σ be a system described by the differential equation (2.1), and assume (2.1) has a unique solution $x(t)$, $t \geq 0$, for any relevant initial condition x_0 and input function u. Then, the system Σ may be regarded as a map $\Sigma : R^n \times C(R^m) \to C(R^n)$ which assigns to each pair $(x_0,u) \in R^n \times C(R^m)$ an output function $x \in C(R^n)$, where $x_0 \in R^n$ is the initial condition and $u \in C(R^m)$ is the input function. Given a subset $A \subset R^n \times C(R^m)$, let $\Sigma(A)$ be the image of the set A through Σ, namely, the set of all output functions generated by the system Σ from elements of A. Sometimes, the initial conditions of the system Σ are known to be restricted to a prescribed subset $S \subset R^n$, and its input functions are known to be restricted to a subset $C \subset C(R^m)$. We shall indicate such restrictions simply by writing $\Sigma : S \times C \to C(R^n)$. Now, let $\omega, \vartheta > 0$ be two real numbers. The set of initial conditions in S that are bounded by ω is clearly $S \cap [-\omega,\omega]^n$, and the set of input functions in C bounded by ϑ is $C \cap C(\vartheta^m)$. The response of Σ to the entire set of such initial conditions and input functions is $\Sigma\{[S \cap [-\omega,\omega]^n] \times [C \cap C(\vartheta^m)]\}$.

(2.5) DEFINITION. A system $\Sigma : S \times C \to C(R^n)$ described by the differential equation (2.1) is *BIBO* (*B*ounded-*I*nput *B*ounded-*O*utput)-*stable* if the following conditions hold: (i) For every initial condition $x_0 \in S$ and every input function $u \in C$, the equation (2.1) has a unique solution $x(t)$, $t \geq 0$; and (ii) For every pair of real numbers $\omega, \vartheta > 0$, there exists a real number $M > 0$ such that $\Sigma\{[S \cap [-\omega,\omega]^n] \times [C \cap C(\vartheta^m)]\} \subset C(M^n)$ ♦

Next, define the norm ρ on the space $R^n \times C(R^m)$ by setting

(2.6) $\rho(x,u) := |x| + \rho(u)$

for all $x \in R^n$ and all $u \in C(R^m)$. The same symbol ρ is used here to simplify notation. The notion of stability employed in the present paper is then as follows.

(2.7) DEFINITION. A system $\Sigma : S \times C \to C(R^n)$ is *stable* if it is BIBO-stable, and if, for every pair of real numbers $\omega, \vartheta > 0$, its restriction $\Sigma : [S \cap [-\omega,\omega]^n] \times [C \cap C(\vartheta^m)] \to C(R^n)$ is a continuous function (with respect to the norm ρ). ♦

An important property of the notion of stability defined in (2.7) is the simplicity it lends to the theory of stabilization of nonlinear systems. Indeed, as the next statement indicates (see HAMMER [1989d] for proof), a system described by a differential equation of the form (2.1) is stable whenever it is BIBO-stable. In other words, boundedness (with respect to the L^∞-norm) of the output functions $x(t)$, $t \geq 0$, implies their continuous dependence on the initial condition x_0 and on the input function u. This is in close analogy to the situation in the discrete-time case (HAMMER [1989a]).

(2.8) PROPOSITION. Let $\Sigma : R^n \times C(R^m) \to C(R^n)$ be a system described by (2.1), where $f : R^n \times R^m \to R^n$ is a continuously differentiable function. Then, if the system Σ is BIBO-stable, it is also stable.

We consider next the effects of inaccuracies on the performance of the closed loop system (1.3). There are two main sources of inaccuracies - the state representation function f of the system Σ and the feedback function σ. To take these inaccuracies into account, we introduce two noise signals ν_1 and ν_2. The noise signal $\nu_1 \in C(\varepsilon^n)$ is injected into the differential equation describing the system Σ to accommodate inaccuracies in f. The equation with the noise becomes

(2.9) $\dot{x}(t) = f(x(t),u(t)) + \nu_1(t)$.

The noise signal $\nu_2 \in C(\varepsilon^m)$ represents inaccuracies in the feedback function σ, in the form

(2.10) $u(t) = \sigma(x(t)) + \nu_2(t)$.

In both cases, the real number $\varepsilon > 0$ represents the level of inaccuracies and noise. Disturbances on the initial condition x_0 are already permitted by the notion of input/output stability discussed earlier, and so need no special consideration.

Formally, we regard ν_1 and ν_2 as external inputs (over which no control is provided). The closed loop system Σ_σ becomes then a map $\Sigma_\sigma : R^n \times C(\varepsilon^n) \times C(\varepsilon^m) \to C(R^n)$, where the terms in the cross product represent the initial condition x_0, the noise ν_1, and the noise ν_2, respectively. We now define the notion of internal stability.

(2.11) DEFINITION. Let $\omega, \vartheta > 0$ be real numbers, and let $S \subset [-\omega,\omega]^n$ be a subset.The closed loop system (1.3) is *internally stable* (over the bounded domain S of initial conditions) if there is a pair of real numbers $\varepsilon, N > 0$ such that the closed loop system Σ_σ has a unique response $x(t)$, $t \geq 0$, for all $x_0 \in S$, $\nu_1 \in C(\varepsilon^n)$, and $\nu_2 \in C(\varepsilon^m)$, and the following hold.

(i) $\Sigma_\sigma \{S \times C(\varepsilon^n) \times C(\varepsilon^m)\} \subset C(N^n)$, and

(ii) The map $\Sigma_\sigma : S \times C(\varepsilon^n) \times C(\varepsilon^m) \to C(R^n)$ is continuous (with respect to ρ).

The number ε is referred to as the *noise level.* ♦

When the noises ν_1 and ν_2 are present, we shall refer to the (α,β)-confinement problem as the *disturbed (α,β)-confinement problem.* To solve it we need to find a continuous function $\sigma : [\alpha,\beta] \to R^m$ which is continuously differentiable over (α,β), and for which the following is valid: the closed loop system Σ_σ of (1.3) is well defined for all relevant initial conditions and noise signals, and satisfies $\Sigma_\sigma \{(\alpha,\beta) \times C(\varepsilon^n) \times C(\varepsilon^m)\} \subset C((\alpha,\beta))$ for some real number $\varepsilon > 0$. Note that the differential equation describing the closed loop system Σ_σ with the noises ν_1 and ν_2 present is given by

(2.12) $\dot{x}(t) = f\big(x(t),\sigma(x(t))+\nu_2(t)\big) + \nu_1(t), \quad x(0) = x_0.$

Regarding ν_1 and ν_2 as (unspecified) input functions, introduce the augmented input vector $w(t) := \big(\nu_1(t),\nu_2(t)\big) \in R^{n+m}$, $t \geq 0$, and define the function

(2.13) $g(x,w) := f\big(x,\sigma(x)+\nu_2\big) + \nu_1.$

Then, the differential equation of the closed loop system becomes

(2.14) $\dot{x}(t) = g(x(t),w(t)),\ x(0) = x_0,$

where the function $g : R^n \times R^{n+m}$ is continuous (or continuously differentiable) wherever the functions f and σ are continuous (or continuously differentiable). By definition, internal stability of the closed loop system Σ_σ simply means stability over the augmented input space $(\alpha,\beta) \times C(\varepsilon^n) \times C(\varepsilon^m)$. By Lemma (2.8) the latter is equivalent to BIBO-stability over the same input space. Now, if σ is a solution of the disturbed (α,β)-confinement problem, the closed loop system Σ_σ has a unique solution $x(t)$, $t \geq 0$, for any initial condition $x_0 \in (\alpha,\beta)$ and for any noise signals $\nu_1 \in C(\varepsilon^n)$ and $\nu_2 \in C(\varepsilon^m)$, and this solution satisfies $x(t) \in (\alpha,\beta)$ for all $t \geq 0$. Recalling that BIBO-stability means existence of a unique solution and boundedness, these facts imply the following conclusion (HAMMER [1989d]).

(2.15) PROPOSITION. Let Σ be a system described by the differential equation (2.1), where $f : R^n \times R^m \to R^n$ is a continuously differentiable function. Let $\alpha < \beta$ be two fixed vectors in R^n, and let $\sigma : R^n \to R^m$ be a continuously differentiable feedback function. If σ is a solution of the disturbed (α,β)-confinement problem for the system Σ, then the closed loop system Σ_σ is internally stable (over the domain (α,β) of initial conditions).

Thus, a solution for the disturbed (α,β)-confinement problem also yields internal stabilization of the given system Σ. This result provides yet another manifestation of the convenience resulting from the use of the norm ρ. Even the rather complex notion of internal stability reduces to a simple condition in this framework.

3. CONFINEMENT AND STABILIZATION.

The present section contains a review the solution of the problem of disturbed (α,β)-confinement derived in HAMMER [1989d]. By Proposition (2.15), such a solution also provides internal stabilization of the system Σ that needs to be controlled. We assume that Σ is described by a differential equation of the form (2.1). First, some notation. Let $\alpha, \beta \in R^n$ be two fixed vectors satisfying $\alpha < \beta$, and let $\Gamma(\alpha,\beta)$ denote the boundary of the rectangular box $[\alpha,\beta]$. In explicit terms, the boundary consists of $2n$ faces, given by

$$\Gamma_i^-(\alpha,\beta) := \{(x_1,...,x_n) \in [\alpha,\beta] : x_i = \alpha_i\}$$

(3.1)

$$\Gamma_i^+(\alpha,\beta) := \{(x_1,...,x_n) \in [\alpha,\beta] : x_i = \beta_i\}$$

where $i = 1, ..., n$, and,

(3.2) $$\Gamma(\alpha,\beta) = \cup_{i=1}^n [\Gamma_i^-(\alpha,\beta) \cup \Gamma_i^+(\alpha,\beta)].$$

Let Σ be a system described by the differential equation $\dot{x}(t) = f(x(t),u(t))$, $x(0) = x_0$. The closed loop system Σ_σ is then represented by the differential equation $\dot{x}(t) = f(x(t),\sigma(x(t)))$, $x(0) = x_0$. Clearly, the state representation function f has n components $f_1, ..., f_n$, each of which represents the derivative of the corresponding coordinate x_i, $i = 1, ..., n$, along the system's trajectory. Now, let $\alpha, \beta \in R^n$ be two fixed vectors with $\alpha < \beta$. Consider the class of feedback functions σ satisfying the following inequalities on the boundary of the box $[\alpha,\beta]$, for some real number $\varsigma > 0$.

(3.3) $$\begin{cases} f_i(x,\sigma(x)) \geq \varsigma \text{ for all } x \in \Gamma_i^-(\alpha,\beta),\ i = 1,...,n, \\ f_i(x,\sigma(x)) \leq -\varsigma \text{ for all } x \in \Gamma_i^+(\alpha,\beta),\ i = 1,...,n. \end{cases}$$

Notice that the conditions (3.3) refer only to the values of the component functions $f_i(x,\sigma(x))$ on the boundary $\Gamma(\alpha,\beta)$, namely, on the faces of the box $[\alpha,\beta]$; the specific values of these functions within the box are not considered. Note also that the feedback function σ needs to be defined only over the domain $[\alpha,\beta]$, since, under (α,β)-confinement, the values of the vector x are confined to this domain during the operation of the closed loop system. Conditions (3.3) have a simple intuitive interpretation. They guaranty that the trajectory $x(t)$ of the closed loop system is 'reflected' back into the box (α,β) whenever it nears the boundary $\Gamma(\alpha,\beta)$. This implies that $x(t) \in (\alpha,\beta)$ for all $t \geq 0$ whenever $x(0) \in (\alpha,\beta)$. The fact that $\varsigma > 0$ assures that this situation is not changed even when the noise ν_1 is present, as long as the noise level is small enough. The following statement, which is reproduced here from HAMMER [1989d], provides a formal indication of the significance of conditions (3.3).

(3.4) PROPOSITION. Let $\sigma : [\alpha,\beta] \to R^m$ be a continuous function solving the disturbed (α,β)-confinement problem. Then, there is a real number $\varsigma > 0$ for which conditions (3.3) are satisfied.

Thus, (3.3) is a necessary condition for the feedback function σ to be a solution of the disturbed (α,β)-confinement problem. As one might expect, it is also a critical ingredient in a sufficient condition for disturbed (α,β)-confinement. We consider next some preliminary implications of this condition on the existence of solutions for our differential equations.

Let $\alpha, \beta \in R^n$ be two fixed vectors satisfying $\alpha < \beta$, and let $\vartheta > 0$ be a real number. Let Σ be a system described by the differential equation

(3.5) $$\dot{x}(t) = g(x(t),w(t)),\ x(0) = x_0,$$

where the dimension of $x(t)$ is n; the dimension of $w(t)$ is p; the input function w is restricted to $C(\vartheta^p)$; and $g : [\alpha,\beta]\times[-\vartheta,\vartheta]^p \to R^n$ is a continuous function which is continuously differentiable over the domain $(\alpha,\beta)\times[-\vartheta,\vartheta]^p$. Assume that there is a real number $\chi > 0$ such that the function g satisfies the following conditions on the boundary $\Gamma(\alpha,\beta)$.

(3.6)

$$\begin{cases} g_i(x,w) \geq \chi \text{ for all } w \in [-\vartheta,\vartheta]^p \text{ and all } x \in \Gamma_i^-(\alpha,\beta),\ i = 1,...,n, \\ g_i(x,w) \leq -\chi \text{ for all } w \in [-\vartheta,\vartheta]^p \text{ and all } x \in \Gamma_i^+(\alpha,\beta),\ i = 1,...,n. \end{cases}$$

Then, as the next statement indicates (HAMMER [1989d]), a unique solution for the differential equation (3.5) is guarantied to exist.

(3.7) PROPOSITION. Let $\alpha < \beta$ be two fixed vectors in R^n, and let $g : [\alpha,\beta]\times[-\vartheta,\vartheta]^p \to R^n$ be a continuous function which is continuously differentiable over the domain $(\alpha,\beta)\times[-\vartheta,\vartheta]^p$. If (3.6) is satisfied for some real number $\chi > 0$, then, for any initial condition $x_0 \in (\alpha,\beta)$ and any input function $w \in C(\vartheta^p)$, the differential equation (3.5) has a unique solution $x(t)$, $t \geq 0$, and $x(t) \in (\alpha,\beta)$ for all $t \geq 0$.

In order to apply the Proposition to the (α,β)-confinement problem, rewrite the differential equation of the closed loop system in the form $\dot{x}(t) = g(x(t),w(t))$, $x(0) = x_0$, of (2.14). Here, the input function w is generated by the noises ν_1 and ν_2 through $w(t) = (\nu_1(t),\nu_2(t))^T \in R^{n+m}$, and the function $g : R^n\times R^{n+m} \to R^n$ is given by $g(x,w) = f(x,\sigma(x)+\nu_2) + \nu_1$, as in (2.13). Clearly, $w \in C(\varepsilon^{n+m})$ whenever $\nu_1 \in C(\varepsilon^n)$ and $\nu_2 \in C(\varepsilon^m)$. Thus, the results of Propositions (3.4) and (3.7) can be combined to obtain necessary and sufficient conditions for the existence of a solution to the disturbed (α,β)-confinement problem. For this purpose we need some notation.

First, given a function $h : R^n \to R$, a subset $S \subset R^n$, and a real number $\varsigma > 0$, let $h\{S\} \geq \varsigma$ indicate the condition $h(y) \geq \varsigma$ for all $y \in S$. Also, Let $\mathcal{B}_\xi(z)$ denote the closed ball in R^m having radius $\xi > 0$ and center at the point z in R^m, namely,

(3.8) $$\mathcal{B}_\xi(z) := \{u \in R^m : |u - z| \leq \xi\}.$$

Note that due to the noise ν_2 in (2.10), the input value u of the system Σ generated by the feedback at the state x may be any element of the ball $\mathcal{B}_\varepsilon(\sigma(x))$, where $\varepsilon > 0$ is the noise level. Conditions (3.3) have to hold, of course, for each such input value. When this fact is taken into account, (3.3) take on the following form, where ξ and ς are two positive real numbers.

(3.9) $$\begin{cases} f_i(x,\mathcal{B}_\xi(\sigma(x))) \geq \varsigma \text{ for all } x \in \Gamma_i^-(\alpha,\beta) \text{ and } i = 1,...,n, \\ f_i(x,\mathcal{B}_\xi(\sigma(x))) \leq -\varsigma \text{ for all } x \in \Gamma_i^+(\alpha,\beta) \text{ and } i = 1,...,n. \end{cases}$$

Propositions (3.4) and (3.7) yield then the following preliminary form of necessary and sufficient conditions for the existence of a feedback function σ that solves the disturbed (α,β)-confinement problem (see HAMMER [1989d] for detailed proof).

(3.10) LEMMA. Let Σ be a nonlinear system described by the differential equation (2.1), where $f : R^n\times R^m \to R^n$ is a continuously differentiable function, and let $\alpha, \beta \in R^n$ be a pair of fixed vectors with $\alpha < \beta$. Then, the disturbed (α,β)-confinement problem by pure feedback has a solution for the system Σ if and only if there is a continuous function $\sigma : [\alpha,\beta] \to R^m$ continuously differentiable over (α,β) and a pair of real numbers $\xi, \varsigma > 0$ for which (3.9) hold.

In fact, the conditions of Lemma (3.10) also guaranty that the closed loop system is internally stable, as one might expect from Proposition (2.15). We state this observation as an independent result (HAMMER [1989d]).

(3.11) PROPOSITION. Let Σ be a nonlinear system described by the differential equation (2.1), where $f : R^n \times R^m \to R^n$ is a continuously differentiable function, and let $\alpha, \beta \in R^n$ be a pair of fixed vectors with $\alpha < \beta$. Let $\sigma : [\alpha,\beta] \to R^m$ be a continuous function which is continuously differentiable over (α,β), and for which (3.9) hold for some real numbers $\xi, \varsigma > 0$. Then, the closed loop system $\Sigma_\sigma : (\alpha,\beta) \to C(R^n)$ is internally stable over the domain (α,β) of initial conditions.

Our next objective is to eliminate the function σ from the conditions of Lemma (3.10), in order to obtain necessary and sufficient conditions for disturbed (α,β)-confinement that involve only the given state representation function f of the system Σ. The new conditions will also yield an explicit method for the construction of feedback functions σ that solve the disturbed (α,β)-confinement problem, whenever such functions exist. With this objective in mind, we introduce some basic quantities (see also HAMMER [1989b]).

As usual, a subset $S \subset R^n \times R^m$ is the *graph* of a function $g : R^n \to R^m$ if $S = \{(x,u) \in R^n \times R^m : u = g(x)\}$. Denote by $\Pi_n : R^n \times R^m \to R^n$ the standard projection onto the first n coordinates, so that $\Pi_n(y_1, ..., y_{n+m})^T = (y_1, ..., y_n)^T$ for every vector $(y_1, ..., y_{n+m})^T \in R^n \times R^m$. Now, let $S \subset R^n \times R^m$ and $X \subset R^n$ be two subsets. Then, S is a *uniform graph on* X if there is a continuous function $g : X \to R^m$ and a real number $\xi > 0$ such that $S = \{(x,u) \in X \times R^m : u \in \mathcal{B}_\xi(g(x))\}$. The function g is then called a *graphing function on* X of the set S, and the number ξ is called a *graphing radius*. Intuitively speaking, a uniform graph is simply a 'thickened' graph of a continuous function. It contains the graphs of all continuous functions $g' : X \to R^m$ satisfying $|g'(x) - g(x)| \le \xi$ for all $x \in X$. The notion of a uniform graph is a natural tool for the discussion of continuous functions whose values are contaminated by noise.

We return now to the problem of disturbed (α,β)-confinement for the system Σ described by (2.1). Let $\varsigma > 0$ be a real number. For each point x of the boundary $\Gamma(\alpha,\beta)$, construct the set of input values

(3.12) $U_{f,\varsigma}(\alpha,\beta,x) :=$

$$\left\{ u \in R^m \;\middle|\; \begin{array}{l} f_i(x,u) \ge \varsigma \text{ for all } i \in \{1,...,n\} \text{ for which } x \in \Gamma^-_i(\alpha,\beta), \\ \text{and} \\ f_i(x,u) \le -\varsigma \text{ for all } i \in \{1,...,n\} \text{ for which } x \in \Gamma^+_i(\alpha,\beta). \end{array} \right\}$$

The set $U_{f,\varsigma}(\alpha,\beta,x) \subset R^m$ is obtained simply by solving a set of inequalities determined by the given state representation function f of the system Σ. For boundary points x that are common to several faces of the box $[\alpha,\beta]$, several of the conditions listed on the right side of (3.12) need to be satisfied simultaneously. By listing each point $x \in \Gamma(\alpha,\beta)$ together with its corresponding set $U_{f,\varsigma}(\alpha,\beta,x)$, we obtain the following subset of $R^n \times R^m$

(3.13) $S_f(\alpha,\beta,\varsigma) :=$

$\{(x,u) \in R^n \times R^m : x \in \Gamma(\alpha,\beta),\ u \in U_{f,\varsigma}(\alpha,\beta,x)\}$.

(3.14) DEFINITION. Let $f : R^n \times R^m \to R^n$ be a continuously differentiable function, and let $\alpha < \beta$ be two fixed vectors in R^n. The function f is *(α,β)-uniformly conductive* if there is a real number $\varsigma > 0$ such that the set $S_f(\alpha,\beta,\varsigma)$ contains a uniform graph on the boundary $\Gamma(\alpha,\beta)$. ♦

Uniform conductivity simply means that a continuous function $g : \Gamma(\alpha,\beta) \to R^m$ exists for which the following holds for all $i = 1, ..., n$: There is a real number $\xi > 0$ such that $f_i(x, \mathcal{B}_\xi(g(x))) \ge \varsigma$ whenever $x \in \Gamma^-_i(\alpha,\beta)$ and $f_i(x, \mathcal{B}_\xi(g(x))) \le -\varsigma$ whenever $x \in \Gamma^+_i(\alpha,\beta)$. To see the implications of uniform conductivity, consider a system Σ described by the differential equation $\dot{x}(t) = f(x(t),u(t))$. Assume there is a continuous feedback function $\sigma : [\alpha,\beta] \to R^m$ solving the disturbed (α,β)-confinement problem for Σ. Then, by Lemma (3.10), there are real numbers $\xi, \varsigma > 0$ for which (3.9) is valid. This directly implies that $\mathcal{B}_\xi(\sigma(x)) \subset U_{f,\varsigma}(\alpha,\beta,x)$ for all $x \in \Gamma(\alpha,\beta)$, so that $S_f(\alpha,\beta,\varsigma)$ contains a uniform graph on $\Gamma(\alpha,\beta)$. Thus, if a solution of the disturbed (α,β)-confinement problem exists, the state representation function f of the system Σ must be (α,β)-uniformly conductive. As it turns out, the converse of this statement is also true, namely, if the given function f is (α,β)-uniformly conductive, then there is a feedback function σ solving the disturbed (α,β)-confinement problem.

Indeed, assume that f is (α,β)-uniformly conductive. Then, on the boundary $\Gamma(\alpha,\beta)$, there is a continuous function $\sigma_\Gamma : \Gamma(\alpha,\beta) \to R^m$ and real numbers $\xi, \varsigma > 0$ such that

(3.15) $\mathcal{B}_\xi(\sigma_\Gamma(x)) \subset U_{f,\varsigma}(\alpha,\beta,x)$ for all $x \in \Gamma(\alpha,\beta)$.

Suppose for a moment that the function $\sigma_\Gamma : \Gamma(\alpha,\beta) \to R^m$ can be extended into a continuous function $\sigma : [\alpha,\beta] \to R^m$ which is continuously differentiable over the interior (α,β). For this extension σ, we have $\mathcal{B}_\xi[\sigma(x)] \subset U_{f,\varsigma}(\alpha,\beta,x)$ for all $x \in \Gamma(\alpha,\beta)$, which means that (3.9) holds for σ with the present values of $\xi, \varsigma > 0$. By Lemma (3.10) this implies that σ is a solution of the disturbed (α,β)-confinement problem. Now, whenever f is uniformly conductive, a continuous boundary function σ_Γ satisfying (3.15) is quite easy to derive from the inequalities (3.12), as a later example will indicate. Thus, the only aspect of the problem that still needs to be considered is the extension of the continuous boundary function σ_Γ into a continuous function $\sigma : [\alpha,\beta] \to R^m$ which is continuously differentiable over the interior (α,β). Such an extension σ can be derived in numerous ways; one way is by using partial differential equations.

For instance, consider the use of the Laplace equation for this purpose. Let $h : R^n \to R : (x_1, ..., x_n)^T \mapsto h(x_1, ..., x_n)$ be a twice continuously differentiable function, and denote by

(3.16) $$\Delta h := \frac{\partial^2 h}{\partial x_1^2} + \frac{\partial^2 h}{\partial x_2^2} + \cdots + \frac{\partial^2 h}{\partial x_n^2}$$

the Laplace operator. Next, let σ_i be the i-th component of our feedback function σ, let $(\sigma_\Gamma)_i$ be the i-th component of the function σ_Γ, and let $\sigma_i|\Gamma(\alpha,\beta)$ denote the values of the function σ_i on the boundary $\Gamma(\alpha,\beta)$. Then, by the continuity of σ_Γ and the form of the boundary $\Gamma(\alpha,\beta)$, it follows from the theory of Laplace equations (e.g., PETROVSKY [1964, Chapter III]) that the subsequent is true. For every $i = 1, ..., m$, the boundary value problem

(3.17) $$\begin{cases} \Delta\sigma_i = 0 \\ \sigma_i|\Gamma(\alpha,\beta) = (\sigma_\Gamma)_i\,, \end{cases}$$

has a unique and continuous solution σ_i over the domain $[\alpha,\beta]$, and this solution is (twice) continuously differentiable over the interior (α,β). When the solutions $\sigma_1, ..., \sigma_m$ are combined into the vector valued function $\sigma = (\sigma_1, ..., \sigma_m)^T$, they create an extension $\sigma : [\alpha,\beta] \to R^m$ of the boundary function σ_Γ. This extension is continuous over $[\alpha,\beta]$ and continuously differentiable over (α,β); by the choice of σ_Γ, it satisfies the conditions (3.9).

Thus, invoking Lemma (3.10), it follows that σ is a solution of the disturbed (α,β)-confinement problem.

Of course, partial differential equations other than (3.17) could also be used to obtain suitable extensions σ, or such extensions could be obtained through other methods, without the use of partial differential equations. In any case, when the discussion of the last few paragraphs is combined with Lemma (3.10) and Proposition (3.11), we obtain the following theorem, which is our main result (HAMMER [1989d]).

(3.18) THEOREM. Let Σ be a system described by the differential equation $\dot{x}(t) = f(x(t),u(t))$, $x(0) = x_0$, where $f : R^n \times R^m \to R^n$ is continuously differentiable, and let $\alpha < \beta$ be two fixed vectors in R^n. Then, the following two statements are equivalent.

(i) There exists a feedback function $\sigma : [\alpha,\beta] \to R^m$ that solves the disturbed (α,β)-confinement problem for the system Σ, with the closed loop system Σ_σ being internally stable for all initial conditions $x_0 \in (\alpha,\beta)$.

(ii) The given state representation function f of Σ is (α,β)-uniformly conductive.

From our discussion, we obtain the following procedure for the solution of the disturbed (α,β)-confinement problem. The given system Σ is described by a differential equation of the form $\dot{x}(t) = f(x(t),u(t))$.

Step 1. Check whether the given state representation function f is (α,β)-uniformly conductive; If it is, find a continuous boundary function $\sigma_r : \Gamma(\alpha,\beta) \to R^m$ which forms a graphing function on $\Gamma(\alpha,\beta)$ for the set $S_f(\alpha,\beta,\varsigma)$. In mathematical terms, this step involves the solution of certain sets of inequalities. Computer programs can be developed to check for uniform conductivity and to compute an appropriate function σ_r whenever it exists. For low dimensional systems, graphical methods may also be employed.

Step 2. Find a continuous extension $\sigma : [\alpha,\beta] \to R^m$ of σ_r which is continuously differentiable on (α,β). This can be done, for instance, by solving the partial differential equation (3.17). Note that since the closed loop configuration Σ_σ is internally stable, approximate solutions for the feedback function σ are also adequate.

Since Theorem (3.18) lists necessary and sufficient conditions, it characterizes all degrees of freedom involved in the solution of the disturbed (α,β)-confinement problem. These consist of the degrees of freedom available in the choice of the boundary function σ_r in Step 1, and the degrees of freedom available in the construction of the extension σ in Step 2. Furthermore, Theorem (3.18) provides a general method for the internal stabilization of nonlinear input/state systems through the use of pure static state feedback. We conclude with a simple example.

(3.19) EXAMPLE. Consider the system Σ with two states and one input, described by the differential equation

$$(3.20)\quad \begin{cases} \dot{x}_1 = 1+[1+(x_2)^2]x_1u, \\ \dot{x}_2 = 1+[1+(x_1)^2]x_2u. \end{cases}$$

For the domain of confinement, we choose $\alpha = (0,0)^T$ and $\beta = (1,1)^T$, so that the system has to be confined within the unit square in the first quadrant. The components of the state representation function f here are

$$\begin{cases} f_1(x_1,x_2,u) = 1+[1+(x_2)^2]x_1u, \\ f_2(x_1,x_2,u) = 1+[1+(x_1)^2]x_2u. \end{cases}$$

We need to compute the sets $U_{f,\varsigma}(\alpha,\beta,x)$ of (3.12) for some $\varsigma > 0$. Choosing $\varsigma = 1$, this simply requires solving for u the inequalities

$f_1(x_1,x_2,u) \geq 1$ for all x_1, x_2 with $x_1 = 0$ and $0 \leq x_2 \leq 1$;

$f_1(x_1,x_2,u) \leq -1$ for all x_1, x_2 with $x_1 = 1$ and $0 \leq x_2 \leq 1$;

$f_2(x_1,x_2,u) \geq 1$ for all x_1, x_2 with $0 \leq x_1 \leq 1$ and $x_2 = 0$;

$f_2(x_1,x_2,u) \leq -1$ for all x_1, x_2 with $0 \leq x_1 \leq 1$ and $x_2 = 1$.

A straightforward calculation yields

$$U_{f,1}(\alpha,\beta,x_1,x_2) = \begin{cases} \{\text{all } u \in R\} \text{ if } x_1 = 0 \text{ and } 0 \leq x_2 < 1, \\ \{\text{all } u \in R \text{ satisfying } u \leq -2/[1+(x_2)^2]\} \text{ if } x_1 = 1 \text{ and } 0 \leq x_2 \leq 1, \\ \{\text{all } u \in R\} \text{ if } 0 < x_1 < 1 \text{ and } x_2 = 0, \\ \{\text{all } u \in R \text{ satisfying } u \leq -2/[1+(x_1)^2]\} \text{ if } 0 \leq x_1 < 1 \text{ and } x_2 = 1. \end{cases}$$

The set $S_f(\alpha,\beta,1)$ is directly determined by the above expression for $U_{f,1}(\alpha,\beta,x)$, and it is easy to see that it contains a uniform graph on the boundary $\Gamma(\alpha,\beta)$. Indeed, observing that the set {all $u \in R$ satisfying $u \leq -2$} is contained in $U_{f,1}(\alpha,\beta,x)$ for all $x \in \Gamma(\alpha,\beta)$, it follows that the constant function $\sigma_r : \Gamma(\alpha,\beta) \to R$ given by $\sigma_r(x) := -3$ is a graphing function for $S_f(\alpha,\beta,1)$, with graphing radius $\xi = 1$. Now, one easy way of extending the function σ_r into a continuous function $\sigma : [\alpha,\beta] \to R$ continuously differentiable over (α,β), is by simply taking the constant function

$$\sigma(x) = -3$$

for all $x \in [\alpha,\beta]$. For this choice of σ, Theorem (3.18) implies that the closed loop system Σ_σ is (α,β)-confined for the present α, β, and internally stable for all initial conditions $x_0 \in (\alpha,\beta)$. ♦

As we can see from the example, the computation of an appropriate feedback function σ is quite simple, and, in many cases, it is not necessary to resort to the solution of the partial differential equation (3.17) for finding an appropriate extension of the boundary function σ_r.

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