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STABILIZATION OF NON-LINEAR SYSTEMS

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Stabilization of non-linear systems†

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The problem of stabilizing a discrete-time non-linear system is considered. For a rather large class of common stabilizable non-linear systems, a procedure leading to the stabilization of a given non-linear system Σ belonging to that class is derived. In this procedure, a pair of compensators is constructed, consisting of a precompensator and an output feedback compensator, which, when connected in closed loop around the system Σ, yield a closed-loop system that is internally stable for bounded input sequences. The procedure allows the construction of infinitely many different pairs of such compensators, thus facilitating the choice of a convenient one.

1. Introduction

In general terms, the problem of stabilizing a given non-linear system Σ is concerned with the construction of a number of other systems, usually called *compensators*, which, when connected to Σ and among themselves in a prescribed way, yield a composite system that exhibits stable behaviour. The compensators control the operation of the system in such a way as to avoid the activation of any instabilities. Owing to the abundance of unstable non-linear systems, the stabilization problem is probably one of the most commonly encountered problems in engineering science and practice. The present paper is devoted to the development of a solution to the stabilization problem for a rather common class of non-linear systems. Similar ideas can be applied to solve the stabilization problem for other classes of systems as well. Our main objective in the present paper is to provide a transparent and concise presentation of the basics of a theory of stabilization for non-linear systems. In some cases, we shall trade off generality of statements for clarity of the exposition. Nonetheless, the restrictions that we impose on the systems under consideration are of a rather mild nature, and they are satisfied in a wide variety of practical applications.

The problem of stabilizing a non-linear system Σ can be perceived as consisting of two subproblems: (i) finding a control configuration that allows stabilization; and (ii) constructing compensators that yield stabilization for this configuration. Certainly, neither of these two subproblems has a unique solution. In choosing the control configuration that we use in our framework, we followed classical patterns in control engineering, which seem to indicate the paramount effectiveness of additive feedback control. We adopt as our basic control configuration the following classical configuration, where Σ is the given non-linear system which has to be stabilized, π is a non-linear causal precompensator, and ϕ is a non-linear causal feedback compensator. We

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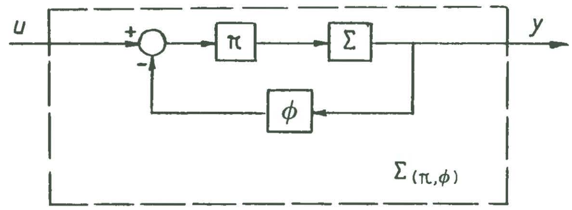
denote by $\Sigma_{(\pi,\phi)}$ the overall composite system described by the diagram



(1.1)

Our main objective is to devise a procedure that, using the parameters of the given system Σ, yields a pair of compensators π and ϕ for which the composite system $\Sigma_{(\pi,\phi)}$ is internally stable. By 'internally stable' we mean that the stability of the composite system $\Sigma_{(\pi,\phi)}$ is not destroyed when small random noises disturb the signals at the entry ports of the subsystems Σ, π and ϕ.

The basic mathematical concept that facilitates the development of our stabilization theory is the concept of rationality. Roughly speaking, we say that a system Σ is *right-rational* if there are stable systems P and Q, where Q is invertible, such that $\Sigma = PQ^{-1}$. The control theoretic significance of the concept of right rationality stems from the fact that, at least in the case of injective systems, only right-rational systems can be internally stabilized (Hammer 1984 b). We say that a system Σ is *left-rational* if there is a pair of stable systems T and R, where T is invertible, such that $\Sigma = T^{-1}R$. Left-rationality also plays a central role in our stabilization theory, as we discuss below.

Returning now to diagram (1.1), we can express the input/output relationship induced by the overall composite system $\Sigma_{(\pi,\phi)}$, when Σ is strictly causal and π and ϕ are causal, by

$$\left.\begin{aligned} \Sigma_{(\pi,\phi)} &= \Sigma\psi_{(\pi,\phi)} \\ \text{where} \qquad & \\ \psi_{(\pi,\phi)} &= \pi[I + \phi\Sigma\pi]^{-1} \end{aligned}\right\} \tag{1.2}$$

and where I denotes the appropriate identity map (for details see e.g. Hammer 1984 b). In order to simplify our computations, we choose the compensators π and ϕ to be of the following particular form:

$$\left.\begin{aligned} \pi &= B^{-1} \\ \phi &= A \end{aligned}\right\} \tag{1.3}$$

where B is a stable and invertible system having a causal inverse B^{-1}, and where A is a causal and stable system. As it turns out (see § 7 below), the systems Σ we are interested in can be stabilized using compensators of this particular form, so there is no urgent need to treat more general forms of compensators at this point. Assume now that there is a right-fraction representation $\Sigma = PQ^{-1}$, where P and Q are stable systems. (We discuss the existence of fraction representations in detail later.) Then, substituting (1.3) into (1.2), we obtain

$$\Sigma_{(\pi,\phi)} = \Sigma_{(B^{-1},A)} = PQ^{-1}B^{-1}[I + APQ^{-1}B^{-1}]^{-1} = P[BQ + AP]^{-1} \tag{1.4}$$

Defining the system

$$M := AP + BQ \tag{1.5}$$

we have

$$\Sigma_{(\pi,\phi)} = PM^{-1} \tag{1.6}$$

Clearly, if we can find A and B for which M has a stable inverse M^{-1}, then, by the stability of P, the overall system $\Sigma_{(\pi,\phi)}$ will be input/output stable. We recall that, in order to be applicable to our stabilization scheme, the systems A and B have to satisfy the appropriate restrictions, namely A has to be stable and causal, and B has to be stable and invertible with a causal inverse B^{-1}. In § 3 we show that if A and B are not merely stable, but also satisfy an additional condition of 'uniform stability' (in an appropriate sense), then the composite system $\Sigma_{(B^{-1},A)}$ will actually be internally stable. Thus the stabilization problem reduces to the problem of finding a pair of stable systems A and B, satisfying the aforementioned requirements, for which $AP + BQ$ has a stable inverse. Our present paper is dedicated to the complete solution of this problem. Generally speaking, the situation is reminiscent of the situation in the theory of linear systems. In close analogy to the linear theory, the existence of A and B is related to a certain notion of coprimeness of non-linear systems, introduced in Hammer (1985 a). We shall review this notion of coprimeness in a later section of this paper. In our present paper, we incorporate the requirements of causality, invertibility and uniform continuity mentioned above into the theory of stability developed in Hammer (1985 a).

We next briefly consider the significance of left-fraction representations to our discussion. First, we recall that, for a pair of systems C, D having the same input and output spaces, the sum $C + D$ is defined, for every input sequence u, by $(C + D)u = Cu + Du$. Consequently, for any system E having an output space that is contained in the input space of C and D, we have $(C + D)E = CE + DE$. Now, let $\Sigma = T^{-1}R$, where T and R are stable systems, be a left-fraction representation, and let h be any stable system for which both of the combinations hT and hR are defined and causal. Using the right-fraction representation $\Sigma = PQ^{-1}$, we have $T^{-1}R = PQ^{-1}$, or

$$TP = RQ$$

and $hTP = hRQ$. Suppose further that we found one pair of maps A, B satisfying $AP + BQ = M$. Then, letting

$$\left.\begin{aligned} A' &:= A + hT \\ B' &:= B - hR \end{aligned}\right\} \tag{1.7}$$

we obtain

$$\begin{aligned} A'P = B'Q &= (A + hT)P + (B - hR)Q = AP + BQ + (hTP - hRQ) \\ &= AP + BQ = M \end{aligned}$$

and we have a new pair of maps A', B' satisfying $A'P + B'Q = M$. Thus, using a left-fraction representation of Σ, we can generate from one pair A, B satisfying the equation $AP + BQ = M$ infinitely many new pairs A', B' satisfying the same equation $A'P + B'Q = M$, each new pair corresponding to one choice of h. This simple procedure will allow us to obtain different pairs of compensators π and ϕ stabilizing Σ, once one such pair of compensators is known. Again, we encounter here a situation closely analogous to the linear theory. A few words of caution in this regard are, however, in place. Notwithstanding the close analogy to the linear theory encountered in the last few paragraphs, one should always be aware of the fundamental distinction that separates the linear and non-linear theories. As we shall see in the course of our ensuing discussion, several time-honoured principles of the theory of linear systems, which at first glance may seem to be of general validity, turn out to be invalid in the non-linear case. When trying to generalize principles of the linear theory to the non-linear theory, each instance has to be given separate careful consideration.

The theory that we develop in the present paper is explicitly presented here for discrete-time systems. A basic assumption that we make is that the stabilized system $\Sigma_{(\pi,\phi)}$ is operated only by bounded inputs. Namely, we assume that there is an arbitrary but fixed positive constant $\theta > 0$, specified in advance, so that all input sequences $u_0, u_1, \ldots, u_i, \ldots$ applied to the closed-loop system $\Sigma_{(\pi,\phi)}$ have all their elements u_i, $i = 0, 1, 2, \ldots$, with norm not exceeding θ. From the practical point of view, this assumption is satisfied in most applications, owing to the inherent natural limitations of physical systems. The mechanism that produces the input signals for $\Sigma_{(\pi,\phi)}$ has an inherent physical limitation on the maximal signal amplitude θ it can produce. Thus it seems that, for most engineering applications, the assumption of bounded inputs has little practical significance. However, from the mathematical point of view, this assumption yields a substantial simplification of our discussion. For more details on the bounded-input assumption and its implications see § 3 below. A detailed description of our stabilization procedure, as well as an accurate description of all the assumptions we make, is provided in § 7 of the paper. Section 2 contains a review of our basic mathematical framework. Sections 4 and 6 are devoted to the development of certain technical concepts that play a central role in our theory. Section 5 contains a discussion of left-fraction representations of non-linear systems. Finally, we remark that even though much of our discussion in the paper concentrates on injective systems, we show in § 7 that injectivity is not an essential assumption as far as stabilization is concerned, and our stabilization theory directly applies to noninjective systems as well.

As we have indicated before, the stabilization theory for non-linear systems developed in our present paper bears, in its appearance, a close resemblance to the stabilization theory for linear systems. Particularly suggestive is the transfer-matrix approach to linear-system stabilization, as presented in Rosenbrock (1970), Desoer and Chan (1972), Hammer (1983 a, b), and in the references cited in these works. Our present paper is a direct continuation of Hammer (1984 a, b, 1985 a, b). Some alternative recent treatments of problems related to stabilization of non linear systems can be found in Vidyasagar (1980), Sontag (1981), Desoer and Lin (1984), and in the references cited in these works.

2. Some basic facts

The basic mathematical framework that we employ in our current paper is taken from Hammer (1984 a, b, 1985 a, b). We devote the present section to a brief review of the fundamentals of this framework, as well as to an exposition of a few additional specific facts which we will need for our discussion in the following sections. Our present exposition is stated for discrete-time systems, and so we start by reviewing the spaces of input sequences.

Let R denote the set of real numbers, and, as usual, for an integer $m \geqslant 1$, let R^m denote the set of all m-tuples of real numbers. We denote by $S_0(R^m)$ the set of all infinite sequences $u := u_0, u_1, u_2, \ldots$, where $u_i \in R^m$ for all integers $i \geqslant 0$. The space $S_0(R^m)$ will serve as the space of input sequences for our systems. For a sequence $u \in S_0(R^m)$ and an integer $i \geqslant 0$, we denote by u_i the ith element of the sequence u, and we regard i as the time marker. Given two integers $a \geqslant b \geqslant 0$, we denote by u_a^b the elements $u_a, u_{a+1}, \ldots, u_b$. The zero sequence in $S_0(R^m)$, consisting of only zero elements, is denoted by 0. A *system* is simply a map $\Sigma: S_0(R^m) \to S_0(R^p)$, transforming input sequences (of m-dimensional vectors) into output sequences (of p-dimensional vectors). Given a subset $S \subset S_0(R^m)$, we denote by $\Sigma[S]$ the set into which S is mapped by Σ. For a sequence $u \in S_0(R^m)$ and a pair of integers $a \geqslant b \geqslant 0$, we denote by $\Sigma u]_a^b$ the elements $y_a, y_{a+1}, \ldots, y_b$, where $y := \Sigma u$ is the corresponding output sequence. If $b < a$, then u_a^b denotes the empty set.

As a concrete example of some systems, consider the recursive systems, defined as follows. A system $\Sigma := S_0(R^m) \to S_0(R^p)$ is *recursive* if there exist a pair of integers $\eta, \mu \geqslant 0$ and a function $f: (R^p)^{\eta+1} \times (R^m)^{\mu+1} \to R^p$ such that, for every input sequence $u \in S_0(R^m)$, the elements of the output sequence $y := \Sigma u$ can be computed recursively from the relation

$$y_{k+\eta+1} = f(y_k, \ldots, y_{k+\eta}, u_k, \ldots, u_{k+\mu}) \tag{2.1}$$

for all integers $k \geqslant 0$. To characterize Σ, the initial conditions $y_0, y_1, \ldots, y_\eta$ have to be specified together with the function f. Unless otherwise stated, we shall (arbitrarily) assume zero initial conditions, i.e. $y_0 = y_1 = \ldots = y_\eta = 0$.

On the space $S_0(R^m)$ we define the operation of addition coefficientwise, as usual, so that the sum $w := u + v$ of two sequences $u, v \in S_0(R^m)$ is the sequence w with elements $w_i := u_i + v_i$ for all integers $i \geqslant 0$. For a pair of systems $\Sigma_1, \Sigma_2: S_0(R^m) \to S_0(R^p)$, we define the *sum* $\Sigma_1 + \Sigma_2: S_0(R^m) \to S_0(R^p)$ pointwise by $(\Sigma_1 + \Sigma_2)u := \Sigma_1 u + \Sigma_2 u$ for all $u \in S_0(R^m)$. Several subsets of $S_0(R^m)$ play an important role in our discussion, and we consider some of them next. For a real number $M \geqslant 0$, let $[-M, M]^m$ denote the set of all m-dimensional real vectors with components in the closed interval $[-M, M]$. We denote by $S_0(M^m)$ the set of all sequences $u \in S_0(R^m)$ with elements u_i belonging to $[-M, M]^m$ for all integers $i \geqslant 0$. Thus $S_0(M^m)$ is the set of all input sequences 'bounded by M'. Referring to classical terminology, we say that a system $\Sigma: S_0(R^m) \to S_0(R^p)$ is *BIBO* (bounded-input bounded-output)-*stable* if, for every real $\theta > 0$, there exists a real $M > 0$ such that $\Sigma[S_0(\theta^m)] \subset S_0(M^p)$.

The main notion of system stability that we employ in our discussion is closely related to the Liapunov notion of stability, which, in turn, is related to the continuity of the map representing the system. In order to discuss continuity of maps, we first define a norm ρ on our spaces of sequences as follows. For a vector $\alpha := (\alpha^1, \ldots, \alpha^m) \in R^m$, we let $|\alpha| := \max\{|\alpha^i|, i = 1, \ldots, m\}$. For a sequence $u \in S_0(R^m)$, we define $\rho(u) := \operatorname{Sup}\{2^{-i}|u_i|, i \geqslant 0\}$. The norm ρ induces, in an evident way, a metric ρ on $S_0(R^m)$ given, for every pair of elements $u, v \in S_0(R^m)$, by $\rho(u, v) := \rho(u - v)$. Throughout our discussion, whenever referring to continuity of maps, we shall always mean continuity with respect to the topology induced by the metric ρ. Under these terms, a system $\Sigma: S_0(R^m) \to S_0(R^p)$ is *stable* if it is BIBO-stable, and if, for every real $\theta > 0$, the restriction $\Sigma: S_0(\theta^m) \to S_0(R^p)$ is a continuous map. A sequence $u \in S_0(R^m)$ is *bounded* if there is a real $\theta > 0$ such that $u \in S_0(\theta^m)$.

The basic tool that makes our discussing possible is the concept of coprimeness, which we review next, using the definition introduced in Hammer (1985 a, § 4). Before citing this definition, we wish to indicate its intuitive significance. Consider two stable maps $P: S \to S_0(R^p)$ and $Q: S \to S_0(R^m)$ having a common domain $S \subset S_0(R^q)$ for some integer $q > 0$. Qualitatively, we say that P and Q are right-coprime if, for every *un*bounded input sequence $u \in S$, at least one of the output sequences Pu or Qu is unbounded. In other words, for coprime maps P and Q, if both of Pu and Qu are bounded sequences, then the input sequence u must be bounded. Using linear terminology, this amounts to the requirement that P and Q have no 'unstable zeros' in common. Formally, it is easiest to express the property of coprimeness in terms of inverse images. For a set $A \subset S_0(R^p)$, let $P^*[A]$ be its inverse image through P, i.e. $P^*[A]$ is the set of all elements $u \in S$ satisfying $Pu \in A$. Then the formal definition of coprimeness is as follows (Hammer 1985 a).

Definition (2.2)

Let $S \subset S_0(R^q)$ be a subset. Two stable maps $P: S \to S_0(R^p)$ and $Q: S \to S_0(R^m)$ are *right-coprime* if the following two conditions hold.

(α) For every real $\tau > 0$ there exists a real $\theta > 0$ such that

$$P^*[S_0(\tau^p)] \cap Q^*[S_0(\tau^m)] \subset S_0(\theta^q)$$

(β) For every real $\tau > 0$ the set $S \cap S_0(\tau^q)$ is complete (i.e. is a closed subset of $S_0(\tau^q)$).

We remark that condition (β) of Definition 2.2 has a natural intuitive interpretation in terms of basic continuity phenomena (see Hammer 1985 b).

One of our main applications of the concept of coprimeness is related to fraction representations of non-linear systems. A system $\Sigma: S_0(R^m) \to S_0(R^p)$ is called *rational* if there exists a pair of stable maps $P: S \to S_0(R^p)$ and $Q: S \to S_0(R^m)$ with common domain S contained in $S_0(R^q)$ for some integer $q \geqslant 1$, where Q is invertible, such that $\Sigma = PQ^{-1}$. We say that the system Σ has a *right-coprime fraction representation* if the stable maps P and Q in the fraction representation $\Sigma = PQ^{-1}$ can be selected as right-coprime maps. The intuitive significance of a right-coprime fraction representation $\Sigma = PQ^{-1}$ is that, for any input sequence $u \in S_0(R^m)$, the intermediate output sequence $Q^{-1}u$ is unbounded if and *only if* the final output sequence Σu is unbounded. Thus, in a right-coprime fraction representation, the instabilities of Q^{-1} are identical to the instabilities of the system Σ, and the map Q contains exactly all the information on the instabilities of Σ. In general, not every non-linear system $\Sigma: S_0(R^m) \to S_0(R^p)$ has a right-coprime fraction representation. A complete characterization of the class of systems Σ possessing a right-coprime fraction representation is given in Hammer (1985 a, § 4). Since this characterization is essential to our present discussion, we briefly review its basics. The concept of a homogeneous system is crucial. Qualitatively, a homogeneous system has the property of being continuous (as a map) over all sets of bounded input sequences for which it produces bounded output sequences. In exact terms, the definition is as follows (Hammer 1985 a, § 4).

Definition (2.3)

A system $\Sigma: S_0(R^m) \to S_0(R^p)$ is *homogeneous* if for every real $\theta > 0$ the following holds. For every subset $S_* \subset S_0(\theta^m)$ for which there exists a real $\tau > 0$ satisfying $\Sigma[S_*] \subset S_0(\tau^p)$, the restriction of Σ to the closure $\bar{S}_*$ of S_* in $S_0(\theta^m)$ is a continuous map $\Sigma: \bar{S}_* \to S_0(\tau^p)$.

The significance of the concept of a homogeneous system is made clear by the following statement, which we reproduce from Hammer (1985 a, § 4).

Theorem (2.4)

An injective system $\Sigma: S_0(R^m) \to S_0(R^p)$ has a right-coprime fraction representation if and only if it is homogeneous.

Of course, one has to consider the question of how common homogeneous systems are. A partial answer to this question is provided by the following observation, which was discussed in detail in Hammer (1985 a, b).

Proposition (2.5)

Let $\Sigma : S_0(R^m) \to S_0(R^p)$ be a recursive system. If Σ has a recursive representation $y_{k+\eta+1} = f(y_k, \ldots, y_{k+\eta}, u_k, \ldots, u_{k+\mu})$ with a continuous recursion function f, then Σ is a homogeneous system.

From the combination of Theorem 2.4 and Proposition 2.5, we see that many commonly encountered systems possess right-coprime fraction representations. Before turning to an examination of the connections between the existence of right-coprime fraction representations and the theory of stabilization of non-linear systems, we wish to review some further facts.

Let $\Sigma : S_0(R^m) \to S_0(R^p)$ be a non-linear system, and assume it has a right-coprime fraction representation $\Sigma = PQ^{-1}$, where $P : S \to S_0(R^p)$ and $Q : S \to S_0(R^m)$ are stable and right-coprime maps, and where $S \subset S_0(R^q)$. We call the space S the *factorization space* of the fraction representation $\Sigma = PQ^{-1}$. As it turns out, the factorization space of a right-coprime fraction representation is uniquely characterized by the system Σ, in the sense that the factorization spaces of any two right-coprime fraction representations of Σ are 'isomorphic'. Moreover, all factorization spaces of right-coprime fraction representations of Σ are 'isomorphic' to the graph of the system Σ (Hammer 1985 a). To make these statements more accurate, we need some terminology. A map $M : S_1 \to S_2$ between two spaces $S_1 \subset S_0(R^\alpha)$ and $S_2 \subset S_0(R^\beta)$ is *unimodular* if it has an inverse M^{-1}, and if both of M and M^{-1} are stable. (Note that, through an evident injection, we can assume that $S_1, S_2 \subset S_0(R^\gamma)$, where $\gamma := \max\{\alpha, \beta\}$.) Two spaces S_1, S_2 among which one can construct a unimodular map $M : S_1 \to S_2$ are said to be *C-morphic*. As usual, the *graph* $G(\Sigma)$ of a system $\Sigma : S_0(R^m) \to S_0(R^p)$ is the set of all ordered pairs $(u, \Sigma u)$, $u \in S_0(R^m)$. In these terms, the structural properties of right-coprime fraction representations of non-linear systems are summarized by the following (Hammer 1985 a, § 4).

Theorem (2.6)

Let $\Sigma : S_0(R^m) \to S_0(R^p)$ be an injective homogeneous system, and let $\Sigma = PQ^{-1}$ and $\Sigma = P_1 Q_1^{-1}$ be two right-coprime fraction representations of Σ, with factorization spaces $S, S_1 \subset S_0(R^q)$ respectively. Then there exists a unimodular map $M : S_1 \to S$ such that $P_1 = PM$ and $Q_1 = QM$. Furthermore, the factorization space of any right coprime fraction representation of Σ is C-morphic to the graph of Σ.

(2.7) *The construction of a right coprime fraction representation*

Let $\Sigma : S_0(R^m) \to S_0(R^p)$ be an injective homogeneous system, and let $G(\Sigma)$ be the graph of Σ. Clearly, $G(\Sigma)$ is a subset of the product space $S_0(R^m) \times S_0(R^p)$. Let $P_1 : S_0(R^m) \times S_0(R^p) \to S_0(R^m)$ and $P_2 : S_0(R^m) \times S_0(R^p) \to S_0(R^p)$ be the standard projections onto the factors of the product space. Define the maps $P : G(\Sigma) \to S_0(R^p)$ and $Q : G(\Sigma) \to S_0(R^m)$ by setting $Px := P_2 x$ and $Qx := P_1 x$ for all $x \in G(\Sigma)$. Then, it is shown in Hammer (1985 a, § 4) that Q is invertible, and that $\Sigma = PQ^{-1}$ is a right-coprime fraction representation of Σ.

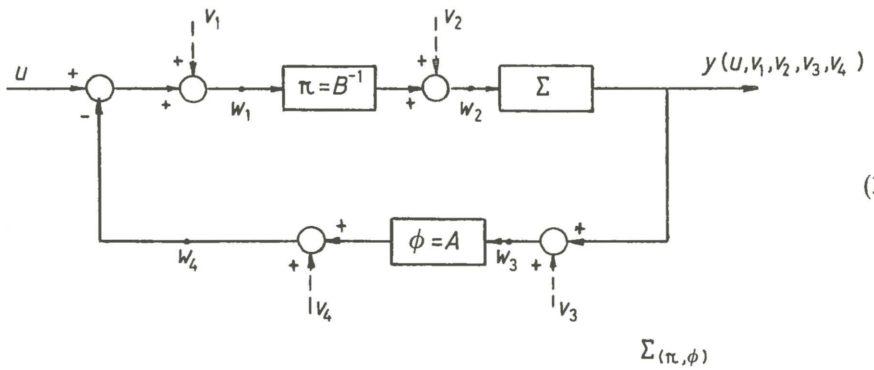
We conclude this section with a review of some standard terminology. A system $\Sigma : S_0(R^m) \to S_0(R^p)$ is *causal* (respectively *strictly causal*) if, for every pair of input sequences $u, v \in S_0(R^m)$ and for every integer $i \geqslant 0$, the equality $u_0^i = v_0^i$ implies that $\Sigma u]_0^i = \Sigma v]_0^i$ (respectively $\Sigma u]_0^{i+1} = \Sigma v]_0^{i+1}$). A system $M: S_0(R^m) \to S_0(R^m)$ is *bicausal* if it has an inverse M^{-1}, and if M and M^{-1} are both causal systems.

Finally, given a real number $\theta > 0$, we say that an element $u \in S_0(R^m)$ is *bounded by* θ if $u \in S_0(\theta^m)$.

3. Bounded-input stabilization

We make a basic assumption regarding the operational circumstances of the systems we consider. We assume that the stabilized system $\Sigma_{(\pi,\phi)}$ (of (1.1)) is operated only by bounded input sequences, namely that there is an arbitrary but fixed real constant $\theta > 0$ such that all input sequences of $\Sigma_{(\pi,\phi)}$ are taken from $S_0(\theta^m)$. We do not assume any correlation between θ and the given system Σ, so that the input bound θ can be fixed arbitrarily. The value of the bound θ has to be specified before the stabilizing compensators π and ϕ are computed. From the practical point of view, it seems that most engineering systems are, in fact, operated under bounded-input circumstances. Indeed, the actual mechanism that generates the input signals feeding the stabilized system $\Sigma_{(\pi,\phi)}$ usually has an inherent structural limitation on the maximal signal amplitude θ it can produce. Thus, from the practical point of view, it seems that, for most applications, the bounded-input assumption is not really restrictive. Since the bounded-input assumption yields some simplifications in the computation of the stabilizing compensators π and ϕ, we adopt it throughout the rest of our present paper.

We start with a discussion of the concept of internal stability. In short, internal stability of the composite systems $\Sigma_{(\pi,\phi)}$ means that the input/output map $\Sigma_{(\pi,\phi)}$ is stable, and that this stability is not disturbed by any small noises that may affect the internal signals of the composite system. In order to provide a formal definition, we redraw our basic control configuration, using the compensators described in (1.3), and incorporating all internal noises. In the diagram (3.1) below, $\Sigma: S_0(R^m) \to S_0(R^p)$ is a given strictly causal system, $B: S_0(R^m) \to S_0(R^m)$ is a stable and invertible system having a causal inverse B^{-1}, and $A: S_0(R^p) \to S_0(R^m)$ is a stable and causal system. (We remark that if Σ is only causal, the composite system $\Sigma_{(\pi,\phi)}$ may not be well defined in some cases.) The signals v_1, v_2, v_3 and v_4 are the possible noise signals, and w_1, w_2, w_3 and w_4 denote the respective internal signals of the composite system. We denote by $y(u, v_1, v_2, v_3, v_4)$ the output sequence generated by the combination of the input sequence u and the internal noise signals v_1, v_2, v_3 and v_4.



(3.1)

Let $\alpha > 0$ be a real number. For conciseness of notation, it is convenient to use for a sequence $x \in S_0(R^m)$ the symbol $|x| < \alpha$ to indicate that $|x_i| < \alpha$, $i = 0, 1, 2, \ldots$ (i.e. that the sequence x is openly bounded by α).

Definition (3.2)

Let $\theta > 0$ be a real number. The system $\Sigma_{(\pi,\phi)}$ of (3.1) is *internally stable* (for input sequences bounded by θ) if the following conditions are satisfied.

(i) The input/output map $\Sigma_{(\pi,\phi)} : S_0(R^m) \to S_0(R^p)$ is stable when restricted to $S_0(\theta^m)$.

(ii) There exists a real $\gamma > 0$ such that, for every input sequence $u \in S_0(\theta^m)$ and for all noise signals $|v_i| \leqslant \gamma$, $i = 1, \ldots, 4$, one has

(*a*) for every real $\varepsilon > 0$ there is a real $\delta > 0$ such that, whenever $\rho(v_i) < \delta$ for all $i = 1, \ldots, 4$, then

$$\rho[y(u, v_1, v_2, v_3, v_4) - y(u, 0, 0, 0, 0)] < \varepsilon$$

(*b*) there is a real $M > 0$ such that the internal signals satisfy $|w_i| \leqslant M$ for all $i = 1, \ldots, 4$.

Qualitatively, condition (ii*a*) above requires continuous dependence of the output signal on the internal noise signals (around the zero noise point), whereas (ii*b*) simply requires the boundedness of all internal signals under any permissible noise conditions. We say that the composite system $\Sigma_{(\pi,\phi)}$ is *input/output stable* if the map $\Sigma_{(\pi,\phi)} : S_0(R^m) \to S_0(R^p)$ is stable.

The following concept, which describes a certain weak form of uniform continuity with respect to the l^∞ norm (see below), plays a fundamental role in our theory.

Definition (3.3)

Let $A : S_0(R^m) \to S_0(R^p)$ be a stable map, and let $\theta > 0$ be a real number. We say that A is *differentially bounded by* θ if there exists a real $\varepsilon > 0$ such that, for every pair of elements $y, y' \in S_0(R^m)$ satisfying $|y - y'| < \varepsilon$ one has $|A(y) - A(y')| < \theta$.

We note that, for a differentially bounded map, a bounded (by ε) fluctuation of the input sequence causes only a bounded (by θ) fluctuation of the output sequence, where θ and ε do not depend on the particular input sequence around which the fluctuation occurs. Thus we may say that a differentially bounded system exhibits 'bounded-input-fluctuation bounded-output-fluctuation' behaviour. In comparison, a BIBO-stable system is guaranteed to exhibit 'bounded-input-fluctuation bounded-output-fluctuation' behaviour only for fluctuations around the zero input sequence. The most common type of differentially bounded maps that we will use is as follows. A map $A : S_0(R^m) \to S_0(R^p)$ is *uniformly* l^∞*-continuous* if, for every real $\varepsilon > 0$, there is a real $\delta > 0$ such that, for every pair of elements $u, v \in S_0(R^m)$ satisfying $|u - v| < \delta$, one has $|A(u) - A(v)| < \varepsilon$. As we can see, a uniformly l^∞-continuous map is differentially bounded by θ for every real $\theta > 0$. Of course, a map that is differentially bounded by θ for some real $\theta > 0$ is not necessarily uniformly l^∞-continuous.

We turn now to a series of results that outline our basic strategy in solving the stabilization problem for non-linear systems. First, we show that, for the particular control configuration depicted in (3.1), input/output stability almost automatically guarantees internal stability, when the stable systems A and B are differentially bounded.

Lemma (3.4)

Let $\Sigma: S_0(R^m) \rightarrow S_0(R^p)$ be a causal system, and let $\theta > 0$ be a real number. Assume that there is a pair of stable systems $A: S_0(R^p) \rightarrow S_0(R^m)$ and $B: S_0(R^m) \rightarrow S_0(R^m)$, where A is causal and B is bicausal, such that the following holds. The composite system $\Sigma_{(B^{-1},A)}$ of (3.1) when restricted to $S_0((5\theta)^m)$, is well defined and input/output stable, and all its internal signals are bounded, say $|w_i| \leqslant N$, $i = 1, \ldots, 4$, for some real $N > 0$. If A and B are differentially bounded by θ, then $\Sigma_{(B^{-1},A)}$ is internally stable when restricted to $S_0(\theta^m)$.

In short, the Lemma states that, for our particular control configuration, input/output stability over $S_0((5\theta)^m)$ implies internal stability over the smaller space $S_0(\theta^m)$, provided A and B are differentially bounded by θ and all internal signals are bounded. The following proof of this fact is based on some simple manipulations of diagram (3.1). Using the continuity and differential-boundedness of A and B, it shows that small internal noises have the same effect on the output as certain small external noises added to the external input u. The latter, however, are known to cause only small fluctuations of the output y, due to input/output stability.

Proof of Lemma 3.4

Let A and B be a pair of maps satisfying the assumptions of the Lemma. To shorten notation, we let $\pi := B^{-1}$, $\phi := A$, so the composite system becomes $\Sigma_{(\pi,\phi)}$. Recalling that A and B are differentially bounded by θ, let $\gamma > 0$ be a real number for which $|A(y+v) - A(y)| < \theta$ and $|B(z+w) - B(z)| < \theta$ whenever $|v| < \gamma$ and $|w| < \gamma$, for all elements $y \in S_0(R^p)$ and $z \in S_0(R^m)$. Define $\varepsilon := \frac{1}{2}\min\{\theta, \gamma\}$, and consider the arbitrary noise signals v_1, v_2, v_4 in $S_0(\varepsilon^m)$ and v_3 in $S_0(\varepsilon^p)$ (ε serves here as our noise-amplitude bound). Fix an external input sequence $u \in S_0((4\theta)^m)$ and, for $i = 1, \ldots, 4$, denote by $y(u, v_i)$ the output y generated by the combination of u and the noise v_i, assuming all other noises to be zero. We consider first the effect of each noise separately.

By inspection of (3.1), it readily follows that

$$y(u, v_1) = \Sigma_{(\pi,\phi)}(u + v_1)$$

$$y(u, v_4) = \Sigma_{(\pi,\phi)}(u - v_4)$$

Since $u \in S_0((4\theta^m))$ and $v_1, v_4 \in S_0(\varepsilon^m) \subset S_0(\theta^m)$, we have $u + v_1, u - v_4 \in S_0((5\theta)^m)$. Thus, by the assumptions of the Lemma regarding the restriction of $\Sigma_{(\pi,\phi)}$ to $S_0((5\theta)^m)$, we obtain directly that v_1 and v_4 do not disturb the boundedness $|w_i| \leqslant N$, $i = 1, \ldots, 4$, and that $y(u, v_1)$ and $y(u, v_4)$ are continuous functions of v_1 and of v_4 respectively over the required domains. Further, to consider the effect of v_3, denote

$$\alpha(v_3) := A(y + v_3) - A(y)$$

so that, by inspection of (3.1), $\alpha(v_3)$ has an effect similar to that of the noise v_4, and we obtain

$$y(u, v_3) = \Sigma_{(\pi,\phi)}[u - \alpha(v_3)] \tag{3.5}$$

Since $v_3 \in S_0(\varepsilon^p)$, we have, by the differential-boundedness of A, that $\alpha(v_3) \in S_0(\theta^m)$. Thus $u - \alpha(v_3) \in S_0((5\theta)^m)$, and it follows by the input/output stability of $\Sigma_{(\pi,\phi)}$ that $y(u, v_3)$ is bounded, i.e. there is a real $a > 0$ such that $y(u, v_3) \in S_0(a^p)$, and consequently $y(u, v_3) + v_3$ belongs to $S_0((a+\varepsilon)^p)$ for all pairs $u \in S_0((4\theta)^m)$ and $v_3 \in S_0(\varepsilon^p)$. Now, let $\delta > 0$ be an arbitrary real number. By the assumed continuity of $\Sigma_{(\pi,\phi)}$ on the compact domain $S_0((5\theta)^m)$, we have uniform continuity there, so there is a real $\delta_1 > 0$ such that $\rho[y(u, v_3) - y(u, 0)] = \rho\{\Sigma_{(\pi,\phi)}[u - \alpha(v_3)] - \Sigma_{(\pi,\phi)}[u]\} < \delta$ for all $u \in S_0((4\theta)^m)$ and all $\alpha(v_3) \in S_0(\theta^m)$ satisfying $\rho[\alpha(v_3)] < \delta_1$. Furthermore, since A is stable on the compact set $S_0((a+\varepsilon)^p)$, it is uniformly continuous there. Consequently there is a real $\delta_2 > 0$ such that $\rho[A(x) - A(z)] < \delta_1$ for all $x, z \in S_0((a+\varepsilon)^p)$ satisfying $\rho(x - z) < \delta_2$. Thus, for all $v_3 \in S_0(\varepsilon^p)$ for which $\rho(v_3) < \delta_2$, we obtain $\rho[\alpha(v_3)] = \rho[A(y + v_3) - A(y)] < \delta_1$ for all $y \in S_0(a^p)$. Summarizing, we have $\rho[y(u, v_3) - y(u, 0)] < \delta$ for all $v_3 \in S_0(\varepsilon^p)$ satisfying $\rho(v_3) < \delta_2$ and for all $u \in S_0((4\theta)^m)$, and it follows that $y(u, v_3)$ depends continuously on v_3 at the origin. Boundedness of all internal signals under the effect of v_3 also is a direct consequence of (3.5), the fact that $u - \alpha(v_3) \in S_0((5\theta)^m)$, and the Lemma assumption on the boundedness of the internal signals.

Finally, to consider the effect of v_2, we denote

$$\beta(v_2) := B[B^{-1}(w_1) + v_2] - B[B^{-1}(w_1)]$$

so that $B^{-1}[\beta(v_2) + w_1] = B^{-1}(w_1) + v_2$, and $\beta(v_2)$ has an effect similar to that of the noise v_1, or

$$y(u, v_2) = \Sigma_{(\pi,\phi)}[u + \beta(v_2)] \tag{3.6}$$

By the differential-boundedness of B, we have $\beta(v_2) \in S_0(\theta^m)$ for all $v_2 \in S_0(\varepsilon^m)$ and all $w_1 \in S_0(R^m)$. Consequently, for any pair $u \in S_0((4\theta)^m)$ and $v_2 \in S_0(\varepsilon^m)$, we obtain $u + \beta(v_2) \in S_0((5\theta)^m)$, and it follows by (3.6) and the Lemma assumptions that any noise $v_2 \in S_0(\varepsilon^m)$ does not disturb the boundedness of the internal signals $|w_i| \leqslant N, i = 1, \ldots, 4$. In particular, since $|w_2| \leqslant N$ and $|v_2| \leqslant \varepsilon$, we have $|B^{-1}(w_1)| \leqslant N + \varepsilon$. The continuous dependence of $y(u, v_2)$ on v_2 at its origin follows then by a uniform-continuity argument applied on a compact domain, similarly to the argument we used in the last paragraph for the effect of v_3.

To conclude our proof, we note that the simultaneous application of all four noises v_1, v_2, v_3 and v_4 will, by the above considerations, result in the equivalent input noise

$$v(v_1, v_2, v_3, v_4) := v_1 - v_4 - \alpha(v_3) + \beta(v_2)$$

so that

$$y(u, v_1, v_2, v_3, v_4) = \Sigma_{(\pi,\phi)}(u + v) \tag{3.7}$$

As we have seen before, whenever $|v_i| \leqslant \varepsilon$, $i = 1, \ldots, 4$, we have $|v_1| < \theta$, $|v_4| < \theta$, $|\alpha(v_3)| < \theta$, and $|\beta(v_2)| < \theta$, independently of the other signals in the composite system, so that then $v \in S_0((4\theta)^m)$, and consequently $u + v \in S_0((5\theta)^m)$ for all $u \in S_0(\theta^m)$. By the Lemma assumptions regarding the restriction of $\Sigma_{(\pi,\phi)}$ to $S_0((5\theta)^m)$, it follows that, whenever $|v_i| \leqslant \varepsilon, i = 1, \ldots, 4$, and $|u| \leqslant \theta$, all internal signals $w_i, i = 1, \ldots, 4$, remain bounded by N, and the output y depends continuously on v. Furthermore, applying the evident relation $\rho[y(u, v_1, v_2, v_3, v_4) - y(u, 0, 0, 0, 0)] \leqslant \rho[y(u, v_1, v_2, v_3, v_4) - y(u, 0, v_2, v_3, v_4)] + \rho[y(u, 0, v_2, v_3, v_4) - y(u, 0, 0, v_3, v_4)] + \rho[y(u, 0, 0, v_3, v_4) - y(u, 0, 0, 0, v_4)] + \rho[y(u, 0, 0, 0, v_4) - y(u, 0, 0, 0, 0)]$, and replacing in each term the fixed internal-noise signals by their external-input equivalents, it follows by the previous parts of this proof that y depends continuously on (v_1, v_2, v_3, v_4) at the origin, as long as $|u| \leqslant \theta$ and $|v_i| \leqslant \varepsilon$, $i = 1, \ldots, 4$. Under these conditions, y evidently also depends continuously on u, and our proof is concluded. □

The basic strategy we employ toward the solution of the stabilization problem for a non-linear system Σ can be briefly outlined as follows. First, we assume that Σ has a right-coprime fraction representation $\Sigma = PQ^{-1}$, and we denote its factorization space by S. As we have discussed in § 2, the existence of a right-coprime fraction representation means that Σ is a homogeneous system, for example a recursive system possessing a continuous recursion function. Next, we need the bounded part of the factorization space S to be 'rich' enough in the following sense. We assume that S contains a subset S' that is C-morphic to $S_0((5\theta)^m)$, where $\theta > 0$ is a real number, and we let $M: S' \simeq S_0(5\theta)^m)$ be a unimodular transformation. (Note that in such cases, the map $M_c := cM$, where $c > 0$ is any positive real number, induces a unimodular transformation $M_c: S' \simeq S_0((5c\theta)^m)$, so that the actual value of θ can be arbitrarily chosen, and it has no structural significance.) We shall discuss in the next section the implications on Σ of the assumption that M exists, and we shall see there that this assumption is not overly restrictive, and that it is satisfied by many commonly encountered stabilizable systems. We shall also discuss in the next section the computation of M. Using the unimodular transformation M, we shall construct in § 7 a pair of stable and differentially bounded systems $A: S_0(R^p) \to S_0(R^m)$ and $B: S_0(R^m) \to S_0(R^m)$, where A is causal and B is bicausal, such that the equation

$$APv + BQv = Mv \tag{3.8}$$

holds for all $v \in S'$. In the next Theorem we show that, on setting $\pi := B^{-1}$ and $\phi := A$, the system $\Sigma_{(\pi,\phi)}$ becomes internally stable, when restricted to $S_0(\theta^m)$. Thus the stabilization problem reduces to the computation of the maps A and B, discussed in § 7 below.

Theorem (3.9)

Let $\Sigma: S_0(R^m) \to S_0(R^p)$ be a causal homogeneous system, and let $\theta > 0$ be a real number. Let $\Sigma = PQ^{-1}$ be a right-coprime fraction representation, and let $S \subset S_0(R^q)$ be its factorization space. Assume S contains a subset S' which is C-morphic to $S_0((5\theta)^m)$, and let $M: S' \simeq S_0((5\theta)^m)$ be a unimodular transformation. Assume there is a pair of stable maps $A: S_0(R^p) \to S_0(R^m)$ and $B: S_0(R^m) \to S_0(R^m)$ satisfying the equation $APv + BQv = Mv$ for all $v \in S'$, where A and B are differentially bounded by θ, A is causal, and B is bicausal. Then the composite system $\Sigma_{(B^{-1},A)}$ is internally stable for all input sequences $u \in S_0(\theta^m)$.

Proof

We refer to diagram (3.1). In view of Lemma 3.4 and the assumptions of the present Theorem, it is enough to show that, for inputs $u \in S_0((5\theta)^m)$ and for zero noise, the composite system $\Sigma_{(B^{-1},A)}$ is input/output stable, and its internal signals satisfy $|w_i| \leqslant N$, $i = 1, \ldots, 4$, for some real $N > 0$. To this end, we assume zero noise, i.e. $v_i = 0$, $i = 1, \ldots, 4$, and we recall that, by the Theorem assumptions, the domain of M^{-1} is $S_0((5\theta)^m)$. Then, using calculations similar to the one employed in the derivation of (1.6), the following formulae follow readily for all input sequences $u \in S_0((5\theta)^m)$:

$$w_1 = u - Ay = [I + A\Sigma B^{-1}]^{-1}u = BQM^{-1}u$$

$$w_2 = B^{-1}w_1 = QM^{-1}u$$

$$y = \Sigma_{(B^{-1},A)}u = \Sigma w_2 = PM^{-1}u$$

$$w_3 = y = PM^{-1}u$$

$$w_4 = Ay = APM^{-1}u$$

By the stability of P, Q, A and B, and by the stability of M^{-1} over $S_0((5\theta)^m)$, the following facts are valid. (i) $\Sigma_{(B^{-1},A)} = PM^{-1}$ is input/output stable when restricted to $S_0((5\theta)^m)$. (ii) There are real numbers $N_i > 0$, $i = 1, \ldots, 4$, such that

$$BQM^{-1}[S_0((5\theta)^m)] \subset S_0(N_1^m)$$

$$QM^{-1}[S_0((5\theta)^m)] \subset S_0(N_2^m)$$

$$PM^{-1}[S_0((5\theta)^m)] \subset S_0(N_3^p)$$

$$APM^{-1}[S_0((5\theta)^m)] \subset S_0(N_4^m)$$

Consequently, denoting $N := \max\{N_1, N_2, N_3, N_4\}$, we obtain that, for all input sequences $u \in S_0((5\theta)^m)$, the internal signals satisfy $|w_i| \leqslant N$, $i = 1, \ldots, 4$. In view of (i), this concludes our proof. □

4. Full stability subspaces

Let $\Sigma : S_0(R^m) \to S_0(R^p)$ be a non-linear system. We say that Σ is *stabilizable* if there exists a real $\theta > 0$ and a pair of causal compensators $\pi : S_0(R^m) \to S_0(R^m)$ and $\phi : S_0(R^p) \to S_0(R^m)$ such that $\Sigma_{(\pi,\phi)}$ is internally stable for input sequences belonging to $S_0(\theta^m)$. In the present section we discuss certain structural requirements the system Σ has to satisfy in order to be stabilizable. In other words, we discuss some necessary conditions for stabilizability. To reveal at the outset the main aspect we are interested in currently, we note that many rather innocently looking simple non-linear systems cannot be stabilized. Indeed, consider the two strictly causal recursive systems $\Sigma_1, \Sigma_2 : S_0(R) \to S_0(R)$ with recursive representations given by

$$\Sigma_1 : y_{k+1} = 2y_k + u_k^2 \qquad (4.1\,a)$$

$$\Sigma_2 : y_{k+1} = 2y_k + \exp(u_k) \qquad (4.1\,b)$$

and with initial condition $y_0 = 0$. (Note that Σ_2 is an injective system.) It is easy to see that, for Σ_1, the identically zero-input sequence $u = 0$ is the only input sequence for which the output sequence y is bounded, whereas for Σ_2 there is no input sequence for which the output sequence is bounded. Consequently, the only way to obtain a bounded output sequence when Σ_1 is connected in the configuration (3.1) is to require $\pi = 0$; but then, any non-zero noise signal v_2 will destroy the output boundedness, owing to the fact that, when $\pi = 0$, the output of π evidently cannot compensate for the effect of the noise. For Σ_2 there is no pair of compensators π and ϕ for which the output of $\Sigma_{2(\pi,\phi)}$ is bounded. Thus Σ_1 and Σ_2 are not stabilizable in the above sense. The intuitive insight that we gain from these two examples is profound. We learn that, for stabilizability of a system Σ, the set of input sequences for which Σ produces bounded output sequences must be a 'rich' enough set, in a suitable sense. The clarification of this intuitive conception, and the discussion of some concrete examples of stabilizable systems, form the core of our present section. Owing to the extensive use of right-coprime fraction representations of systems in the other sections of the paper, we shall limit our attention in this section also to systems having right-coprime fraction representations, i.e. to homogeneous systems.

Let $\Sigma : S_0(R^m) \to S_0(R^p)$ be a homogeneous system. A subset $S_s \subset S_0(R^m)$ is a *stability subspace* of Σ if there exists a pair of real numbers $\theta, N > 0$ such that $S_s \subset S_0(\theta^m)$ and $\Sigma[S_s] \subset S_0(N^p)$. Simply stated, a stability subspace is a set of bounded-input sequences generating bounded-output sequences. The interest in stability subspaces arises from the fact that, if $\Sigma : S_0(R^m) \to S_0(R^p)$ is a homogeneous system and if $S_s \subset S_0(\theta^m)$ is a stability subspace of Σ then, by homogeneity, the restriction of Σ to the closure $\bar{S}_s$ of S_s in $S_0(\theta^m)$ is a continuous map. In other words, the restriction of Σ to a stability subspace is a stable system, and the closure of a stability subspace of Σ is again a stability subspace of Σ. We also note that, as a direct consequence of the definition, every subset of a stability subspace is a stability subspace as well. We next discuss the structure of the stability subspaces of stabilizable systems.

Let $\Sigma : S_0(R^m) \to S_0(R^p)$ be an injective, homogeneous and causal system, and assume there is a pair of causal compensators π and ϕ, where π is injective, such that $\Sigma_{(\pi,\phi)}$ is well defined and internally stable when restricted to $S_0(\theta^m)$, for some real $\theta > 0$. We require the injectivity of π in order to ensure that the stabilized system $\Sigma_{(\pi,\phi)}$ inherits the injectivity of the given system Σ (see (1.2)). Referring again to (1.2), we have $\Sigma_{(\pi,\phi)} = \Sigma \psi_{(\pi,\phi)}$. We next show that the subspace $S_* := \psi_{(\pi,\phi)}[S_0(\theta^m)]$ is a stability subspace of Σ, and that it is C-morphic to $S_0(\theta^m)$. Indeed, referring to (3.1), we have $w_2 = \psi_{(\pi,\phi)} u$, so that, by internal stability, there is a real $N_1 > 0$ such that $\psi_{(\pi,\phi)}[S_0(\theta^m)] \subset S_0(N_1^m)$, or $S_* \subset S_0(N_1^m)$. Moreover, by the stability of $\Sigma_{(\pi,\phi)}$, there is a real $N > 0$ such that $\Sigma_{(\pi,\phi)}[S_0(\theta^m)] \subset S_0(N^p)$, so that $\Sigma[S_*] = \Sigma \psi_{(\pi,\phi)}[S_0(\theta^m)] = \Sigma_{(\pi,\phi)}[S_0(\theta^m)] \subset S_0(N^p)$, and it follows that S_* is a stability subspace of Σ. Further, by homogeneity, Σ is continuous on the compact set $\bar{S}_* \subset S_0(N_1^m)$, and consequently, by injectivity, Σ induces a homeomorphism $\Sigma : \bar{S}_* \to \Sigma[\bar{S}_*]$. But then, since clearly $\psi_{(\pi,\phi)} u = \Sigma^{-1} \Sigma_{(\pi,\phi)} u$ for all $u \in S_0(\theta^m)$, it follows by the continuity of $\Sigma_{(\pi,\phi)}$ on $S_0(\theta^m)$ and the continuity of Σ^{-1} on $\Sigma[S_*]$ that $\psi_{(\pi,\phi)}$ is continuous when restricted to $S_0(\theta^m)$. By the injectivity of $\psi_{(\pi,\phi)}$ and the compactness of $S_0(\theta^m)$, the latter implies that $\psi_{(\pi,\phi)} : S_0(\theta^m) \to S_*$ is actually a homeomorphism. Thus we conclude that the stability subspace S_* is C-morphic to $S_0(\theta^m)$, and Σ has a stability subspace C-morphic to $S_0(\theta^m)$. This completes the proof of the following statement.

Proposition (4.2)

Let $\Sigma : S_0(R^m) \to S_0(R^p)$ be an injective, homogeneous and causal system, and let $\theta > 0$ be a real number. Assume there is a pair of causal compensators $\pi : S_0(R^m) \to S_0(R^m)$ and $\phi : S_0(R^p) \to S_0(R^m)$, where π is injective, such that $\Sigma_{(\pi,\phi)}$ is well defined and internally stable when restricted to $S_0(\theta^m)$. Then Σ has a stability subspace which is C-morphic to $S_0(\theta^m)$.

Returning now to our previous vague remark following (4.1) regarding the 'richness' of the set of bounded-input sequences producing bounded-output sequences (i.e. the 'richness' of the stability subspaces), we see from Proposition 4.2 that this set must indeed be 'rich' enough to contain a subset that is C-morphic to $S_0(\theta^m)$. Proposition 4.2 gives a precise meaning to our qualitative opening remarks.

From Proposition 4.2 we see that a necessary condition for the stabilization of an injective homogeneous system Σ is that Σ possess a stability subspace S_s for which there is a unimodular transformation $M:S_s \simeq S_0(\theta^m)$, for some real $\theta > 0$. We note that, owing to compactness of $S_0(\theta^m)$, the maps M and M^{-1} are both uniformly continuous with respect to ρ. A significant simplification of our ensuing discussion results if we assume that, in addition, M also is uniformly l^∞-continuous. This is a slightly stronger requirement than the actual necessary condition, but, nevertheless, as we show below, it is satisfied by a wide variety of common stabilizable non-linear systems. The exact requirement that we make is that Σ possess a 'full stability subspace', defined as follows.

Definition (4.3)

Let $\Sigma: S_0(R^m) \to S_0(R^p)$ be an injective homogeneous system, let $S_s \subset S_0(\theta^m)$ be a stability subspace of Σ, and let $\bar{S}_s$ be the closure of S_s in $S_0(\theta^m)$, where $\theta > 0$ is a real number. We say that S_s is a *full stability subspace* if there is a bicausal, stable and uniformly l^∞-continuous map $M: S_0(R^m) \to S_0(R^m)$ such that $M[\bar{S}_s] = S_0(\beta^m)$ for some real $\beta > 0$.

In order to clarify the connection between Definition 4.3 and our previous discussion, we state the following.

Lemma (4.4)

Let $M: S_0(R^m) \to S_0(R^p)$ be a causal and uniformly l^∞-continuous map. Let $\theta > 0$ be a real number, and let $S \subset S_0(\theta^m)$ be a subset for which $M[S] \subset S_0(\beta^p)$ for some real $\beta > 0$. Then the restriction of M to S is a continuous map (with respect to ρ).

Proof

We use the notation of the Lemma. Let $\varepsilon > 0$ be a real number. Since M is uniformly l^∞-continuous, there exists a real $\delta > 0$ such that $|M(u) - M(v)| < \varepsilon$ for all elements $u, v \in S_0(R^m)$ for which $|u - v| < \delta$. Let $a \geqslant 0$ be any integer. Then, by the causality of M and its uniform l^∞-continuity over $S_0(R^m)$, we have that $|M(u)]_0^a - M(v)]_0^a| < \varepsilon$ for any pair of elements $u, v \in S$ for which $|u_0^a - v_0^a| < \delta$. In particular, choose a so that $2^{-a}\beta < \varepsilon$. Then, recalling the definition of ρ, we have $\rho[M(u) - M(v)] \leqslant \max\{|M(u)]_0^a - M(v)]_0^a|, 2^{-a-1}\beta\} < \varepsilon$ for all elements $u, v \in S$ satisfying $\rho(u - v) < 2^{-a}\delta$. But this implies that the restriction of M to S is continuous (with respect to ρ). □

Returning now to Definition 4.3, we see that the map M is continuous when restricted to $\bar{S}_s$. By the injectivity of M and the compactness of $\bar{S}_s$, the map M actually forms a homeomorphism $M:\bar{S}_s \simeq S_0(\beta^m)$. Thus the stability subspace $\bar{S}_s$ is C-morphic to $S_0(\beta^m)$, and we see that the existence of a full stability subspace is a somewhat stronger version of the necessary condition for stabilization mentioned in Proposition 4.2. The relevance of full stability subspaces to our theory originates from the fact that they are instrumental in the construction of the stable systems A and B mentioned in the discussion of (3.8). As is implied by that discussion, the systems A and B form the key to our stabilization procedure. The exact way in which stability subspaces aid in the construction of these systems will become clear in § 7, where we provide the detailed description of the stabilization procedure. Meanwhile, we devoted the remaining part of this section to the discussion of some concrete examples of families of systems possessing full stability subspaces. As we shall see, the existence of full stability subspaces is related to a certain controllability notion, and many common stabilizable systems do indeed possess full stability subspaces.

As a broad example for the investigation of stability subspaces, we consider the family of systems having continuous realizations in the following sense. (We remark that similar ideas can be applied to other classes of systems as well.) Let $\Sigma : S_0(R^m) \to S_0(R^n)$ be a system. We say that Σ has a *continuous realization* if there exists a pair of continuous functions $f: R^p \times R^m \to R^p$ and $h: R^p \times R^m \to R^n$ such that, for each input sequence $u \in S_0(R^m)$ the output sequence $y := \Sigma u$ can be computed recursively from the equations

$$\Sigma_r\colon x_{k+1} = f(x_k, u_k), \quad x_0 = 0 \tag{4.4 a}$$

$$y_k = h(x_k, u_k) \tag{4.4 b}$$

for all integers $k \geqslant 0$. We denote by Σ_r the recursive system of (4.4 *a*) (together with the appropriate initial condition $x_0 = 0$). Owing to the continuity of h, every stability subspace of Σ_r is a stability subspace of Σ as well. We shall consider then the existence of stability subspaces for recursive systems of the form (4.4 *a*). For this purpose, we define the following terminology.

Let $\Sigma : S_0(R^m) \to S_0(R^p)$ be a recursive system with representation $x_{k+1} = f(x_k, u_k)$, where $f: R^p \times R^m \to R^p$ is a continuous function. For each integer $j \geqslant 1$ we denote by $f_j(x, u_0, u_1, \ldots, u_j)$ the *jth iterate* of the function f, defined recursively by

$$f_1(x, u_0) := f(x, u_0),$$

$$f_{j+1}(x, u_0, \ldots, u_j) := f_j(f(x, u_0), u_1, \ldots, u_j)$$

As we can see, the function f_j provides the output value x_{k+j}, when the output value $x := x_k$ and the input values $u_k, u_{k+1}, \ldots, u_{k+j-1}$ are specified. By the continuity of f, all the iterates f_j, $i = 1, 2, 3, \ldots$, are continuous functions.

The *reachable set* $\mathscr{R}(\Sigma)$ of the system $\Sigma : S_0(R^m) \to S_0(R^p)$ is a subset of R^p, defined by

$$\mathscr{R}(\Sigma) := \bigcup_{u \in S_0(R^m)} \bigcup_{k=0}^{\infty} \{y_k \mid y = \Sigma u\}$$

and it simply consists of all possible output values of Σ.

Further, given a pair of real numbers $\alpha, \beta > 0$, an integer $j \geqslant 0$, and a point $x \in \mathscr{R}(\Sigma) \cap [-\alpha, \alpha]^p$, we denote by $C_j(\alpha, \beta, x)$ the set of all input values $(u_0, u_1, \ldots, u_j)$, where $u_i \in [-\beta, \beta]^m$, $i = 0, 1, \ldots, j$, which produce at time $j+1$ an output value $x_{j+1} := f_{j+1}(x, u_0, \ldots, u_j)$ belonging to $[-\alpha, \alpha]^p$. More directly, let f^*_{j+1} be the inverse set-function of f_{j+1}, and let $p_u : R^p \times (R^m)^{j+1} \to (R^m)^{j+1} : p_u(x, u_0, \ldots, u_j) \mapsto (u_0, \ldots, u_j)$ be the standard projection onto the factor of the product space. Then we have $C_j(\alpha, \beta, x) = p_u\{(f^*_{j+1}[-\alpha, \alpha]^p) \cap (\{x\} \times [-\beta, \beta]^{m(j+1)})\}$. We refer to $C_j(\alpha, \beta, x)$ as a *j-step controllability set*. Clearly, $C_j(\alpha, \beta, x)$ is a subset of $[-\beta, \beta]^{m(j+1)}$, and it consists of all input values (bounded by β) that take a state x bounded by α into another state also bounded by α, $j+1$ steps later. There is no specific requirement for the output values at the intermediate times; however, by the continuity of the functions $f_1, f_2, \ldots, f_j$, the intermediate output values will all be uniformly bounded.

To discuss the significance of our controllability sets, fix a pair of real numbers $\alpha, \beta > 0$ and an integer $d \geqslant 0$, and assume that the d-step controllability set $C_d(\alpha, \beta, x)$ is non-empty for every $x \in [-\alpha, \alpha]^p$. Starting at $t = 0$ from the initial condition $x_0 = 0$, each set of input values $(u_0, \ldots, u_d) \in C_d(\alpha, \beta, 0)$ leads to a state $x_{d+1} = f_{d+1}(0, u_0, \ldots, u_d)$ bounded by α. Further, each set of input values $(u_{d+1}, \ldots, u_{2d+1}) \in C_d(\alpha, \beta, x_{d+1})$ yields a state $x_{2d+2} = f_{2d+2}(0, u_0, \ldots, u_{2d+1})$ again bounded by α. In general, every input sequence $u \in S_0(R^m)$ for which $(u_0, \ldots, u_d) \in C_d(\alpha, \beta, 0)$ and $(u_{j(d+1)}, \ldots, u_{(j+1)(d+1)-1}) \in C_d(\alpha, \beta, f_{j(d+1)}(0, u_0, \ldots, u_{j(d+1)-1}))$ for all integers $j \geqslant 1$, generates an output sequence $x := \Sigma u$ for which all elements $x_{j(d+1)}, j = 0, 1, 2, \ldots$, are bounded by α. Now, by definition, all such input sequences u are bounded by β. Moreover, using the continuity of the functions $f_1, f_2, \ldots, f_d$ and the evident relation $x_{j(d+1)+k} = f_k(x_{j(d+1)}, u_{j(d+1)}, \ldots, u_{j(d+1)+k-1})$, $k = 1, 2, \ldots, d$, it follows readily from our previous observations that there is a real number $\gamma \geqslant \alpha$ such that the whole sequence x is bounded by γ, for every input sequence u satisfying the above requirements. Thus the set of input sequences u, constructed as before from the d-step controllability set $C_d(\alpha, \beta, x)$, forms a stability subspace of Σ. Furthermore, this stability subspace can be computed directly from the recursive representation of Σ (see examples below).

Consider a family of functions $\{g_j\}, j = 0, 1, 2, \ldots$, where, for each integer $j \geqslant 0$, $g_j : (R^m)^{j+1} \to R^p : (u_0, \ldots, u_j) \mapsto g_j(u_0, \ldots, u_j)$. We say that the family of functions $\{g_j\}$ is *uniformly continuous over* $S_0(R^m)$ if, for every real $\varepsilon > 0$, there is a real $\delta > 0$ such that, for all integers $j \geqslant 0$ and for any pair of sequences $u, v \in S_0(R^m)$, one has $|g_j(v_0, v_1, \ldots, v_j) - g_j(u_0, u_1, \ldots, u_j)| < \varepsilon$ whenever $|(v_0, v_1, \ldots, v_j) - (u_0, u_1, \ldots, u_j)| < \delta$.

We define now a class of systems that, as we show below, possess full stability subspaces.

Definition (4.5)

Let $\Sigma : S_0(R^m) \to S_0(R^p)$ be a recursive system with a recursive representation $x_{k+1} = f(x_k, u_k)$, $x_0 = 0$, where $f : R^p \times R^m \to R^p$ is a continuous function. The system Σ is *uniformly controllable* if there exists a real number $\varepsilon > 0$, an integer $d \geqslant 0$, and a pair of real numbers $\alpha, \beta > 0$ such that the following holds. For every element $x \in \mathscr{R}(\Sigma) \cap [-\alpha, \alpha]^p$, the d-step controllability set $C_d(\alpha, \beta, x)$ contains a closed ball $\bar{B}_\varepsilon(c(x))$ in $(R^m)^{d+1}$ of fixed radius ε, centred at the point $c(x) \in (R^m)^{d+1}$, where the function $c(x)$ is a uniformly continuous function of the input values leading to x from within the balls, as follows. The family of functions $g_j(u_0, \ldots, u_{j(d+1)-1}) := c(f_{j(d+1)}(0, u_0, \ldots, u_{j(d+1)-1})), j = 1, 2, 3, \ldots$, is uniformly continuous over the set of all input sequences $u \in S_0(R^m)$ satisfying $(u_0, \ldots, u_d) \in \bar{B}_\varepsilon(c(0))$ and $(u_{j(d+1)}, \ldots, u_{(j+1)(d+1)-1}) \in \bar{B}_\varepsilon[c(f_{j(d+1)}(0, u_0, \ldots, u_{j(d+1)-1}))]$ for all integers $j \geqslant 1$.

Qualitatively speaking, uniform controllability means that, for each state x bounded by α, there is a ball of input values bounded by β, all of which produce, after $d + 1$ steps starting from x, an output value bounded again by α. The centre $c(x)$ of this ball of input values may vary from one state x to another, but it must depend in a uniformly continuous way on the input values leading to that state from the initial condition $x_0 = 0$. The radius ε of this ball must remain fixed. Before discussing Definition 4.5 in more detail, we provide several examples of uniformly controllable systems.

Examples (4.6)

Consider the recursive system $\Sigma_1 : S_0(R) \to S_0(R)$ given by

$$\Sigma_1 : x_{k+1} = 2x_k + u_k^3, \quad x_0 = 0, \quad k = 0, 1, 2, \ldots$$

Let $\alpha > 0$ be a real number. On choosing $\beta \geq (3\alpha)^{1/3}$, we obtain by a simple computation that

$$C_0(\alpha, \beta, x) = [(-\alpha - 2x)^{1/3}, (\alpha - 2x)^{1/3}]$$

A further simple computation shows that, when minimizing over all values $x \in [-\alpha, \alpha]$, the minimal radius of the interval $C_0(\alpha, \beta, x)$ is $r_m := \frac{1}{2}(3^{1/3} - 1)\alpha^{1/3}$. Consequently, for every value of $x \in [-\alpha, \alpha]$, the set $C_0(\alpha, \beta, x)$ contains a ball (in R) of radius $r_m > 0$. The centre of this ball is at $c(x) := \frac{1}{2}[(-\alpha - 2x)^{1/3} + (\alpha - 2x)^{1/3}]$. Using the fact that Σ_1 is linear in u_k^3, and the fact that all output values are bounded by α for the input sequences we consider, it can be readily shown that the family of functions $\{g_j\}$ mentioned in Definition 4.5 is uniformly continuous over the required set of input sequences. Thus Σ_1 is a uniformly controllable system.

As another simple example, consider the recursive system $\Sigma_2 : S_0(R) \to S_0(R)$ given by

$$\Sigma_2 : x_{k+1} = x_k^2 u_k, \quad x_0 \neq 0, \quad k = 0, 1, 2, \ldots$$

It is easy to see that here, for any pair of real numbers $\alpha, \beta > 0$,

$$C_0(\alpha, \beta, x) = \begin{cases} [-\beta, \beta] & \text{for } x = 0 \\ \left[-\frac{\alpha}{x^2}, \frac{\alpha}{x^2} \right] \cap [-\beta, \beta] & \text{for } x \neq 0 \end{cases}$$

If we choose $\alpha \leq 1$, $\beta \geq 1$, the set $C_0(\alpha, \beta, x)$ contains the interval $[-1, 1]$ for any value of $x \in [-\alpha, \alpha]$. Thus we can choose $\varepsilon = 1$ and $c(x) = 0$ for all $x \in [-\alpha, \alpha]$, and Σ_2 is evidently uniformly controllable. In this case, we see that the centre of the ball $\bar{B}_\varepsilon(c(x))$ is fixed as well, in contrast with the situation in our previous example. In both of these examples, we were able to satisfy the uniform-controllability requirement already at the step $d = 0$. This may not be possible in general, especially in the case of multivariable systems, as can be seen from the following example:

$$\Sigma_3 : \begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} 2x_k + u_k^3 \\ x_k + 2y_k \end{pmatrix}, \quad \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which is uniformly controllable (for $d = 1$).

As a final example, we consider the recursive system $\Sigma_4 : S_0(R) \to S_0(R)$ given by

$$\Sigma_4 : x_{k+1} = x_k + x_k u_k + u_k, \quad x_0 = 0, \quad k = 0, 1, 2, \ldots$$

Choosing $\alpha = 1$ and $\beta = 1$, we have

$$C_0(1, 1, x) = \left[\frac{-1 - x}{1 + x}, \frac{1 - x}{1 + x} \right] \cap [-1, 1] \text{ for } x \neq -1, \text{ and } C_0(1, 1, -1) = [-1, 1]$$

A simple minimization with respect to x shows that $[-1, 0] \subset C_0(1, 1, x)$ for all $x \in [-1, 1]$. Thus we can choose $\varepsilon = \frac{1}{2}$ and $c(x) = -\frac{1}{2}$, and the conditions of uniform controllability for Σ_4 are satisfied. In this case, again, the centre of the input values ball could be left fixed. As we have seen before, this is not always possible.

As the above examples show, the condition of uniform controllability is, in many cases, rather easy to check explicitly. When reflecting on the implications of Examples (4.6) and (4.1), it seems intuitively that the condition of uniform controllability is satisfied by most systems that one would expect to be stabilizable. In order to obtain a more concrete feeling on how common uniformly controllable systems are, we provide the following statement, which shows that, generically, a recursive system having a twice continuously differentiable recursion function is uniformly controllable.

Proposition (4.7)

Let $\Sigma: S_0(R^m) \to S_0(R^p)$ be a recursive system with representation $x_{k+1} = f(x_k, u_k)$, $x_0 = 0$, where $f: R^p \times R^m \to R^p$ and $f(0, 0) = 0$. Assume that f is a twice continuously differentiable function having continuous second-order partial derivatives (in some neighbourhood of 0). Let $J_f(0)$ be the jacobian matrix of f at the point 0, and write $J_f(0) = (A, B)$, where A is a $p \times p$ matrix and B is a $p \times m$ matrix. Assume that the pair (A, B) is controllable, and that the eigenvalues $\lambda_1, \ldots, \lambda_p$ of A are distinct and satisfy $|\lambda_i| \neq 1$ for all $i = 1, \ldots, p$. Then the system Σ is uniformly controllable.

Proof

We consider first the controllable linear system

$$\Sigma_0: x_{k+1} = Ax_k + Bu_k =: L(x_k, u_k)$$

In view of the linearity and the controllability of Σ_0, thetre is an integer $d \geqslant 0$ for which the following holds. For every real $\alpha > 0$ there is a pair of real numbers $\beta, \varepsilon_\alpha > 0$ such that, for every $x \in [-\alpha, \alpha]^p$, there is a closed ball $\bar{B}(c(x)) \subset (R^m)^{d+1}$ of radius ε_α contained in $([-\beta, \beta]^m)^{d+1}$ and centred at the point $c(x) \in ([-\beta, \beta]^m)^{d+1}$ such that $L_{d+1}(x, u_0, \ldots, u_d) \in [-0{\cdot}9\alpha, 0{\cdot}9\alpha]^p$ for all $(u_0, \ldots, u_d) \in \bar{B}(c(x))$, where L_{d+1} is the $(d+1)$-iterate of L. By linearity, it can be readily seen that one can choose $\beta = r\alpha$, where r is a fixed number, and we can evidently take $r \geqslant 1$ (since the increase of β has no effect). Furthermore, again by linearity, it can be readily verified that the function $c(x)$ is a linear function of x.

Returning now to the original system Σ, let $s := (x, u_0, \ldots, u_d)$ and $t := (y, v_0, \ldots, v_d)$ be two points, and consider the linear approximation of the $(d+1)$-iterate around the point s:

$$f_{d+1}(t) - f_{d+1}(s) = L_s(t - s) + E_s(t)$$

where L_s is a linear function, and $\lim |t - s|^{-1} E_s(t) = 0$ as $|t - s| \to 0$. We write $L_s(t - s) = A_s(y - x) + B_s(v_0 - u_0, \ldots, v_d - u_d)$, and we notice that $A_0 = A^{d+1}$. Let $\lambda'_1, \ldots, \lambda'_p$ be the eigenvalues of A_0, let $a_i := ||\lambda'_i| - 1|, i = 1, \ldots, p$, and denote $a := \frac{1}{2} \min \{1, a_1, \ldots, a_p\}$. Then, by our assumptions, $a > 0$. We define the norm $|H|$ of a matrix H by $|H| := \max \{|Hx|, |x| = 1\}$, so that $|Hx| \leqslant |H|\,|x|$ for all x. Considering now the linear approximation at the origin $x_{(j+1)(d+1)} = A_0 x_{j(d+1)} + B_0(u_{j(d+1)}, \ldots, u_{(j+1)(d+1)-1}) + E_0(x_{j(d+1)}, u_{j(d+1)}, \ldots, u_{(j+1)(d+1)-1})$, we use a linear transformation to change the coordinates of x so that A_0 becomes a diagonal matrix. All our following discussion is in these coordinates, although, for conciseness, we continue to use the same notation for x, f, A_s, B_s, E_s as before. The input sequence u is not affected by this transformation.

By the continuity of the partial derivatives of f, there is a compact neighbourhood N of the origin such that $|A_s - A_0| \leqslant \frac{1}{8}a$ for all $s \in N$. Let $\gamma := \min \{0{\cdot}1, \frac{1}{8}a\}$. Using the Taylor approximation formula, the fact that f_{d+1} has continuous second-order derivatives, and the compactness of N, it follows that there is a real $\delta > 0$ such that $|E_s(t)| < (\gamma/r)|t - s|$ for all points $s, t \in N$ for which $|t - s| < \delta$. Since N was a neighbourhood of the origin, it contains a ball centred at the origin, say of radius $\delta' > 0$.

Returning now to the first paragraph of this proof, we let $\alpha := \min\{\delta/2r, \gamma/2r, \delta'/2r\}$ and $\beta := r\alpha$. Then, since $\gamma \leqslant 0{\cdot}1$, we have by our construction that $f_{d+1}(x, u_0, \ldots, u_d) \in [-\alpha, \alpha]^p$ for all $x \in [-\alpha, \alpha]^p$ and for all $(u_0, \ldots, u_d) \in \bar{B}(c(x))$. This implies that $\bar{B}(c(x)) \subset C_d(\alpha, \beta, x)$, where $C_d(\alpha, \beta, x)$ is the d-step controllability set for the original system Σ. (We use for x here the coordinates in which A_0 is diagonal.) Thus our proof will conclude upon showing that the family of functions $\{g_j\}$ of Definition 4.5 is uniformly continuous.

Let $S \subset S_0(\beta^m)$ be the set of all input sequences $u \in S_0(R^m)$ satisfying $(u_0, \ldots, u_d) \in \bar{B}(c(0))$ and $(u_{j(d+1)}, \ldots, u_{(j+1)(d+1)-1}) \in \bar{B}[c(f_{j(d+1)}(0, u_0, \ldots, u_{j(d+1)-1}))]$. In view of the fact that $c(x)$ is a linear function of x, the uniform continuity of the family $\{g_j\}$ follows if we show that the restriction of Σ to S is uniformly l^∞-continuous. We next prove the latter.

Let $b_s := |B_s|$ and $b := \max\{b_s, s \in N\}$, so $|B_s(u_0, \ldots, u_d)| \leqslant b|(u_0, \ldots, u_d)|$ for all applicable input values. Consider now two input sequences $u, v \in S$, let $x := \Sigma u$, $y := \Sigma v$, $z := y - x$ and $w := v - u$, and note that z is a bounded sequence. Denote $s := (x_{j(d+1)}, u_{j(d+1)}, \ldots, u_{(j+1)(d+1)-1})$. Then,

$$\begin{aligned} z_{(j+1)(d+1)} &= L_s(z_{j(d+1)}, w_{j(d+1)}, \ldots, w_{(j+1)(d+1)-1}) \\ &\quad + E_s(y_{j(d+1)}, v_{j(d+1)}, \ldots, v_{(j+1)(d+1)-1}) \\ &= A_0 z_{j(d+1)} + (A_s - A_0) z_{j(d+1)} \\ &\quad + B_s(w_{j(d+1)}, \ldots, w_{(j+1)(d+1)-1}) + E_s \end{aligned} \tag{*}$$

By our construction, $|E_s| \leqslant \frac{1}{8}a|(z_{j(d+1)}, w_{j(d+1)}, \ldots, w_{(j+1)(d+1)-1})| \leqslant \frac{1}{8}a|z_{j(d+1)}| + \frac{1}{8}a|w_{j(d+1)}, \ldots, w_{(j+1)(d+1)-1})|$, and $|(A_s - A_0)z_{j(d+1)}| \leqslant \frac{1}{8}a|z_{j(d+1)}|$. Now fix the input sequence $u \in S$, and consider the set of all input sequences $v \in S$ for which $|w| = |v - u| < \xi$ for some real $\xi > 0$. Decompose $z_{k(d+1)} = z'_{k(d+1)} + z''_{k(d+1)}$ into a direct sum, where $z'_{k(d+1)}$ corresponds to eigenvalues of A_0 having absolute value less than one, and $z''_{k(d+1)}$ corresponds to eigenvalues of A_0 having absolute value greater than one, and recall that A_0 is diagonal. Let $|z''| := \sup\{|z''_{j(d+1)}|, j = 0, 1, 2, \ldots\}$, and note that, since z is a bounded sequence, $|z''|$ exists. Then, using (*), we obtain

$$\begin{aligned} |z'_{(j+1)(d+1)}| &\leqslant (1-a)|z'_{j(d+1)}| + b\xi + 2(a/8)|z'_{j(d+1)}| + 2(a/8)|z''_{j(d+1)}| + (a/8)\xi \\ &\leqslant (1 - \tfrac{3}{4}a)|z'_{j(d+1)}| + (b + \tfrac{1}{8}a)\xi + \tfrac{1}{4}a|z''| \end{aligned}$$

Using $z'_0 = 0$ and computing the series sum, we have

$$|z'_{k(d+1)}| \leqslant \frac{4}{3a}\left[\left(b + \frac{1}{8}a\right)\xi + \frac{1}{4}a|z''|\right]$$

for all integers $k \geqslant 0$. Similarly, again using (*),

$$\begin{aligned} |z''_{(j+1)(d+1)}| &\geqslant (1+a)|z''_{j(d+1)}| - b\xi - 2(a/8)|z''_{j(d+1)}| - 2(a/8)|z'_{j(d+1)}| - (a/8)\xi \\ &\geqslant (1 + \tfrac{3}{4}a)|z''_{j(d+1)}| - \tfrac{1}{12}a|z''| - \tfrac{4}{3}(b + \tfrac{1}{8}a)\xi \end{aligned} \tag{**}$$

where, in the last step, we substituted the bound for $|z'_{j(d+1)}|$ derived before. Denote $e := b + \frac{1}{8}a$ and choose an integer $j \geqslant 0$ for which $(1 + \frac{3}{4}a)|z''_{j(d+1)}| = (1 + \frac{3}{4}a)|z''| - \varepsilon$, where $\varepsilon \geqslant 0$. By the definition of $|z''|$, we have $|z''_{(j+1)(d+1)}| \leqslant |z''|$ for all integers $j \geqslant 0$, so we must have from (**)

$$(1 + \tfrac{2}{3}a)|z''| - \varepsilon - \tfrac{4}{3}e\xi \leqslant |z''|$$

or

$$|z''| \leqslant \frac{2e}{a}\xi + \frac{3}{2a}\varepsilon$$

However, in the last inequality, ε can be chosen arbitrarily close to zero, so we obtain $|z''| \leqslant (2e/a)\xi$. Substituting into the previous bound, we obtain $|z'_{k(d+1)}| \leqslant (4/3a)(e+\frac{1}{2})\xi$, and it follows that there is a constant $\sigma > 0$ such that $|z|_{d+1} < \sigma\xi$, where $|z|_{d+1} := \text{Sup}\,\{|z_{j(d+1)}|, j = 0, 1, 2, \ldots\}$. By the continuity of the iterated functions $f_1, f_2, \ldots, f_d$, this implies that for every real $\zeta > 0$ there is a real $\xi > 0$ such that $|\Sigma u - \Sigma v| < \zeta$ for all elements $u, v \in S$ satisfying $|u - v| < \xi$, and it follows that the restriction of Σ to S is uniformly l^∞-continuous. In view of the above arguments, this concludes our proof. □

The requirements listed in Proposition 4.7 are not really necessary conditions for uniform controllability, as can be seen from some of Examples (4.6). Nevertheless, Proposition 4.7 provides us with a strong indication that many of the commonly encountered systems that one expects to be stabilizable, are indeed uniformly controllable.

We now show that every uniformly controllable system has a full stability subspace.

Theorem (4.8)

Let $\Sigma : S_0(R^m) \to S_0(R^p)$ be a recursive system with a recursive representation $x_{k+1} = f(x_k, u_k)$, $x_0 = 0$, where $f : R^p \times R^m \to R^p$ is a continuous function. If Σ is uniformly controllable then it has a full stability subspace.

Proof

We assume that Σ is uniformly controllable, and we employ the notation of Definition 4.5. Using the balls $\bar{B}_\varepsilon(c(x))$, we construct recursively, for every integer $j \geqslant 1$, the subsets $S_j \subset (R^m)^{j(d+1)}$ as follows. We let $S_1 := \bar{B}_\varepsilon(c(0))$, and, having constructed S_j, we let S_{j+1} be the set of all points $(u_0, \ldots, u_{(j+1)(d+1)-1}) \in (R^m)^{(j+1)(d+1)}$ for which $(u_0, \ldots, u_{j(d+1)-1}) \in S_j$ and $(u_{j(d+1)}, \ldots, u_{(j+1)(d+1)-1}) \in \bar{B}_\varepsilon(g_j(u_0, \ldots, u_{j(d+1)-1}))$. For notational convenience, we take $S_0 := \varnothing$, the empty set. Further, we let $S_* \subset S_0(R^m)$ be the set of all sequences $u \in S_0(R^m)$ for which $(u_0, \ldots, u_{j(d+1)-1}) \in S_j$ for all integers $j \geqslant 1$. By definition of the balls $\bar{B}_\varepsilon(c(x))$, we have $S_* \subset S_0(\beta^m)$ and, as we have seen in the two paragraphs preceding Definition 4.5, there is a real $\gamma \geqslant \alpha > 0$ such that $\Sigma[S_*] \subset S_0(\gamma^p)$, so that S_* is a stability subspace of Σ. We show now that S_* is actually a full stability subspace of Σ, by constructing a unimodular, bicausal and uniformly l^∞-continuous transformation $M : S_0(R^m) \to S_0(R^m)$ for which $M[S_*] = S_0(\varepsilon^m)$, where $\varepsilon > 0$ is the radius of the balls $\bar{B}_\varepsilon(c(x))$.

Recalling that, by Definition 4.5, $\{g_j\}$ is a uniformly continuous family of functions over S_*, assume, for a moment, that, for each integer $j \geqslant 1$, the function $g_j : S_j \to ([-\beta, \beta]^m)^{d+1}$ can be extended into a function $h_j : (R^m)^{j(d+1)} \to ([-\beta, \beta]^m)^{d+1}$ in such a way that $\{h_j\}$ is a uniformly continuous family of functions over the entire space $S_0(R^m)$. For $j = 0$ let h_0 be the constant function $h_0 := c(0)$. Using the functions $\{h_j\}$, we can define a function $M : S_0(R^m) \to S_0(R^m)$ as follows. For every $u \in S_0(R^m)$, the elements of the sequence $y := Mu$ are given by

$$(y_{j(d+1)}, \ldots, y_{(j+1)(d+1)-1}) := (u_{j(d+1)}, \ldots, u_{(j+1)(d+1)-1}) - h_j(u_0, \ldots, u_{j(d+1)-1})$$

$j = 0, 1, 2, \ldots$. As a direct consequence of this definition, it follows that M is bicausal, and, by the uniform continuity of the family $\{h_j\}$, the function M is uniformly l^∞-continuous. In view of the fact that $|h_j| \leqslant \beta$ for all integers $j \geqslant 0$, we obtain by Lemma 4.4 that M is unimodular. Furthermore, since $h_j = g_j$ on S_j for all integers $j \geqslant 0$ (i.e. h_j is the centre of the ball $\bar{B}_\varepsilon(g_j)$), it follows by our construction of M that $M[S_*] = S_0(\varepsilon^m)$. Thus, recalling that S_* is a stability subspace, the last formula implies that S_* is a full stability subspace, and consequently our assertion will be proved upon the construction of the extensions $\{h_j\}$.

We construct the extensions $\{h_j\}$ as follows. For $j = 0$ we let $h_0 := c(0)$, a constant. For each integer $j \geqslant 1$ we note that $S_j \subset (R^m)^{j(d+1)}$ is a closed set, and we construct the extension $h_j : (R^m)^{j(d+1)} \to ([-\beta, \beta]^m)^{d+1}$ of the function $g_j : S_j \to ([-\beta, \beta]^m)^{d+1}$ using the standard construction employed in the proof of the Tietze Extension Theorem (see e.g. Kuratowski 1961). We show below that the family $\{h_j\}$ constructed in this way inherits the uniform continuity of the family $\{g_j\}$. We start with a review of the construction employed in the Tietze Theorem.

For an integer $k = 1, \ldots, m(d+1)$, denote by $g_j^k : S_j \to [-\beta, \beta]$ the real-valued function that forms the kth component of g_j, i.e. $g_j = (g_j^1, \ldots, g_j^{m(d+1)})$, and, similarly, let $h_j^k : (R^m)^{j(d+1)} \to [-\beta, \beta]$ be the kth component of h_j. It is evidently enough to show that, for each fixed k, the family of real-valued functions $\{h_j^k\}_{j=1}^\infty$ is uniformly continuous over the whole space $S_0(R^m)$, so we fix an integer $k \in \{1, \ldots, m(d+1)\}$. Given a point $x \in (R^m)^{j(d+1)}$ and a subset $A \subset (R^m)^{j(d+1)}$, we denote by $\alpha(x, A) := \inf \{|x - y|, y \in A\}$ the distance of x from A. When A is the empty set $\varnothing$, we let $\alpha(x, \varnothing) := 1$ for all x. For a fixed integer $j \geqslant 1$, and for all integers $i \geqslant 0$, we define, recursively, the function $h_{j,i}^k : (R^m)^{j(d+1)} \to R$ as follows. For $i = 0$ set $h_{j,0}^k := 0$; having constructed $h_{j,i}^k$ for some integer $i \geqslant 0$, we define the two sets

$$A_i^j := \left\{ x \in S_j \,\middle|\, \left[g_j^k(x) - \sum_{l=0}^{i} h_{j,l}^k(x) \right] \leqslant -\frac{1}{3}\left(\frac{2}{3}\right)^i \beta \right\}$$

$$B_i^j := \left\{ x \in S_j \,\middle|\, \left[g_j^k(x) - \sum_{l=0}^{i} h_{j,l}^k(x) \right] \geqslant \frac{1}{3}\left(\frac{2}{3}\right)^i \beta \right\}$$

and we define the function $h_{j,i+1}^k : (R^m)^{j(d+1)} \to R$ by

$$h_{j,i+1}^k(x) := \frac{1}{3}\left(\frac{2}{3}\right)^i \frac{\beta[\alpha(x, A_i^j) - \alpha(x, B_i^j)]}{\alpha(x, A_i^j) + \alpha(x, B_i^j)}$$

It can then be shown that the following holds for all integers $i \geqslant 0$ (Kuratowski 1961):

(a) $|h_{j,i+1}^k(x)| \leqslant \frac{1}{3}(\frac{2}{3})^i \beta$ for all $x \in (R^m)^{j(d+1)}$

(b) $\left| g_j^k(x) - \sum_{l=0}^{i} h_{j,l}^k(x) \right| \leqslant \left(\frac{2}{3}\right)^i \beta$ for all $x \in S_j$

(c) the series $h_j^k := \sum_{i=0}^{\infty} h_{j,i}^k$ is uniformly convergent on $(R^m)^{j(d+1)}$, and it forms a continuous extension $h_j^k : (R^m)^{j(d+1)} \to [-\beta, \beta]$ of g_j^k.

We now show that the family of functions $\{h_j^k\}_{j=1}^\infty$ is uniformly continuous over the whole space $S_0(R^m)$.

Let $\xi > 0$ be a real number. We have to show that there is a real $\delta > 0$ such that $|h_j^k(x) - h_j^k(y)| < \xi$ for all integers $j \geqslant 1$, whenever the elements $x, y \in (R^m)^{j(d+1)}$ satisfy $|x - y| < \delta$. To this end, let $n \geqslant 0$ be an integer such that $\Sigma_{i=n+1}^\infty (\frac{2}{3})^i \beta < \frac{1}{4}\xi$, and, for an integer $l \geqslant 0$, denote $f_{j,l}^k := \Sigma_{i=0}^{l} h_{j,i}^k$. Then, in view of (*a*)–(*c*) above, the existence of δ will follow if we show that there is a real $\delta > 0$ such that, for all integers $j \geqslant 1$, one has $|f_{j,n}^k(x) - f_{j,n}^k(y)| < \frac{1}{2}\xi$ for all $x, y \in (R^m)^{j(d+1)}$ satisfying $|x - y| < \delta$. This, in turn, holds if we show that the family of functions $\{f_{j,n}^k\}_{j=1}^{\infty}$ is uniformly continuous over the whole space $S_0(R^m)$. We prove the latter by recursion on n. First, since $f_{j,0}^k = h_{j,0}^k = 0$, the family $\{f_{j,0}^k\}_{j=1}^{\infty}$ is evidently uniformly continuous. By recursion, assume that, for some integer $l \geqslant 0$, the family $\{f_{j,l}^k\}_{j=1}^{\infty}$ is uniformly continuous over the whole space, and consider the functions $f_{j,l+1}^k, j = 1, 2, 3, \ldots$. In view of the recursion assumption and the fact that $f_{j,l+1}^k = f_{j,l}^k + h_{j,l+1}^k$, we clearly have that the family $\{f_{j,l+1}^k\}_{j=1}^{\infty}$ is uniformly continuous over the whole space if the family $\{h_{j,l+1}^k\}_{j=1}^{\infty}$ is uniformly continuous over the whole space. We conclude our proof by proving the latter.

By the uniform-continuity assumptions on the two families of functions $\{g_j^k\}_{j=1}^{\infty}$ and $\{f_{j,l}^k\}_{j=1}^{\infty}$, there is a real $\zeta > 0$ such that

$$|[g_j^k(x) - f_{j,l}^k(x)] - [g_j^k(y) - f_{j,l}^k(y)]| < \tfrac{1}{3}(\tfrac{2}{3})^l \beta$$

for all elements $x, y \in S_j$ satisfying $|x - y| < \zeta$, and for all integers $j \geqslant 1$. Consequently, by the definition of the sets A_l^j and B_l^j, we have $\alpha(A_l^j, B_l^j) \geqslant \zeta$ for all integers $j \geqslant 1$, which implies that $[\alpha(x, A_l^j) + \alpha(x, B_l^j)] \geqslant \zeta$ for all $x \in (R^m)^{j(d+1)}$ and for all integers $j \geqslant 1$. Now, for any pair of elements $x, v \in (R^m)^{j(d+1)}$, denote $A := A_l^j$, $B := B_l^j$, $\sigma := \frac{1}{3}(\frac{2}{3})^l \beta$, and $\varepsilon_{\pm} := [\alpha(x+v, A) \pm \alpha(x+v, B)] - [\alpha(x, A) \pm \alpha(x, B)]$, and note that $|\varepsilon_{\pm}| \leqslant 2|v|$ for all $x \in (R^m)^{j(d+1)}$. Then we have

$$\begin{aligned}
&|h_{j,l+1}^k(x+v) - h_{j,l+1}^k(x)| \\
&= \frac{\sigma|\{[\alpha(x, A) - \alpha(x, B) + \varepsilon_-][\alpha(x, A) + \alpha(x, B)] - [\alpha(x, A) + \alpha(x, B) + \varepsilon_+][\alpha(x, A) - \alpha(x, B)]\}|}{|[\alpha(x+v, A) + \alpha(x+v, B)][\alpha(x, A) + \alpha(x, B)]|} \\
&= \sigma\left|\frac{\varepsilon_-}{\alpha(x+v, A) + \alpha(x+v, B)} + \frac{\varepsilon_+}{\alpha(x+v, A) + \alpha(x+v, B)}\,\frac{\alpha(x, A) - \alpha(x, B)}{\alpha(x, A) + \alpha(x, B)}\right| \\
&\leqslant \frac{\sigma[|\varepsilon_-| + |\varepsilon_+|]}{\zeta} \leqslant \frac{4\sigma|v|}{\zeta}
\end{aligned}$$

where we used the fact that $|[\alpha(x, A) - \alpha(x, B)]/[\alpha(x, A) + \alpha(x, B)]| \leqslant 1$. Since the term $4\sigma|v|/\zeta$ is independent of j, this proves that the family $\{h_{j,l+1}^k\}_{j=1}^{\infty}$ is uniformly continuous over the whole space $S_0(R^m)$, and our proof concludes. □

From the combination of Proposition 4.7 and Theorem 4.8, and from Examples (4.6), we see that many of the systems commonly encountered in practice do indeed possess full stability subspaces. The stabilization theory that we develop in the following sections is for such systems.

5. Left-fraction representations of non-linear systems

Let $\Sigma: S_0(R^m) \to S_0(R^p)$ be a non-linear system. We say that Σ has a *left-fraction representation* if there exists an integer $q > 0$, a subset $S \subset S_0(R^q)$ and a pair of stable maps $T: \operatorname{Im} \Sigma \to S$ and $R: S_0(R^m) \to S$, where T is invertible, such that $\Sigma = T^{-1}R$. The subset S is called the *factorization space* of that representation. Our present interest in left-fraction representations of non-linear systems is motivated by the fact that they are instrumental in the solution of the stabilization problem, as we have discussed in § 1. Given a left-fraction representation of Σ and one pair of compensators π and ϕ that stabilize Σ through the configuration (3.1), one can easily construct infinitely many new pairs of compensators stabilizing Σ, by using a left-fraction representation of Σ and the straightforward pattern described in (1.7). In other words, left-fraction representations can be used to parametrize solutions of the stabilization problem in a particularly transparent way, and thus they become a tool of central importance in the theory of stabilization. In the present section, we discuss a few basic questions related to the existence and the construction of left-fraction representations of non-linear systems. We shall provide a more detailed exposition of the theory of left-fraction representations of non-linear systems in a separate report. Our present discussion augments the study of recursive left-fraction representations given in Hammer (1984 a).

From general principles in duality theory, the construction of left-fraction representations of systems is dual to the construction of right-fraction representations of systems. Let $\Sigma : S_0(R^m) \to S_0(R^p)$ be an injective system. As we have seen in § 2, the basic space used in the construction of right-fraction representations of Σ is the direct product of the Image of Σ by the Domain of Σ, i.e. the space $(\text{Im } \Sigma) \times S_0(R^m)$. For instance, the factorization space of a right-coprime fraction representation of Σ is C-morphic to the graph of Σ, which is a subset of $(\text{Im } \Sigma) \times S_0(R^m)$. As we know from duality theory, the dual of a domain is a codomain, the dual of a codomain is a domain, the dual of a projection is an injection, and the dual of a direct product of two sets is a disjoint union of two sets (see e.g. MacLane 1972). (We recall that a disjoint union $S_1 \dot{\cup} S_2$ of two sets S_1 and S_2 is the union of disjoint copies of S_1 and S_2.) Thus, from duality considerations, we would expect the disjoint union

$$U(\Sigma) := (\text{Im } \Sigma) \dot{\cup} S_0(R^m)$$

to play a central role in the theory of left-fraction representations of nonlinear systems. We shall need to identify $U(\Sigma)$ with a subset of $S_0(R^{q+1})$, where $q := \max\{p, m\}$, in the following way. Denoting $\alpha := q - m$ and $\beta := q - p$, we identify $S_0(R^{q+1}) = S_0(R^m) \times S_0(R^\alpha) \times S_0(R)$ and $S_0(R^{q+1}) = S_0(R^p) \times S_0(R^\beta) \times S_0(R)$ where, for notational convenience, we let $S_0(R^0) := 0$. Then, using two arbitrary, but distinct, elements $u_1, u_2 \in S_0(\theta)$, where $\theta > 0$ is a fixed real number, we construct the two sets $S' := (\text{Im } \Sigma) \times 0 \times u_1$, where $\text{Im } \Sigma \subset S_0(R^p)$ and $0 \in S_0(R^\beta)$, and $S'' := S_0(R^m) \times 0 \times u_2$, where $0 \in S_0(R^\alpha)$. Clearly, S' and S'' are two disjoint subsets of $S_0(R^{q+1})$, and we identify

$$U(\Sigma) := S' \cup S''$$

We denote by

$$i_\Sigma : \text{Im } \Sigma \to U(\Sigma), \quad i_D : S_0(R^m) \to U(\Sigma)$$

the natural injections satisfying $i_\Sigma[\text{Im } \Sigma] = S'$ and $i_D[S_0(R^m)] = S''$, and we call i_Σ and i_D the *canonical injections*. Whenever referring to $U(\Sigma)$, we shall always mean this particular construction with its canonical injections. In the next result we show that $U(\Sigma)$ is indeed the fundamental underlying space for the construction of left-fraction representations of Σ. We need the following notation. Given a function $g : S_1 \to S_2$ and an element $u \in S_1$, we denote by $K_g(u)$ the equivalence class of all elements $x \in S_1$ for which $g(x) = g(u)$ (i.e. the respective equivalence class in kernel g).

Theorem (5.1)

Let $\Sigma : S_0(R^m) \to S_0(R^p)$ be an injective system. Let $U(\Sigma)$ and the canonical injections $i_\Sigma : \text{Im } \Sigma \to U(\Sigma)$ and $i_D : S_0(R^m) \to U(\Sigma)$ be as defined above. Then the following hold true.

(i) Let $r>0$ be an integer. Each stable function $g: U(\Sigma) \rightarrow S_0(R^r)$ satisfying $K_g(i_D u) = \{i_D u, i_\Sigma \Sigma u\}$ for all elements $u \in S_0(R^m)$, gives rise to a left-fraction representation $\Sigma = T^{-1}R$ having the factorization space $\operatorname{Im} g$, where $T := gi_\Sigma : \operatorname{Im} \Sigma \rightarrow \operatorname{Im} g$ is an invertible stable map and where $R := gi_D : S_0(R^m) \rightarrow \operatorname{Im} g$ is a stable map.

(ii) If $\Sigma = T_1^{-1} R_1$ is any left-fraction representation of Σ with factorization space S_1 then there is a stable and surjective function $g_1 : U(\Sigma) \rightarrow S_1$ satisfying $K_{g_1}(i_D u) = \{i_D u, i_\Sigma \Sigma u\}$ for all elements $u \in S_0(R^m)$, and $T_1 = g_1 i_\Sigma$, $R_1 = g_1 i_D$.

As we can see from the Theorem, left-fraction representations of Σ are obtained from stable functions g having $U(\Sigma)$ as their domain, and satisfying the requirement that every equivalence class in kernel g consists of exactly two points—a point $u \in S_0(R^m)$ and its counterpart $\Sigma u \in \operatorname{Im} \Sigma$ (both appropriately injected into $U(\Sigma)$).

Proof of Theorem 5.1

(i) Let $g: U(\Sigma) \rightarrow S_0(R^r)$ be a stable function satisfying $K_g(i_D u) = \{i_D u, i_\Sigma \Sigma u\}$ for all elements $u \in S_0(R^m)$. Then the functions $T := gi_\Sigma : \operatorname{Im} \Sigma \rightarrow \operatorname{Im} g$ and $R := gi_D : S_0(R^m) \rightarrow \operatorname{Im} g$ are basically the restrictions of g to the sets S' and S'' respectively, and consequently are stable by the stability of g and the construction of $U(\Sigma)$. Since $K_g(i_D u) = \{i_D u, i_\Sigma \Sigma u\}$ for all $u \in S_0(R^m)$, we clearly have that T is injective and that $\operatorname{Im} T = \operatorname{Im} g$, so T is invertible. The same fact also implies that $gi_D u = gi_\Sigma y$ if and only if $y = \Sigma u$. Equivalently, $Ru = Ty$ if and only if $y = \Sigma u$ and, since T is invertible, we obtain $T^{-1}Ru = \Sigma u$ for all $u \in S_0(R^m)$, which concludes the proof of (i).

(ii) Let $\Sigma = T_1^{-1} R_1$ be a left-fraction representation of Σ with factorization space S_1. Define the function $g: U(\Sigma) \rightarrow S_1$ as follows. $g(i_D u) := R_1 u$ for all $u \in S_0(R^m)$, and $g(i_\Sigma y) := T_1 y$ for all $y \in \operatorname{Im} \Sigma$. Then g is a stable function defined on $U(\Sigma)$, by the stability of R_1 and of T_1 and by the construction of $U(\Sigma)$. To check kernel g, we note that the restrictions of g to $i_D[S_0(R^m)]$ and to $i_\Sigma[\operatorname{Im} \Sigma]$ are both injective by the injectivity of R_1 and of T_1 respectively (R_1 is injective because Σ is). Consequently, if $gv_1 = gv_2$ for a pair of elements $v_1 \neq v_2$ then we must have either $v_1 \in i_\Sigma[\operatorname{Im} \Sigma]$ and $v_2 \in i_D[S_0(R^m)]$, or $v_2 \in i_\Sigma[\operatorname{Im} \Sigma]$ and $v_1 \in i_D[S_0(R^m)]$. Now, let $u \in S_0(R^m)$ and $y \in \operatorname{Im} \Sigma$ be a pair of elements. Then, clearly, $g(i_D u) = g(i_\Sigma y)$ if and only if $R_1 u = T_1 y$, or, equivalently, if and only if $y = T_1^{-1} R_1 u = \Sigma u$. Thus $K_g(i_D u) = \{i_D u, i_\Sigma \Sigma u\}$ for all $u \in S_0(R^m)$, and, taking $g_1 := g$, our proof concludes. □

Before continuing with our discussion of left-fraction representations, we need to introduce a certain map $R: S_0(R^m) \rightarrow S_0(R^m)$ that has several important applications in our theory. The main use of this map is to transform functions that are continuous with respect to ρ into functions which are continuous with respect to l^∞. We shall refer to R as the *metric transformation*, and it is defined as follows. For every sequence $u \in S_0(R^m)$, the sequence $y := Ru \in S_0(R^m)$ is given elementwise by

$$y_i := 2^{-i} u_i, \quad i = 0, 1, 2, \ldots \tag{5.2}$$

It is easy to see that R is uniformly continuous with respect to ρ as well as with respect to l^∞, and it is bicausal and invertible. Moreover, as the definition states, R is actually linear, and it can be implemented by a simple linear time-varying system. Our interest in R arises from the following.

Lemma (5.3)

Let $\Sigma: S_0(R^m) \to S_0(R^p)$ be a map that is uniformly continuous (with respect to ρ) on a set $S \subset S_0(R^m)$, and let $R: S_0(R^p) \to S_0(R^p)$ be the metric transformation. Then the map $R\Sigma: S_0(R^m) \to S_0(R^p)$ is uniformly l^∞-continuous on the set S.

Proof

Let $\varepsilon > 0$ be a real number. In view of the uniform continuity of Σ on S, there exists a real $\delta > 0$ such that $\rho(\Sigma u' - \Sigma v') < \varepsilon$ for all elements $u', v' \in S$ for which $\rho(u' - v') < \delta$. By the definition of ρ, we clearly have that $\rho(u - v) < \delta$ for all elements $u, v \in S$ for which $|u - v| < \delta$, so that, by the previous observation, $\rho(\Sigma u - \Sigma v) < \varepsilon$ for all elements u, $v \in S$ for which $|u - v| < \delta$. Furthermore, by the definition of R, the inequality $\rho(\Sigma u - \Sigma v) < \varepsilon$ implies that $|R(\Sigma u - \Sigma v)| < \varepsilon$, and, by the linearity of R, we obtain $|R\Sigma u - R\Sigma v| < \varepsilon$. Thus, for every real $\varepsilon > 0$, there is a real $\delta > 0$ such that $|R\Sigma u - R\Sigma v| < \varepsilon$ for all elements u, $v \in S$ satisfying $|u - v| < \delta$, and $R\Sigma$ is uniformly l^∞-continuous on S. □

In our construction of the stabilizing compensators for a system Σ we shall need a left-fraction representation of Σ in which both the numerator and the denominator functions are uniformly l^∞-continuous. The reason why such a need arises will become clear in § 7, where the construction of the stabilizing compensators is discussed. Presently, we show that, using the metric transformation R, such a left-fraction representation can be readily constructed on any stability subspace of an injective homogeneous system Σ. Indeed, let $\Sigma: S_0(R^m) \to S_0(R^p)$ be an injective homogeneous system. let $S \subset S_0(R^m)$ be a stability subspace of Σ, and let $\theta > 0$ be a real number for which $S \subset S_0(\theta^m)$. By homogeneity, Σ is continuous on the closure $\bar{S}$ of S in $S_0(\theta^m)$ (§ 2). Consequently, by the compactness of $\bar{S}$ and the injectivity of Σ, the restriction $\Sigma_{\bar{S}}: \bar{S} \to \Sigma[\bar{S}]$ of Σ actually is a homeomorphism, and $\Sigma[\bar{S}]$ is compact. Thus the inverse map $\Sigma_{\bar{S}}^{-1}: \Sigma[\bar{S}] \to \bar{S}$ is uniformly continuous (with respect to ρ). Now, let $R: S_0(R^m) \to S_0(R^m)$ be the metric transformation. Then, by Lemma 5.3, the map $T_{\bar{S}} := R\Sigma_{\bar{S}}^{-1}: \Sigma[\bar{S}] \to R[\bar{S}]$ is uniformly l^∞-continuous. Furthermore, since $T_{\bar{S}}$ is invertible, we obtain the left-fraction representation $\Sigma_{\bar{S}} = T_{\bar{S}}^{-1} R$, where $T_{\bar{S}}$ and R are both stable and uniformly l^∞-continuous. The factorization space of this representation is, as we see, $R[\bar{S}]$. Recalling that R is bicausal and uniformly l^∞-continuous over the whole space $S_0(R^m)$, we obtain the following.

Proposition (5.4)

Let $\Sigma: S_0(R^m) \to S_0(R^p)$ be an injective homogeneous system, and let S be a stability subspace of Σ. Denote by $\Sigma_S: S \to \Sigma[S]$ the restriction of Σ to S. Then there exists a left-fraction representation $\Sigma_S = T_S^{-1} R$, where $T_S: \Sigma[S] \to R[S]$ and $R: S \to R[S]$ are stable and uniformly l^∞-continuous, and where R has an extension into a bicausal and uniformly l^∞-continuous map: $S_0(R^m) \to S_0(R^m)$.

6. Extension of maps and causality

The theory of stabilization for a system $\Sigma: S_0(R^m) \to S_0(R^p)$ critically depends on the extension of certain stable and causal maps, which are originally defined over some subspace of $S_0(R^m)$, and which have to be extended into stable and causal maps defined over the whole space $S_0(R^m)$. This problem of extending stable and causal maps is inherent to stabilization theory, and it is encountered in the linear case as well (see e.g. Hammer 1983 a, b). In the present section we derive the extension theorems for stable and causal maps necessary for our construction of the stabilizing compensators π and ϕ in the next section. We start with the following.

Theorem (6.1)

Let $\theta > 0$ be a real number, and let S be a closed subset of $S_0(\theta^m)$. Let $F: S \to S_0(R^p)$ be a stable and causal map. There then exists a causal and stable extension $F_e: S_0(\theta^m) \to S_0(R^p)$ of F.

Proof

For every integer $k \geqslant 0$ we define two projections $p_k: S_0(R^m) \to (R^m)^{k+1}$ and $q_k: S_0(R^m) \to R^m$ as follows: $p_k u := u_0^k$ for all sequences $u \in S_0(R^m)$, namely the projection onto the first $k+1$ elements; and $q_k u := u_k$ for all sequences $u \in S_0(R^m)$, namely the projection onto the kth element. Now, let $S_k := p_k[S]$, $k = 0, 1, 2, \ldots$, and note that, since $S \subset S_0(\theta^m)$, we have $S_k \subset ([-\theta, \theta]^m)^{k+1}$. By the causality of F, we can define, for every integer $k \geqslant 0$, a function $F_k: S_k \to R^p$ such that, for every element $u \in S$, the value $F_k(p_k u) := q_k F u$. Then, for every element $u \in S$, the output sequence $y := Fu$ is given elementwise by $y_k = F_k p_k u$, $k = 0, 1, 2, \ldots$, and the set of functions $\{F_k\}_{k=0}^{\infty}$ completely characterizes, and is characterized by, the system F. In view of the continuity of F, each one of the functions F_k is continuous. By the stability of F and the fact that $S \subset S_0(\theta^m)$, there is a real number $\alpha > 0$ such that $F[S] \subset S_0(\alpha^p)$, which implies that $F_k[S_k] \subset [-\alpha, \alpha]^p$ for all integers $k \geqslant 0$. Also, since $S \subset S_0(\theta^m)$ and is a closed subset, each one of the sets S_k, $k = 0, 1, 2, \ldots$, is a closed subset of $([-\theta, \theta]^m)^{k+1}$. Consequently, by the Tietze Extension Theorem (see e.g. Kuratowski 1961), there is, for every integer $k \geqslant 0$, a continuous extension $F_k^e: ([-\theta, \theta]^m)^{k+1} \to [-\alpha, \alpha]^p$ of the function F_k. We now define the system $F_e: S_0(\theta^m) \to S_0(\alpha^p)$ as follows. For every sequence $u \in S_0(\theta^m)$ the output sequence $y := F_e u$ is given, elementwise, by $y_k := F_k^e p_k u$, $k = 0, 1, 2, \ldots$. The function F_e is then clearly an extension of F, and F_e is causal by its construction, since y_k is determined by u_0^k. Thus it only remains to show that F_e is continuous as well. To this end, choose a real number $\varepsilon > 0$. Let $n \geqslant 0$ be an integer for which $2^{-(n+1)}\alpha < \frac{1}{2}\varepsilon$. By the continuity of the functions $F_0, F_1, \ldots, F_n$ on their respective (compact) domains, there are real numbers $\delta_i > 0$, $i = 0, \ldots, n$, such that $|F_i u - F_i v| < \varepsilon$ for all elements $u, v \in ([-\theta, \theta]^m)^{i+1}$ satisfying $|u - v| < \delta_i$. Let $\delta := \min\{\delta_0, \ldots, \delta_n\}$. Then, by the definition of our metric ρ and by the choice of n, we obtain that $\rho(F_e x - F_e y) \leqslant \max\{|F_0 x_0 - F_0 y_0|, \ldots, 2^{-n}|F_n x_0^n - F_n y_0^n|, 2^{-n}\alpha\} < \varepsilon$ for all elements $x, y \in S_0(\theta^m)$ satisfying $\rho(x - y) < 2^{-n}\delta$, and it follows that F_e is continuous (with respect to ρ) over $S_0(\theta^m)$. This concludes our proof. □

As we can see, Theorem 6.1 generates a causal and stable extension to the domain $S_0(\theta^m)$. In order to extend further from $S_0(\theta^m)$ to the whole space $S_0(R^m)$, we shall use composition with the map E considered in the following simple Lemma.

Lemma (6.2)

Let $\theta > 0$ be a real number, and let $I: S_0(\theta^m) \to S_0(\theta^m)$ be the identity map. There exists a uniformly l^∞-continuous extension $E: S_0(R^m) \to S_0(\theta^m)$ of I.

Proof

We first define the function $e: R^m \to [-\theta, \theta]^m$ as follows. For every vector $(x_1, \ldots, x_m) \in R^m$, we set $e(x_1, \ldots, x_m) := (\alpha_1, \ldots, \alpha_m)$, where, for each $i = 1, \ldots, m$, we let $\alpha_i := x_i$ if $|x_i| \leqslant \theta$ and $\alpha_i := \theta(\text{sign } x_i) \exp(\theta - |x_i|)$ if $|x_i| > \theta$. The function e is evidently uniformly continuous on R^m, and the system $E: S_0(R^m) \to S_0(\theta^m)$ having the recursive representation

$$E: \quad y_k = e(u_k), \quad k = 0, 1, 2, \ldots$$

$y = Eu$, clearly satisfies the requirements of the Lemma. We note that E is actually a static system. □

We interrupt now our discussion of the extension of maps with a discussion of certain causality properties of systems which are needed in the sequel. Let λ be an integer, and let D^λ be the λ-step shift operator, defined, on any sequence u, by $D^\lambda u]_k := u_{k-\lambda}$ for all integers k for which $u_{k-\lambda}$ exists. Turning now to causality properties, let n be any integer, and let $\Sigma: S_1 \to S_2$ be a system, where $S_1 \subset S_0(R^m)$ and $S_2 \subset D^n[S_0(R^p)]$. We say that Σ has *latency of at least* λ, denoted $\mathscr{L}(\Sigma) \geqslant \lambda$, if there is an integer λ such that, for every pair of input sequences $u, v \in S_1$ and for every integer $k \geqslant 0$, the equality $u_0^k = v_0^k$ implies $\Sigma u]_n^{k+\lambda} = \Sigma v]_n^{k+\lambda}$. We say that Σ has *well-defined latency* if there is an integer λ such that $\mathscr{L}(\Sigma) \geqslant \lambda$. Intuitively, the latency is a 'time delay' incurred in the propagation of changes from the input of Σ to the output of Σ. The latency may be either positive, when the system induces a delay, or negative, when the system exhibits anticipatory behaviour for some of its inputs. By definition, a system Σ is causal if and only if $\mathscr{L}(\Sigma) \geqslant 0$, and it is strictly causal if and only if $\mathscr{L}(\Sigma) \geqslant 1$. We list now a few simple properties of latency.

Proposition (6.3)

Let $\Sigma_1: S_1 \to S_2$ and $\Sigma_2: S_2 \to S_3$, where $S_1 \subset S_0(R^m)$, $S_2 \subset D^{\lambda_1}[S_0(R^p)]$ and $S_3 \subset D^{\lambda_1+\lambda_2}[S_0(R^q)]$, be systems with well-defined latency $\mathscr{L}(\Sigma_1) \geqslant \lambda_1$ and $\mathscr{L}(\Sigma_2) \geqslant \lambda_2$. Then the composition $\Sigma := \Sigma_2\Sigma_1$ has well-defined latency, and $\mathscr{L}(\Sigma) \geqslant \lambda_2 + \lambda_1$. In particular if $\lambda_2 + \lambda_1 \geqslant 0$ then Σ is a causal system.

Proof

Let $u, v \in S_1$ be a pair of input sequences. We show that, for any integer $k \geqslant 0$, the equality $u_0^k = v_0^k$ implies that $\Sigma u]_{\lambda_1+\lambda_2}^{k+\lambda_1+\lambda_2} = \Sigma v]_{\lambda_1+\lambda_2}^{k+\lambda_1+\lambda_2}$. This fact has the clear consequence that $\mathscr{L}(\Sigma) \geqslant \lambda_1 + \lambda_2$, which is what we need to prove. Now assume that $u_0^k = v_0^k$. Then, since λ_1 is the latency of Σ_1, we have $\Sigma_1 u]_{\lambda_1}^{k+\lambda_1} = \Sigma_1 v]_{\lambda_1}^{k+\lambda_1}$; since λ_2 is the latency of Σ_2, it further follows that $\Sigma_2(\Sigma_1 u)]_{\lambda_1+\lambda_2}^{k+\lambda_1+\lambda_2} = \Sigma_2(\Sigma_1 v)]_{\lambda_1+\lambda_2}^{k+\lambda_1+\lambda_2}$, and our assertion follows. □

For the case of a recursive system Σ, one can easily derive the following bound for the latency.

Proposition (6.4)

Let $\Sigma: S_1 \to S_2$, where $S_1 \subset S_0(R^m)$ and $S_2 \subset S_0(R^p)$, be a recursive system with a recursive representation $y_{k+\eta+1} = f(y_k^{k+\eta}, u_k^{k+\mu})$ (and some fixed initial conditions). Then Σ has well-defined latency, and $\mathscr{L}(\Sigma) \geqslant \eta + 1 - \mu$.

A simple, but important, consequence of well-defined latency is the following simple remark. Let $\Sigma: S_1 \to S_2$ be a system with well-defined latency, say $\mathscr{L}(\Sigma) \geqslant \lambda$. Then it is a direct consequence of the definitions that the system $D^{-\lambda}\Sigma$ is a causal system. In other words, a system having well-defined latency can be transformed into a causal system simply by composing it with an appropriate time shift. We state this fact formally.

Proposition (6.5)

A system $\Sigma: S_1 \to S_2$, where $S_1 \subset S_0(R^m)$ and $S_2 \subset S_0(R^p)$, has well-defined latency if and only if there exists an integer λ such that the system $D^{-\lambda}\Sigma: S_1 \to D^{-\lambda}[S_2]$ is a causal system.

Let $\Sigma: S_0(R^m) \to S_0(R^p)$ be a causal system, let $\Sigma': S_0(R^m) \to \operatorname{Im} \Sigma$ be the restriction of Σ, and let $\Sigma^*: \operatorname{Im} \Sigma \to S_0(R^m)$ be a right-inverse of Σ'. We say that the system Σ is *normal* if Σ^* can be chosen so that it has well-defined latency. Most systems of common interest are normal. For instance, if Σ is a recursive system then Σ^* can be chosen as a recursive system (see e.g. Hammer 1984 a), in which case Σ^* has well-defined latency by Proposition 6.4. Thus a recursive system is normal.

The following is a specific result that we need for the construction of the stabilizing compensators in the next section. It is based on Proposition 5.4.

Lemma (6.5)

Let $\Sigma: S_0(R^m) \to S_0(R^p)$ be an injective, homogeneous, causal and normal system. Let S_s be a stability subspace of Σ, and let $\Sigma_s: S_s \to \Sigma[S_s]$ be the restriction of Σ. There then exists a pair of stable and uniformly l^∞-continuous maps $R: S_0(R^m) \to S_0(R^m)$ and $T: S_0(R^p) \to S_0(R^m)$, where R is bicausal and T has well-defined latency, such that the restriction $T_s: \Sigma[S_s] \to T\Sigma[S_s]$ of T is an invertible map, and $\Sigma_s = T_s^{-1} R$.

Proof

We modify the proof of Proposition 5.4 (stated immediately preceding that Proposition), using the same notation. Let $F := \Sigma_{\bar{S}}^{-1}: \Sigma[\bar{S}_s] \to \bar{S}_s$, and let $\theta > 0$ be a real number such that $S_s \subset S_0(\theta^m)$. Since S_s is a stability subspace and Σ is homogeneous, there is a real $\alpha > 0$ such that $\Sigma[\bar{S}_s] \subset S_0(\alpha^p)$, and $\Sigma[\bar{S}_s]$ is a compact, and hence closed, subspace of $S_0(\alpha^p)$. Since Σ is normal and causal, there is an integer $\lambda \leqslant 0$ such that the map $F_1 := D^{-\lambda}F: \Sigma[\bar{S}_s] \to D^{-\lambda}[\bar{S}_s] \subset D^{-\lambda}[S_0(\theta^m)]$ is causal. By (a slight modification of) Theorem 6.1, there is a causal and continuous extension $F_e: S_0(\alpha^p) \to D^{-\lambda}[S_0(\theta^m)]$ of F_1 (to obtain this result, the proof of Theorem 6.1 has to be slightly modified so that the extensions F_k^e are constructed only for all integers $k \geqslant -\lambda$, instead of for all integers $k \geqslant 0$). Since $S_0(\alpha^p)$ is compact, F_e is uniformly continuous. Denoting $G := D^{\lambda}F_e: S_0(\alpha^p) \to S_0(\theta^m)$, we have that G is still uniformly continuous over $S_0(\alpha^p)$, has well-defined latency $\mathscr{L}(G) \geqslant \lambda$, and the restriction of G to $\Sigma[\bar{S}_s]$ is equal to $\Sigma_{\bar{S}}^{-1}$, i.e. $Gx = \Sigma_{\bar{S}}^{-1}x$ for all $x \in \Sigma[\bar{S}_s]$. Now, let $R: S_0(R^m) \to S_0(R^m)$ be the metric transformation defined in § 5. Then, by Lemma 5.3, the map $RG: S_0(\alpha^p) \to S_0(\theta^m)$ is uniformly l^∞-continuous. Let $E: S_0(R^p) \to S_0(\alpha^p)$ be the extension of the identity map $I: S_0(\alpha^p) \to S_0(\alpha^p)$ constructed in Lemma 6.2. Then the map $T := RGE: S_0(R^p) \to S_0(\theta^m)$ is clearly uniformly l^∞-continuous and stable. In view of the facts that R is bicausal, E is causal and $\mathscr{L}(G) \geqslant \lambda$, it follows that also

$\mathcal{L}(T) \geqslant \lambda$. Finally, the restriction $T_{\bar{S}}: \Sigma[\bar{S}_s] \to T\Sigma[\bar{S}_s]$ of T is clearly equal to $R\Sigma_{\bar{S}}^{-1}$, and consequently, in view of the proof of Proposition 5.4, $T_{\bar{S}}$ is invertible, and the left-fraction representation $\Sigma_{\bar{S}} = T_{\bar{S}}^{-1} R$ satisfies the requirements of the present Lemma over $\bar{S}_s$, and hence also over S_s. □

We close this section with the following handy result.

Proposition (6.6)

Let $M: S_0(R^m) \to S_0(R^m)$ be a bicausal map, and let $F: S_0(R^m) \to S_0(R^m)$ be a strictly causal map. Then the map $G := M + F: S_0(R^m) \to S_0(R^m)$ is bicausal.

Proof

Define the map $G_1 := GM^{-1} = (M + F)M^{-1} = I + FM^{-1}$, where I is the identity map on $S_0(R^m)$. Then, since FM^{-1} is still strictly causal, the argument provided in the proof of Lemmas 1 and 2 of Hammer (1984 b, § 2) implies that G_1 is bicausal. But then, since $G = G_1 M$, we obtain that G is bicausal as well. □

7. The stabilization procedure

We are now in a position to provide the description of a procedure that leads to the stabilization of a given non-linear system $\Sigma: S_0(R^m) \to S_0(R^p)$. We use (3.1) as our basic control configuration, and we construct pairs of compensators π and ϕ that, when connected in closed loop around the system Σ, yield a system $\Sigma_{(\pi,\phi)}$ that is internally stable for input sequences from $S_0(\theta^m)$, where $\theta > 0$ is an arbitrary real number specified in advance (see the discussion of bounded-input stabilization in § 3). We shall construct a whole family of pairs of compensators π and ϕ yielding stabilization, thus allowing freedom of choice in selecting the pair most convenient for each implementation.

Naturally, our procedure is not valid for all conceivable non-linear systems $\Sigma: S_0(R^m) \to S_0(R^p)$. The diversity of singular behaviour possible for a non-linear system is boundless, and, as we have discussed in the opening of § 4, certain non-linear systems are not even stabilizable in the usual sense of the word. The restrictions that we shall impose on the system Σ are listed below, and they were discussed in detail in the previous sections of this paper. As we have noted in those sections, the restrictions are satisfied in many common practical applications. A system $\Sigma: S_0(R^m) \to S_0(R^p)$ that can be stabilized through the procedure described in the present section must satisfy the following requirements (in parentheses we list the section in which the named property was discussed):

(7.1) Σ must be homogeneous (§ 2);

(7.2) Σ must possess a full stability subspace (§ 4);

(7.3) Σ must be causal and normal (§ 6);

(7.4) Σ must be injective.

The first three requirements are of a fundamental nature, and our stabilization procedure inherently depends on their validity. The last requirement—that Σ be injective—however, is not really fundamental, and our theory can be readily extended to the non-injective case as well, as we show in a moment. We use here the injectivity assumption just for convenience, since it leads to the simplification of some technical arguments.

The non-injective case

One way of circumventing the injectivity requirement (7.4) is the following. Suppose $\Sigma : S_0(R^m) \to S_0(R^q)$ is a strictly causal system satisfying requirements (7.1)–(7.3), but not necessarily (7.4). Let $p := \max\{m, q\}$, and define the identity embedding maps $\mathscr{I}_1 : S_0(R^m) \to S_0(R^p)$ and $\mathscr{I}_2 : S_0(R^q) \to S_0(R^p)$ as follows. If $q \geqslant m$ write $S_0(R^p) = S_0(R^q) = S_0(R^m) \times S_0(R^{q-m})$, let $\mathscr{I}_1 : S_0(R^m) \to S_0(R^p)$: $\mathscr{I}_1[S_0(R^m)] = S_0(R^m) \times 0$ be the obvious identity injection, and let $\mathscr{I}_2 : S_0(R^q) \to S_0(R^p)\,(= S_0(R^q))$ be the identity map. If $q < m$ write $S_0(R^p) = S_0(R^m) = S_0(R^q) \times S_0(R^{m-q})$, let $\mathscr{I}_2 : S_0(R^q) \to S_0(R^p) : \mathscr{I}_2[S_0(R^q)] = S_0(R^q) \times 0$ be the obvious identity injection, and let $\mathscr{I}_1 : S_0(R^m) \to S_0(R^p)(= S_0(R^m))$ be the identity map. Then, clearly, the system $\Sigma' := (\mathscr{I}_1 + \mathscr{I}_2\Sigma) : S_0(R^m) \to S_0(R^p)$, which is a simple modification of the given system Σ, is injective by the strict causality of Σ, and it still satisfies (7.1)–(7.3) because Σ satisfies these requirements. Consequently, our stabilization procedure can be used to stabilize the system Σ'. But this will yield the stabilization of the original system Σ in a configuration that is a slight modification of (3.1). Basically, the only modification occurring in (3.1) in this way is that the input of the feedback compensator ϕ becomes $y + w_2$ instead of just y. Thus the injectivity assumption (7.4) does not significantly limit the scope of our stabilization theory. Of course, more direct methods of eliminating the injectivity assumption could also be used.

Several simple examples of systems for which our theory applies are listed in (4.6); Proposition 4.7 and Theorem 4.8 provide an example of a class of such systems. Generally speaking, most practical stabilizable systems can be stabilized using the procedure described in this section.

We turn now to a step-by-step description of the procedure for stabilizing a system $\Sigma : S_0(R^m) \to S_0(R^p)$ that satisfies (7.1)–(7.4). We let $\theta > 0$ be a specified real number, serving as the bound for the input sequences feeding the stabilized system $\Sigma_{(\pi,\phi)}$. The following steps 1–7 lead to the construction of a precompensator π and a feedback compensator ϕ that, when connected in the closed loop (3.1) around Σ, yield a system $\Sigma_{(\pi,\phi)}$ that is internally stable for input sequences belonging to $S_0(\theta^m)$.

Step 1

Choose a full and closed stability subspace S_s of Σ. The existence of a full stability subspace is required in (7.2), and the implications of this requirement are discussed in § 4. The stability subspace can be chosen closed as a result of the homogeneity of Σ, required in (7.1) (see § 2). Section 4 also contains a discussion of how a full stability subspace can be computed for some common types of systems Σ (see, in particular, Example 4.6, and the proofs of Proposition 4.7 and Theorem 4.8).

Step 2

Choose a bicausal, stable and uniformly l^∞-continuous map $M : S_0(R^m) \to S_0(R^m)$ satisfying $M[S_s] = S_0((5\theta)^m)$. The existence of M is a direct consequence of the definition of a full stability subspace and the discussion in the paragraph preceding (3.8). The computation of M for some common types of systems Σ is described in the proof of Theorem 4.8.

Step 3

Let $R:S_0(R^m)\to S_0(R^m)$ and $T:S_0(R^p)\to S_0(R^m)$ be the pair of maps constructed in Lemma 6.5 for our present system Σ and the stability subspace S_s (the assumptions of the Lemma are satisfied by (7.1), (7.3) and (7.4)). As stated in the Lemma, R and T are both stable and uniformly l^∞-continuous, R is bicausal and T has well-defined latency, say $\mathscr{L}(T)\geqslant\lambda$ for some integer λ. By the causality of Σ and the construction described in the proof of Lemma 6.5, we can take $\lambda\leqslant 0$.

Step 4

Choose an arbitrary causal, stable and uniformly l^∞-continuous map $h_1:S_0(R^m)\to S_0(R^m)$. Let D be the one-step time-delay operator (see § 6), and define the map $h:S_0(R^m)\to S_0(R^m)$, for all elements $u\in S_0(R^m)$, by $hu]_i=0$ for $i=0,\ldots,-\lambda$, and $hu]_i=D^{1-\lambda}h_1u]_i$ for all integers $i\geqslant 1-\lambda$.

Step 5

Define the maps

$$A:=hT:S_0(R^p)\to S_0(R^m)$$

$$B:=M-hR:S_0(R^m)\to S_0(R^m)$$

Clearly, A and B are stable and uniformly l^∞-continuous. Furthermore, by the choice of λ, the maps hT and hR are both strictly causal. Consequently, A is strictly causal, and, since M is bicausal, B is bicausal by Proposition 6.6. The latter implies that B has an inverse B^{-1}.

Step 6

Compute the inverse $B^{-1}:S_0(R^m)\to S_0(R^m)$.

Step 7

Set

$$\pi:=B^{-1}:S_0(R^m)\to S_0(R^m)$$

$$\phi:=A:S_0(R^p)\to S_0(R^m)$$

In this way, we obtain a family of pairs of compensators π and ϕ, one pair for each choice of the function h_1 in Step 4. (Of course, additional degrees of freedom are available in the construction of the functions M, T and R.) We now show that, for each such pair of compensators, the system $\Sigma_{(\pi,\phi)}$ is internally stable for all input sequences bounded by θ.

Theorem (7.5)

Let $\Sigma:S_0(R^m)\to S_0(R^p)$ be a homogeneous, causal, normal and injective system, having a full stability subspace. Let $\theta>0$ be a real number, and let $\pi:S_0(R^m)\to S_0(R^m)$ and $\phi:S_0(R^p)\to S_0(R^m)$ be a pair of compensators constructed in Steps 1–7 above. Then the system $\Sigma_{(\pi,\phi)}$ is internally stable for all input sequences $u\in S_0(\theta^m)$.

Proof

We show that the conditions of Theorem 3.9 are satisfied. By the homogeneity of Σ and Theorem 2.4, there is a right-coprime fraction representation $\Sigma=PQ^{-1}$, where $P:S\to S_0(R^p)$ and $Q:S\to S_0(R^m)$ are stable and right-coprime maps, and where

$S \subset S_0(R^q)$ is the factorization space. Let S_s be the closed stability subspace chosen in Step 1. By the definition of stability subspaces, there is a pair of real numbers $\alpha > 0$ and $\beta > 0$ such that $S_s \subset S_0(\alpha^m)$ and $\Sigma[S_s] \subset S_0(\beta^p)$. By the coprimeness of P and Q, there is a real $\gamma > 0$ such that $P^*[S_0(\beta^p)] \cap Q^{-1}[S_0(\alpha^m)] \subset S_0(\gamma^q)$, and, since clearly $Q^{-1}[S_s] \subset P^*[S_0(\beta^p)] \cap Q^{-1}[S_0(\alpha^m)]$, we obtain that $Q^{-1}[S_s] \subset S_0(\gamma^q)$. Let $S_1 := Q^{-1}[S_s] \subset S$, and let $\bar{S}_1$ be the closure of S_1 in $S_0(\gamma^q)$. By the coprimeness requirement, $\bar{S}_1 \subset S$. Further, by the continuity and injectivity of Q, the restriction of Q to $\bar{S}_1$ is a homeomorphism by the compactness of $\bar{S}_1$. Since $Q[\bar{S}_1] = \bar{S}_s$ and since S_s was closed, we have $\bar{S}_s = S_s$, so that the injectivity of Q implies that also $\bar{S}_1 = S_1$. Recalling from Step 2 that $M[S_s] = S_0((5\theta)^m)$, let $M_1 : S_s \to M[S_s] = S_0((5\theta)^m)$ be a restriction of M. Then, M_1 is unimodular. Let $Q_1 : S_1 \to Q[S_1] = S_s$ be the restriction of Q. Then, as we saw before, Q_1 is unimodular, and, consequently, the map $M_* := M_1 Q_1 : S_1 \to S_0((5\theta)^m)$ is unimodular as well. Consider now the maps T and R of Step 3. Using the notation of Lemma 6.5, we have $\Sigma_S = T_S^{-1} R$ on S_s, or $PQ^{-1} = T_S^{-1} R$ on S_s, or $T_S P u = R Q_1 u$ for all elements $u \in Q^{-1}[S_s] = S_1$. Using the maps A and B of Step 5, we obtain, for all elements $u \in S_1$, that $APu + BQu = hTPu + (M - hR)\, Qu = hT_S Pu + (M_1 - hR)\, Q_1 u = hT_S Pu - hRQ_1 u + M_1 Q_1 u = hT_S Pu - hT_S Pu + M_1 Q_1 u = M_1 Q_1 u =: M_* u$, so that

$$APu + BQu = M_* u$$

for all elements $u \in S_1$, where the unimodular transformation M_* satisfies $M_*[S_1] = S_0((5\theta)^m)$. Moreover, A and B are stable, and, being uniformly l^∞-continuous, they are differentially bounded by θ (for any real $\theta > 0$). Also, as we noted in Step 5, A is causal and B is bicausal. Thus all conditions of Theorem 3.9 are satisfied, whence $\Sigma_{(B^{-1},A)}$ is internally stable on $S_0(\theta^m)$. This concludes our proof. □

An interesting consequence of Theorem 7.5 is that, when using non-linear compensators, an unstable system Σ can be made internally stable through precompensation alone. This occurs, for instance, when the function h_1 of Step 4 is chosen as the zero function, which results in $\phi = 0$ in Step 7. As is well known from the linear theory, an unstable linear system cannot be rendered internally stable by the application of linear precompensation alone. The fact that the situation is different when the linearity restriction on the compensators is released is a manifestation of the large freedom that one has in designing non-linear compensators. Finally, we remark that, even though injectivity of the system Σ is required in the statement of Theorem 7.5, our present theory also yields a stabilization procedure for non-injective systems, as described earlier in this section.

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