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State feedback control of nonlinear systems: a simple approach

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A simple methodology for the design of state feedback controllers for nonlinear continuous-time systems is introduced. The objective is to develop controllers that drive a nonlinear system from an initial condition into an assigned region of state space. It is shown that such state feedback controllers can be derived by solving a set of inequalities obtained directly from quantities given in the controlled system's equation. The results are applied to the design of state feedback controllers that achieve robust asymptotic stabilisation for a rather general class of nonlinear systems.

Keywords: nonlinear control; robust control; state feedback

1. Introduction

Techniques for the design of state feedback controllers for nonlinear systems have received considerable attention in the scientific literature for at least three quarters of a century. Important progress has been made, but, to this day, the process of deriving feedback controllers for nonlinear systems remains a daunting one for many classes of systems. Important difficulties persist in conceptual as well as in computational aspects of the theory of nonlinear feedback control. This paper revisits state feedback control of nonlinear systems from a perspective that enhances intuitive insight and yields a relatively simple process for the derivation of state feedback controllers.

We concentrate on the problem of designing state feedback controllers that drive a nonlinear system Σ from an initial condition into a specified region of state space, a problem that includes asymptotic stabilisation. We show that such feedback controllers can be derived simply by solving a set of inequalities that is obtained directly from quantities given in the equation of the controlled system. These inequalities provide necessary and sufficient conditions for the existence of appropriate feedback controllers as well as computational means for controller design.

The discussion concentrates on 'autonomous' controllers, namely, on controllers that operate on their own with no need for external intervention. The control configuration is described in Figure 1, where Σ is the system being controlled and C is a controller. It is assumed that Σ is an input/state system, namely, that Σ has its state as output, so that C is a state feedback controller. We denote by Σ_c the system induced by the closed-loop configuration. In formal terms, our control objective can be described as follows.

Problem 1.1: Let Σ be an input/state system, let X_0 be a set of potential initial conditions of Σ, and let D_0 be an open domain in R^n. Referring to Figure 1, find necessary and sufficient conditions for the existence a state feedback controller C that takes Σ_c from every initial condition in X_0 into D_0 in finite time. If such a controller exists, provide a method for its derivation.

The domain D_0 of Problem 1.1 is called the *target domain*; it is the domain into which Σ must be driven by the state feedback controller C.

We show in Section 3 that state feedback controllers that fulfill the design objective of Problem 1.1 can be derived from the solution of a set of inequalities; the inequalities are obtained directly from quantities that appear in the differential equation of the controlled system Σ. Furthermore, we show that, whenever achievable, the objective of Problem 1.1 can be achieved by *static* state feedback controllers, namely, by controllers that are described by a state feedback function, rather than being described by a differential equation. All controllers derived in this paper are robust in the sense that they continue to perform their task in the presence small deviations or errors.

In Section 6, we specialise our discussion to the case where the target domain D_0 is a small neighbourhood of the state-space origin and consider the existence and the design of state feedback controllers that take the controlled system Σ closer and closer to the origin of its state space as time progresses. In other words, we examine the existence and the design of state feedback controllers that asymptotically stabilise a given nonlinear system Σ. As before, we show that such controllers can be derived from the solution of a set of inequalities obtained directly from quantities given in the differential equation of Σ.

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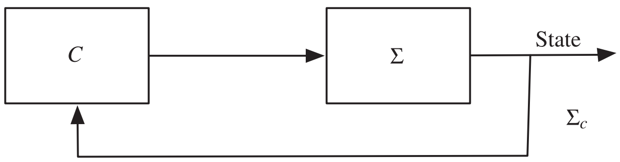

Figure 1. State feedback.

To be specific, consider a time-invariant input/state system Σ described by the differential equation

$$\Sigma : \begin{array}{l} \dot{x}(t) = f(x(t), u(t)),\ t \geq 0, \\ x(0) = x_0, \end{array} \tag{1.1}$$

where $x(t) \in R^n$ is the state of Σ and $u(t) \in R^m$ is the input of Σ at the time t. Here, f: $R^n \times R^m \to R^n$ is a continuous function, and x_0 is the initial state of Σ. (In Section 6, the function f is required to be twice continuously differentiable.) Small uncertainties about the exact values of f and of x_0 are allowed.

Referring to Problem 1.1, our objective is to derive (whenever possible) a state feedback controller C that takes Σ from its initial condition x_0 into the target domain D_0. We show in Section 3 that, if such a state feedback controller exists, it can be chosen as a static time-invariant state feedback controller, namely, as a member of the simplest class of controllers. Such a controller is characterised by a state feedback function φ: $R^n \to R^m$ that generates the input signal $u(t)$ of Σ according to the equation

$$u(t) = \varphi(x(t)). \tag{1.2}$$

The corresponding control configuration is shown in Figure 2, where Σ_φ denotes the closed-loop system. The differential equation of Σ_φ is given by

$$\Sigma_\varphi : \begin{array}{l} \dot{x}(t) = f(x(t), \varphi(x(t))),\ t \geq 0, \\ x(0) = x_0. \end{array} \tag{1.3}$$

As Σ_φ includes no external input, it as an *autonomous system*. In these terms, Problem 1.1 reduces to the following: find necessary and sufficient conditions for the existence of a state feedback function φ such that, for every initial state $x_0 \in X_0$, there is a time $\tau > 0$ at which the state of Σ_φ satisfies $x(\tau) \in D_0$.

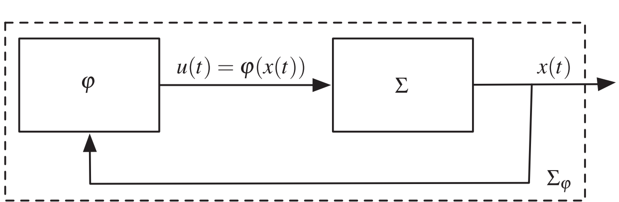

Figure 2. Static state feedback control.

An important aspect of the present approach is its conceptual simplicity. We show in Sections 2 and 3 that an appropriate state feedback function φ can be calculated from the solution of a set of inequalities derived directly from the function f that appears in the differential equation (1.1) of the controlled system Σ. Furthermore, Problem 1.1 has a solution if and only if this set of inequalities has a solution. In this sense, the methodology presented here is reminiscent of linear state feedback theory, where state feedback functions are derived directly from matrices that appear in the controlled system's differential equation. Notwithstanding, the present methodology is very different from its linear counterpart; it does not yield exclusively linear feedback functions when the controlled system Σ is linear.

When discussing feedback controllers that take a system Σ from an initial condition into an assigned target domain in state space, it is natural to contemplate the form of the path taken by the closed-loop system as it makes its way. We focus here on paths formed by broken straight lines, namely, paths that consist of successions of straight line segments. The state feedback function φ is designed to guide the closed-loop system Σ_φ along a broken line from an initial condition x_0 to the target domain D_0. Of course, errors and disturbances, which are permitted, may distort the form of the path. Other categories of paths can also be employed.

The construction of state feedback functions φ is described in detail in Sections 2 and 3. We provide here a simplified (and somewhat inaccurate) overview of this construction. At a state $x \in R^n$, let $U_1(x)$ be the set of all input values $u \in R^m$ for which the vector $f(x, u)$ points from x to the target domain D_0. Let D_1 be the set of all states $x \in R^n$ at which the set $U_1(x)$ is not empty. The set D_1 is derived by solving an inequality induced by the function f of (1.1), namely, by the function that appears in the differential equation of the controlled system Σ.

At each point $x \in D_1$, choose a value $u(x) \in U_1(x)$ and define the state feedback function $\varphi(x) := u(x)$. Then, in view of (1.3), the path derivative $\dot{x}(t) = f(x(t), \varphi(x(t)))$ of the closed-loop system Σ_φ is directed toward D_0 at every point $x(t) \in D_1$. As a result, the state $x(t)$ of the closed-loop system Σ_φ moves toward the target domain D_0 as time progresses. In the special case when the target domain D_0 is a single point, the fact that $\dot{x}(t)$ points toward D_0 at every point of D_1 implies that the closed-loop system Σ_φ moves within D_1 in constant direction toward D_0, thus tracing a straight line segment.

Having built the set D_1, we look at the difference set $D'_2 := R^n \setminus D_1$, namely, the set of remaining states. At a state $x \in D'_2$, denote by $U_2(x)$ the set of all input values $u \in R^m$ for which the vector $f(x, u)$ points from x to a point of the set D_1. Let D_2 be the set of all points $x \in D'_2$ at which $U_2(x)$ is not empty. As before, at each point $x \in D_2$, choose a value $u(x) \in U_2(x)$ and define the state feedback function $\varphi(x) := u(x)$. The set $U_2(x)$ and the domain D_2

are calculated by solving an inequality based on the given function f of (1.1). Considering (1.3), we conclude that the derivative $\dot{x}(t) = f(x(t), \varphi(x(t)))$ points toward the set D_1 at all points $x(t)$ of D_2. Consequently, the state trajectory $x(t)$ takes Σ_φ from every point of D_2 toward a point of D_1. Once a point of D_1 is reached, the values of the state feedback function φ previously defined on D_1 take Σ_φ to the target domain D_0. Thus, the resulting state feedback function φ takes the closed-loop system Σ_φ to the target domain D_0 from all points of the union $D_1 \cup D_2$.

Continuing with this process, we derive in a recursive manner a sequence of domains $D_1, D_2, \ldots \subseteq R^n$, as follows. Let $i \geq 1$ be an integer, and assume that the domains $D_1, D_2, \ldots D_i \subseteq R^n$ have been derived. Then, define the difference set

$$D'_{i+1} := R^n \setminus \left(\bigcup_{j=0,\ldots,i} D_j \right)$$

that consists of all points outside these domains. At each point $x \in D'_{i+1}$, let $U_{i+1}(x)$ be the set of all input values $u \in R^m$ for which the vector $f(x, u)$ points from x to a point of the set D_i. Denote by D_{i+1} the set of all points $x \in D'_{i+1}$ at which $U_{i+1}(x)$ is not empty. The domain D_{i+1} is obtained from the solution of an inequality based on the given function f of (1.1). At each point $x \in D_{i+1}$, choose a value $u(x) \in U_{i+1}(x)$ and define the state feedback function $\varphi(x) := u(x)$. Then, considering (1.3), the path $x(t)$ of the closed-loop system Σ_φ points toward the set D_i at all points of D_{i+1}. As a result, Σ_φ moves from any point of D_{i+1} toward a point of D_i. Once a point of D_i is reached, the previously defined values of φ on D_i take the closed-loop system into the set D_{i-1}. From there, previously defined values of φ assure that Σ_φ progresses to the set D_{i-2}, and so on, until Σ_φ reaches the target domain D_0.

Schematically, the progression of the closed-loop system Σ_φ toward its target domain can be described as depicted in Figure 3. Note that, in general, the domain D_i may not be a connected set for some or for all integers $i = 1, 2, \ldots$

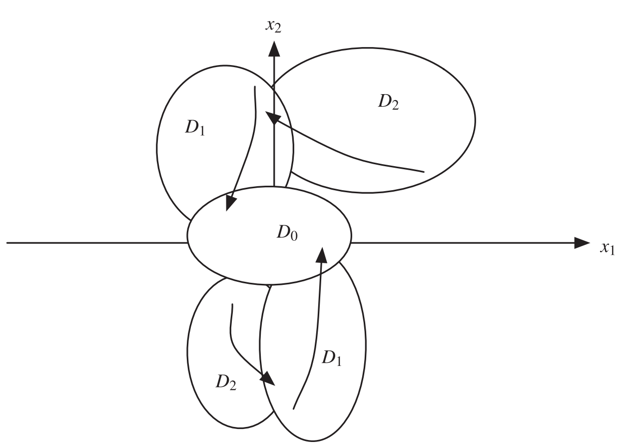

Figure 3. Domains and paths.

In summary, the static state feedback function φ so constructed takes the system Σ to the target domain D_0 from any point of the union:

$$S(D_0) := \bigcup_{i \geq 0} D_i.$$

Furthermore, we show in Section 3 that this is an exclusive feature of the set $S(D_0)$: there is no state feedback controller, not static nor dynamic, which, in finite time, can take Σ into the target domain D_0 from an initial state outside $S(D_0)$. Thus, recalling that X_0 denotes the set of all potential initial conditions of Σ, we conclude that there is a state feedback controller solving Problem 1.1 if and only if

$$X_0 \subseteq S(D_0). \tag{1.4}$$

We show in Section 3 that, whenever it exists, a state feedback controller C that solves Problem 1.1 can be implemented as a static state feedback function φ. The state feedback function φ is obtained from the solution of a set of inequalities based on the function f given in the differential equation (1.1) of the controlled system Σ.

As mentioned earlier, the tools presented here can be used to derive feedback controllers that asymptotically stabilise a nonlinear system Σ. To this end, simply choose the target domain D_0 as a tight neighbourhood of the origin and derive a state feedback function φ that takes Σ into D_0. Then, expand φ onto D_0 by patching it together with a linear state feedback function that asymptotically stabilises a linearisation of Σ at the origin. The resulting state feedback function provides asymptotic stabilisation of Σ over the domain $S(D_0)$. This process yields a simple approach to global stabilisation of nonlinear system by state feedback. The critical computational step involves the solution of a set of inequalities based on the function f given in the differential equation (1.1) of Σ. A detailed discussion of asymptotic stabilisation is provided in Section 6.

As mentioned earlier, we show in Section 3 that, whenever solvable, Problem 1.1 can be solved by a static state feedback controller. Thus, in the context of Problem 1.1, dynamic state feedback is not necessary. Considering the discussion of the previous paragraph, it also follows that asymptotic stabilisation of nonlinear systems, whenever possible, can be achieved by static state feedback. This is not to say that dynamic state feedback controllers are to be ignored, as they may offer broader capabilities of molding the dynamical behaviour of the closed-loop system Σ_c. The static state feedback controllers derived in this report can help calculate desired dynamic controllers. Indeed, a stabilising static state feedback controller can be used to obtain a fraction representation of the controlled system Σ, and such a fraction representation can be utilised to derive dynamical state feedback controllers that assign desirable dynamical behaviour to the closed-loop system (Hammer, 1984, 1985, 1989, 1994).

To conclude, static state feedback controllers that lead a nonlinear system Σ into a prescribed target domain D_0 in state space can be derived from the solution of a set of inequalities obtained directly from the function f given in the differential equation (1.1) of the controlled system Σ. If this set of inequalities has no solution, then there is no state feedback controller, not static nor dynamic, that takes Σ into D_0 from the desired initial condition. These facts provide a simple and transparent foundation for the solution of a wide range of problems in nonlinear control. In Sections 2 and 3, these principles are formulated within a framework that assures robustness of the closed-loop feedback system Σ_φ, guaranteeing that small errors in the function f or in the implementation of the state feedback function φ do not breach control objectives.

Alternative approaches to the control of nonlinear systems can be found in Lasalle and Lefschetz (1961), Lefschetz (1965), Hammer (1984, 1985, 1989, 2004, 1994), Desoer and Kabuli (1988), Verma (1988), Sontag (1989), Chen and de Figueiredo (1990), Paice and Moore (1990), Verma and Hunt (1993), Sandberg (1993), Paice and van der Schaft (1994), Baramov and Kimura (1995), Georgiou and Smith (1997), Logemann, Ryan, and Townley (1999), Hammer (2004), the references cited in these publications, and others.

The paper is organised as follows. Section 2 introduces basic concepts and notation, while Section 3 expands on these concepts and utilises them to derive state feedback controllers that solve Problem 1.1. Section 4 derives necessary and sufficient conditions for robust state feedback control of nonlinear systems. Section 5 consists of two examples that demonstrate the computation of robust state feedback controllers using the formalism developed in Section 4. The paper concludes in Section 6 with the derivation of state feedback controllers that provide robust asymptotic stabilisation of nonlinear systems.

2. Basics

In this section, we introduce some of the notions that underlie our discussion. Let Σ be an input/state system described by the differential equation (1.1) with a continuous function $f: R^n \times R^m \to R^n$, and consider the static state feedback configuration of Figure 2. We concentrate on necessary and sufficient conditions for the existence of a state feedback function $\varphi : R^n \to R^m$ for which the closed-loop system Σ_φ proceeds from an initial state x_0 into a specified target domain D_0 in state space. First, some notation.

Notation: As usual, the superscript ^{T} indicates the transpose. Throughout our discussion, we employ the Euclidian norm $|\cdot|$ given, for an m-dimensional vector $u = (u_1, u_2, \ldots, u_m)^T \in R^m$, by

$$|u| := \sqrt{u_1^2 + u_2^2 + \cdots + u_m^2}.$$

In practice, systems often have bounds on the maximal input signal amplitude they can tolerate. These bounds are determined by structural limitations of a system's components. To incorporate such bounds into our considerations, we assume that the controlled system Σ allows only input signals whose amplitude does not exceed a specified magnitude $M > 0$. It is convenient to state this fact formally for future reference.

Assumption 2.1: *The controlled system Σ permits only input signals u of magnitude $|u| \le M$, where $M > 0$ is a specified real number.*

Note that when Assumption 2.1 is valid, the solution $x(t)$ of the controlled system's differential equation (1.1) is a continuous function of time. We take implicit advantage of this continuity in our forthcoming discussion.

Given a point $s \in R^n$ and a real number $\rho > 0$, the open ball $B(s, \rho)$ of centre s and radius ρ is, as usual, the set

$$B(s, \rho) := \{x \in R^n : |x - s| < \rho\}.$$

With every non-zero vector $z \in R^n$, we associate a *unit vector* $\hat{z}$ in the direction of z; we take $\hat{z}$ to be the zero vector when $z = 0$. Formally,

$$\hat{z} := \begin{cases} z/|z| & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases} \tag{2.1}$$

A slight reflection shows that the function $\hat{\ }$ is continuous everywhere, except at the origin.

An object that is frequently used in our discussion is the straight line segment $\ell(y, z)$ that connects two distinct points $y, z \in R^n$; it consists of the set of points:

$$\ell(y, z) := \{\alpha(z - y) + y : \alpha \in [0, 1]\}. \tag{2.2}$$

Using such straight line segments, we build the following body in R^n.

Definition 2.2: Given two points $z, s \in R^n$ and a real number $\rho > 0$, the *ball-cone* $\chi(z, s, \rho)$ is a body in R^n consisting of all straight line segments that start in the open ball $B(s, \rho)$ and end at z, namely,

$$\begin{aligned}\chi(z, s, \rho) : &= \bigcup_{y \in B(s,\rho)} \ell(y, z) \\ &= \{x \in R^n : x = \alpha(z - y) \\ &\quad + y, \alpha \in [0, 1], y \in B(s, \rho)\}.\end{aligned}$$

The point z is the *apex* of the ball-cone $\chi(z, s, \rho)$, while s is its *centre of base*.

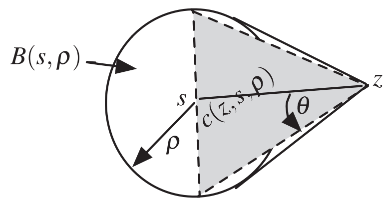

Figure 4. A ball-cone $\chi(z, s, \rho)$.

Note that, when $s = z$, the ball-cone reduces to the ball $\chi(s, s, \rho) = B(s, \rho)$.

Visually, the ball-cone is akin to a cone, except that instead of a cone's 'flat' base, the ball-cone has a ball, as depicted in Figure 4. The shaded area in the figure shows a standard right circular cone $c(z, s, \rho)$ whose vertex is at z, the centre of its base is at s, and the radius of its base is ρ. The angle between the generator and the axis of the right circular cone $c(z, s, \rho)$ is denoted by θ in the figure. We refer to θ as the *opening angle* of the ball-cone $\chi(z, s, \rho)$. Note that the angle between the generator and the axis of $\chi(z, s, \rho)$ is larger than θ, since a ball-cone has the entire n-dimensional ball $B(s, \rho)$ as its 'base'.

For the right circular cone $c(z, s, \rho)$, we have

$$\rho = |s - z| \tan\theta. \tag{2.3}$$

2.1 Directional error

Consider a system Σ described by the differential equation (1.1), and assume that the system is at a state $z \in R^n$. Suppose that it is necessary to make Σ proceed from the state z to a state $s \neq z \in R^n$ along the straight line segment that connects the two states, namely, along the segment $\ell(z, s)$ of (2.2). For Σ to proceed from z to s along this straight line segment, the trajectory $x(t)$ of Σ must have the following feature: the derivative $\dot{x} = f(x, u)$ must point in the direction from z to s at all points $x \in \ell(z, s)$. As the direction of the derivative $\dot{x}$ is given by the function f, there must be, at every state $x \in \ell(z, s)$, an input value $u(x)$ for which $f(x, u(x))$ points in the direction of the unit vector $\widehat{(s-z)}$. Considering the representation (2.2) of the line segment, every point x of $\ell(z, s)$ is characterised by the triplet z, s, α; hence, we can express the appropriate input value at each point as a function $u(z, s, \alpha)$. In these terms, it follows that Σ can be driven along the straight line segment from z to s if an only if there are input values $u(z, s, \alpha) \in R^m$ such that

$$\hat{f}(\alpha(z - s) + s, u(z, s, \alpha)) = \widehat{(s-z)} \text{ for all } \alpha \in [0, 1]. \tag{2.4}$$

The requirement (2.4) implies complete accuracy and, as a result, cannot be implemented in practice. We must modify the requirement into a form that allows for small errors and leads to a robust implementation. To this end, we replace (2.4) by the following somewhat weaker condition: rather than pointing exactly at s, we allow the derivative $\dot{x} = f(x, u)$ to point to the vicinity of s. Permitting such deviation in the direction of motion removes the burden of absolute accuracy and facilitates a robust implementation.

Formally, let $\varepsilon > 0$ be a real number that describes the available directional accuracy. Then, (2.4) is replaced by the requirement that there be an input function $u(z, s, \alpha) \in R^m$ satisfying

$$\left|\hat{f}(\alpha(z - s) + s, u(z, s, \alpha)) - \widehat{(s-z)}\right| < \varepsilon \text{ for all } \alpha \in [0, 1]. \tag{2.5}$$

The error ε may be caused by a combination of factors, including inaccuracies of the function f and errors in the implementation of the input $u(z, s, \alpha)$. We say that (2.5) represents a *directional error* of ε. As usual, the term 'error' indicates an unpredictable event. The expression 'an error of ε' refers to all errors whose magnitude does not exceed ε.

To be meaningful, condition (2.5) must be valid at all states through which the system might pass on its way from z to s. Due to the directional error of ε, the motion of Σ induced by the input values $u(z, s, \alpha)$ may not be confined to the straight line segment connecting z to s. To characterise the states through which the system may travel under condition (2.5), let $\theta(\varepsilon) \geq 0$ be the supremal angle between the two unit vectors $\hat{f}(\alpha(z - s) + s, u(z, s, \alpha))$ and $\widehat{(s-z)}$ that is consistent with a directional error of ε; namely, $\theta(\varepsilon)$ is the angle between the two unit vectors $\hat{f}(\alpha(z - s) + s, u(z, s, \alpha))$ and $\widehat{(s-z)}$ when $|\hat{f}(\alpha(z - s) + s, u(z, s, \alpha)) - \widehat{(s-z)}| = \varepsilon$. Build a ball-cone with opening angle $\theta(\varepsilon)$ and base centre s, as depicted in Figure 5. Letting $\rho(z, s, \varepsilon)$ be the base radius of this ball-cone, it follows by (2.3) that

$$\rho(z, s, \varepsilon) = |z - s| \tan\theta(\varepsilon). \tag{2.6}$$

Letting $\Gamma(z, s, \varepsilon)$ denote the resulting ball-cone, we have

$$\begin{aligned}\Gamma(z, s, \varepsilon) &= \bigcup_{y \in B(s, \rho(z,s,\varepsilon))} \ell(z, y) \\ &= \{x \in R^n : x = \alpha(z - y) + y, \alpha \in [0, 1], \\ &\quad y \in B(s, \rho(z, s, \varepsilon))\},\end{aligned} \tag{2.7}$$

where the radius $\rho(z, s, \varepsilon)$ is given by (2.6).

To express our quantities directly in terms of ε, consider the supremal case when

$$\left|\hat{f}(\alpha(z - s) + s, u(z, s, \alpha)) - \widehat{(s-z)}\right| = \varepsilon.$$

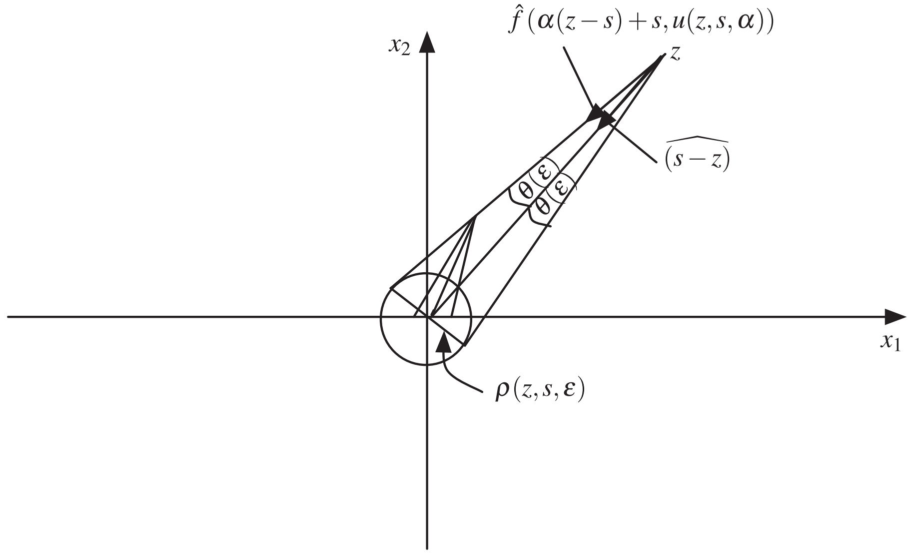

Figure 5. Directional errors.

Then, the triangle formed by the two unit vectors $\hat{f}(\alpha(z-s)+s, u(z,s,\alpha))$ and $\widehat{(s-z)}$, when pinned together at their tail and connected by a straight line segment at their heads, is a bilateral triangle with sides of length 1 (the unit vectors) and base of length ε. The height h of this triangle is determined by the Pythagorean theorem to be $h = \sqrt{1-\varepsilon^2/4}$, and the apex angle of this triangle is

$$\theta(\varepsilon) = 2\arctan\frac{\varepsilon/2}{h} = 2\arctan\frac{\varepsilon/2}{\sqrt{1-\varepsilon^2/4}}. \qquad (2.8)$$

A direct examination of Figures 4 and 5 shows that $\theta(\varepsilon)$ is the opening angle of the ball-cone $\Gamma(z,s,\varepsilon)$. Note that $\theta(\varepsilon)$ is completely determined by ε.

In view of (2.6), the base radius $\rho(z,s,\varepsilon)$ of the ball-cone $\Gamma(z,s,\varepsilon)$ is given by

$$\rho(z,s,\varepsilon) = |z-s|\tan\left(2\arctan\frac{\varepsilon/2}{\sqrt{1-\varepsilon^2/4}}\right). \qquad (2.9)$$

For $\varepsilon \to 0$, this yields the first-order approximations:

$$\theta(\varepsilon) \doteq \varepsilon,$$
$$\rho(z,s,\varepsilon) \doteq |z-s|\varepsilon.$$

As we discuss next, condition (2.5) must be valid within the entire ball-cone $\Gamma(z,s,\varepsilon)$, if it is to be meaningful.

2.2 Ball-cones and invariance

Note that the ball-cone $\Gamma(z,s,\varepsilon)$ of (2.7) is not an open set, since it does not include a neighbourhood of the apex z. However, a slight reflection shows that, except for z, all points of $\Gamma(z,s,\varepsilon)$ are interior points. Given a subset $S \subseteq R^n$, denote by $\overline{S}$ the closure of S in R^n. We start now the process of showing that the closure of a ball-cone $\overline{\Gamma}(z,s,\varepsilon)$ forms an invariant set in the sense that, as long as the directional error does not exceed ε, the trajectory of the closed-loop system does not exit $\overline{\Gamma}(z,s,\varepsilon)$ before reaching a vicinity of s.

Proposition 2.3: *Let $z, s \in R^n$ be two distinct points, and let $\varepsilon > 0$ be a directional error for which the opening angle of the ball-cone $\Gamma(z,s,\varepsilon)$ satisfies $\theta(\varepsilon) < \pi/4$. Then, $\overline{\Gamma}(z',s,\varepsilon) \subseteq \overline{\Gamma}(z,s,\varepsilon)$ for all $z' \in \overline{\Gamma}(z,s,\varepsilon)$.*

Proof: For a point $z' \in \overline{\Gamma}(z,s,\varepsilon)$, we can write

$$\overline{\Gamma}(z',s,\varepsilon) = \{x \in R^n : x = \alpha(z'-y')+y', \alpha \in [0.1], y' \in \overline{B}(s,\rho(z',s,\varepsilon))\}, \qquad (2.10)$$

where, according to (2.6), we have $\rho(z',s,\varepsilon) = |z'-s|\tan\theta(\varepsilon)$. Considering that $z' \in \overline{\Gamma}(z,s,\varepsilon)$, it follows from (2.7) that

$$z' = \beta(z-y)+y \qquad (2.11)$$

for some $\beta \in [0,1]$ and $y \in \overline{B}(s,\rho(z,s,\varepsilon))$. Substituting into (2.10), we get

$$\overline{\Gamma}(z',s,\varepsilon) = \{x \in R^n : x = \alpha(\beta(z-y)+y-y') + y', \alpha, \beta \in [0.1], y \in \overline{B}(\rho(z,s,\varepsilon), y' \in \overline{B}(\rho(z',s,\varepsilon)\}. \qquad (2.12)$$

We claim that the point

$$x = \alpha(\beta(z-y)+y-y')+y' \qquad (2.13)$$

of (2.12) is a member of $\overline{\Gamma}(z,s,\varepsilon)$ for all $\alpha, \beta \in [0,1]$.

To this end, note first that when $\alpha = \beta = 1$, we get $x = z$, so that $x \in \overline{\Gamma}(z, s, \varepsilon)$, and our claim is valid in this case. Next, consider the case where α and β are not both equal to 1. Then, x can be rewritten in the form

$$x = \alpha\beta(z - \eta) + \eta, \tag{2.14}$$

where a comparison of (2.13) and (2.14) yields, after some simplification, that

$$\eta = y - \frac{1-\alpha}{1-\alpha\beta}(y - y') \text{ for all } \alpha, \beta \in [0, 1] \text{ satisfying } \alpha\beta \neq 1. \tag{2.15}$$

Now, assume for a moment that $\eta \in \overline{B}(s, \rho(z, s, \varepsilon))$. Then, setting $\delta := \alpha\beta \in [0, 1]$, we can rewrite (2.14) in the form $x = \delta(z - \eta) + \eta$. In view of (2.7), this implies that $x \in \overline{\Gamma}(z, s, \varepsilon)$. As the latter is true for every point x of the ball-cone $\overline{\Gamma}(z', s, \varepsilon)$ and for all $z' \in \overline{\Gamma}(z, s, \varepsilon)$, it follows that $\overline{\Gamma}(z', s, \varepsilon) \subseteq \overline{\Gamma}(z, s, \varepsilon)$ for all $z' \in \overline{\Gamma}(z, s, \varepsilon)$. Thus, our proof will conclude upon showing that $\eta \in \overline{B}(s, \rho(z, s, \varepsilon))$.

To prove the latter, examine the coefficient of (2.15)

$$\gamma(\alpha, \beta) := \frac{1-\alpha}{1-\alpha\beta},$$

where $\alpha, \beta \in [0, 1]$ and $\alpha\beta \neq 1$. Considering that

$$\frac{\partial\gamma(\alpha, \beta)}{\partial\beta} = \frac{(1-\alpha)\alpha}{(1-\alpha\beta)^2} > 0 \text{ for all } \alpha, \beta \in [0, 1], \alpha\beta \neq 1,$$

it follows that $\gamma(\alpha, \beta)$ is a monotone increasing function of β for all $\alpha, \beta \in [0, 1]$, $\alpha\beta \neq 1$. Since we have $\gamma(\alpha, 0) = (1-\alpha)$ and $\gamma(\alpha, 1) = 1$, we conclude that $0 \leq \gamma(\alpha, \beta) \leq 1$ for all $\alpha, \beta \in [0, 1]$, $\alpha\beta \neq 1$. Thus, we can rewrite (2.15) in the form

$$\eta = \gamma(\alpha, \beta)(y' - y) + y, \gamma(\alpha, \beta) \in [0, 1],$$

which implies that η is a point on the straight line segment connecting y and y'.

Now, recall that $y \in \overline{B}(s, \rho(z, s, \varepsilon))$; if we can show that also $y' \in \overline{B}(s, \rho(z, s, \varepsilon))$, then, since $\overline{B}(s, \rho(z, s, \varepsilon))$ is a convex set, it would follow that $\eta \in \overline{B}(s, \rho(z, s, \varepsilon))$, and our proof would conclude. Further, since $y' \in \overline{B}(s, \rho(z', s, \varepsilon))$ by (2.12), and since the two balls $\overline{B}(s, \rho(z', s, \varepsilon))$ and $\overline{B}(s, \rho(z, s, \varepsilon))$ are concentric (with centre at s), the inclusion $y' \in \overline{B}(s, \rho(z, s, \varepsilon))$ would follow from the inequality $\rho(z', s, \varepsilon) \leq \rho(z, s, \varepsilon)$. By (2.9), this inequality would be a consequence of the inequality $|z' - s| \leq |z - s|$. Thus, our proof will conclude upon showing that the last inequality is valid.

Using the Proposition's assumption that $\theta(\varepsilon) < \pi/4$, we infer from (2.6) that

$$\rho(z, s, \varepsilon) < |z - s|. \tag{2.16}$$

Subtracting s from both sides of (2.11), we obtain $z' - s = \beta(z - y) + (y - s) = \beta(z - s) + (1 - \beta)(y - s)$. As $0 \leq \beta \leq 1$, we can write

$$|z' - s| = |\beta(z - s) + (1 - \beta)(y - s)| \leq \beta|z - s| + (1 - \beta)|y - s|. \tag{2.17}$$

Now, since $y \in \overline{B}(s, \rho(z, s, \varepsilon))$, we have $|y - s| \leq \rho(z, s, \varepsilon)$; using (2.16) this yields $|y - s| < |z - s|$. Substituting into (2.17), we obtain $|z' - s| < \beta|z - s| + (1 - \beta)|z - s| = |z - s|$, and our proof concludes. □

Proposition 2.3 is an important component of our forthcoming discussion. As is it valid for $\theta(\varepsilon) < \pi/4$, we restrict our attention from now on to cases that satisfy this requirement. In light of (2.8), this assumption is not overly restrictive; a direct calculation shows that it corresponds to $0 < \varepsilon < 0.7654$. Normally, directional errors are small, and their corresponding opening angles much smaller than $\pi/4$.

Assumption 2.4: *The directional error satisfies $\theta(\varepsilon) < \pi/4$.*

The next statement is a technical result. It will help us later to show that a system pointed toward a point s with directional error of ε, remains within the closed ball-cone $\overline{\Gamma}(z, s, \varepsilon)$ until reaching a vicinity of s.

Notation 2.5: For a real number $\beta > 0$, the symbol $0(\beta)$ represents the set of all functions $\omega: R \to R^n$ for which $\lim_{\beta \to 0}|\omega(\beta)|/\beta = 0$.

Lemma 2.6: *Let $x, x', s, z \in R^n$ be points and assume that $x' - x = \beta\widehat{a} + \mu(x', x)$, where $\beta > 0$ is a real number, $\widehat{a}$ is a unit vector satisfying $|\widehat{a} - \widehat{(s - x)}| < \varepsilon$, and $\mu(x', x) \in 0(\beta)$. If $x \in \overline{\Gamma}(z, s, \varepsilon)$, then also $x' \in \overline{\Gamma}(z, s, \varepsilon)$ for sufficiently small $\beta > 0$.*

Proof: Consider x as a fixed point, so that x' is a function of β. Note that, if $x = s$, then x is the centre of the ball $B(s, \rho(z, s, \varepsilon)) \subseteq \overline{\Gamma}(z, s, \varepsilon)$; in such case, every point x' sufficiently close to x is in $B(s, \rho(z, s, \varepsilon))$ and, consequently, in $\overline{\Gamma}(z, s, \varepsilon)$. Thus, the statement is valid when $x = s$.

Next, examine the case where $x \neq s$. Let $\theta(\varepsilon)$ be the opening angle of the ball-cone $\Gamma(z, s, \varepsilon)$. Since $x \in \overline{\Gamma}(z, s, \varepsilon)$, it follows by Proposition 2.3 that $\overline{\Gamma}(x, s, \varepsilon) \subseteq \overline{\Gamma}(z, s, \varepsilon)$. This implies that $\overline{\Gamma}(z, s, \varepsilon)$ includes the straight line segment $\ell(x, s)$ as well as any straight line segment that connects x to a point of the ball $\overline{B}(s, \rho(x, s, \varepsilon))$.

Applying magnitude to the equation in the Lemma's statement, we obtain $0 \leq |x' - x| \leq \beta|\widehat{a}| + |\mu(x', x)| = \beta(1 + |\mu(x', x)|/\beta)$, since $\hat{a}$ is a unit vector. As

$\mu(x', x) \in 0(\beta)$, this implies that $\lim_{\beta\to 0}|x'-x| = 0$. Keeping in mind that $x \neq s$, the latter implies that there is a real number $\beta_0 > 0$ such that

$$|x' - x| < |s - x|/2 \text{ for all } 0 < \beta < \beta_0. \tag{2.18}$$

Further, define the real number $\xi := |\widehat{a} - \widehat{(s-x)}| < \varepsilon$, and let $\theta(\xi)$ be the angle between the unit vectors $\hat{a}$ and $\widehat{(s-x)}$. Then, as the function (2.8) is strictly monotone increasing over our range and $\xi < \varepsilon$, we have that $\theta(\xi) < \theta(\varepsilon)$. Let $\psi(x')$ be the angle formed between the vector $x'-x$ and the unit vector $\widehat{(s-x)}$. Using the expression $x' - x = \beta\hat{a} + \mu(x, x')$ and the fact that $\widehat{(x-s)}$ is a unit vector, we can write

$$\begin{aligned}\psi(x') &= \arccos\left(\frac{(x'-x)\cdot\widehat{(s-x)}}{|x'-x|}\right)\\ &= \arccos\left(\frac{\beta\hat{a}\cdot\widehat{(s-x)} + \mu(x,x')\cdot\widehat{(s-x)}}{\sqrt{\beta^2 + \mu(x,x')\cdot\mu(x,x') + 2\beta\mu(x,x')\cdot\hat{a}}}\right)\\ &= \arccos\left(\frac{\hat{a}\cdot\widehat{(s-x)} + \mu(x,x')\cdot\widehat{(s-x)}/\beta}{\sqrt{1 + \mu(x,x')\cdot\mu(x,x')/\beta^2 + 2\mu(x,x')\cdot\hat{a}/\beta}}\right).\end{aligned}$$

Considering that $\lim_{\beta\to 0}|\mu(x, x')|/\beta = 0$, we conclude that

$$0 \le \lim_{\beta\to 0}\psi(x') = \arccos(\hat{a}\cdot\widehat{(s-x)}) = \theta(\xi) < \theta(\varepsilon). \tag{2.19}$$

Now, denote $\zeta := \theta(\varepsilon) - \theta(\xi)$; in view of (2.19), there is a real number $\beta_1 > 0$ such that $\psi(x') < \theta(\varepsilon) - \zeta/2$ for all $0 < \beta < \beta_1$. Then, for all $0 < \beta < \min\{\beta_0, \beta_1\}$, the vector $x'-x$ forms an angle smaller than $\theta(\varepsilon)$ with the line segment $\ell(x, s)$, and, by (2.18), the length of $x'-x$ does not exceed $|x-s|$. This shows that $x' \in \overline{\Gamma}(x, s, \varepsilon)$, and, since $x \in \overline{\Gamma}(z, s, \varepsilon)$, the lemma follows by Proposition 2.3. □

In Section 3, we examine state feedback functions φ that take a system Σ from a state z to a state s, while continually aiming at s with directional error of ε. At that point, Lemma 2.6 will help us show that such state feedback keeps the closed-loop system Σ_φ within the ball-cone $\overline{\Gamma}(z, s, \varepsilon)$ until a vicinity of s is reached. This fact turns out to be critical to our discussion.

2.3 Directional uncertainty and interception

We start by characterising the set of all states from which a system Σ with directional error of ε can reach the vicinity of a specific target state. For a state $s \in R^n$, let $\Delta(s, \varepsilon)$ be the set of all states of Σ at which the trajectory of Σ can be pointed in the direction of s with a directional error of $\varepsilon > 0$. Recalling Assumption 2.1, we have

$$\Delta(s,\varepsilon) = \Big\{x \in R^n \setminus s : |\hat{f}(x, u(x)) - \widehat{(s-x)}| < \varepsilon \text{ for some } u(x) \in R^m \text{ satisfying } |u(x)| \le M\Big\}. \tag{2.20}$$

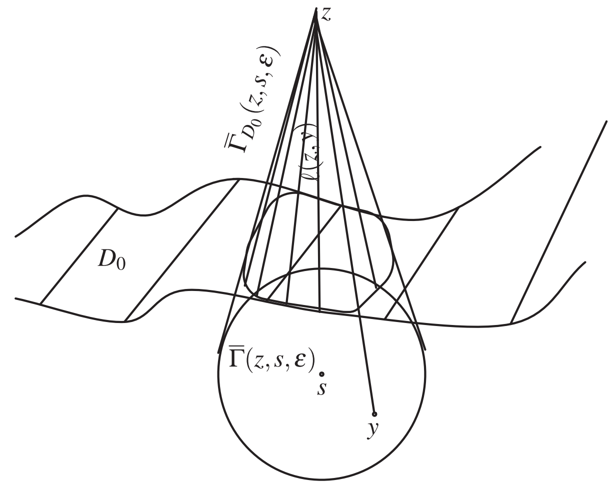

Figure 6. Interception.

Note that pointing the path of Σ toward s does not guarantee that Σ can reach s from x, since it might not be possible to maintain this direction all the way from x to s. The set of all states from which Σ can actually reach the vicinity of s will be characterised shortly.

Considering that our objective is to move the state of Σ in the direction of s within $\Delta(s, \varepsilon)$, we have to make sure that $f(x, u(x)) \neq 0$ at all states x along the way, since otherwise a stationary point is met at $(x, u(x))$ and the system stops progress. Now, substituting $f(x, u(x)) = 0$ into (2.20) yields $|\widehat{(s-x)}| < \varepsilon$, or $1 < \varepsilon$, since $\hat{f}(x, u(x)) = 0$ whenever $f(x, u(x)) = 0$ (see (2.1)). Thus, Assumption 2.4 guaranties that $f(x, u(x)) \neq 0$ at all points of $\Delta(s, \varepsilon)$, and stationary points are avoidable in $\Delta(s, \varepsilon)$.

Problem 1.1 requires the design of a feedback controller that takes Σ from an initial state $x_0 = z$ into a target domain D_0. The path that takes Σ from z to D_0 is, generally speaking, unpredictable, since the closed-loop system is subject to a directional error of ε. Nevertheless, as Proposition 2.3 hints (this connection is made precise in Proposition 3.1), the closed-loop system remains confined to the closed ball-cone $\overline{\Gamma}(z, s, \varepsilon)$. Thus, if every path through $\overline{\Gamma}(z, s, \varepsilon)$ meets the target domain D_0, we are assured that the objective of entering D_0 is achievable, despite possible directional errors. These considerations lead us to the following notion.

Definition 2.7: An open domain $D_0 \subseteq R^n$ *intercepts* the ball-cone $\Gamma(z, s, \varepsilon)$ if $\ell(z, y) \cap D_0 \neq \varnothing$ for all $y \in \overline{B}(s, \rho(z, s, \varepsilon))$.

When the target domain D_0 intercepts the ball-cone $\Gamma(z, s, \varepsilon)$, then every ray within the ball-cone from the apex z meets D_0. We show later that this implies that every path taking Σ from z to s with directional error of ε must enter D_0. The situation is depicted schematically in Figure 6. In particular, note that D_0 intercepts the ball-cone $\Gamma(z, s, \varepsilon)$

whenever $z \in D_0$; this is, of course, the degenerate case here.

Our interest in the system's progress ends upon entering the target domain D_0. Therefore, in Figure 6, our interest is confined to the 'upper' part of the ball-cone, namely to the part between the apex z and the set D_0. Formally, this is described by the following notion.

Definition 2.8: Let D_0 be an open subset of R^n that intercepts the ball-cone $\Gamma(z, s, \varepsilon)$, and denote by $\check{D}_0 := \overline{D}_0 \setminus D_0$ the boundary of D_0. Then, the *restriction* $\overline{\Gamma}_{D_0}(z, s, \varepsilon)$ is the set

$$\overline{\Gamma}_{D_0}(z, s, \varepsilon) := \begin{cases} \{\bigcup \ell(y, z) : y \in \overline{\Gamma}(z, s, \varepsilon) \cap \check{D}_0 \\ \quad \text{and } \ell(y, z) \cap D_0 = \varnothing\} & \text{if } z \notin D_0, \\ \varnothing & \text{if } z \in D_0. \end{cases} \tag{2.21}$$

When $z \notin D_0$, the restriction $\overline{\Gamma}_{D_0}(z, s, \varepsilon)$ consists of all points of the closed ball-cone $\overline{\Gamma}(z, s, \varepsilon)$ that are between the apex z of the ball-cone and the target domain D_0, including z and the 'upper' boundary of D_0. The restriction is represented graphically in Figure 6. It is a closed and bounded set in R^n, as follows.

Lemma 2.9: *Let $D_0 \subseteq R^n$ be an open set that intercepts the ball-cone $\Gamma(z, s, \varepsilon)$. Then, the restriction $\overline{\Gamma}_{D_0}(z, s, \varepsilon)$ is a compact set.*

Proof: According to Definition 2.8, the restriction $\overline{\Gamma}_{D_0}(z, s, \varepsilon)$ is a subset of $\overline{\Gamma}(z, s, \varepsilon)$, and hence is a bounded set. Thus, it remains to show that the restriction is a closed set. To this end, consider a sequence of points $x_1, x_2, \ldots \in \overline{\Gamma}_{D_0}(z, s, \varepsilon)$ that converges to a point $x \in R^n$. We have to show that $x \in \overline{\Gamma}_{D_0}(z, s, \varepsilon)$. By Definition 2.8, there is a sequence of points $y_1, y_2, \ldots \in \overline{\Gamma}(z, s, \varepsilon) \cap \check{D}$ such that $x_i \in \ell(y_i, z)$ and $\ell(y_i, z) \cap D_0 = \varnothing$, $i = 1, 2, \ldots$ As $\overline{\Gamma}(z, s, \varepsilon) \cap \check{D}$ is a closed subset of a bounded set, it is a compact set, and whence the sequence $y_1, y_2, \ldots$ has a convergent subsequence that converges to a point $y \in \overline{\Gamma}(z, s, \varepsilon) \cap \check{D}$. It follows then that $\ell(y, z) = \lim_{i \to \infty} \ell(y_i, z)$, so that $\ell(y, z) \cap D_0 = (\lim_{i\to\infty} \ell(y_i, z)) \cap D_0 = \lim_{i\to\infty}(\ell(y_i, z) \cap D_0) = \varnothing$. Referring again to Definition 2.8, we conclude that $\ell(y, z) \subseteq \overline{\Gamma}_{D_0}(z, s, \varepsilon)$. This implies that, if $x \in \ell(y, z)$, then $x \in \overline{\Gamma}_{D_0}(z, s, \varepsilon)$, and it follows that $\overline{\Gamma}_{D_0}(z, s, \varepsilon)$ is a closed set. Thus, our proof will conclude upon showing that $x \in \ell(y, z)$.

To show the latter, assume by contradiction that $x \notin \ell(y, z)$. Then, considering that $\ell(y, z)$ is a closed set, the complement of $\ell(y, z)$ is an open set, and hence there is a real number $\mu > 0$ such that the ball $B(x, \mu)$ includes no points of $\ell(y, z)$. As the sequence $\{x_i\}$ converges to x, there is an integer $N > 0$ such that $|x - x_i| < \mu/2$ for all $i \geq N$. This implies that none of the points $\{x_i\}_{i=N}^{\infty}$ gets closer than $\mu/2$ to the segment $\ell(y, z)$. Considering that the segments $\{\ell(y_i, z)\}_{i=1}^{\infty}$ all have the common apex z, this implies that none of the points $\{y_i\}_{i=N}^{\infty}$ gets closer than $\mu/2$ to y, contradicting the fact that y is a limit point of the sequence $\{y_i\}$. Thus, we must have that $x \in \ell(y, z)$, and our proof concludes. □

3. Existence and construction of state feedback functions

3.1 Reaching the target domain

We start our investigation of the existence of state feedback functions that solve Problem 1.1 by considering states from which the controlled system can be driven into the target domain along a path related to a straight line.

Proposition 3.1: *Let Σ be a system described by (1.1), where the function f is continuous. Let $z, s \in R^n$ be a pair of points, let $\varepsilon > 0$ be a real number, let $\Delta(s, \varepsilon)$ be given by (2.20), and let $D_0 \subseteq R^n$ be an open domain that intercepts $\Gamma(z, s, \varepsilon)$. If $\overline{\Gamma}_{D_0}(z, s, \varepsilon) \subseteq \Delta(s, \varepsilon)$, then there is a state feedback function φ with directional error of ε that takes Σ from z into D_0 in finite time.*

Proof: As the proposition is clearly true when $z \in D_0$, we consider the case where $z \notin D_0$. Let $x(t)$ be the state of the system Σ at the time t, and let $u(x(t))$ be the input of Σ at t. Use the initial state $x(0) = z$, so that $x(0) \in \overline{\Gamma}_{D_0}(z, s, \varepsilon)$ by Definition 2.8. As $\overline{\Gamma}_{D_0}(z, s, \varepsilon) \subseteq \Delta(s, \varepsilon)$ by the proposition's assumption, it follows that $x(0) \in \Delta(s, \varepsilon)$. Consequently, there is an input value $u(x(0)) \in R^m$ such that

$$|\hat{f}(x(0), u(x(0))) - (\widehat{s - x(0)})| < \varepsilon. \tag{3.1}$$

Now, let $\tau \geq 0$ be a real number for which the following is true: there is a state feedback function $u(x)$ that drives Σ with a directional error of ε in such a way that the state satisfies $x(t) \in \overline{\Gamma}_{D_0}(z, s, \varepsilon)$ for all $t \in [0, \tau]$. We are interested in the upper limit of τ, namely, in the quantity

$$T := \sup \tau. \tag{3.2}$$

In view of (3.1), we have $\tau \geq 0$, so also $T \geq 0$.

By (1.1), we can write for a real number $\delta > 0$ that

$$x(t + \delta) = x(t) + f(x(t), u(x(t)))\,\delta + 0(\delta). \tag{3.3}$$

Setting $t = 0$ and considering that $x(0) = z \in \overline{\Gamma}(z, s, \varepsilon)$, it follows by Lemma 2.6 that there is a real number $\delta_0 > 0$ such that $x(\delta) \in \overline{\Gamma}(z, s, \varepsilon)$ for all $0 < \delta < \delta_0$. Clearly, if $x(\delta) \in D_0$ for some $0 < \delta < \delta_0$, then our proof is complete, as the target domain has been reached. Otherwise, it follows by Definition 2.8 that $x(\delta) \in \overline{\Gamma}_{D_0}(z, s, \varepsilon)$ for all $0 < \delta < \delta_0$. Also, since $\overline{\Gamma}_{D_0}(z, s, \varepsilon) \subseteq \Delta(s, \varepsilon)$ by the proposition's assumption, it follows that $x(\delta) \in \Delta(s, \varepsilon)$ for all $0 < \delta < \delta_0$.

Next, by (3.2), there is a sequence of times $t_1, t_2, \ldots \in [0, T]$ converging to T such that $x(t_i) \in \overline{\Gamma}_{D_0}(z, s, \varepsilon)$ for all

$i = 1, 2, \ldots$ As $\overline{\Gamma}_{D_0}(z, s, \varepsilon)$ is compact by Lemma 2.9, the sequence $\{x(t_i)\}_{i=1}^{\infty}$ has a convergent subsequence $\{x(t_{i_k})\}_{k=1}^{\infty}$ and $\lim_{k\to\infty} x(t_{i_k}) \in \overline{\Gamma}_{D_0}(z, s, \varepsilon)$. Considering that $x(t)$ is the solution of the differential equation (1.1) with a bounded input function (Assumption 2.1), it follows that $x(t)$ is a continuous function of time; thus, $\lim_{k\to\infty} t_{i_k} = T$ implies that $\lim_{k\to\infty} x(t_{i_k}) = x(T)$, and whence $x(T) \in \overline{\Gamma}_{D_0}(z, s, \varepsilon)$.

Further, since $\overline{\Gamma}_{D_0}(z, s, \varepsilon) \subseteq \Delta(s, \varepsilon)$ by assumption, $x(T) \in \Delta(s, \varepsilon)$. Thus, there is an input value $u(x(T)) \in R^m$ such that $|\hat{f}(x(T), u(T)) - (\widehat{s - x(T)})| < \varepsilon$. Using (3.3) with $t = T$ and Lemma 2.6, this implies that there is a real number $\delta_1 > 0$ such that $x(T + \delta) \in \overline{\Gamma}(z, s, \varepsilon)$ for all $0 < \delta < \delta_1$. Selecting one such value of δ, we are left with the following two options: *(i)* $x(T+\delta) \notin D_0$, or *(ii)* $x(T+\delta) \in D_0$. Case *(i)* implies that $x(T + \delta) \in \overline{\Gamma}_{D_0}(z, s, \varepsilon)$, violating the fact that T is the supremum given by (3.2). Thus, *(ii)* must be valid, and our proof concludes. □

Proposition 3.1 forms the foundation of our solution of Problem 1.1. The solution is based on a concept introduced in the next subsection.

3.2 The expansion set

Let D_0 be an open domain in R^n serving as the target domain for the system Σ. Let $\varepsilon > 0$ be a real number, let $f: R^n \times R^m \to R^n$ be a continuous function, and let $\Delta(s, \varepsilon)$ be given by (2.20). Then, in view of Proposition 3.1, the system Σ can be driven into the target domain D_0 by a state feedback function with directional error of ε from any point of the set

$$E^1_f(D_0, \varepsilon) := \left\{ z \in R^n \,\middle|\, \begin{array}{c} D_0 \text{ intercepts } \Gamma(z, s, \varepsilon) \\ \text{for some } s \in R^n \text{ and} \\ \overline{\Gamma}_{D_0}(z, s, \varepsilon) \subseteq \Delta(s, \varepsilon). \end{array} \right\} \quad (3.4)$$

Definition 3.2: $E^1_f(D_0, \varepsilon)$ is the *expansion set* of D_0 relative to f with directional error of ε.

In view of Definitions 2.7 and 2.8, we have that

$$D_0 \subseteq E^1_f(D_0, \varepsilon). \quad (3.5)$$

Proposition 3.1 can then be restated in the following form.

Proposition 3.3: *Let Σ be a system described by (1.1) with a continuous function f. Let $\varepsilon > 0$ be a real number, let $D_0 \subseteq R^n$ be an open domain, and let $E^1_f(D_0, \varepsilon)$ be the expansion set. Then, there is a state feedback function φ with directional error of ε that takes Σ from every initial state $z \in E^1_f(D_0, \varepsilon)$ into D_0 in finite time.*

A partial inverse of Proposition 3.3 is provided by the following.

Proposition 3.4: *Let Σ be a system described by the differential equation (1.1) with a continuous function f. Let D_0 be an open domain in R^n, let $\varepsilon > 0$ be a real number, and let $E^1_f(D_0, \varepsilon)$ be the expansion set. If the difference set $E^1_f(D_0, \varepsilon)\backslash D_0$ is empty, then there is no state feedback function with directional error of ε that drives Σ from a state outside of D_0 into D_0 in finite time.*

Proof: The proof is by contradiction. Assume that $E^1_f(D_0, \varepsilon)\backslash D_0 = \varnothing$, but there is a state $z \notin D_0$ from which Σ can be driven into D_0 by a state feedback function with directional error of ε. Let $\varphi: R^n \to R^m$ be such a state feedback function, and let $s \in D_0$ be a point reached by the closed-loop system Σ_φ from z. Recalling that a directional error of ε includes all directional errors smaller than ε, it follows by the proof of Proposition 3.1 that D_0 intercepts $\Gamma(z, s, \varepsilon)$.

Now, let $x(t)$ be the state of the closed loop system Σ_φ at the time t, as it progresses from z to s. Set $x(0) = z$, and let $t_1 > 0$ be a time at which $x(t_1) = s \in D_0$. Considering that D_0 is an open set, $x(t_1)$ is an interior point of D_0. In view of Assumption 2.1, all input values of Σ are uniformly bounded, and, as a result, $x(t)$ is a continuous function of t. Define the time

$$t^* := \inf\{t \geq 0 \,|\, x(t) \in D_0\}; \quad (3.6)$$

as $x(t_1)$ is an interior point of D_0, it follows that $t_1 - t^* \geq 0$. We claim that $x(t^*)$ is a boundary point of D_0.

Indeed, assume, by contradiction, that $x(t^*) \in D_0$. As D_0 is an open set by the proposition's assumptions, there is a real number $\delta > 0$ such that the ball $B(x(t^*), \delta)$ of radius δ and centre $x(t^*)$ is included in D_0. By the continuity of $x(t)$, there is a real number $\mu > 0$ such that $|x(t^*) - x(t)| < \delta/2$ for all times t satisfying $|t^* - t| < \mu$. But then, $x(t^* - \mu) \in B(x(t^*), \delta)$ and, since $B(x(t^*), \delta) \subseteq D_0$, it follows that $x(t^* - \mu) \in D_0$, in contradiction to the definition of t^* as the infimum (3.6). Thus, $x(t^*) \notin D_0$ and $x(t^*)$ is a boundary point of the open domain D_0.

Consider next the vector $f(x(t^*), \varphi(x(t^*)))$. We claim that this vector has the following two properties: *(a)* it is not zero, and *(b)* it points toward a point of D_0. To prove this claim, note that, if $f(x(t^*), \varphi(x(t^*))) = 0$, then $x(t^*)$ is a stationary point of the closed-loop system Σ_φ. In such case, Σ_φ remains at the state $x(t^*)$ indefinitely, and, as $x(t^*) \notin D_0$, the closed-loop system Σ_φ never reaches a point of D_0. This is in contradiction to the fact that $x(t_1) \in D_0$, since $t_1 \geq t^*$. Thus, $f(x(t^*), \varphi(x(t^*))) \neq 0$ and *(a)* is valid.

Regarding *(b)*, it follows by (3.6) and the continuity of the function $x(t)$ that there is a real number $\eta_0 > 0$ such that $x(t^* + \eta) \in D_0$ for all $0 < \eta < \eta_0$. Using the differential equation (1.1) of Σ, we can write $x(t^* + \eta) =$

$x(t^*) + \eta f(x(t^*), \varphi(x(t^*))) + 0(\eta)$, or

$$x(t^* + \eta) - x(t^*) = \eta[f(x(t^*), \varphi(x(t^*))) + 0(\eta)/\eta]. \tag{3.7}$$

Considering that $x(t^* + \eta)$ is an interior point of D_0 for all $0 < \eta < \eta_0$, it follows that the vector

$$f_\eta := f(x(t^*), \varphi(x(t^*))) + 0(\eta)/\eta$$

points from $x(t^*)$ to an interior point of D_0 for all $0 < \eta < \eta_0$. As all directions may include a directional error of ε, there is then a point $s' \in D_0$ such that $|\widehat{f_\eta} - (\widehat{x(t^*) - s'})| < \varepsilon$. Denote $\xi(\eta) := |\widehat{f_\eta} - (\widehat{x(t^*) - s'})| < \varepsilon$.

Using the facts that $\lim_{\eta\to 0} 0(\eta)/\eta = 0$ and that $f(x(t^*), \varphi(x(t^*))) \neq 0$, it follows that the unit vector

$$\widehat{f_\eta} := \frac{f(x(t^*), \varphi(x(t^*))) + 0(\eta)/\eta}{|f(x(t^*), \varphi(x(t^*))) + 0(\eta)/\eta|}$$

satisfies the condition $\lim_{\eta\to 0} \widehat{f_\eta} = \hat{f}(x(t^*), \varphi(x(t^*)))$. Therefore, for every real number $\zeta > 0$, there is a real number $0 < \eta_1 < \eta_0$ such that

$$|\widehat{f_\eta} - \hat{f}(x(t^*), \varphi(x(t^*)))| < \zeta \tag{3.8}$$

for all $0 < \eta < \eta_1$. In particular, we can choose $0 < \zeta < \varepsilon$. Then, the vector $f(x(t^*), \varphi(x(t^*)))$ points in the direction of a point of D_0 with a directional error not exceeding ε. By (3.7) and (3.8), we obtain that $x(t^*) \in \Delta(x(t^* + \eta), \varepsilon)$, and by the definition of η we have $x(t^* + \eta) \in D_0$. Thus, $x(t^*) \in E^1_f(D_0, \varepsilon)$ by (3.4). Recalling that $x(t^*) \notin D_0$, this contradicts the assumption that $E^1_f(D_0, \varepsilon) \setminus D_0 = \varnothing$, and our proof concludes. □

Our ensuing discussion depends on the fact that the expansion set is an open set, as follows.

Lemma 3.5: *Let Σ be a system described by* (1.1) *with a continuous function f, and let D_0 be an open domain in R^n. Then, the expansion set $E^1_f(D_0, \varepsilon)$ is an open set for every $\varepsilon > 0$.*

Proof: First, we show that the set $\Delta(s, \varepsilon)$ is an open set. Indeed, consider a point $s \in R^n$ at which $\Delta(s, \varepsilon) \neq \varnothing$. Then, since $\varepsilon < 1$ by Assumption 2.4, it follows in (2.20) that $f(x, u(x)) \neq 0$ for all $x \in \Delta(s, \varepsilon)$. Consequently, the unit vector $\hat{f}(y, u(x))$ is a continuous function of y in a neighbourhood of x at any $x \in \Delta(s, \varepsilon)$. Now, consider a point $x \in \Delta(s, \varepsilon)$ and let $u(x) \in R^m$ be an input value at which $|\hat{f}(x, u(x)) - (\widehat{s-x})| < \varepsilon$. Denote $\delta := \varepsilon - |\hat{f}(x, u(x)) - (\widehat{s-x})| > 0$. By continuity, there are real numbers $\zeta_1, \zeta_2 > 0$ such that $|\hat{f}(x', u(x)) - \hat{f}(x, u(x))| < \delta/3$ for all $|x' - x| < \zeta_1$ and $|(\widehat{s - x''}) - (\widehat{s-x})| < \delta/3$ for all $|x'' - x| < \zeta_2$. Let $\zeta := \min\{\zeta_1, \zeta_2\}$. Then, for all $y \in B(x, \zeta)$, we have $|\hat{f}(y, u(x)) - (\widehat{s - y})| = |[\hat{f}(y, u(x)) - \hat{f}(x, u(x))] + \hat{f}(x, u(x)) - [(\widehat{s - y}) - (\widehat{s-x})] - (\widehat{s-x})| \leq |\hat{f}(y, u(x)) - \hat{f}(x, u(x))| + |(\widehat{s - y}) - (\widehat{s-x})| + |\hat{f}(x, u(x)) - (\widehat{s-x})| \leq \delta/3 + \delta/3 + |\hat{f}(x, u(x)) - (\widehat{s-x})| < \varepsilon$. This shows that $B(x, \zeta) \subseteq \Delta(s, \varepsilon)$ (the constant input value $u := u(x)$ is used for all $y \in B(x, \zeta)$). Consequently, $\Delta(s, \varepsilon)$ is an open set.

Now, consider a point $z \in E^1_f(D_0, \varepsilon)$. By (3.4), there is a point $s \in R^n$ such that D_0 intercepts $\Gamma(z, s, \varepsilon)$ and $\overline{\Gamma}_{D_0}(z, s, \varepsilon) \subseteq \Delta(s, \varepsilon)$. The fact that D_0 and $\Delta(s, \varepsilon)$ are both open sets implies that there is a real number $\rho_1 > 0$ such that D_0 intercepts $\Gamma(y, s, \varepsilon)$ and $\overline{\Gamma}_{D_0}(y, s, \varepsilon) \subseteq \Delta(s, \varepsilon)$ for all $y \in B(z, \rho_1)$. Consequently, $B(z, \rho_1) \subseteq E^1_f(D_0, \varepsilon)$. As this argument is valid at every point $z \in E^1_f(D_0, \varepsilon)$, it follows that $E^1_f(D_0, \varepsilon)$ is an open set. □

3.3 Iterating expansion sets

Iterating the construction (3.4) of an expansion set, we build a sequence of sets $E^0_f(D_0, \varepsilon), E^1_f(D_0, \varepsilon), E^2_f(D_0, \varepsilon) \ldots$ by setting

$$\begin{aligned} E^{i+1}_f(D_0, \varepsilon) &:= E^1_f(E^i_f(D_0, \varepsilon), \varepsilon), i = 1, 2, \ldots \\ E^0_f(D_0, \varepsilon) &:= D_0. \end{aligned} \tag{3.9}$$

According to (3.5), we have the relationship

$$E^i_f(D_0) \subseteq E^{i+1}_f(D_0), i = 0, 1, 2, \ldots,$$

so that we have obtained a monotone increasing sequence of sets. Translating Proposition 3.3 to the current notation, we obtain

Proposition 3.6: *Let $\varepsilon > 0$ be a real number, and let D_0 be an open domain in R^n. There is a static state feedback controller with directional error of ε that drives Σ from every initial state $z \in E^{i+1}_f(D_0, \varepsilon)$ into $E^i_f(D_0, \varepsilon)$.*

Iterating Lemma 3.5 yields the following.

Lemma 3.7: *Let Σ be a system described by the differential equation* (1.1) *with a continuous function f, and let D_0 be an open domain in R^n. Then, the expansion set $E^i_f(D_0, \varepsilon)$ is an open set for all $\varepsilon > 0$ and all $i = 1, 2, \ldots$*

We can introduce now the main notion of our discussion.

Definition 3.8: Let D_0 be an open domain in R^n, let $f : R^n \times R^m \to R^n$ be a continuous function, and let $\varepsilon > 0$ be a real number. The *expansion* $E_f(D_0, \varepsilon)$ of D_0 with respect to f and ε is given by

$$E_f(D_0, \varepsilon) := \bigcup_{i \geq 0} E^i_f(D_0, \varepsilon). \tag{3.10}$$

Considering that the union of open sets is an open set, we conclude from Lemma 3.7 that the following is true.

Corollary 3.9: *Let $f: R^n \times R^m \to R^n$ be a continuous function and let $\varepsilon > 0$ be a real number. Then, for every open domain $D_0 \subseteq R^n$, the expansion $E_f(D_0, \varepsilon)$ is an open set.*

The definition of a union directly yields the next statement.

Lemma 3.10: *If $x \in E_f(D_0, \varepsilon)$, then there is a first integer $i \geq 0$ such that $x \in E^i_f(D_0, \varepsilon)$.*

Our interest in the expansion set $E_f(D_0, \varepsilon)$ originates from the following statement, which is one of the main results of our discussion.

Theorem 3.11: *Let Σ be a system described by the differential equation* (1.1) *with a continuous function f, and let D_0 be an open domain in R^n. Then, (i) and (ii) are equivalent.*

(i) There is a static state feedback controller with directional error of ε that drives Σ from a state $z \in R^n$ into D_0 in finite time.
(ii) $z \in E_f(D_0, \varepsilon)$.

Furthermore,

(iii) For a state $z \notin E_f(D_0, \varepsilon)$, there is no state feedback controller – not static nor dynamic – that drives Σ from z into D_0 in finite time with a directional error of ε.

Proof: First, consider a point $z \in E_f(D_0, \varepsilon)$. According to Lemma 3.10, there is a first integer i such that $z \in E^i_f(D_0, \varepsilon)$. In view of Proposition 3.6, this implies that there is a state feedback function φ with a directional error of ε that drives Σ from z to a point $z_1 \in E^{i-1}_f(D_0, \varepsilon)$ in finite time. The same Proposition implies further that the state feedback function φ can be extended to take Σ from z_1 to a point $z_2 \in E^{i-2}_f(D_0, \varepsilon)$ in finite time, and so on, until Σ reaches a point $z_i \in D_0$. As the number of such segments is finite and each segment is traversed in finite time, the total time is also finite. Hence, *(ii)* implies *(i)*.

Conversely, assume by contradiction that *(i)* is valid, but $z \notin E_f(D_0, \varepsilon)$. Considering (3.10), the latter implies that

$$z \notin E^i_f(D_0, \varepsilon) \text{ for any } i = 0, 1, 2, \ldots \tag{3.11}$$

Now, let $\varphi: R^n \to R^m$ be a state feedback function satisfying *(i)*, and denote by $x(t)$ the trajectory of the closed-loop system Σ_φ from the initial state $x(0) = z$ to a point $s \in D_0$. Let T be the time at which Σ_φ reaches s. As $x(t)$ is a solution of the differential equation (1.1) with bounded input (Assumption 2.1), it follows that $x(t)$ is a continuous function of t. Further, as $D_0 \subseteq E_f(D_0, \varepsilon)$ by definition, we conclude that $s \in E_f(D_0, \varepsilon)$. By replacing the open set D_0 by the open set $E_f(D_0, \varepsilon)$ in the proof of Proposition 3.4, we conclude that the assumption at the start of the current paragraph leads to a contradiction. Hence, *(i)* implies *(ii)*, and *(i)* and *(ii)* are equivalent.

Finally, to prove *(iii)*, consider again a state $z \notin E_f(D_0, \varepsilon)$, and assume, by contradiction, that there is a state feedback controller C that takes Σ from z to a point s of D_0 in finite time T. By the previous part of the proof, it follows that C is not a static state feedback controller. Then, C must be a dynamic state feedback controller. Denote by $x(t)$, $0 \leq t \leq T$ the path of the closed-loop system Σ_c from $x(0) = z$ to $x(T) = s$. Note that, irrespective of the nature of the controller C, the system Σ receives an input value $u(t)$ at the state $x(t)$ at every time $0 \leq t \leq T$, and this input function takes the closed-loop system Σ_c from z into D_0. As the latter is accomplished through the values of $f(x(t), u(t))$, $0 \leq t \leq T$, the argument used in the previous paragraph also shows that our current assumption leads to a contradiction. Therefore, *(iii)* is valid, and the proof concludes. □

So far, we have given no consideration to continuity features of the state feedback function φ that guides Σ toward the target domain D_0. We show next that a piecewise continuous implementation of φ can be used.

Theorem 3.12: *Let Σ be a system described by the differential equation* (1.1) *with a continuous function f. Let D_0 be an open domain in R^n, let $\varepsilon > 0$ be a real number consistent with Assumption* 2.4, *and assume that the initial state of Σ satisfies $x_0 \in E_f(D_0, \varepsilon)$. Then, there is a piecewise continuous state feedback function φ that takes Σ from x_0 into D_0 in finite time, with a directional error of ε.*

Proof: The proof is an elaboration on the proofs of Theorem 3.11 and Lemma 3.5. Assume that the closed-loop system has reached a point $x_1 \in E^i_f(D_0, \varepsilon)$, $i \geq 1$. There is then an input value $u(x_1) \in R^m$ for which the vector $f(x_1, u(x_1))$ points to a point $y \in E^{i-1}_f(D_0, \varepsilon)$. As $E^{i-1}_f(D_0, \varepsilon)$ is an open set by Lemma 3.7, the state y is an interior point of $E^{i-1}_f(D_0, \varepsilon)$. Recall from the proof of Lemma 3.5 that $f(x, \varphi(x)) \neq 0$ at all points of $E_f(D_0, \varepsilon)\backslash D_0$, since the closed-loop system cannot have a stationary point there. Combining this with the continuity of the function f, it follows that there is a real number $\delta'_1 > 0$ such that the vector $f(x', u(x_1))$ points to a point of $E^{i-1}_f(D_0, \varepsilon)$ for all $x' \in B(x_1, \delta'_1)$. In addition, as $E^i_f(D_0, \varepsilon)$ is an open set, there is a real number $\delta''_1 > 0$ for which the ball $B(x_1, \delta''_1)$ is included in $E^i_f(D_0, \varepsilon)$. Letting $\delta_1 := \min\{\delta'_1, \delta''_1\}$, we conclude that the constant value $\varphi_c(x') := u(x_1)$ can be used as a state feedback function at all states $x' \in B(x_1, \delta_1)$. When we reach a point x_2 in the boundary of the ball $B(x_1, \delta_1)$, we repeat this process to obtain a radius δ_2 analogous to δ_1; extend the function φ_c with the corresponding new constant value. Continuing in this manner, one of the following two option must result: *(a)* the closed-loop system Σ_{φ_c} reaches a point of the set $E^{i-1}_f(D_0, \varepsilon)$; or *(b)* the sequence of radii $\delta_1, \delta_2, \ldots$ converges to zero for all possible choices of $\delta_1, \delta_2, \ldots$

In case *(a)*, the state feedback function φ_c takes the closed-loop system Σ_{φ_c} into $E_f^{i-1}(D_0, \varepsilon)$; as φ_c is a piecewise constant function, it is piecewise continuous, and our theorem follows by recursion on i. In case *(b)*, let z be a limit point of the sequence $x_1, x_2, \ldots$. As the argument of the previous paragraph applies to the point z in the same way it applied to the point x_1, we conclude that there is a real number $\delta > 0$ such that the ball $B(z, \delta)$ has the same features as the ball $B(x_1, \delta_1)$, contradicting the possibility that $\lim_{i\to\infty}\delta_i = 0$ for all possible choices of the radii δ_1, $\delta_2, \ldots$. Hence, case *(b)* leads to a contradiction, and our proof concludes. □

The proof of Theorem 3.11 outlines a simple and effective method for calculating state feedback functions that drive a given system Σ into a desired domain in state space with a directional error of ε: at each state x of a domain $E^i{}_f(D_0, \varepsilon)$, choose a state feedback function φ for which the vector $f(x, \varphi(x))$ points to a point of $E_f^{i-1}(D_0, \varepsilon)$. Such a value of φ is obtained by solving an inequality based on the function f – the function that is given in the differential equation of the controlled system Σ. Section 5 demonstrates this process of deriving state feedback functions on two examples.

4. Robust control

The question often arises as to whether a system can be controlled toward an assigned objective under conditions of imperfect accuracy, with no specific inaccuracy specified. The motivation behind this question originates from the fact that controller accuracy can frequently be improved at additional cost. Formally, this leads us to the notion of robustness, which addresses the issue of whether an imperfect controller can be effective. If an imperfect controller can be effective, then one can proceed further to estimate the maximal tolerable controller error. In more precise terms, we refer to the following.

Definition 4.1: A *robust implementation* of a state feedback controller is an implementation with a non-zero directional error.

Our investigation of robust implementations requires the next notion.

Definition 4.2: Let $f : R^n \times R^m \to R^n$ be a continuous function, and let D_0 be an open domain in R^n. The *super extension set* of D_0 with respect to f is

$$E_f(D_0) := \bigcup_{\varepsilon>0} E_f(D_0, \varepsilon). \tag{4.1}$$

Combining this notion with Theorem 3.11 leads to the following characterisation of the circumstances under which a robust implementation can be effective.

Theorem 4.3: *Let Σ be a system described by the differential equation* (1.1) *with a continuous function f. Let D_0 be an open domain in R^n, and let $E_f(D_0)$ be the super expansion set. Then, (i) and (ii) are equivalent.*

(i) There is a robust implementation of a static state feedback controller that drives Σ from a state $z \in R^n$ into D_0 in finite time.
(ii) $z \in E_f(D_0)$.

Furthermore,

(iii) For a state $z \notin E_f(D_0)$, there is no robust implementation of a state feedback controller – not a static nor a dynamic controller – that takes Σ from z into D_0 in finite time.

Proof: Since $z \in E_f(D_0)$, it follows from (4.1) that there is a real number $\varepsilon > 0$ such that $z \in E_f(D_0, \varepsilon)$. Consequently, parts *(i)* and *(ii)* follow directly from Theorem 3.11*(i)* and *(ii)*. Regarding *(iii)*, the fact that $z \notin E_f(D_0)$ means that there is no $\varepsilon > 0$ for which $z \in E_f(D_0, \varepsilon)$. Thus, *(iii)* is a consequence of Theorem 3.11*(iii)*. □

Considering that a union of open sets is an open set, the following is a consequence of Corollary 3.9.

Corollary 4.4: *Let f: $R^n \times R^m \to R^n$ be a continuous function, and let D_0 be an open domain in R^n. Then, the super expansion set $E_f(D_0)$ is an open set.*

5. Examples

In this section, we demonstrate the computation of robust controllers. We start with an example where the controlled system has a one-dimensional state space.

Example 5.1: Consider the system

$$\dot{x} = x^2 + xu$$

with the target domain $D_0 = B(0, 0.1)$, namely, a ball of radius 0.1 around the origin. Considering that this is a one-dimensional system, it follows by Theorem 3.11 that we are looking for a state feedback function $\varphi : R \to R$ that satisfies the following inequalities:

(i) $x^2 + x\varphi(x) < 0$ for $x \geq 0.1$; and
(ii) $x^2 + x\varphi(x) > 0$ for $x \leq -0.1$.

There are, in fact, infinitely many state feedback functions $\varphi(x)$ that satisfy these inequalities at all states $x \in R$ in a robust manner. To demonstrate one such state feedback

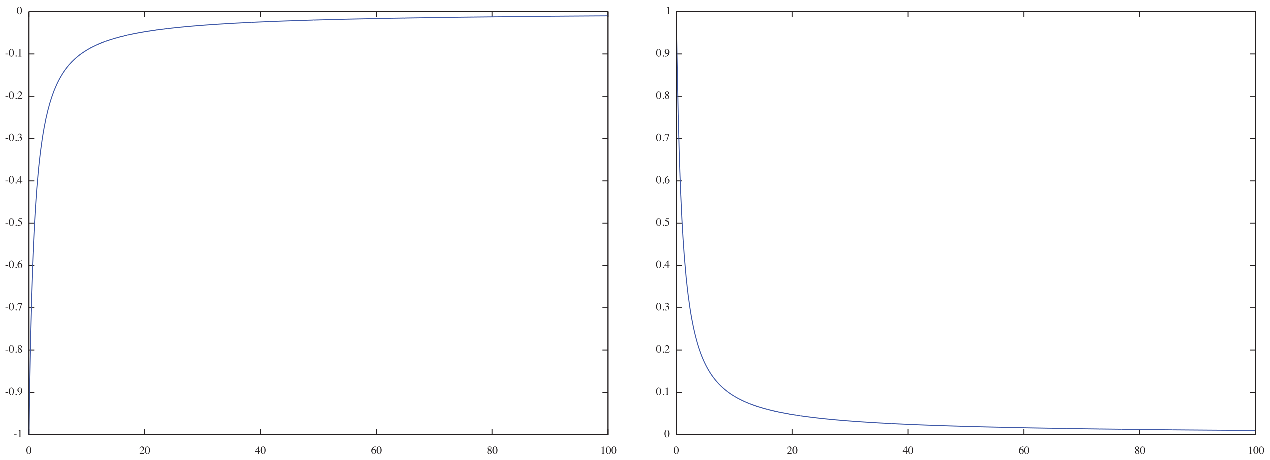

Figure 7. Closed-loop trajectory as a function of time for Example 5.1.

function, define

$$\text{sgn}(x) := \begin{cases} 1 \text{ for } x > 0, \\ 0 \text{ for } x = 0, \\ -1 \text{ for } x < 0. \end{cases}$$

Then, for an input magnitude bound of $M > 0$, the state feedback function given by

$$\varphi(x) := -x(1 + \text{sgn}(x))$$

satisfies the requirements for all $|x| \leq M/2$. A brief examination shows that this function tolerates non-zero implementation errors, and hence provides robust control. According to (1.3), this state feedback function yields a closed-loop system Σ_φ with the differential equation $\dot{x} = -x^2\text{sgn}(x)$. A slight reflection shows that this closed-loop system approaches the origin asymptotically. Figure 7 shows the trajectory of Σ_φ as a function of time, starting from two initial states: $x(0) = -1$ (left side of the figure) and $x(0) = 1$ (right side of the figure).

Next, we demonstrate the construction of a state feedback function for a nonlinear system with a two-dimensional state space.

Example 5.2: Consider a system Σ with a two-dimensional state space and a one-dimensional input described by the differential equation

$$\Sigma : \begin{array}{l} \dot{x} = x^2 - y^2 \\ \dot{y} = u \end{array} = f(x, y, u).$$

We use again the target domain $D_0 = B(0, 0.1)$. For the sake of transparency, we ignore here the input magnitude bound M; it can be readily incorporated. Note that the first component of f cannot be directly affected by a state feedback function, since the input u does not appear in it. Whenever $x^2 > y^2$, we have $\dot{x} > 0$, so the state will move generally to the right, as indicated by the arrows in Figure 8 . Similarly, when $x^2 < y^2$, we have $\dot{x} < 0$, and the state will move generally toward the left, as indicated in the same figure. Seeing that $\dot{y} = u$, it follows that the tilt of the state's trajectory can be assigned as desired by selecting the value of the input u. Combining the observations of the last three sentences, we conclude that f can be directed toward the origin only in the domains marked A in Figure 8. Consequently, the domains marked A form the domain $E^1_f(D_0)$ in this case. In explicit form, we have

$$E^1_f(D_0) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : y^2 > x^2 \text{ and } x > 0; \text{ or } y^2 < x^2 \text{ and } x < 0 \right\}.$$

A slight reflection shows that, in the remaining parts of the plane, the direction of the vector f can always be

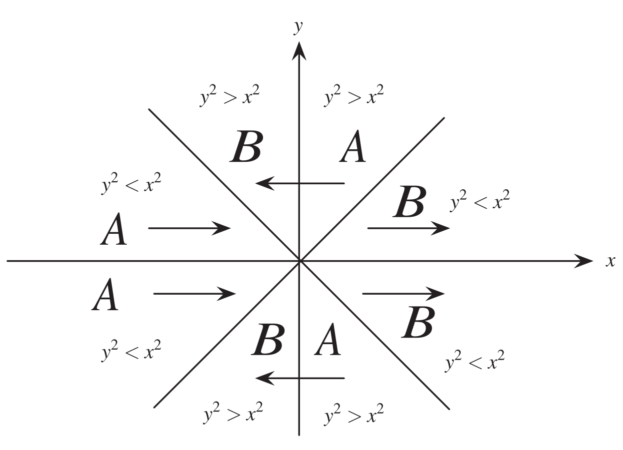

Figure 8. Orientations of motion for Example 5.2.

oriented toward a point of $E^1_f(D_0)$ by choosing an appropriate input value u. Furthermore, within $E^1_f(D_0)$, the vector f can also be oriented toward a point of $E^1_f(D_0)$ by choosing an appropriate input value u. Consequently,

$$E^2_f(D_0) \supseteq R^2 \setminus E^1_f(D_0),$$

and $E^2_f(D_0)$ can be taken as an open set that includes the set $R^2 \setminus E^1_f(D_0)$. Specifically, $E^2_f(D_0)$ is formed by an open set that includes the domains marked as B in Figure 8. Combining our conclusions, we obtain in this case that

$$E_f(D_0) = E^1_f(D_0) \cup E^2_f(D_0) = R^2$$

(ignoring input magnitude bounds). Thus, according to Theorem 4.3, there is a state feedback function φ that takes Σ to a close vicinity of the origin from every bounded domain in state space (with suitable input magnitude bound).

We construct now one such state feedback function (as usual, there are infinitely many appropriate state feedback functions). To simplify the form of our state feedback function, define the domain

$$A' := \{(x, y) : x^2 - y^2 < 0, x > 0, |y/x| < 100 \text{ or } x^2 - y^2 > 0, x < 0, |y/x| < 100\}.$$

This domain is obtained from the domain A by excluding a thin angle around the y-axis. All remaining points of the plane are included in the domain

$$B' := R^2 \setminus A'.$$

Then, a slight reflection shows that the following state feedback function assigns directions to the vector $f(x, y, \varphi(x, y))$ that point to the origin within A', and point to A' within B'. Consequently, this feedback function takes Σ into D_0 in finite time from every state (x, y) in the plane (with input magnitude bound determined by $|(x, y)|$).

$$\varphi(x, y) := \begin{cases} (x^2 - y^2)y/x \text{ for all } (x, y) \in A', \\ (5(x^2 - y^2) + 1) \text{ for all } (x, y) \notin A' \\ \qquad \text{with } (y \geq 0 \text{ and } x > 0), \\ (5(x^2 - y^2) - 1) \text{ for all } (x, y) \notin A' \\ \qquad \text{with } (y > 0 \text{ and } x < 0), \\ (5(y^2 - x^2) - 1) \text{ for all } (x, y) \notin A' \\ \qquad \text{with } (y \leq 0 \text{ and } x > 0), \\ (5(y^2 - x^2) + 1) \text{ for all } (x, y) \notin A' \\ \qquad \text{with } (y < 0 \text{ and } x < 0). \end{cases}$$

Figure 9 shows the path of the closed-loop system Σ_φ in state space, starting from four different initial conditions – one initial condition in each quadrant; the origin is at the centre of each subfigure.

6. Asymptotic stabilisation

The feedback methodology developed in the previous sections can be used to achieve global asymptotic stabilisation of nonlinear systems by static state feedback under rather general conditions. In fact, Examples 5.1 and 5.2 demonstrate static state feedback controllers that achieve global asymptotic stabilisation. To consider this issue in general, let Σ be a system described by the differential equation (1.1), where f is twice continuously differentiable. Given a state feedback function φ, denote by $\Sigma_\varphi(x_0, t)$ the response of the closed-loop system as a function of the time t, when started at the initial state x_0. We aim to derive a state feedback function φ that takes Σ asymptotically from x_0 to the origin; namely, we look for a state feedback function φ for which $\lim_{t\to\infty}\Sigma_\varphi(x_0, t) = 0$. It is convenient to assume in this section that Σ has a stationary point at the origin, i.e., that $f(0, 0) = 0$.

Clearly, a state feedback function φ that achieves global asymptotic stabilisation of Σ can be built in two steps:

(i) Use the technique of Theorem 4.3 to derive a state feedback function φ_1 that brings Σ from the initial state x_0 into a close vicinity $V = B(0, \rho)$ of the origin, where $\rho > 0$ is a small radius.

(ii) Use the linear approximation of Σ at the origin

$$\dot{x}(t) = \frac{\partial f(0, 0)}{\partial x}x(t) + \frac{\partial f(0, 0)}{\partial u}u(t) \tag{6.1}$$

to derive a linear state feedback function φ_2 that takes Σ asymptotically to the origin from within V.

Patching φ_1 and φ_2 together into one function φ yields a state feedback function that drives Σ asymptotically from an initial state x_0 to the origin, thus achieving asymptotic stabilisation.

As step *(ii)* involves the use of a linear state feedback to asymptotically stabilises Σ near the origin, we must assume that the linear approximation (6.1) is stabilisable.

Preliminary consideration must be given to the question of how close must Σ be taken to the origin, before a linear state feedback function can be activated. The proof of the next statement includes an estimate of the radius ρ^* of a ball $B(0, \rho^*)$ from within which a linear state feedback function can take Σ asymptotically to the origin.

Proposition 6.1: *Let Σ be a system with the differential equation $\dot{x}(t) = f(x(t), u(t))$, where f is twice continuously differentiable and $f(0, 0) = 0$, and assume that the linear approximation (6.1) of Σ at the origin is stabilisable. Then, there is a real number $\rho^* > 0$ and an $m \times n$ constant matrix F such that the solution $x(t)$ of the differential equation $\dot{x}(t) = f(x(t), Fx(t))$ satisfies $\lim_{t\to\infty}x(t) = 0$ for all initial conditions $x(0) \in B(0, \rho^*)$.*

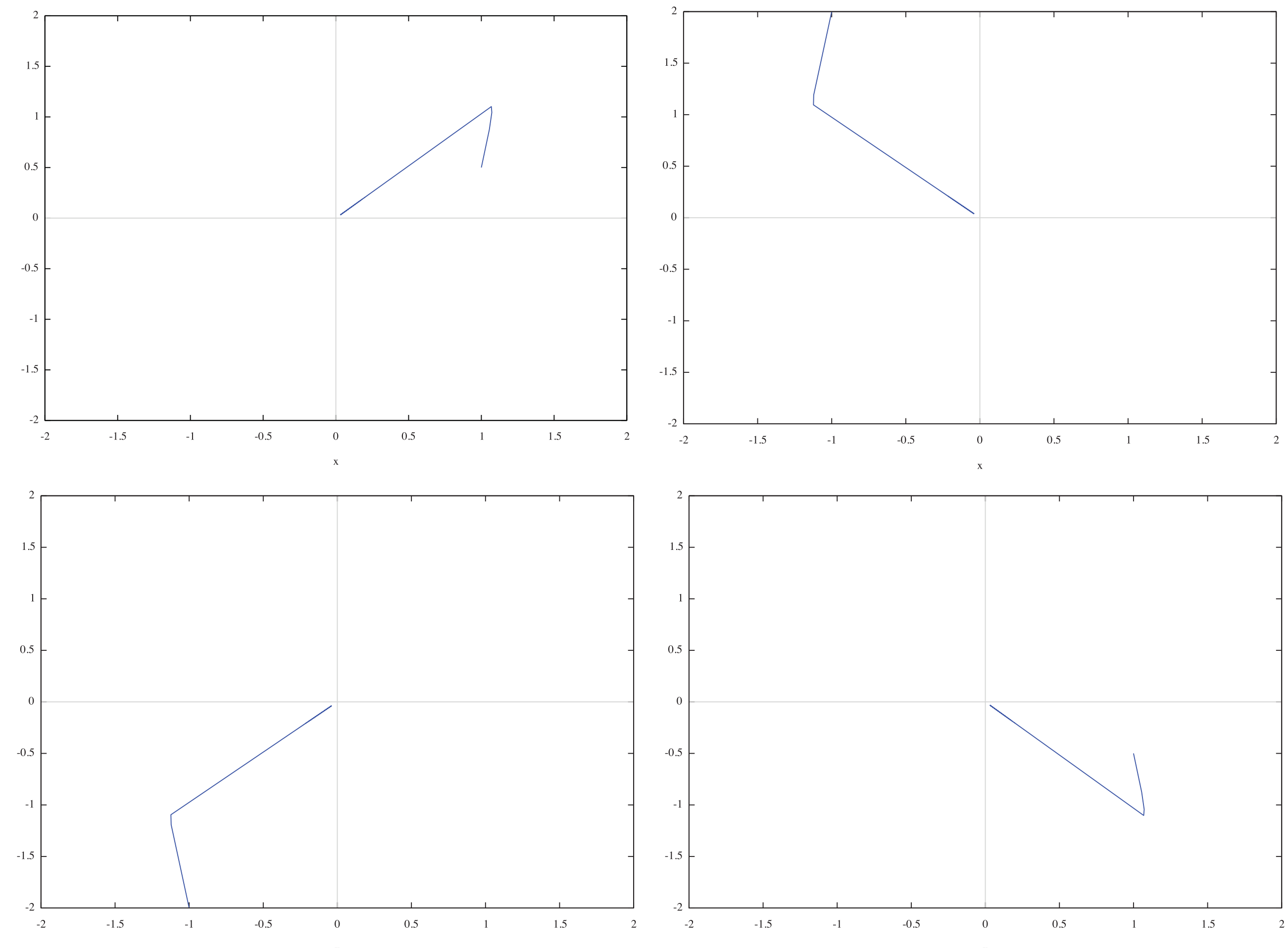

Figure 9. Trajectories of the closed-loop system of Example 5.2.

Proof: As $f(0,0) = 0$ and the linear system (6.1) is stabilisable, there is an $m \times n$ constant matrix F for which every solution $z(t)$ of the linear differential equation

$$\dot{z}(t) = \frac{\partial f(0,0)}{\partial x} z(t) + \frac{\partial f(0,0)}{\partial u} F z(t) =: D z(t) \qquad (6.2)$$

satisfies $\lim_{t\to\infty} z(t) = 0$; here,

$$D := \partial f(0,0)/\partial x + (\partial f(0,0)/\partial u) F \qquad (6.3)$$

is a constant $n \times n$ matrix. As this linear system is asymptotically stable, it follows by Lyapunov's theory that there is a constant symmetric positive definite matrix P such that

$$\frac{d}{dt}[z^T(t) P z(t)] < 0 \qquad (6.4)$$

for all $z(t) \neq 0$. Invoking the derivative, we get $\dot{z}^T(t) P z(t) + z^T(t) P \dot{z}(t) < 0$ for all $z(t) \neq 0$. Substituting from (6.2) yields

$$z^T(t) D^T P z(t) + z^T(t) P D z(t) < 0$$

for all $z(t) \neq 0$. Consequently, the symmetric matrix

$$Q := -(D^T P + P D) \qquad (6.5)$$

is positive definite.

Turning now to the nonlinear system Σ, define the function

$$g(x) := f(x, Fx).$$

Then, the closed-loop system around Σ with the linear state feedback function F is described by the differential equation

$$\dot{x}(t) = g(x(t)), x(0) = x_0. \qquad (6.6)$$

Using the fact that f is twice continuously differentiable and expanding it around the origin yields, according to Taylor's theorem, that

$$g(x) = Dx + \frac{1}{2}\begin{pmatrix} x \\ \vdots \\ x \end{pmatrix}^T g^{(2)}(\xi)\begin{pmatrix} x \\ \vdots \\ x \end{pmatrix}, \tag{6.7}$$

where D is given by (6.3),

$$g^{(2)}(\xi) := \begin{pmatrix} H(g_1(\xi)) \\ \vdots \\ H(g_n(\xi)) \end{pmatrix}$$

is an $n^2 \times n^2$ matrix with $H(g_i(\xi))$ being the $n \times n$ Hessian of the i-th component of g, and $\xi \in R^n$ is an appropriate point determined according to the error term of Taylor's theorem. To simplify notation, define the n-dimensional vector

$$\mu(x, \xi) := \frac{1}{2}\begin{pmatrix} x \\ \vdots \\ x \end{pmatrix}^T g^{(2)}(\xi)\begin{pmatrix} x \\ \vdots \\ x \end{pmatrix},$$

so that

$$g(x) = Dx + \mu(x, \xi). \tag{6.8}$$

Substituting the solution $x(t)$ of (6.6) into (6.4) instead of $z(t)$, and using the form (6.8) of g, we obtain

$$\begin{aligned}\frac{d}{dt}[x^T(t)Px(t)] &= g^T(x(t))Px(t) + x^T(t)Pg(x(t)) \\ &= (Dx(t) + \mu(x(t), \xi(t)))^T Px(t) \\ &\quad + x^T(t)P(Dx(t) + \mu(x(t), \xi(t))) \\ &= -x^T(t)Qx(t) + [\mu^T(x(t), \xi(t))Px(t) \\ &\quad + x^T(t)P\mu(x(t), \xi(t))] \\ &= x^T(t)Qx(t) \\ &\times\left[-1 + \frac{\mu^T(x(t), \xi(t))Px(t) + x^T(t)P\mu(x(t), \xi(t))}{x^T(t)Qx(t)}\right], \\ & x(t) \neq 0,\end{aligned} \tag{6.9}$$

where Q is given by (6.5). Now, according to Taylor's theorem and the fact that f is twice continuously differentiable, $\mu(x, \xi)$ is a second-order term in x. Combining this with the fact that Q is positive definite, it follows that, for a real number $\rho > 0$, the quantity

$$e(\rho) := \sup_{x,\xi \in B(0,\rho), x \neq 0} \left|\frac{\mu^T(x, \xi)Px + x^T P\mu(x, \xi)}{x^T Qx}\right|$$

satisfies

$$\lim_{\rho \to 0} e(\rho) = 0.$$

Consequently, for every real number $0 < \alpha < 1$, there is a radius $\rho_\alpha > 0$ such that $0 < e(\rho) < \alpha$ for all $0 < \rho \leq \rho_\alpha$. Substituting this fact into (6.9), we obtain that

$$\frac{d}{dt}\left[x^T(t)Px(t)\right] < 0 \tag{6.10}$$

for all $x(t) \neq 0$ satisfying $x(t) \in B(0, \rho)$, where $0 < \rho \leq \rho_\alpha$ and $0 < \alpha < 1$.

Next, let p_1 be the smallest eigenvalue of P and let p_2 be the largest eigenvalue of P. Then, since P is a positive definite matrix, we have

$$p_2 \geq p_1 > 0. \tag{6.11}$$

By classical features of eigenvalues of positive definite matrices, we can write

$$p_1|x|^2 \leq x^T Px \leq p_2|x|^2 \tag{6.12}$$

for all $x \in R^n$. Now, select a real number $0 < \alpha < 1$ and consider the positive number

$$\rho^* := \frac{p_1}{p_2}\rho_\alpha. \tag{6.13}$$

Note that by (6.11), we have $\rho^* \leq \rho_\alpha$.

An initial state $x(0) \in B(0, \rho^*)$ satisfies $x^T(0)Px(0) \leq p_2|x(0)|^2 \leq p_2\rho^* = p_1\rho_\alpha$. In view of (6.10) and the fact that $\rho^* \leq \rho_\alpha$, this implies that $x^T(t)Px(t) \leq p_1\rho_\alpha$ for all $t \geq 0$. But then, by (6.12), we have $p_1|x(t)|^2 \leq p_1\rho_\alpha$, or $|x(t)|^2 \leq \rho_\alpha$ for all $t \geq 0$. In other words, when $x(0) \in B(0, \rho^*)$, then the state of Σ stays within the ball $B(0, \rho_\alpha)$ for all $t \geq 0$, and (6.10) is valid for all $t \geq 0$. Lyapunov's theorem then guarantees that $\lim_{t \to \infty} x(t) = 0$. This concludes our proof. □

The proof of Proposition 6.1 proves the following statement.

Corollary 6.2: *Under the notation and conditions of Proposition 6.1, the system Σ can be asymptotically stabilised by linear state feedback within a ball of radius ρ^* around the origin, where ρ^* is given by (6.13).*

In view of Proposition 6.1 and Corollary 6.2, we can obtain asymptotic stabilisation of the system Σ if we can find a state feedback function that takes Σ from its initial state into the ball $B(0, \rho^*)$, where ρ^* is given by (6.13). Combining this with Theorem 4.3 and recalling that linear state feedback is robust, yields the following, which is the main result of this section.

Theorem 6.3: *Let Σ be a system described by the differential equation (1.1), where the function f is twice continuously differentiable and $f(0, 0) = 0$. Assume that the linear approximation (6.1) of Σ at the origin forms a stabilisable linear system. Let $X_0 \subseteq R^n$ be the set of all potential initial states of Σ, let ρ^* be given by (6.13), and let $E_f(B(0, \rho^*))$ be the super expansion set. Then, the following two statements are equivalent.*

(i) Σ *is robustly and asymptotically stabilisable over the domain X_0 of initial states.*
(ii) $X_0 \subseteq E_f(B(0, \rho^*))$.

7. Conclusion

In summary, we have described a general framework for the construction of state feedback controllers for nonlinear systems. The main step of this construction involves the solution of a set of inequalities based on the given function f that appears in the differential equation (1.1) of the controlled system Σ. These inequalities are induced by requiring the vector $f(x, u)$ to point in an appropriate range of directions. As the examples of Section 5 demonstrate, the calculation of stabilising feedback controllers is rather simple in this framework.

In the special case of systems with states of dimensions 1, 2, and 3, the selection of state feedback functions can be visualised. However, visualisation is, of course, not necessary; the corresponding inequalities can be solved computationally in any dimension, and appropriate state feedback functions are derived from these solutions.

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