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Stability and Nonsingular Stable Precompensation: An Algebraic Approach*

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Abstract. The module theoretic framework in linear time invariant system theory is extended to include stability considerations as well. The resulting setup is then applied to an investigation of nonsingular, causal, and stable precompensation. The main issues are resolved through the introduction of two sets of integer invariants-the *stability indices* and the *pole indices*. The stability indices characterize the dynamical properties of all the stable systems that can be obtained from a specified system through the application of nonsingular, causal, and stable precompensation. The pole indices characterize the dynamical properties of all the nonsingular, causal, and stable precompensators that stabilize a specified system.

1. Introduction

The classical theory of modules was first employed in a system theoretic context by Kalman [1965], in his algebraic theory of linear system realization. The module theoretic approach to the realization problem (see also Kalman [1968] and Kalman, Falb and Arbib [1969]) has revealed, on a fundamental level, the connection between the "external" and the "internal" descriptions of linear time invariant systems. Later, the module theoretic approach allowed a reconciliation of the the theories of system realization and of state feedback into one uniform algebraic framework (Hautus and Heymann [1978]). More recently, module theory was found to form a natural mathematical framework for the accommodation of linear dynamic output feedback, a framework that lead to the concept of latency (Hammer and Heymann [1981]). So far, however, stability considerations have not been incorporated into the module theoretic setup. Stability properties

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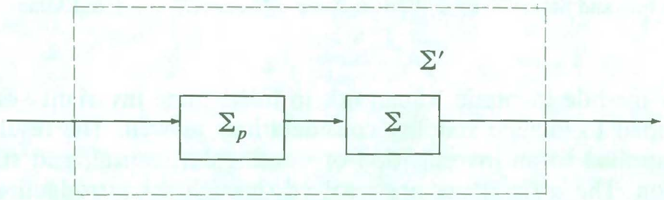
of systems were usually studied through either classical state space methods (e.g., Kwakernaak and Sivan [1974]), or polynomial matrix fractions methods (Rosenbrock [1970], Wolovich [1974]), or geometric methods (Wonham [1974]), or generalized matrix fractions (Desoer, Liu, Murray, and Saeks [1980]).

Our main objective in the present paper is to show that stability theory can be naturally accommodated within the module theoretic approach. Thus, when suitably generalized, algebraic realization theory will lead to certain "canonical stability representations" of systems, which play a dominant role in stability considerations. We then apply our stability framework to an examination of nonsingular, causal, and stable precompensation.

Specifically, we refer to the following situation. Let Σ and Σ_p be linear time invariant systems,and consider the following diagram,



(1.1)

where Σ' is the system resulting from the series combination $\Sigma\Sigma_p$, and the system Σ_p is called a *precompensator*. Of particular importance in applications is the case when the following conditions are satisfied: (i) the resulting system Σ' is stable, (ii) the precompensator Σ_p is causal and stable, and (iii) the precompensator Σ_p is nonsingular. The nonsingularity of Σ_p ensures that no degrees of freedom of the control variables are being destroyed by the precompensator, so that the final system Σ' has the same control capabilities as the original one Σ. Conditions (i), (ii), and (iii) are essential for actual implementation of precompensation schemes, e.g., when one wishes to represent Σ_p as an internally stable output feedback configuration (see Hammer [1981]). Whenever conditions (i), (ii), and (iii) are met, we shall refer to the situation as *nonsingular*, *causal*, *and stable precompensation*.

In the present paper we show that the structural properties of the problem of nonsingular, causal, and stable precompensation can be characterized through two ordered sets of integers $\sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_m$, and $\rho_1 \leq \rho_2 \leq \cdots \leq \rho_m$, derived from the original system Σ. We call these invariants the *stability indices*, and the *pole indices* of Σ. As it turns out, the stability indices and the pole indices play a major role in the theory of linear control. They determine the possibilities of pole assignment by internally stable *output* feedback control configurations, similarly to the way in which the reachability indices determine the possibilities of pole assignment by *state* feedback (Hammer [1983]). Since every actual control configuration has to be internally stable, it seems that the stability indices and the pole indices are natural system invariants from a theoretical as well as from a practical point of view.

The main properties of the stability indices and of the pole indices can be derived from a study of the above mentioned problem of nonsingular, causal, and stable precompensation, without need to refer to the full conditions for internal stability. In the present paper we study these properties, whereas the application of the stability indices and of the pole indices to internally stable control is considered in Hammer [1983]. We next interpret the basic definitions of these indices in classical state space terms, and afterwards we shall give a qualitative explanation of their origin in the module theoretic framework.

Let $S(\Sigma)$ denote the set of all stable systems obtainable from Σ through nonsingular, causal, and stable precompensation. Let $\Sigma' \in S(\Sigma)$ be an element in this set, and let

$$\dot{x} = Fx + Gu,$$

$$y = Hx,$$

be a realization of Σ'. The dynamical properties of Σ' are, of course, determined by the pair (F, G), and we shall refer to (F, G) as a *semirealization* of Σ'. As usual, we say that the pair (F, G) is *stable* if all roots of the characteristic polynomial of F lie in a prescribed region of the complex plane, and we say that it is *reachable* if the column vectors of $G, FG, F^2G, F^3G, \ldots,$ span X. We shall say that (F, G) is a *canonical semirealization* of Σ' if there exists a matrix $H: X \to Y$ such that (F, G, H) is a canonical realization of Σ'. In this terminology, we prove the following. Let (A, B) be an arbitrary stable and reachable pair with reachability indices $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m$. Then, there exists a system $\Sigma' \in S(\Sigma)$ for which (A, B) is a canonical semirealization if and only if $\lambda_i \geq \sigma_i$ for all $i = 1, \ldots, m$, where $\sigma_1, \ldots, \sigma_m$ are the stability indices of Σ. Thus, every stable dynamics can be *canonically* assigned by nonsingular stable precompensation, subject only to the above integers inequality, and the stability indices completely characterize the dynamical properties of all systems in $S(\Sigma)$. The stability indices can be explicitly calculated from the transfer matrix of Σ.

Symmetrically, let $P(\Sigma)$ denote the set of all nonsingular, causal, and stable precompensators Σ_p for which the series combination $\Sigma\Sigma_p$ is stable, and let (F, G) be an arbitrary stable and reachable pair with reachability indices $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m$. Then, we show that there exists a precompensator $\Sigma_p \in P(\Sigma)$ having a canonical semirealization (F, G) if and only if $\lambda_i \geq \rho_i$ for all $i = 1, \ldots, m$, where $\rho_1, \ldots, \rho_m$ are the pole indices of Σ. Thus, the pole indices $\rho_1, \ldots, \rho_m$ completely characterize the dynamical properties of all the precompensators in $P(\Sigma)$. The pole indices can be explicitly computed from the transfer matrix of Σ. As we show, the stability indices and the pole indices have the same algebraic origin as the latency indices (Hammer and Heymann [1981]) and the reduced reachability indices (Hammer and Heymann [1983]). All these different kinds of invariants turn out to be just different manifestations of a uniform underlying algebraic structure.

The module theoretic stability framework that we develop in the present paper is in close analogy to the framework of modules over polynomial rings employed in realization theory. In both cases there is a strong connection with certain canonical fraction representations of transfer matrices. We give now a qualitative interpretation of our approach in the single variable case. Consider a linear, time-invariant, discrete time, single-input single-output system f. Every input (or output) sequence to f can be regarded as a formal Laurent series $\sum_{t=t_0}^{\infty} k_t z^{-t}$, where $t_0 \neq -\infty$, and where $\{k_t\}$ are real numbers which represent the input (or output) value at the time instant t. Denoting by ΛR the set of all such formal Laurent series, where t_0 ranges over all integers, we obtain that the system f induces a map $f\colon \Lambda R \to \Lambda R$. Further, we denote by ΩR the set of all polynomials with real coefficients, so that clearly ΩR is a subset of ΛR. An element $r \in \Lambda R$ is called *rational* if it can be expressed as a quotient $r = \alpha/\beta$ of two polynomials $\alpha, \beta \in \Omega R$. A rational element $r = \alpha/\beta$, where α and β are coprime polynomials, is called *stable* if all the roots of β are on the left hand side of the complex plane. We denote by $\Omega_0 R$ the set of all stable elements in ΛR. As is well known, ΩR is a principal ideal domain, and it can be shown that so is also $\Omega_0 R$.

To each element $d \in \Lambda R$ we assign now the equivalence class $d + \Omega R$, which consists of all possible sums of d with an element in ΩR. The set of all such equivalence classes forms then the quotient ΩR-module $\Lambda R/\Omega R$, and we denote the projection by

$$\pi\colon \Lambda R \to \Lambda R/\Omega R\colon d \mapsto d + \Omega R.$$

Similarly, we assign to every element $d \in \Lambda R$ the equivalence class $d + \Omega_0 R$, and the set of all such equivalence classes again forms a quotient module $\Lambda R/\Omega_0 R$, this time over the ring $\Omega_0 R$, of course. We define then a projection

$$\pi_0\colon \Lambda R \to \Lambda R/\Omega_0 R\colon d \mapsto d + \Omega_0 R.$$

Consider now the ΩR-module $\ker \pi f$, which consists of *all* input sequences that lead to polynomial outputs, that is, to outputs which are identically zero for all $t \geq 1$ ("the future"). From this module one can obtain the classical Kalman [1965] realization module Δ, which consists of all *past* inputs that lead to zero future outputs, simply by intersecting

$$\Delta = \ker \pi f \cap \Omega R. \tag{1.2}$$

The ΩR-module Δ possesses a generator $d \in \Omega R$, so that

$$\Delta = d[\Omega R].$$

Clearly, since $d \in \Delta$, d is a polynomial and $n := fd \in f[\ker \pi f] \subset \Omega R$ is a polynomial as well. Thus, Δ induces a polynomial fraction representation

$$f = n/d, \tag{1.3}$$

and it can be shown that the polynomials n, d are *coprime*. Further, with the module Δ one can associate the integer

$$\mu(\Delta) := \deg d \tag{1.4}$$

(see Hautus and Heymann [1978]), and then the *reachability index* λ of f is given by

$$\lambda = \mu(\Delta).$$

Turning now to stability, we introduce, in analogy to $\ker \pi f$, the $\Omega_0 R$-module $\ker \pi_0 f$. This module consists of all input sequences that lead to *stable* output sequences. Then, the notion of the realization module (1.2) can be carried over to the stability context in the following three ways, each of which has a significant system theoretic interpretation.

$$\Delta_0^0 := \ker \pi_0 f \cap \Omega_0 R, \tag{1.5}$$

$$\Delta^0 := \ker \pi f \cap \Omega_0 R, \tag{1.6}$$

$$\Delta_0 := \ker \pi_0 f \cap \Omega R, \tag{1.7}$$

where Δ_0^0 consists of all stable inputs that lead to stable outputs; Δ^0 consists of all stable inputs that lead to polynomial outputs; and Δ_0 consists of all polynomial inputs that lead to stable outputs. It is worthwhile to note the symmetry between Δ^0 and Δ_0, and that Δ^0 and Δ_0 are ΩR-modules, whereas Δ_0^0 is an $\Omega_0 R$-module. It can be shown that each of these three modules is of rank one over the appropriate ring, so that there are elements $d_1, d_2, d_3 \in \Lambda R$ such that

$$\Delta_0^0 = d_1[\Omega_0 R], \qquad \Delta^0 = d_2[\Omega R], \qquad \Delta_0 = d_3[\Omega R].$$

Similarly to (1.3), we define $n_i := fd_i$, $i = 1,2,3$, and we obtain fraction representations $f = n_i/d_i$, $i = 1,2,3$. In order to see the explicit interpretation of these fractions, we refer to the coprime polynomial fraction $f = n/d$, and we let

$$d = d^+ d^- \quad \text{and} \quad n = n^+ n^-$$

be polynomial factorizations, where the polynomials d^- and n^- contain exactly all the *stable* roots of d and n, respectively. Then, letting α be an arbitrary unit of the ring $\Omega_0 R$, it can be shown that

$$d_1 = \alpha d, \quad d_2 = d/n^-, \quad \text{and} \quad d_3 = d^+.$$

Thus, we obtain that the fraction representations $f = n_i/d_i$, $i = 1,2,3$, have the following explicit form

$$f = n_1/d_1, \qquad n_1 := \alpha n \qquad \text{(stable/stable)} \tag{1.8}$$

$$f = n^+/d_2, \qquad \text{(polynomial/stable)} \tag{1.9}$$

$$f = n_3/d^+, \qquad n_3 := n/d^- \qquad \text{(stable/polynomial)} \tag{1.10}$$

Now, the numerator n^+ in (1.9) exactly characterizes the unstable zeros of f, and, therefore, (1.9) is called a *zero representation* of f. In (1.10), the denominator d^+ exactly characterizes the unstable *poles* of f, and (1.10) is called a *pole representa-*

tion of f. These two are particular cases of the more general fraction representation (1.8), which we call a *canonical stability representation* of f.

Finally, in analogy to (1.4), we define the integers

$$\mu(\Delta^0) := \deg d_2 = \deg d - \deg n^-, \tag{1.11}$$

$$\mu(\Delta_0) := \deg d_3 = \deg d^+. \tag{1.12}$$

Then, $\mu(\Delta^0)$ is the number of those zeros of f which are unstable or at infinity, and $\mu(\Delta_0)$ is the number of unstable poles of f. Referring now back to our introductory discussion, the stability degree σ of f is given by $\sigma = \mu(\Delta^0)$, and the pole degree ρ of f is given by $\rho = \mu(\Delta_0)$. The multivariable cases are discussed within the paper.

Many aspects of dynamic compensation were considered in the literature. Thus, pole shifting by dynamic compensation was treated in Brasch and Pearson [1970], and invariants under dynamic compensation were considered in Wolovich and Falb [1976], Morse [1975], Hammer and Heymann [1981 and 1983] and Khargonekar and Emre [1980]. The investigation reported in Morse [1975] includes stability restrictions on the allowable precompensators, and, in fact, we shall employ in our examination a result obtained there.

A certain class of precompensators (namely, the *bicausal* precompensators) can be represented as static state feedback in a suitable realization (Hautus and Heymann [1978]). Static state feedback, under various restrictions of stability on the resulting feedback system, was studied in Wonham [1974].

The present paper is organized as follows. In Section 2 we survey our basic notation and terminology, and in Sections 3, 4, and 5 we study the connection between modules and fractional representations in the stability sense. Sections 6 and 7 are devoted, respectively, to the study of the pole indices and of the stability indices.

2. ΛK-Linear Spaces and Stability Rings

In this section we review and extend the terminology and framework of Hammer and Heymann [1981 and 1983]. We let K be a field, S a K-linear space, and denote by ΛS the set of all Laurent series of the form

$$s = \sum_{t=t_0}^{\infty} s_t z^{-t} \tag{2.1}$$

where, for all t, $s_t \in S$. Then, under coefficientwise addition and convolution as scalar multiplication, ΛK is endowed with a field structure, and ΛS forms a ΛK-linear space. Moreover, when the K-linear space S is of some finite dimension n, then so is also ΛS as a ΛK-linear space.

Now, let U and Y be K-linear spaces, and let Σ be a linear system admitting U-valued inputs and having Y-valued outputs. For intuitive convenience, we assume that Σ is a discrete time system. Then every element $u = \sum_{t=t_0}^{\infty} u_t z^{-t} \in \Lambda U$ can be interpreted as an input time-sequence to Σ, the index t being identified as the time marker. The corresponding output sequence is then an element in ΛY, so that Σ induces a K-linear map $f\colon \Lambda U \to \Lambda Y$. Assume further that f is also ΛK-linear. Then we evidently have $fzu = zfu$ for every $u \in \Lambda U$, which implies that the original system Σ is time invariant. Thus, *a* ΛK*-linear map* $f\colon \Lambda U \to \Lambda Y$ *represents a* $(K-)$ *linear time invariant system* (Wyman [1972], Hautus and Heymann [1978]). Throughout our discussion we shall consider ΛK-linear maps $f\colon \Lambda U \to \Lambda Y$, where U and Y are of finite dimension. We shall denote $m = \dim_K U$ and $p = \dim_K Y$.

Let $f\colon \Lambda U \to \Lambda Y$ be a ΛK-linear map. Then, f can, of course, be represented as a matrix relative to specified bases $u_1,\ldots,u_m$ in ΛU and $y_1,\ldots,y_p$ in ΛY. If $u_1,\ldots,u_m$ are in the K-linear space U and $y_1,\ldots,y_p$ are in Y, then the matrix representation of f is called a *transfer matrix*.

We consider next some particular types of ΛK-linear maps which appear in our discussion, but, for this, we need a more detailed examination of the space ΛS. First we note that ΛS contains, as subsets, the set ΩS of all (polynomial) elements of the form $\sum_{t=t_0}^{0} s_t z^{-t}$, $t_0 \le 0$, and the set $\Omega^- S$ of all (power series) elements of the form $\sum_{t=0}^{\infty} s_t z^{-t}$. In particular, it can be readily verified that, under the operations defined in ΛK, the both of the sets ΩK and $\Omega^- K$ are endowed with a principal ideal domain structure. Also, ΩS forms an ΩK-module, and $\Omega^- S$ forms an $\Omega^- K$-module, and both of these modules have rank equal to $\dim_K S$. Further, since the ΛK-linear space ΛS is evidently both an ΩK-module and an $\Omega^- K$-module, we can consider the quotient modules $\Gamma S := \Lambda S/\Omega S$ and $\Lambda S/\Omega^- S$. We shall need the following notation.

$$\begin{aligned} &j\colon \Omega S \to \Lambda S\colon s \mapsto s && \text{(natural injection)} \\ &\pi\colon \Lambda S \to \Gamma S && \text{(canonical projection).} \end{aligned} \tag{2.2}$$

Let $s = \sum_{t=t_0}^{\infty} s_t z^{-t} \in \Lambda S$ be an element. The *order* of s is defined by $\operatorname{ord} s := \min_t \{s_t \neq 0\}$ if $s \neq 0$ and $\operatorname{ord} s := \infty$ if $s = 0$. The *leading* coefficient $\hat{s}$ of s is $\hat{s} := s_{\operatorname{ord} s}$ if $s \neq 0$ and $\hat{s} := 0$ if $s = 0$.

Let $f\colon \Lambda U \to \Lambda Y$ be a ΛK-linear map. Then, f is called *causal* (respectively *strictly causal*) if $\operatorname{ord} fu \ge \operatorname{ord} u$ (respectively $\operatorname{ord} fu > \operatorname{ord} u$) for all $u \in \Lambda U$. Also, a ΛK-linear map $\ell\colon \Lambda U \to \Lambda U$ is called *bicausal* if it has an inverse ℓ^{-1} and if both of ℓ and ℓ^{-1} are causal (Hautus and Heymann [1978]). It is then readily seen that f is *causal* (*respectively strictly causal*) *if and only if* $f[\Omega^- U] \subset \Omega^- Y$ (*respectively* $f[\Omega^- U] \subset z^{-1}\Omega^- Y$). Still, equivalently, f is causal if and only if all entries in its transfer matrix belong to $\Omega^- K$.

Further, we say that a ΛK-linear map $f\colon \Lambda U \to \Lambda Y$ is a *polynomial map* if it can be restricted to the set of polynomials, that is, if $f[\Omega U] \subset \Omega Y$. Explicitly, f is polynomial if and only if all entries in its transfer matrix belong to ΩK. A

ΛK-linear map $f: \Lambda U \to \Lambda Y$ is called *rational* if there exists a nonzero element $\Psi \in \Omega K$ such that Ψf is polynomial. Finally, a strictly causal and rational ΛK-linear map is called a *linear input/output map*.

Next, we turn to stability. Let $\theta \subset \Omega K$ be a multiplicative set (i.e., for every pair of elements $k_1, k_2 \in \theta$, also $k_1 k_2 \in \theta$). We say that θ is a *stability set* if (i) $0 \notin \theta$, and (ii) there exists an element $a \in K$ such that $(z+a) \in \theta$ (see Morse [1975]). The following terminology will be repeatedly used.

Definition 2.3. Let $f: \Lambda U \to \Lambda Y$ be a ΛK-linear map and let $\theta \subset \Omega K$ be a stability set. Then, f is *input/output stable* (in the sense of θ) if there exists an element $\Psi \in \theta$ such that Ψf is a polynomial map.

We note that, when K is the field of real numbers, the above definition includes the classical notion of stability in linear control theory, where all the roots of the characteristic polynomial are required to lie in a specified region of the complex plane (which intersects the real line). We now fix the stability set θ, and all our considerations below are in the sense of θ.

The definition of input/output stability leads to the consideration of a class of subrings of ΛK as follows. Let $\Omega_\theta K$ be the set of all rational elements $\alpha \in \Lambda K$ which can be expressed as a polynomial fraction $\alpha = \beta/\gamma$, with $\beta \in \Omega K$ and $\gamma \in \theta$. Equivalently, $\Omega_\theta K$ is the set of all input/output stable elements in ΛK. It can be readily verified that $\Omega_\theta K$ is a ring. Moreover, the following is true

Proposition 2.4. *$\Omega_\theta K$ is a principal ideal domain.*

Proof. Let $A \subset \Omega_\theta K$ be a nonzero ideal, and let $A^+ = A \cap \Omega K$. Then, $A^+ \neq 0$, is an ideal in ΩK, and, since ΩK is a principal ideal domain, there exists an element $\alpha \in \Omega K$ such that $A^+ = \alpha[\Omega K]$. But then, we also have $A = \alpha[\Omega_\theta K]$ and A is a principal ideal. □

We note that Proposition 2.4 is a particular case of a much more general and well established result in localization theory, see Zariski and Samuel [1958].

The space ΛS is, of course, an $\Omega_\theta K$-module as well. We denote by $\Omega_\theta S$ the minimal $\Omega_\theta K$-submodule of ΛS containing S. Explicitly, let $s_1, \ldots, s_m$ be a basis of the K-linear space S. Then we have

$$\Omega_\theta S = \left\{ s \in \Lambda S : s = \sum_{i=1}^{m} \alpha_i s_i,\ \alpha_1, \ldots, \alpha_m \in \Omega_\theta K \right\}, \tag{2.5}$$

which is, of course, the same for every basis $s_1, \ldots, s_m \in S$. As a consequence, we have $\operatorname{rank}_{\Omega_\theta K} \Omega_\theta S = \dim_K S$.

We denote by

$$j_\theta : \Omega_\theta S \to \Lambda S \qquad \text{(canonical injection)}$$

the injection which maps each element in $\Omega_\theta S$ into the same element in ΛS. Since ΛS is evidently an $\Omega_\theta K$-module as well, we can define an $\Omega_\theta K$-module projection

$$\pi_\theta : \Lambda S \to \Lambda S/\Omega_\theta S \qquad \text{(canonical projection)}$$

which maps each element $s \in \Lambda S$ into the equivalence class $s + \Omega_\theta S$ in $\Lambda S/\Omega_\theta S$.

Returning now to the definition of input/output stability, we obtain that a ΛK-linear map $f\colon \Lambda U \to \Lambda Y$ is input/output stable if and only if all the entries in its transfer matrix belong to $\Omega_\theta K$. Equivalently, *f is input/output stable if and only if $f[\Omega_\theta U] \subset \Omega_\theta Y$.*

We shall also need to consider ΛK-linear maps which are both input/output stable and causal. For this purpose one defines an additional ring. Let $\Omega_\theta^- K := \Omega_\theta K \cap \Omega^- K$, that is, the set of all elements in ΛK which are both input/output stable and causal. By a direct calculation, it follows then that $\Omega_\theta^- K$ is again a ring. The following stronger result was proved by Morse [1975].

Proposition 2.6. *$\Omega_\theta^- K$ is a principal ideal domain.*

As before, $\Omega_\theta^- S := \Omega_\theta S \cap \Omega^- S$ is an $\Omega_\theta^- K$-module spanned by any basis of S, and we have $\operatorname{rank}_{\Omega_\theta^- K} \Omega_\theta^- S = \dim_K S$. A ΛK-linear map $f\colon \Lambda U \to \Lambda Y$ is both input/output stable and causal if and only if all entries in its transfer matrix are in $\Omega_\theta^- K$ or, equivalently, if and only if $f[\Omega_\theta^- U] \subset \Omega_\theta^- Y$.

Finally, let $\ell\colon \Lambda U \to \Lambda U$ be a ΛK-linear map. We shall say that ℓ is $\Omega^- K$- (respectively, ΩK-, $\Omega_\theta K$-, $\Omega_\theta^- K$-) *unimodular* if ℓ has an inverse ℓ^{-1} and if both of ℓ and ℓ^{-1} are causal (respectively, polynomial, input/output stable, input/output stable and causal). Thus an $\Omega^- K$-unimodular map is the bicausal map. An ΩK-unimodular map is the usual polynomial unimodular map. We also note that every $\Omega_\theta^- K$-unimodular map is necessarily bicausal.

3. Stability Representations

In the present section we extend the module theoretic approach to include stability theory, and we use the resulting framework to construct several types of matrix fraction representations of transfer matrices, each of which plays a different role in the theory of stability of linear time invariant systems. Our constructions will be in close analogy to the construction of polynomial matrix fraction representations in classical realization theory, and, in order to emphasize this connection, we start with a brief review of realization theory, following Kalman [1965], Kalman, Falb, and Arbib [1969], and Hautus and Heymann [1978]. Let $f\colon \Lambda U \to \Lambda Y$ be a rational ΛK-linear map. One associates with f a *restricted* map $\tilde{f}$ given by

$$\tilde{f} := \pi f j \colon \Omega U \to \Gamma Y. \tag{3.1}$$

Intuitively speaking, the map $\tilde{f}$ associates with each *past* input sequence the *future* part of the corresponding output sequence. The map $\tilde{f}$ is clearly an ΩK-homomorphism, and its kernel

$$\Delta := \ker \tilde{f} \subset \Omega U \tag{3.2}$$

is the classical Kalman [1965] realization module. This module consists of all past input sequences that lead to zero future output. Using the fact that f is rational, one can prove the following (see Fuhrmann [1976], Hautus and Heymann [1978]).

Lemma 3.3. *Let $f\colon \Lambda U \to \Lambda Y$ be a rational ΛK-linear map. Then,* $\ker f$ *contains a basis of the ΛK-linear space ΛU.*

Now, since ΩK is a principal ideal domain, the inclusion $\Delta \subset \Omega U$ implies that $\operatorname{rank}_{\Omega K}\Delta \leq \operatorname{rank}_{\Omega K}\Omega U = \dim_K U =: m$. Combining this fact with Lemma 3.3, we obtain that $\operatorname{rank}_{\Omega k}\Delta = m$, so that there exists a nonsingular polynomial matrix $D\colon \Lambda U \to \Lambda U$ such that

$$\Delta = D[\Omega U].$$

Letting $d_1,\ldots,d_m$ be the columns of D, and noting that $d_1,\ldots,d_m \in \Delta$, we have that $N_i := fd_i \in f[\Delta] \subset \Omega Y$, so that the matrix $N := [N_1,\ldots,N_m]$ is a polynomial matrix, and $N = fD$. Thus, we obtain a polynomial matrix fraction representation

$$f = ND^{-1},$$

where the denominator matrix D generates Δ. Moreover, in view of the following statement, this representation is *canonical* (see Hautus and Heymann [1978]).

Proposition 3.4. *Let $f\colon \Lambda U \to \Lambda Y$ be a linear input/output map, and let $f = ND^{-1}$, where $N\colon \Lambda U \to \Lambda Y$ and $D\colon \Lambda U \to \Lambda U$ are polynomial. Then, N and D are right coprime if and only if* $\ker \bar{f} = D[\Omega U]$.

An analog procedure can also be applied in the stability context, as we next discuss. First, we associate with the rational ΛK-linear map $f\colon \Lambda U \to \Lambda Y$ the *stability restriction* $\bar{f}^\sigma$ of f, given by

$$\bar{f}^\sigma := \pi_\theta f j_\theta \colon \Omega_\theta U \to \Lambda Y/\Omega_\theta Y. \tag{3.5}$$

The map $\bar{f}^\sigma$ is an $\Omega_\theta K$-homomorphism which assigns to each stable input sequence $u \in \Omega_\theta U$ the equivalence class $fu + \Omega_\theta Y$. Qualitatively speaking, one can decompose $fu = y_u + y_s$, where y_u is the unstable part of fu, and y_s is its stable part, and one clearly has then that $\bar{f}^\sigma u = fu + \Omega_\theta Y = y_u + \Omega_\theta Y$. Thus, $\bar{f}^\sigma$ can be regarded as a map that associates with each stable input sequence the unstable part of the corresponding output sequence. In particular, in case f is input/output stable, we have $f[\Omega_\theta U] \subset \Omega_\theta Y$ so that $\bar{f}^\sigma = 0$, and, conversely, if $\bar{f}^\sigma = 0$ then necessarily $f[\Omega_\theta U] \subset \Omega_\theta Y$ and f is input/output stable. This proves the following.

Lemma 3.6. *Let $f\colon \Lambda U \to \Lambda Y$ be a rational ΛK-linear map. Then, f is input/output stable if and only if $\bar{f}^\sigma = 0$ (or, equivalently, if and only if* $\ker \bar{f}^\sigma = \Omega_\theta U$*).*

Next, similarly to (3.2), we define the *stability realization module* Δ_θ^θ by

$$\Delta_\theta^\theta := \ker \bar{f}^\sigma \subset \Omega_\theta U. \tag{3.7}$$

It consists of all stable input sequences (to f) that lead to stable output sequences (from f). Clearly, Δ_θ^θ is an $\Omega_\theta K$-module, and, since $\Omega_\theta K$ is a principal ideal domain and $\Delta_\theta^\theta \subset \Omega_\theta U$, it follows that $\operatorname{rank}_{\Omega_\theta K}\Delta_\theta^\theta \leq \operatorname{rank}_{\Omega_\theta K}\Omega_\theta U = m$. Noting that $\ker \bar{f} \subset \Delta_\theta^\theta$, it follows by the last observation and Lemma 3.3 that $\operatorname{rank}_{\Omega_\theta K}\Delta_\theta^\theta = m$.

Thus, there exists a nonsingular matrix $D_\theta: \Lambda U \to \Lambda U$ such that

$$\Delta_\theta^\theta = D_\theta[\Omega_\theta U].$$

Now, since $\Delta_\theta^\theta \subset \Omega_\theta U$, we have that $D_\theta[\Omega_\theta U] \subset \Omega_\theta U$, so that D_θ is input/output stable. Furthermore, defining the ΛK-linear map

$$N_\theta := fD_\theta : \Lambda U \to \Lambda Y,$$

we have that $N_\theta[\Omega_\theta U] = fD_\theta[\Omega_\theta U] = f[\Delta_\theta^\theta] \subset f[\ker \pi_\theta f] \subset \Omega_\theta Y$, so that N_θ is input/output stable as well. Thus, we obtain a matrix fraction representation

$$f = N_\theta D_\theta^{-1}, \tag{3.8}$$

where both of N_θ and D_θ are input/output stable. This representation is distinguished by a canonical property, which we next examine.

Let $N: \Lambda U \to \Lambda Y$ and $D: \Lambda U \to \Lambda Y'$ be input/output stable maps, that is, all entries in the transfer matrices of N and D belong to the principal ideal domain $\Omega_\theta K$. As usual, an input/output stable map $R: \Lambda U \to \Lambda U$ is a *common right θ-divisor* of N and D if there exist input/output stable maps $N': \Lambda U \to \Lambda Y$ and $D': \Lambda U \to \Lambda Y'$ such that $N = N'R$ and $D = D'R$. The maps N and D are *right θ-coprime* if all their common right θ-divisors are $\Omega_\theta K$-unimodular. Equivalently, N and D are right θ-coprime if and only if there exist input/output stable maps $A: \Lambda Y \to \Lambda U$ and $B: \Lambda Y' \to \Lambda U$ such that $AN + BD = I$, the identity map (see MacDuffee [1934]).

We next show, in analogy to Proposition 3.4, that, in the representation $f = N_\theta D_\theta^{-1}$ of (3.8), the maps N_θ and D_θ are right θ-coprime. For this purpose we need the following.

Lemma 3.9. *Let $f: \Lambda U \to \Lambda Y$ be a rational ΛK-linear map, and assume that $f = ND^{-1}$, where $N: \Lambda U \to \Lambda Y$ and $D: \Lambda U \to \Lambda U$. Then, both of N and D are input/output stable if and only if $D[\Omega_\theta U] \subset \operatorname{Ker} \tilde{f}^\sigma$.*

Proof. Assume first that $D[\Omega_\theta U] \subset \operatorname{Ker} \tilde{f}^\sigma$. Then, evidently, $D[\Omega_\theta U] \subset \Omega_\theta U$, so that D is input/output stable. Also, $N[\Omega_\theta U] = f[D\Omega_\theta U] \subset f[\operatorname{Ker} \tilde{f}^\sigma] \subset \Omega_\theta Y$, and N is input/output stable as well. Conversely, assume that both of N and D are input/output stable, and let $\Delta := D[\Omega_\theta U]$. Then, since D is input/output stable, $\Delta \subset \Omega_\theta U$, and, since N is input/output stable, $f[\Delta] = N[\Omega_\theta U] \subset \Omega_\theta Y$, so that also $\Delta \subset \operatorname{Ker} \pi_\theta f$. Hence $\Delta \subset \operatorname{Ker} \pi_\theta f \cap \Omega_\theta U = \operatorname{Ker} \tilde{f}^\sigma$, concluding our proof. □

Theorem 3.10. *Let $f: \Lambda U \to \Lambda Y$ be a rational ΛK-linear map, and let $f = ND^{-1}$, where both of $N: \Lambda U \to \Lambda Y$ and $D: \Lambda U \to \Lambda Y$ are input/output stable. Then, N and D are right θ-coprime if and only if $D[\Omega_\theta U] = \operatorname{Ker} \tilde{f}^\sigma$.*

Proof. Assume first that $D[\Omega_\theta U] = \operatorname{Ker} \tilde{f}^\sigma$, and let R be a common right θ-divisor of N and D. Then, R is nonsingular, both of DR^{-1} and NR^{-1} are still input/output stable, and $f = (NR^{-1})(DR^{-1})^{-1}$. Hence, by Proposition 4.3,

$DR^{-1}[\Omega_\theta U] \subset \operatorname{Ker} \tilde{f}^\sigma = D[\Omega_\theta U]$, so that $R^{-1}[\Omega_\theta U] \subset \Omega_\theta U$, and R^{-1} is input/output stable. Thus, R is $\Omega_\theta K$-unimodular, and N, D are right θ-coprime. Conversely, assume that D and N are right θ-coprime. Now, by Lemma 3.9, $D[\Omega_\theta U] \subset \operatorname{Ker} \tilde{f}^\sigma$, and whence, letting $\operatorname{Ker} \tilde{f}^\sigma = D_\theta[\Omega_\theta U]$, there exists a nonsingular input/output stable map $R: \Lambda U \to \Lambda U$ such that $D = D_\theta R$. Then, $NR^{-1}[\Omega_\theta U] = fDR^{-1}[\Omega_\theta U] = f[\operatorname{Ker} \tilde{f}^\sigma] \subset \Omega_\theta Y$, so that $N_\theta := NR^{-1}$ is input/output stable, and $N = N_\theta R$. Thus, R is a common right θ-divisor of D and N, so that, by the coprimeness assumption, R is $\Omega_\theta K$-unimodular. But then $D[\Omega_\theta U] = D_\theta[\Omega_\theta U] = \operatorname{Ker} \tilde{f}^\sigma$, concluding our proof. □

Let $f: \Lambda U \to \Lambda Y$ be a rational ΛK-linear map, and let $f = ND^{-1}$, where N and D are input/output stable. If N and D are right θ-coprime, then we say that $f = ND^{-1}$ is a *canonical stability representation of* f *(in the sense of* θ*)*. We note that, if $f = ND^{-1}$ is a canonical stability representation, then, by Lemma 3.6 and Theorem 3.10, *f is input/output stable if and only if D is $\Omega_\theta K$-unimodular.*

Formula (3.7) is only one of several distinct possibilities to extend the notion of the realization module (3.2) to the stability framework. In order to explore the other possibilities, we need a deeper insight into the underlying system structure. Let $f: \Lambda U \to \Lambda Y$ be a rational ΛK-linear map, and consider the ΩK-module $\operatorname{Ker} \pi f \subset \Lambda U$. Intuitively speaking, this module consists of *all* the input sequences that lead to zero future output, and its properties were studied in detail in Hammer and Heymann [1983]. By definition, we have that

$$f[\operatorname{Ker} \pi f] = \operatorname{Im} f \cap \Omega Y. \tag{3.11}$$

The module $\operatorname{Ker} \pi f$ forms an extension of the classical Kalman realization module Δ of (3.2), and the relationship between the two is simply

$$\Delta = \operatorname{Ker} \pi f \cap \Omega U. \tag{3.12}$$

Similarly, we can consider the $\Omega_\theta K$-module $\operatorname{Ker} \pi_\theta f$, which consists of all input sequences that lead to stable output sequences. Then, the module Δ_θ^θ of (3.7) is given by

$$\Delta_\theta^\theta = \operatorname{Ker} \pi_\theta f \cap \Omega_\theta U \qquad (\subset \Omega_\theta U), \tag{3.13}$$

and, as expected, it is in close resemblance to (3.12). However, the expression (3.12) leads to the following two additional candidates for a central role in the stability context:

$$\Delta^\theta := \operatorname{Ker} \pi f \cap \Omega_\theta U \qquad (\subset \Omega_\theta U), \tag{3.14}$$

$$\Delta_\theta := \operatorname{Ker} \pi_\theta f \cap \Omega U \qquad (\subset \Omega U). \tag{3.15}$$

Being an intersection between ΩK- and $\Omega_\theta K$-modules, Δ^θ is clearly an ΩK-module, and it consists of all the stable input sequences that lead to polynomial output sequences. Similarly, Δ_θ is, again, an ΩK-module, and it consists of all the polynomial input sequences that lead to stable output sequences. Thus, Δ^θ and Δ_θ are, in a sense, duals of each other. It turns out, as we show below, that these two modules characterize the structure of the solution to the problem of nonsingular, causal, and stable precompensation. We proceed now to show that each one of Δ^{θ} and Δ_{θ} leads to its own particular type of matrix fraction representation of f.

4. Pole Representations

Consider first the ΩK-module Δ_{θ} of (3.15). This module is clearly contained in Δ_{θ}^{θ} of (3.13), and, as we shall see in a moment, it constitutes a minimal description of the unstability of f. By definition we have that $\operatorname{Ker}\tilde{f} \subset \Delta_{\theta} \subset \Omega U$, and, in view of Lemma 3.3 and the fact that ΩK is a principal ideal domain, these inclusions imply that $\operatorname{rank}_{\Omega K}\Delta_{\theta} = m$. Whence, there exists a nonsingular matrix $D_p: \Lambda U \to \Lambda U$ such that

$$\Delta_{\theta} = D_p[\Omega U],$$

and, since $\Delta_{\theta} \subset \Omega U$, the matrix D_p is polynomial. Define now the map

$$N_p := fD_p : \Lambda U \to \Lambda Y.$$

Then, since $D_p[\Omega U] = \Delta_{\theta} \subset \operatorname{Ker}\tilde{f}^{\sigma}$, and since $\operatorname{Ker}\tilde{f}^{\sigma}$ is an $\Omega_{\theta}K$-module, it follows that also $D_p[\Omega_{\theta}U] \subset \operatorname{Ker}\tilde{f}^{\sigma}$, and, thus, $N_p[\Omega_{\theta}U] = fD_p[\Omega_{\theta}U] \subset f[\operatorname{Ker}\tilde{f}^{\sigma}] \subset \Omega_{\theta}Y$, so that N_p is input/output stable. In this way we obtain the matrix fraction representation

$$f = N_p D_p^{-1}, \tag{4.1}$$

where the denominator is polynomial and the numerator is input/output stable. We call the representation (4.1) a *pole representation* of f. The matrix D_p is called a *pole matrix* of f, and, as we show later, it exactly characterizes the unstable poles of f. We start our study of pole representations with the following

Lemma 4.2. *A pole representation of a rational ΛK-linear map is a canonical stability representation.*

Proof. In view of Theorem 3.10, we have to show that $D_p[\Omega_{\theta}U] = \operatorname{Ker}\tilde{f}^{\sigma}$, and, since we have noted above that $D_p[\Omega_{\theta}U] \subset \operatorname{Ker}\tilde{f}^{\sigma}$, our proof will be complete upon showing that also $\operatorname{Ker}\tilde{f}^{\sigma} \subset D_p[\Omega_{\theta}U]$. Now, we have that $\operatorname{Ker}\tilde{f}^{\sigma} = D_{\theta}[\Omega_{\theta}U]$, and since D_{θ} is input/output stable, there exists an element ψ in the stability set θ such that ψD_{θ} is a polynomial matrix. But then, clearly, $\psi D_{\theta}[\Omega U] \subset \operatorname{Ker}\tilde{f}^{\sigma}$ and $\psi D_{\theta}[\Omega U] \subset \Omega U$, so that $\psi D_{\theta}[\Omega U] \subset \operatorname{Ker}\tilde{f}^{\sigma} \cap \Omega U = \Delta_{\theta} = D_p[\Omega U]$. Whence, also $\psi D_{\theta}[\Omega_{\theta}U] \subset D_p[\Omega_{\theta}U]$, and, since $\psi \in \theta$, $\psi^{-1}D_p[\Omega_{\theta}U] = D_p[\Omega_{\theta}U]$. Thus, we finally obtain that $\operatorname{Ker}\tilde{f}^{\sigma} = D_{\theta}[\Omega_{\theta}U] \subset \psi^{-1}D_p[\Omega_{\theta}U] = D_p[\Omega_{\theta}U]$, so that $\operatorname{Ker}\tilde{f}^{\sigma} \subset D_p[\Omega_{\theta}U]$, completing our proof. □

In the next several statements we give the system theoretic interpretation of the pole representation, namely, that the pole matrix D_p exactly characterizes the unstable poles of f. We start by showing that D_p has no stable (proper) divisors.

Proposition 4.3. *Let $f: \Lambda U \to \Lambda Y$ be a rational ΛK-linear map, and let D_p be a pole matrix of f. If $P: \Lambda U \to \Lambda U$ is any polynomial right divisor of D_p, then either P is polynomial unimodular, or P^{-1} is not input/output stable.*

Proof. Let P be a right polynomial divisor of D_p, and assume that P^{-1} is input/output stable. Our proof will conclude upon showing that P is unimodular. Now, let $D: \Lambda U \to \Lambda U$ be the polynomial matrix satisfying $D_p = DP$. Then, since P^{-1} is input/output stable, P is $\Omega_\theta K$-unimodular, so that

$$D[\Omega U] \subset D[\Omega_\theta U] = DP[\Omega_\theta U] = D_p[\Omega_\theta U] = \operatorname{Ker} \tilde{f}^\sigma,$$

where the last equality is by Lemma 4.2 and Theorem 3.10. But then, since D is polynomial, also $D[\Omega U] \subset \Omega U$, so that we have $D[\Omega U] \subset \operatorname{Ker} \tilde{f}^\sigma \cap \Omega U = D_p[\Omega U]$. Since P is polynomial, we evidently also have that $D_p[\Omega U] \subset D[\Omega U]$. Hence, $D[\Omega U] = D_p[\Omega U]$, so that $P[\Omega U] = \Omega U$, and P is polynomial unimodular, concluding our proof. □

In view of Proposition 4.3, it is convenient to distinguish a certain type of polynomial maps, as follows. Let $P: \Lambda U \to \Lambda Y$ be a polynomial map. We say that P is *completely unstable* if the (polynomial) invariant factors of the transfer matrix of P are polynomially coprime with every element in the stability set θ. By Proposition 4.3 we have that *every pole matrix D_p is completely unstable*. The converse of this statement is also true, in the following sense.

Proposition 4.4. *Let $f: \Lambda U \to \Lambda Y$ be a rational ΛK-linear map, and let $f = ND^{-1}$ be a canonical stability representation, with D a polynomial matrix. Then, $f = ND^{-1}$ is a pole representation if and only if D is completely unstable.*

Proof. In view of our previous discussion, it only remains to prove the "if" direction. Assume then that $f = ND^{-1}$ is a canonical stability representation, and that D is a completely unstable polynomial matrix. Then, since D is polynomial, $D[\Omega U] \subset \Omega U$, and by Lemma 3.9 also $D[\Omega U] \subset \operatorname{Ker} \tilde{f}^\sigma$, so that $D[\Omega U] \subset \operatorname{Ker} \tilde{f}^\sigma \cap \Omega U = \Delta_\theta$. Now, let D_p be a pole matrix of f. Then, $D_p[\Omega U] = \Delta_\theta$, so that, since $D[\Omega U] \subset \Delta_\theta$, there exists a polynomial matrix $R: \Lambda U \to \Lambda U$ such that $D = D_p R$. By Theorem 3.10 and Lemma 4.2 we further have that $D[\Omega_\theta U] = \operatorname{Ker} \tilde{f}^\sigma$ and $D_p[\Omega_\theta U] = \operatorname{Ker} \tilde{f}^\sigma$. Whence, $D_p[\Omega_\theta U] = D_p R[\Omega_\theta U]$, so that R is $\Omega_\theta K$-unimodular. But then, R^{-1} is input/output stable, and, since R is a divisor of the completely unstable D, it follows that R is polynomial unimodular. Thus, $D[\Omega U] = D_p[\Omega U] = \Delta_\theta$, and ND^{-1} is a pole representation. □

The pole matrix can also be characterized as the minimal possible *polynomial* denominator of a stability representation, as follows.

Theorem 4.5. *Let $f: \Lambda U \to \Lambda Y$ be a rational ΛK-linear map, and let $f = ND^{-1}$ be a fraction representation, where $N: \Lambda U \to \Lambda Y$ is input/output stable and $D: \Lambda U \to \Lambda U$ is polynomial. Then, the following are equivalent:*

(i) *$f = ND^{-1}$ is a pole representation.*

(ii) *For every fraction representation $f = AB^{-1}$, where $A: \Lambda U \to \Lambda Y$ is input/output stable and $B: \Lambda U \to \Lambda U$ is polynomial, D is a left polynomial divisor of B.*

Proof. (i) → (ii): Since $f = ND^{-1}$ is a pole representation, $D[\Omega U] = \operatorname{Ker} \pi_\theta f \cap \Omega U =: \Delta_\theta$. Also, since B is polynomial, $B[\Omega U] \subset \Omega U$, and, since both of A and B are input/output stable, we have by Lemma 3.9 that $B[\Omega_\theta U] \subset \operatorname{Ker} \tilde{f}^\sigma$. Consequently, $B[\Omega U] \subset \operatorname{Ker} \tilde{f}^\sigma \cap \Omega U = \Delta_\theta = D[\Omega U]$, and, whence, there exists a polynomial matrix R such that $B = DR$.

(ii) → (i): Let $f = N_p D_p^{-1}$ be a pole representation of f. Then, by (ii), D is a left polynomial divisor of D_p. On the other hand, it follows by the previous part of the proof that D_p is a left polynomial divisor of D as well. Thus, there exists a polynomial unimodular matrix $M: \Lambda U \to \Lambda U$ such that $D = D_p M$, so that $D[\Omega U] = D_p[\Omega U] = \Delta_\theta$, and $f = ND^{-1}$ is a pole representation. □

Consider a pole representation $f = N_p D_p^{-1}$. Now, since N_p is input/output stable, all the unstable poles of f originate from D. Also, by Proposition 4.4, D causes only unstable poles, and, in view of Theorem 4.5, all these poles are actual poles of f. Thus, the pole matrix D_p, or, equivalently, the ΩK-module $\Delta_\theta = D_p[\Omega U]$, gives a minimal description of the unstability of f. The map f is input/output stable if and only if D_p is polynomial unimodular, or, equivalently, if and only if $\Delta_\theta = \Omega U$. Before turning to our next topic, it is worthwhile to note the following property of completely unstable matrices, which can be readily verified.

Lemma 4.6. *Let $P: \Lambda U \to \Lambda U$ and $Q: \Lambda U \to \Lambda Y$ be polynomial maps, where P is nonsingular and completely unstable. If $P^{-1}Q$ is input/output stable, then it is a polynomial map.*

Remark 4.7. *Explicit construction of pole representations*: In view of our previous discussion, a pole representation can be explicitly constructed as follows. Let $f = ND^{-1}$ be a polynomial matrix fraction representation, where $N: \Lambda U \to \Lambda Y$ and $D: \Lambda U \to \Lambda U$ are right coprime polynomial matrices. Further, let $M_1, M_2: \Lambda U \to \Lambda U$ be polynomial unimodular matrices such that $\delta := M_1 D M_2$ is in Smith canonical form, say $\delta = \operatorname{diag}(\delta_1, \dots, \delta_m)$. We now factor $\delta_i = \delta_i' \delta_i''$, $i = 1, \dots, m$, where $\delta_i'^{-1}$ is input/output stable, and δ_i'' is coprime with every element of the stability set θ. Then, we define the polynomial matrices $D_p := M_1[\operatorname{diag}(\delta_1'', \dots, \delta_m'')]$ and $D_1 := [\operatorname{diag}(\delta_1', \dots, \delta_m')] M_2$, (so that $D = D_p D_1$), and the input/output stable matrix $N_p := N D_1^{-1}$. Then, it can be seen that $f = N_p D_p^{-1}$ is a pole representation of f. The factorization $D = D_p D_1$ is actually a somewhat weaker form of the classical left standard factorization of D (see Gokhberg and Krein [1960], and Youla [1961]).

It is intuitively clear that, when stabilizing a system by stable precompensation, one has to pay attention only to the denominator of a canonical stability representation, whereas the numerator has no effect. This is the contents of the following.

Proposition 4.8. *Let $f: \Lambda U \to \Lambda Y$ be a rational ΛK-linear map, and let $f = ND^{-1}$ be a canonical stability representation. Also, let $\mathscr{L}: \Lambda U \to \Lambda U$ be a nonsingular input/output stable ΛK-linear map. Then, $f\mathscr{L}$ is input/output stable if and only if $D^{-1}\mathscr{L}$ is input/output stable.*

Proof. If $D^{-1}\ell$ is input/output stable, then, since N is input/output stable, so is also $f\ell$. Conversely, assume that $f\ell$ is input/output stable. Now, by definition, $\operatorname{Ker}\pi_\theta D^{-1} = D[\Omega_\theta U] = \operatorname{Ker}\tilde{f}^\sigma$, and, also, since ℓ is input/output stable, $\ell^{-1}[\Omega_\theta U] \supset \Omega_\theta U$. Hence, $\operatorname{Ker}\pi_\theta D^{-1}\ell = \ell^{-1}[\operatorname{Ker}\pi_\theta D^{-1}] = \ell^{-1}[\operatorname{Ker}\pi_\theta f \cap \Omega_\theta U]$ $= \ell^{-1}[\operatorname{Ker}\pi_\theta f] \cap \ell^{-1}[\Omega_\theta U] \supset \operatorname{Ker}\pi_\theta f\ell \cap \Omega_\theta U = \Omega_\theta U$, where the last step is by the input/output stability of $f\ell$. Thus, $\operatorname{Ker}\pi_\theta D^{-1}\ell \supset \Omega_\theta U$, or $D^{-1}\ell[\Omega_\theta U] \subset \Omega_\theta U$, and $D^{-1}\ell$ is input/output stable. □

5. Zero Representations

We turn now to an examination of the ΩK-module $\Delta^\theta = \operatorname{Ker}\pi f \cap \Omega_\theta U$ of (3.14). We show that this module is "dual" to Δ_θ in the sense that it leads to a characterization of the unstable *zeros* of f, whereas Δ_θ characterizes, as we have seen, the unstable poles. Since, Δ^θ originates from the ΩK-module $\operatorname{Ker}\pi f$, we have to review a few facts related to the latter. Let $f\colon \Lambda U \to \Lambda Y$ be a rational ΛK-linear map. It is readily seen that $\operatorname{Ker} f \subset \operatorname{Ker}\pi f$, and, since $\operatorname{Ker} f$ is a ΛK-linear space, it follows that, when $\operatorname{Ker} f \neq 0$, the ΩK-module $\operatorname{Ker}\pi f$ is not finitely generated. However, when $\operatorname{Ker} f = 0$, it is finitely generated, as stated in the following (Hammer and Heymann [1983]).

Lemma 5.1. *Let $f\colon \Lambda U \to \Lambda Y$ be a rational ΛK-linear map. Then, there exists a nonsingular map $D\colon \Lambda U \to \Lambda U$ such that* $\operatorname{Ker}\pi f = D[\Omega U]$ *if and only if f is injective.*

Thus, it will be more convenient to start our discussion with the injective case. Assume then that $f\colon \Lambda U \to \Lambda Y$ is an injective rational ΛK-linear map. Now, since clearly $\operatorname{Ker}\tilde{f}(=\operatorname{Ker}\pi f \cap \Omega U) \subset \Delta^\theta$, we have the inclusions $\operatorname{Ker}\tilde{f} \subset \Delta^\theta \subset \operatorname{Ker}\pi f$ so that, in view of Lemmas 3.3, 5.1, and the fact that ΩK is a principal ideal domain, it follows that $\operatorname{rank}_{\Omega K}\Delta^\theta = m$. Whence, there exists a nonsingular matrix $D_0\colon \Lambda U \to \Lambda U$ such that

$$\Delta^\theta = D_0[\Omega U],$$

and, since $\Delta^\theta \subset \Omega_\theta U$, we obtain that D_0 is input/output stable. As we have done before, we define the map

$$N_0 := fD_0\colon \Lambda U \to \Lambda Y,$$

and, since $N_0[\Omega U] = fD_0[\Omega U] \subset f[\operatorname{Ker}\pi f] \subset \Omega Y$, it follows that N_0 is a *polynomial* map. Thus, we obtain a matrix fraction representation

$$f = N_0 D_0^{-1}, \tag{5.2}$$

in which the numerator is polynomial and the denominator is input/output stable. We call this fraction representation a *zero representation* of f. The matrix N_0 is called a *zeros matrix* of f, and we shall see in a few moments that it exactly characterizes the unstable zeros of f. First, we have to show that the zero representation is a canonical stability representation.

Lemma 5.3. *Let $f: \Lambda U \to \Lambda Y$ be a rational ΛK-linear map. Then,* (i) $\Delta^\theta \subset \operatorname{Ker} \tilde{f}^\sigma$, *and* (ii) $\operatorname{Ker} \tilde{f}^\sigma$ *is the minimal $\Omega_\theta K$-module containing the ΩK-module Δ^θ.*

Proof. (i) follows by the evident inclusion $\operatorname{Ker} \pi f \subset \operatorname{Ker} \pi_\theta f$, since $\Delta^\theta = \operatorname{Ker} \pi f \cap \Omega_\theta U \subset \operatorname{Ker} \pi_\theta f \cap \Omega_\theta U = \operatorname{Ker} \tilde{f}^\sigma$, so we turn to (ii). Let $\Delta \subset \Lambda U$ be an $\Omega_\theta K$-module, and assume that $\Delta^\theta \subset \Delta$. We have to show that then also $\operatorname{Ker} \tilde{f}^\sigma \subset \Delta$. Now, let $u \in \operatorname{Ker} \tilde{f}^\sigma (\subset \Omega_\theta U)$ be any element. Then, since $fu \in \Omega_\theta Y$, there exists an element $\Psi \in \theta$ such that $\Psi fu \in \Omega Y$. But then, $\Psi u \in \operatorname{Ker} \pi f$, and since also $\Psi u \in \Omega_\theta U$, we have $\Psi u \in \Delta^\theta \subset \Delta$. Whence, since Ψ is invertible in $\Omega_\theta K$, it follows that $u \in \Delta$, so that $\operatorname{Ker} \tilde{f}^\sigma \subset \Delta$, and our proof concludes. □

Now, considering the zero representation $f = N_0 D_0^{-1}$, it follows by Lemma 5.3 that $D_0[\Omega_\theta U] = \operatorname{Ker} \tilde{f}^\sigma$, so that, applying Theorem 3.10, we obtain that indeed the *zero representation is a canonical stability representation.*

The system theoretic interpretation of the zero representation is made explicit in Theorem 5.5, where we show that the zero matrix N_0 is a minimal numerator for a stable-matrix fraction representation of f. It is worthwhile to note the duality between Theorems 5.5 and 4.5. Before turning to these issues, we have to reproduce from Hammer and Heymann [1983] the following polynomial factorization theorem, which characterizes the algebraic role of $\operatorname{Ker} \pi f$.

Theorem 5.4. *Let $f, f': \Lambda U \to \Lambda Y$ be rational ΛK-linear maps. Then, the following hold:* (i) *There exists a polynomial map $P: \Lambda Y \to \Lambda Y$ such that $f' = Pf$ if and only if* $\operatorname{Ker} \pi f \subset \operatorname{Ker} \pi f'$. (ii) *There exists a polynomial unimodular map $M: \Lambda Y \to \Lambda Y$ such that $f' = Mf$ if and only if* $\operatorname{Ker} \pi f = \operatorname{Ker} \pi f'$.

Theorem 5.5. *Let $f: \Lambda U \to \Lambda Y$ be a rational ΛK-linear map, and let $f = ND^{-1}$ be a fraction representation, where $N: \Lambda U \to \Lambda Y$ is polynomial and $D: \Lambda U \to \Lambda U$ is input/output stable. Then, $f = ND^{-1}$ is a zero representation if and only if the following holds: For every fraction representation $f = N'D'^{-1}$, where $N': \Lambda U \to \Lambda Y$ is polynomial and $D': \Lambda U \to \Lambda U$ is input/output stable, N is a left polynomial divisor of N'.*

Proof. Assume first that $f = ND^{-1}$ is a zero representation, and let $f = N'D'^{-1}$ be any fraction representation, where $N': \Lambda U \to \Lambda Y$ is polynomial and $D': \Lambda U \to \Lambda U$ is input/output stable. Then, since N' is polynomial, we have by Theorem 5.4 that $\operatorname{Ker} \pi D'^{-1} \subset \operatorname{Ker} \pi f$, so that $D'[\Omega U]$ $(= D'[\operatorname{Ker} \pi] = \operatorname{Ker} \pi D'^{-1}) \subset \operatorname{Ker} \pi f$. Also, since D' is input/output stable, $D'[\Omega U] (\subset D'[\Omega_\theta U]) \subset \Omega_\theta U$, and, whence, we obtain that $D'[\Omega U] \subset \operatorname{Ker} \pi f \cap \Omega_\theta U = \Delta^\theta = D[\Omega U]$. Thus, there exists a polynomial map $R: \Lambda U \to \Lambda U$ such that $D' = DR$, and, by injectivity, $N' = NR$.

Conversely, assume that for every representation $f = N'D'^{-1}$, where N' is polynomial and D' is input/output stable, N is a left polynomial divisor of N'. In particular, choosing $f = N'D'^{-1}$ as a zero representation, it follows by the previous part of the proof that N' is a left polynomial divisor of N as well. By injectivity, there exists then a polynomial unimodular map $M: \Lambda U \to \Lambda U$ such that $N = N'M$ and $D = D'M$, so that $D[\Omega U] = D'[\Omega U]$. But then, since $f = N'D'^{-1}$ is a zero representation, we obtain that $D[\Omega U] = D'[\Omega U] = \Delta^\theta$, so that $f = ND^{-1}$ is a zero representation as well and our proof concludes. □

In further analogy to the pole representation, the zero representation also has the following properties.

Proposition 5.6. *Let $f: \Lambda U \to \Lambda Y$ be an injective rational ΛK-linear map, let $f = ND^{-1}$ be a zero representation of f, and let $R: \Lambda U \to \Lambda U$ be a nonsingular polynomial map. If R is a right divisor of N, then either R is unimodular, or R^{-1} is not input/output stable.*

Proof. Assume that R is a right polynomial divisor of N, and that R^{-1} is input/output stable. Our proof will conclude upon showing that then R is necessarily polynomial unimodular. Defining $N' := NR^{-1}$ and $D' := DR^{-1}$, we have that $f = N'D'^{-1}$, the map N' is polynomial, and D' is input/output stable. But then, since $f = ND^{-1}$ is a zero representation, it follows by Theorem 5.5 that N is a left polynomial divisor of N'. By injectivity, this implies that R^{-1} is polynomial, so that R is indeed polynomial unimodular. □

Corollary 5.7. *Let $f: \Lambda U \to \Lambda Y$ be an injective rational ΛK-linear map, and let $f = ND^{-1}$ be a canonical stability representation, where $N: \Lambda U \to \Lambda Y$ is polynomial (and $D: \Lambda U \to \Lambda U$ is input/output stable). Then, $f = ND^{-1}$ is a zero representation if and only if N is completely unstable.*

In the next statement we show that the zero representation is related to the factorization of a polynomial map into a multiple of polynomial maps, in which one factor is completely unstable and the other factor has a stable inverse. Thus, zero representations are related to the classical problem of spectral factorization of matrices as considered in Gokhberg and Krein [1960], and Youla [1961].

Proposition 5.8. *Let $f: \Lambda U \to \Lambda Y$ be an injective polynomial map, and let $f = ND^{-1}$ be a zero representation of f. Then, D^{-1} is a polynomial map.*

Proof. By assumption, f is polynomial, so that $\Omega U \subset \operatorname{Ker} \pi f$, and, whence, $\Omega U \subset \operatorname{Ker} \pi f \cap \Omega_\theta U = \Delta^\theta$. But then $\Omega U \subset D[\Omega U]$, so that $D^{-1}[\Omega U] \subset \Omega U$, and D^{-1} is polynomial. □

Remark 5.9. *Explicit calculation of zero representations.* Let $f: \Lambda U \to \Lambda Y$ be an injective rational ΛK-linear map, and let $f = PQ^{-1}$ be a right coprime polynomial matrix fraction representation of f. Let $P = NR$ be a spectral factorization of P into a multiple of polynomial matrices, where $N: \Lambda U \to \Lambda Y$ is completely unstable, and where $R: \Lambda U \to \Lambda U$ is nonsingular and has a stable inverse. Such a factorization can be obtained through an application of the Smith canonical form theorem to P, and a suitable factorization of the invariant factors. It can then be readily seen that N is a zeros matrix of f, and that $f = ND^{-1}$, where $D := QR^{-1}$, is a zero representation of f. The factorization $P = NR$ is actually a somewhat weaker form of the classical left standard factorization of P (see Gokhberg and Krein [1960] and Youla [1961]). This factorization was applied to the numerator matrix P also in Pernebo [1980], where N was called a "left structure matrix". Finally we note that $\Delta^\theta = D[\Omega U]$, and that any canonical stability representation of f is then of the form $f = AB^{-1}$, where $A = N\mathscr{L}$, $B = D\mathscr{L}$, and $\mathscr{L}$: $\Lambda U \to \Lambda U$ is an arbitrary $\Omega_\theta K$-unimodular map. □

Before proceeding with our discussion of stability representations, we examine the effect of the ΛK-linear space $\operatorname{Ker} f$ on such representations. In particular, we show that the above discussion on zero representations remains almost unchanged when the injectivity assumption is released. Let f: $\Lambda U \to \Lambda Y$ be a rational ΛK-linear map. We say that f has a *static kernel* whenever $\operatorname{Ker} f = \Lambda U_0$, where $U_0 \subset U$ is a K-linear subspace (Hammer and Heymann [1981]). Now, let f: $\Lambda U \to \Lambda Y$ be any rational ΛK-linear map, let $n := \dim_{\Lambda K} \operatorname{Ker} f$, and assume that $n < m$ $(= \dim_K U)$ (that is, $f \neq 0$). By rationality, it follows that there exists a nonzero element $\Psi \in \Omega_\theta^- K$ such that the transfer matrix of Ψf has all its entries in $\Omega_\theta^- K$. Since $\Omega_\theta^- K$ is a principal ideal domain, it follows (see e.g. MacDuffee [1934]) that there exists a $\Omega_\theta^- K$-unimodular map $\mathscr{L}$: $\Lambda U \to \Lambda U$ such that $(\Psi f)\mathscr{L} = (\Psi f_0, 0)$, where $f_0 : \Lambda K^{m-n} \to \Lambda Y$ is injective (and 0 denotes the $p \times n$ zero matrix). Cancelling out Ψ, we obtain that $f\mathscr{L}$ has a static kernel. This proves

Lemma 5.10. *Let f: $\Lambda U \to \Lambda Y$ be a rational ΛK-linear map. Then, there exists an $\Omega_\theta^- K$-unimodular map $\mathscr{L}$: $\Lambda U \to \Lambda U$ such that $f\mathscr{L}$ has a static kernel.*

We continue our examination of the representation $f\mathscr{L} = (f_0, 0)$ derived above. In particular, we show that the canonical stability representation of f is essentially determined by such a representation of the injective map f_0, and the dimension of $\operatorname{Ker} f$ (i.e., n). This observation will then allow us to restrict our attention to the case of injective maps.

Let $U_0, U_1 \subset U$ be K-linear subspaces such that $\operatorname{Ker} f\mathscr{L} = \Lambda U_0$ and ΛU_1 is the domain of f_0. Then, evidently, $U_0 \oplus U_1 = U$ is a direct sum, we have

$$\operatorname{Ker} \pi_\theta(f\mathscr{L}) = \operatorname{Ker} \pi_\theta f_0 \oplus \Lambda U_0, \tag{5.11}$$

and, in view of (2.5),

$$\operatorname{Ker} \pi_\theta f\mathscr{L} \cap \Omega_\theta U = \operatorname{Ker} \tilde{f}_0^\sigma \oplus \Omega_\theta U_0. \tag{5.12}$$

Also, since $\mathscr{L}$ is clearly $\Omega_\theta K$-unimodular as well, we have

$$\mathscr{L}[\operatorname{Ker} \pi_\theta(f\mathscr{L}) \cap \Omega_\theta U] = \mathscr{L}[\operatorname{Ker} \pi_\theta f\mathscr{L}] \cap \mathscr{L}[\Omega_\theta U] = \operatorname{Ker} \pi_\theta f \cap \Omega_\theta U. \tag{5.13}$$

Thus, applying $\mathscr{L}$ to both sides of (5.12), we obtain

$$\operatorname{Ker} \tilde{f}^\sigma = \mathscr{L}[\operatorname{Ker} \tilde{f}_0^\sigma \oplus \Omega_\theta U_0]. \tag{5.14}$$

Turning now to matrix representations, let $f_0 = N_0 D_0^{-1}$ be a canonical stability representation of f_0. Also, let $D_1 : \Lambda U_0 \to \Lambda U_0$ (where $\dim_K U_0 = \dim_{\Lambda K} \operatorname{Ker} f$) be any $\Omega_\theta K$-unimodular map, and let $D_* : \Lambda U_0 \to \Lambda U_1$ be any input/output stable map. Then, using (5.14), it can be readily verified that a canonical stability representation of f is given by

$$f = (N_0,0)(\ell D)^{-1}, \quad \text{where } D = \begin{bmatrix} D_0 & 0 \\ D_* & D_1 \end{bmatrix}. \tag{5.15}$$

Any other stability representation $f = N_x D_x^{-1}$ of f satisfies $N_x = (N_0,0)\ell'$ and $D_x = \ell D \ell'$, where $\ell' : \Lambda U \to \Lambda U$ is $\Omega_\theta K$-unimodular. Consequently, the injective map f_0 and the dimension of $\operatorname{Ker} f$ essentially determine the canonical stability structure of f.

In particular, assume that $f_0 = N_0 D_0^{-1}$ is a *zero* representation of f_0. We then define the representation constructed in (5.15) as a *zero representation of* f, thus generalizing our previous definition to the noninjective case. One can then readily verify that *all properties of the zero representation mentioned in our earlier discussion of the injective case, continue to hold in the noninjective case as well.*

In the study of zero representations, it is convenient to employ a certain type of maps, which we next introduce. Let $f: \Lambda U \to \Lambda Y$ be a rational ΛK-linear map. We say that f is *θ-invertible* if there exists an input/output stable map (in the sense of θ) $f' : \Lambda Y \to \Lambda U$ such $f'f = I$ (the identity). Evidently, every θ-invertible map is injective, but, of course, not every injective map is θ-invertible. We shall need a more explicit characterization of θ-invertible maps, but before stating it, we state the following, which can be easily verified.

Lemma 5.16. *Let $f: \Lambda U \to \Lambda Y$ be a rational ΛK-linear map. Then, f is input/output stable if and only if $f[\Omega U] \subset \Omega_\theta Y$.*

Proposition 5.17. *Let $f: \Lambda U \to \Lambda Y$ be a rational ΛK-linear map. Then, f is θ-invertible if and only if* $\operatorname{Ker} \pi f \subset \Omega_\theta U$.

Proof. First, assume that f is θ-invertible, and let $f' : \Lambda Y \to \Lambda U$ be an input/output stable left inverse of f. Then, since $f[\operatorname{Ker} \pi_\theta f] \subset \Omega_\theta Y$ and $f'[\Omega_\theta Y] \subset \Omega_\theta U$, it follows that $f'f[\operatorname{Ker} \pi_\theta f] \subset \Omega_\theta U$, so that $\operatorname{Ker} \pi_\theta f \subset \Omega_\theta U$. Whence, since clearly $\operatorname{Ker} \pi f \subset \operatorname{Ker} \pi_\theta f$, we have $\operatorname{Ker} \pi f \subset \Omega_\theta U$.

Conversely, we assume that $\operatorname{Ker} \pi f \subset \Omega_\theta U$. Then, $\operatorname{Ker} \pi f$ contains no nonzero ΛK-linear subspace, and hence, since $\operatorname{Ker} f \subset \operatorname{Ker} \pi f$, it follows that $\operatorname{Ker} f = 0$, so that, by Lemma 5.1, there exists a nonsingular map $D: \Lambda U \to \Lambda U$ such that $\operatorname{Ker} \pi f = D[\Omega U]$. Then, since $\operatorname{Ker} \pi f \subset \Omega_\theta U$, also $D[\Omega U] \subset \Omega_\theta U$, and Lemma 5.16 implies that D is input/output stable. Thus, there exists an element $\Psi \in \theta$ such that ΨD is polynomial. But then, $\Psi[\operatorname{Ker} \pi f] \subset \Omega U$, so that $\operatorname{Ker} \pi f \Psi^{-1} \subset \Omega U = \operatorname{Ker} \pi I$, and Theorem 5.4 guarantees the existence of a polynomial map $P: \Lambda U \to \Lambda U$ such that $(P\Psi^{-1})f = I$. Recalling that $\Psi \in \theta$, we have that $P\Psi^{-1}$ is input/output stable, so that f is θ-invertible. □

In the above proof, we showed that if f is θ-invertible, then $\operatorname{Ker} \pi_\theta f \subset \Omega_\theta U$. Conversely, if $\operatorname{Ker} \pi_\theta f \subset \Omega_\theta U$, then also $\operatorname{Ker} \pi f (\subset \operatorname{Ker} \pi_\theta f) \subset \Omega_\theta U$, so that, by Proposition 5.17, f is θ-invertible. Thus, we obtained the following

Corollary 5.18. *Let $f: \Lambda U \to \Lambda Y$ be a rational ΛK-linear map. Then, f is θ-invertible if and only if* $\operatorname{Ker} \pi_\theta f \subset \Omega_\theta U$.

Corollary 5.18 indicates a very close analogy between θ-invertible maps and the concept of strictly observable maps of Hammer and Heymann [1983]. A rational ΛK-linear map $f: \Lambda U \to \Lambda Y$ is called *strictly observable* whenever $\operatorname{Ker} \pi f \subset \Omega U$, and it can be shown that f is strictly observable if and only if there exists a polynomial map $P: \Lambda Y \to \Lambda U$ such that $Pf = I$. Thus, in our present stability framework, the θ-invertible map replaces the strictly observable map of the polynomial framework of Hammer and Heymann [1983]. Evidently, a strictly observable map is θ-invertible as well, for every stability set θ.

Using θ-invertible maps, we can give an alternative interpretation of stability representations. Let $f: \Lambda U \to \Lambda Y$ be a rational ΛK-linear map, and let $f = ND^{-1}$ be a stability representation of f. Defining $g := D^{-1}$, we clearly have that g is θ-invertible and $f = Ng$. Thus, the stability representation is, in particular, a factorization $f = Ng$, where N is input/output stable and g is θ-invertible. In case N is a completely unstable polynomial map (and g is θ-invertible), then $f = Ng$ is a zero representation. Also, if g is strictly observable and completely unstable (and N is input/output stable), then $f = Ng$ is a pole representation of f.

6. The Pole Indices

Our discussion in the remaining part of this paper strongly depends on the notion of proper bases, which we next review. Let S be a K-linear space, and let $s_1, \ldots, s_n \in \Lambda S$ be a set of elements. We say that $s_1, \ldots, s_n$ are *properly independent* if their leading coefficients $\hat{s}_1, \ldots, \hat{s}_n \in S$ are K-linearly independent (see Wedderburn [1936], Wolovich [1974], Forney [1975], Hautus and Heymann [1978]). A basis consisting of properly independent elements is called a *proper basis*. A proper basis $d_1, \ldots, d_n \in \Lambda S$ satisfying $\operatorname{ord} d_i \geq \operatorname{ord} d_{i+1}$ for all $i = 1, \ldots, n-1$ is said to be *ordered*. In the ΛK-linear context, proper bases behave similarly to usual linear bases, as evidenced by the following statement (reproduced from Hammer and Heymann [1981]).

Theorem 6.1. *Let $W \subset \Lambda S$ be a nonzero ΛK-linear subspace. Then,* (i) *W has a proper basis, and* (ii) *every properly independent subset of W can be extended into a proper basis of W.*

In linear system theory, applications of proper bases are mostly in relation to causality, involving their following two properties (see Wolovich [1974]; for proofs in the present framework see Hammer and Heymann [1983]).

Theorem 6.2. *Let $f: \Lambda U \to \Lambda Y$ be a ΛK-linear map and let $u_1, \ldots, u_m$ be a proper basis of ΛU. Then, f is causal if and only if, for all $i = 1, \ldots, m$,* $\operatorname{ord} fu_i \geq \operatorname{ord} u_i$.

Proposition 6.3. *Let $\ell: \Lambda U \to \Lambda U$ be a ΛK-linear map, and let $u_1, \ldots, u_m$ be a proper basis of ΛU. Then, ℓ is bicausal if and only if* (i) *$\ell u_1, \ldots, \ell u_m$ form a proper basis of ΛU, and* (ii) $\operatorname{ord} \ell u_i = \operatorname{ord} u_i$, $i = 1, \ldots, m$.

Proper bases also play an important role in the structure of modules over polynomial rings (see Forney [1975], Hautus and Heymann [1978]). We next review this topic from Hammer and Heymann [1983]. Let $\Delta \subset \Lambda S$ be an ΩK-module. For every integer k, we denote by S_k the K-linear subspace (of S) spanned by the leading coefficients of all elements $s \in \Delta$ satisfying $\operatorname{ord} s \geq k$. We thus obtain a chain $\ldots \supset S_{-1} \supset S_0 \supset S_1 \supset \ldots$ of K-linear subspaces of S, called the *order chain* of Δ. We also denote by $\hat{\Delta} \subset S$ the K-linear subspace spanned by the leading coefficients of *all* elements in Δ, and call it the *leading subspace* of Δ. Finally, the set of integers $\eta_k := \dim_K S_k$, $k = \ldots, -1, 0, 1, \ldots$ is called the *order list* of Δ.

Further, let $\Delta \subset \Lambda S$ be an ΩK-module. We say that Δ is *bounded* if there exists an integer b such that, for every nonzero $s \in \Delta$, $\operatorname{ord} s \leq b$. One example of bounded modules is, of course, the case when $\Delta \subset \Omega S$. Another example, which we shall frequently use, is given by the following (see Hammer and Heymann [1983]).

Lemma 6.4. *Let $f\colon \Lambda U \to \Lambda Y$ be a ΛK-linear map. Then, f is injective if and only if* $\operatorname{Ker} \pi f$ *is bounded.*

For a nonzero and bounded ΩK-module $\Delta \subset \Lambda S$, we define the *degree indices* $\mu_1, \ldots, \mu_n$ of Δ as follows: Let $\{\eta_k\}$ be the order list of Δ. Then for every integer j satisfying $\eta_i \leq j < \eta_{i-1}$, the degree index $\mu_j := -i$ (see also Hautus and Heymann [1978]). The following statement (reproduced from Hammer and Heymann [1983]) shows that the degree indices are actually the degrees of the elements of an ordered proper basis of Δ.

Theorem 6.5. *Let $\Delta \subset \Lambda S$ be a nonzero and bounded ΩK-module with degree indices $\mu_1, \ldots, \mu_n$. Then, the following hold*: (i) *Δ has an ordered proper basis.* (ii) *If $d_1, \ldots, d_n$ is any ordered proper basis of Δ, then* $\operatorname{ord} d_i = -\mu_i$, $i = 1, \ldots, n$. (iii) $\operatorname{rank}_{\Omega K} \Delta = \dim_K \hat{\Delta}$, *where $\hat{\Delta}$ is the leading subspace of Δ.*

The notion of degree indices is closely related to the notion of reachability indices, which was extensively treated in the system theoretic literature (see Brunovski [1970], Rosenbrock [1970], Kalman [1971], Münzner and Prätzel-Wolters [1978]). The relationship between these two notions is the following (see Hautus and Heymann [1978]). Let $f\colon \Lambda U \to \Lambda Y$ be a rational ΛK-linear map. *The reachability indices of (a canonical realization of) f are the degree indices of the Kalman realization module* $\operatorname{Ker} \tilde{f}$. We are now in a position to define the following set of invariants.

Definition 6.6. Let $f\colon \Lambda U \to \Lambda Y$ be a rational ΛK-linear map. The *pole indices* $\rho_1 \leq \rho_2 \leq \cdots \leq \rho_m$ of f are the degree indices of the ΩK-module $\Delta_\theta = \operatorname{Ker} \pi_\theta f \cap \Omega U$.

The system theoretic significance of the pole indices, and their role in the context of the nonsingular, causal and stable precompensation problem are given by Proposition 6.8 and by the converse direction of it, which will be stated later in this section. We note that the pole indices can be explicitly computed as follows. Let $f = N_p D_p^{-1}$ be a pole representation of f. The matrix D_p is polynomial and nonsingular, and there exists a polynomial unimodular matrix $M\colon \Lambda U \to \Lambda U$ such that the columns $d_1, \ldots, d_m$ of $D_p M$ form an ordered proper basis. The pole indices $\rho_1, \ldots, \rho_m$ of f are then simply $\rho_i = -\operatorname{ord} d_i$, $i = 1, \ldots, m$.

The fact that $\operatorname{Ker} \tilde{f} \subset \Delta_\theta(f)$ directly implies, via Lemma 6.9 below, the following magnitude evaluation of the pole indices.

Proposition 6.7. *Let $f: \Lambda U \to \Lambda Y$ be a rational ΛK-linear map with pole indices $\rho_1 \leq \rho_2 \leq \cdots \leq \rho_m$ and reachability indices $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m$. Then, $\rho_i \leq \lambda_i$ for all $i = 1, \ldots, m$.*

The pole indices characterize the dynamical properties of all the nonsingular, causal, and input/output stable precompensators that stabilize the system f. We now start our discussion of this point.

Proposition 6.8. *Let $f: \Lambda U \to \Lambda Y$ be a rational ΛK-linear map with pole indices $\rho_1 \leq \rho_2 \leq \cdots \leq \rho_m$, and let $\ell: \Lambda U \to \Lambda U$ be a nonsingular, causal, and input/output stable precompensator. If $f\ell$ is input/output stable, then the reachability indices $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m$ of ℓ satisfy $\lambda_i \geq \rho_i$ for all $i = 1, \ldots, m$.*

Before proving Proposition 6.8, we need several auxiliary results which are related to the comparison of the degree indices of different ΩK-modules. First, we reproduce from Hammer and Heymann [1983] the following two statements. (We recall that an ΩK-module $\Delta \subset \Lambda U$ is called *full* if it contains a basis of the ΛK-linear space ΛU.)

Lemma 6.9. *Let $\Delta' \subset \Delta$ be nonzero and bounded ΩK-submodules of ΛS, with degree indices $\mu'_1, \ldots, \mu'_n$ and $\mu_1, \ldots, \mu_n$, respectively. Then, $\mu'_i \geq \mu_i$ for all $i = 1, \ldots, n$. Moreover, if $\mu'_i = \mu_i$ for all $i = 1, \ldots, n$, then $\Delta' = \Delta$.*

Lemma 6.10. *Let $\Delta, \Delta' \subset \Lambda S$ be bounded and full ΩK-modules with degree indices $\mu_1, \ldots, \mu_m$ and $\mu'_1, \ldots, \mu'_m$, respectively. Then, there exists a bicausal ΛK-linear map $\ell: \Lambda U \to \Lambda U$ such that $\Delta' = \ell[\Delta]$ if and only if $\mu'_i = \mu_i$ for all $i = 1, \ldots, m$.*

When the restriction of bicausality in Lemma 6.10 is released, one arrives at the following

Lemma 6.11. *Let $\Delta, \Delta' \subset \Lambda S$ be bounded and full ΩK-submodules with order indices $\mu_1, \ldots, \mu_m$ and $\mu'_1, \ldots, \mu'_m$, respectively. Then, there exists a causal ΛK-linear isomorphism $v: \Lambda S \to \Lambda S$ such that $\Delta' = v[\Delta]$ if and only if $\mu'_i \leq \mu_i$ for all $i = 1, \ldots, m$.*

Proof. Assume first that $\mu'_i \leq \mu_i$ for all $i = 1, \ldots, m$. By Theorem 6.5, the module Δ (respectively Δ') has an ordered proper basis $d_1, \ldots, d_m$ (respectively $d'_1, \ldots, d'_m$), and $\operatorname{ord} d_i = -\mu_i$ (respectively $\operatorname{ord} d'_i = -\mu'_i$), $i = 1, \ldots, m$. But then the ΛK-linear map $v: \Lambda U \to \Lambda U$ defined by its values as $vd_i = d'_i$, $i = 1, \ldots, m$, is evidently nonsingular, is causal by Theorem 6.2, and satisfies $\Delta' = v[\Delta]$.

Conversely, assume that there exists a causal ΛK-linear isomorphism $v: \Lambda U \to \Lambda U$ such that $\Delta' = v[\Delta]$. Let $\{\eta_k\}$ (respectively $\{\eta'_k\}$) be the order list of Δ (respectively Δ'), and let $d_1, \ldots, d_m$ be an ordered proper basis of Δ. For every integer k, let $\Delta_k \subset \Delta$ (respectively $\Delta'_k \subset \Delta'$) be the ΩK-module generated by all elements $u \in \Delta$ (respectively $u \in \Delta'$) with $\operatorname{ord} u \geq k$ (see Hautus and Heymann [1978]). Now, by Theorem 6.5 (iii), we have $\operatorname{rank} \Delta_k = \eta_k$ and $\operatorname{rank} \Delta'_k = \eta'_k$. Since v is causal, $v[\Delta_k] \subset \Delta'_k$, so that, since v is also nonsingular, $\operatorname{rank} \Delta'_k \geq \operatorname{rank} \Delta_k$. But then, $\eta'_k \geq \eta_k$ for all integers k, and it follows by definition that the degree indices $\mu'_i \leq \mu_i$ for all $i = 1, \ldots, m$. □

Proof of Proposition 6.8. We assume that ℓ is nonsingular, causal, and input/output stable, and that $f\ell$ is input/output stable. Let $f = N_pD_p^{-1}$ be a pole representation of f, and let $\ell = ND^{-1}$ be a coprime polynomial matrix fraction representation of ℓ. By Proposition 4.8 we have that $D_p^{-1}ND^{-1}$ is input/output stable, and whence so is also $D_p^{-1}N$. But then, since D_p is completely unstable, we obtain by Lemma 4.6 that the map $N_1 := D_p^{-1}N$ is polynomial, and $N = D_pN_1$. Further, by Proposition 3.4, $\operatorname{Ker}\tilde{\ell} = D[\Omega U] =: \Delta$, and, defining $\Delta_1 := N[\Omega U]$, we obtain that $\Delta_1 = \ell[\Delta] \subset D_p[\Omega U] = \Delta_\theta$. Finally, since $\lambda_1, \ldots, \lambda_m$ are the degree indices of Δ and $\rho_1, \ldots, \rho_m$ are those of Δ_θ, the inclusion $\ell[\Delta] \subset \Delta_\theta$ implies, by Lemmas 6.9 and 6.11, that $\lambda_i \geq \rho_i$ for all $i = 1, \ldots, m$, and our proof concludes. □

Proposition 6.8 is only a manifestation of a stronger result, which is our next objective. We start with a brief review of some terminology in realization theory. Let $f: \Lambda U \to \Lambda Y$ be a linear input/output map. An (*abstract*) *realization* of f is a triple (X, g, h), where X is an ΩK-module, and $g: \Omega U \to X$ and $h: X \to \Gamma Y$ are ΩK-homomorphisms, such that the following diagram is commutative (Kalman [1965], Kalman *et. al* [1969])

$$\begin{tikzcd} \Omega U \arrow[rr, "\tilde{f}"] \arrow[dr, "g"'] & & \Gamma Y \\ & X \arrow[ur, "h"'] & \end{tikzcd} \tag{6.12}$$

The module X is called the *state space*, and is, of course, a K-linear space as well. The realization (X, g, h) is called *reachable* if g is surjective, *observable* if h is injective, and *canonical* if it is both reachable and observable. We shall always assume that X is finite dimensional as a K-linear space, in which case one has that $\operatorname{rank}_{\Omega K} \operatorname{Ker} g = \dim_K U$ (see e.g. Fuhrmann [1976]). In this case there exists a nonsingular polynomial matrix $D: \Lambda U \to \Lambda U$ such that $\operatorname{Ker} g = D[\Omega U]$.

Now, let $g: \Omega U \to X$ be an ΩK-homomorphism. The pair (X, g) is called a *semirealization* of f if there exists an ΩK-homomorphism $h: X \to \Gamma Y$ such that (X, g, h) is a realization of f. (We note that a semirealization is equivalent to a pair of matrices (F, G), as mentioned in the introduction.) The semirealization (X, g) is called *reachable* if g is surjective, and *canonical* if (X, g, h) is a canonical realization of f. For a reachable semirealization, we evidently have that X is ΩK-isomorphic to the quotient ΩK-module $\Omega U/\operatorname{Ker} g$, so that a reachable semirealization is determined by the ΩK-module $\operatorname{Ker} g \subset \Omega U$ up to a state space isomorphism. It can be shown that a reachable semirealization of f is canonical if and only if $\operatorname{Ker} g = \operatorname{Ker} \tilde{f}$ (see Hautus and Heymann [1978]). The *reachability indices* of a reachable semirealization (X, g) are the degree indices of the ΩK-module $\operatorname{Ker} g$. Finally, let (X, g) be a reachable semirealization, and let $\operatorname{Ker} g = D[\Omega U]$. Then, we say that (X, g) is *stable* if D^{-1} is input/output stable.

The significance of a semirealization is most clearly demonstrated through polynomial matrix fraction representations. Indeed, let (X, g) be a canonical semirealization of f, and let $\operatorname{Ker} g = D[\Omega U]$. Then, also $\operatorname{Ker} \tilde{f} = D[\Omega U]$, and, by Proposition 3.4, there exists a polynomial matrix $N: \Lambda U \to \Lambda Y$, right coprime with D, such that $f = ND^{-1}$. Thus, a canonical semirealization basically is a denominator of a right coprime polynomial matrix fraction representation of f.

We turn now to the main result of the present section. Let $f\colon \Lambda U \to \Lambda Y$ be a rational ΛK-linear map. We denote by $P(f)$ the set of all nonsingular, causal, and input/output stable precompensators $\ell\colon \Lambda U \to \Lambda U$ for which $f\ell$ is input/output stable.

Theorem 6.13. *Let $f\colon \Lambda U \to \Lambda Y$ be a rational ΛK-linear map with pole indices $\rho_1 \le \rho_2 \le \cdots \le \rho_m$, and let (X, g) be any stable and reachable semirealization with reachability indices $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_m$. Then, there exists a precompensator $\ell \in P(f)$ having (X, g) as a canonical semirealization if and only if $\lambda_i \ge \rho_i$ for all $i = 1,\dots,m$.*

Proof. The "only if" direction is stated in Proposition 6.8 above. Conversely, assume that $\lambda_i \ge \rho_i$ for all $i = 1,\dots,m$, and let $\operatorname{Ker} g = D[\Omega U]$. We can assume that the columns $d_1,\dots,d_m$ of D are properly independent, and that $\operatorname{ord} d_i = -\lambda_i$, $i = 1,\dots,m$. Further, let $f = N_p D_p^{-1}$ be a pole representation of f, where D_p has ordered properly independent columns $d'_1,\dots,d'_m$, so that $\operatorname{ord} d'_i = -\rho_i$, $i = 1,\dots,m$. Then, since $\lambda_i \ge \rho_i$ for all $i = 1,\dots,m$, it follows by Theorem 6.2 that the map $\ell := D_p D^{-1}$ is causal, and it is easily seen that $\ell \in P(f)$. Moreover, since D^{-1} is input/output stable, it follows by Proposition 4.3 that D_p and D are right polynomially coprime. Thus, by Proposition 3.4, $\operatorname{Ker} \tilde{\ell} = D[\Omega U]$, so that $\operatorname{Ker} g = \operatorname{Ker} \tilde{\ell}$, and (X, g) is a canonical semirealization of ℓ. □

In particular, it follows by Theorem 6.13 that all the precompensators of minimal MacMillan degree in $P(f)$ have reachability indices equal to the pole indices $\rho_1,\dots,\rho_m$ of f.

Among all precompensators, the bicausal ones are of particular interest, since they are related to feedback, and they do not affect the internal delay of the system (see Hammer and Heymann [1981]). Every causal feedback, be it state feedback or output feedback, induces an equivalent bicausal precompensator. Our next objective is to show that Theorem 6.13 can be sharpened in such a way that the precompensator ℓ there will also have the property of being bicausal. This can be done under a mild assumption on the stability set θ, as follows. Let θ be a stability set. We say that θ is a *strict stability set* if there exists an element $\beta \in K$ such that $(z + \beta)$ is coprime with every element Ψ of θ. Clearly, in case K is the field of real numbers, the set of all polynomials having their roots in a region of the complex plane which includes part of, but not all of, the real line, forms a strict stability set. Given a strict stability set θ, we denote by θ^c the multiplicative set generated by all primes $p \in \Omega K$ which are coprime with every element Ψ of θ. Then, since $(z + \beta) \in \theta^c$, we have that θ^c is nonempty, and it forms a stability set.

Theorem 6.14. *Assume that θ is a strict stability set, and let $f\colon \Lambda U \to \Lambda Y$ be a rational ΛK-linear map with pole indices $\rho_1 \le \rho_2 \le \cdots \le \rho_m$. Also, let (X, g) be any stable and reachable semirealization with reachability indices $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_m$. Then, there exists a* bicausal *precompensator $\ell \in P(f)$ having (X, g) as a canonical semirealization if and only if $\lambda_i \ge \rho_i$ for all $i = 1,\dots,m$.*

Proof. The "only if" direction is evidently stated in Theorem 6.13. Conversely, assume that $\lambda_i \geq \rho_i$ for all $i = 1,\ldots,m$, and let $\zeta_i := \lambda_i - \rho_i$ (≥ 0), $i = 1,\ldots,m$. Since θ is a strict stability set, there is an element $\beta \in K$ such that $(z+\beta)$ is coprime with every element in θ. Define the matrix $\zeta := \operatorname{diag}((z+\beta)^{\zeta_1},\ldots,(z+\beta)^{\zeta_m})$. Then, using the notation and argumentation of the proof of Theorem 6.13, it can be readily seen that the map $\mathscr{L}_c := D_p \zeta D^{-1}: \Lambda U \to \Lambda U$ is bicausal, and that it satisfies our assertion. □

Example. Computation of pole indices: For the sake of simplicity, we assume that the stability set θ consists of all powers of $(z+1)$, and we consider the map

$$f := \left(\frac{z+2}{(z+1)(z+3)}, \frac{z+4}{z+5} \right) : \Lambda K^2 \to \Lambda K.$$

Defining

$$N_p := \left(\frac{(z+2)(z+5)}{(z+1)}, \frac{(z+4)(z+3)}{1} \right)$$

and $D_p := (z+3)(z+5)$, we obtain that $f = N_p D_p^{-1}$ is a pole representation of f. The pole index of f is then $\rho = -\operatorname{ord}(z+3)(z+5) = 2$.

7. The Stability Indices

In the previous section we directed our attention to the dynamical properties of the precompensators that stabilize a given system. In the present section, we consider the dynamical properties of the resulting stabilized systems. For the sake of simplicity, we start our discussion with the injective case, and later we show that all our statements are valid for the general noninjective case as well. It will be convenient to compare our present situation with the one encountered in Hammer and Heymann [1983], where we studied general precompensation, and for this purpose we reproduce from there the following

Definition 7.1. Let $f: \Lambda U \to \Lambda Y$ be an injective linear input/output map. The *reduced reachability indices* $\nu_1,\ldots,\nu_m$ of f are the degree indices of the ΩK-module $\operatorname{Ker} \pi f$.

Let $f: \Lambda U \to \Lambda Y$ be an injective linear input/output map, and let $\mathfrak{L}(f)$ be the set of all linear input/output maps of the form $f' = f\mathscr{L}$, where $\mathscr{L}: \Lambda U \to \Lambda U$ is a rational bicausal precompensator. Then, the dynamical properties of the systems in $\mathfrak{L}(f)$ are characterized in terms of the reduced reachability indices of f as follows (Hammer and Heymann [1983]).

Theorem 7.2. *Let $f: \Lambda U \to \Lambda Y$ be an injective linear input/output map, with reduced reachability indices $\nu_1 \leq \nu_2 \leq \cdots \leq \nu_m$. Then, the following hold true.*

(i) *Let $f' \in \mathfrak{L}(f)$ be any element, and let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m$ be the reachability indices of any reachable realization of f'. Then, for all $i = 1,\ldots,m$, $\lambda_i \geq \nu_i$.*

(ii) *Let (X, g) be any reachable semirealization with reachability indices $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m$, and assume that $\lambda_i \geq \nu_i$ for all $i = 1,\ldots,m$. Then, there exists a linear input/output map $f' \in \mathfrak{L}(f)$ such that (X, g) is a semi-realization of f'. Moreover, if the field K is infinite, then there exists a linear input/output map $f'' \in \mathfrak{L}(f)$ such that (X, g) is a canonical semirealization of f''.*

Our main objective in the present section is to show that a similar situation also holds in the case of stable precompensation, when the reduced reachability indices are replaced by the following set of invariants.

Definition 7.3. Let $f: \Lambda U \to \Lambda Y$ be an injective rational ΛK-linear map. The *stability indices* of f (in the sense of θ), $\sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_m$, are the order indices of the ΩK-module Δ^θ (:= $\operatorname{Ker} \pi f \cap \Omega_\theta U$).

In more explicit terms, let $f: \Lambda U \to \Lambda Y$ be an injective rational ΛK-linear map, and let $f = ND^{-1}$ be a zero representation of f. Also, let $M: \Lambda U \to \Lambda U$ be a polynomial unimodular matrix such that the columns $d_1,\ldots,d_m$ of DM are ordered and properly independent. Then, the stability indices $\sigma_1,\ldots,\sigma_m$ of f are $\sigma_i = -\operatorname{ord} d_i$, $i = 1,\ldots,m$. We now examine some properties of stability indices, starting with a magnitude evaluation.

Proposition 7.4. *Let $f: \Lambda U \to \Lambda Y$ be an injective rational ΛK-linear map with reduced reachability indices $\nu_1 \leq \nu_2 \leq \cdots \leq \nu_m$, and let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m$ be the reachability indices of any reachable realization of f. Then, the stability indices of f, $\sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_m$, satisfy for all $i = 1,\ldots,m$, $\nu_i \leq \sigma_i \leq \lambda_i$.*

Proof. Let (X, g, h) be any reachable realizations of f. Then, by (6.12), $\operatorname{Ker} g \subset \operatorname{Ker} \tilde{f}$. Also, by definition, $\operatorname{Ker} \tilde{f} \subset \Delta^\theta(f) \subset \operatorname{Ker} \pi f$, so that $\operatorname{Ker} g \subset \Delta^\theta(f) \subset \operatorname{Ker} \pi f$. But then, our assertion follows directly by Lemma 6.9. □

One consequence of Proposition 7.4 is that, when f is causal, its stability indices are nonnegative integers. This is implied by the fact that, when f is causal, its reduced reachability indices are nonnegative (see Hammer and Heymann [1983], Section 5). The connection between the stability indices and the reduced reachability indices is made clear through the notion of θ-invertible maps (see end of Section 5) as follows.

Proposition 7.5. *Let $f: \Lambda U \to \Lambda Y$ be an injective rational ΛK-linear map with reduced reachability indices $\nu_1 \leq \nu_2 \leq \cdots \leq \nu_m$ and stability indices $\sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_m$. Then, $\sigma_i = \nu_i$ for all $i = 1,\ldots,m$ if and only if f is θ-invertible.*

Proof. If f is θ-invertible, then, by Proposition 5.17, $\operatorname{Ker} \pi f \subset \Omega_\theta U$, so that $\Delta^\theta = \operatorname{Ker} \pi f$ and $\sigma_i = \nu_i$ for all $i = 1,\ldots,m$. Conversely, assume that $\sigma_i = \nu_i$, $i = 1,\ldots,m$. Then, the submodule $\Delta^\theta \subset \operatorname{Ker} \pi f$ has the same degree indices as $\operatorname{Ker} \pi f$, so that, by Lemma 6.9, $\Delta^\theta = \operatorname{Ker} \pi f$. But then, since $\Delta^\theta \subset \Omega_\theta U$, also $\operatorname{Ker} \pi f \subset \Omega_\theta U$, and f is θ-invertible by Proposition 5.17. □

Next, we show that the stability indices cannot be reduced by stable precompensation.

Proposition 7.6. *Let $f: \Lambda U \to \Lambda Y$ be an injective rational ΛK-linear map with stability indices $\sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_m$. Also, let $\ell: \Lambda U \to \Lambda U$ be a nonsingular, causal, and input/output stable ΛK-linear map. Then, the stability indices $\sigma_1' \leq \sigma_2' \leq \cdots \leq \sigma_m'$ of $f\ell$ satisfy $\sigma_i' \geq \sigma_i$ for all $i = 1, \ldots, m$.*

Proof. Let $f' := f\ell$. Then, $\ell[\operatorname{Ker} \pi f'] = \operatorname{Ker} \pi f$ and, since ℓ is input/output stable, also $\ell[\Omega_\theta U] \subset \Omega_\theta U$. Now, let $\Delta := \ell[\Delta^\theta(f')]$, let $\alpha_1, \ldots, \alpha_m$ be the order indices of Δ, and note that, by Lemma 6.11, we have $\sigma_i' \geq \alpha_i$ for all $i = 1, \ldots, m$. Using the facts that ℓ is nonsingular and input/output stable, we obtain $\Delta = \ell[\operatorname{Ker} \pi f' \cap \Omega_\theta U] = \ell[\operatorname{Ker} \pi f'] \cap \ell[\Omega_\theta U] \subset \operatorname{Ker} \pi f' \ell^{-1} \cap \Omega_\theta U = \operatorname{Ker} \pi f \cap \Omega_\theta U = \Delta^\theta(f)$. Hence, $\Delta \subset \Delta^\theta(f)$, so that by Lemma 6.9, $\alpha_i \geq \sigma_i$ for all $i =, \ldots, m$, and since also $\sigma_i' \geq \alpha_i$, we conclude that $\sigma_i' \geq \sigma_i$, $i =, \ldots, m$. □

A reconsideration of our proof leads to the following.

Corollary 7.7. *Let $f: \Lambda U \to \Lambda Y$ be an injective rational ΛK-linear map, and let $\ell: \Lambda U \to \Lambda U$ be $\Omega_\theta^- K$-unimodular. Then, f and $f\ell$ have the same stability indices.*

Combining Propositions 7.4 and 7.6, we directly obtain the following bound on the reachability indices of the stabilized system.

Corollary 7.8. *Let $f: \Lambda U \to \Lambda Y$ be an injective ΛK-linear map with stability indices $\sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_m$. Also, let $\ell: \Lambda U \to \Lambda U$ be a nonsingular, causal, and input/output stable ΛK-linear map, and let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m$ be the reachability indices of any reachable realization of $f\ell$. Then, $\lambda_i \geq \sigma_i$ for all $i = 1, \ldots, m$.*

Thus, the stability indices form a lower bound for the sets of reachability indices attainable under stability restrictions. We now show that this bound is tight, and leads to a stable-dynamics assignment theorem. We denote by $\Sigma(f)$ the set of all *input/output stable* ΛK-linear maps of the form $f\ell$, where $\ell: \Lambda U \to \Lambda U$ is a nonsingular, causal, and input/output stable precompensator.

Theorem 7.9. *Let $f: \Lambda U \to \Lambda Y$ be an injective rational ΛK-linear map with stability indices $\sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_m$, and let (X, g) be any stable and reachable semirealization with reachability indices $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m$. Then, there exists a map $f' \in \Sigma(f)$ having (X, g) as its canonical semirealization if and only if $\lambda_i \geq \sigma_i$ for all $i = 1, \ldots, m$.*

Proof. If (X, g) is a canonical semirealization of a map $f' \in \Sigma(f)$, then it follows by Corollary 7.8 that $\lambda_i \geq \sigma_i$ for all $i = 1, \ldots, m$. Conversely, assume that $\lambda_i \geq \sigma_i$ for all $i = 1, \ldots, m$, and let $f = N_0 D_0^{-1}$ be a zero representation of f, where D_0 was chosen with ordered and properly independent columns. Also, let $\operatorname{Ker} g = D[\Omega U]$, where, again, D was chosen with ordered and properly independent columns. Then, the degrees of the columns of D_0 are $\sigma_1, \ldots, \sigma_m$, and those of D are $\lambda_1, \ldots, \lambda_m$, so that, since $\lambda_i \geq \sigma_i$ for all $i = 1, \ldots, m$, it follows by Theorem 6.2 that the map $\ell := D_0 D^{-1}$ is causal. Since both of D_0 and D^{-1} are input/output stable, so is also ℓ. Clearly, ℓ is nonsingular as well, and the map $f' := f\ell = N_0 D^{-1}$ is input/output stable. Whence, $f' \in \Sigma(f)$. In view of Proposition 5.6, the maps N_0 and D are right polynomially coprime, so that, by Proposition 3.4, $\operatorname{Ker} \tilde{f}' = D[\Omega U]$.

Thus $\operatorname{Ker} g = \operatorname{Ker} \tilde{f}'$, and (X, g) is a canonical semirealization of the map $f' \in \Sigma(f)$. □

When the stability set θ is a strict stability set we can, in analogy to Theorem 6.14, add in Theorem 7.9 the additional requirement that the precompensators be bicausal. This leads to the following statement, the proof of which is a modified version of the proof of Theorem 7.9, using the construction of the proof of Theorem 6.14. (We recall that $P(f)$ is the set of all nonsingular, causal, and input/output stable precompensators $\ell: \Lambda U \to \Lambda U$ for which $f\ell$ is input/output stable.)

Theorem 7.10. *Assume that θ is a strict stability set, and let $f: \Lambda U \to \Lambda Y$ be an injective rational ΛK-linear map with stability indices $\sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_m$. Also, let (X, g) be any stable and reachable semirealization with reachability indices $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m$. Then, there exists a* bicausal *precompensator $\ell \in P(f)$ such that (X, g) is a canonical semirealization of $f\ell$ if and only if $\lambda_i \geq \sigma_i$ for all $i = 1, \ldots, m$.*

We note the analogy between Theorem 7.10 and Theorem 7.2. Through the use of strict stability sets, we were able to overcome the requirement of infinite fields in Theorem 7.2 (ii).

We conclude our discussion of the injective case with an examination of the connection between our present situation and the theory of strict observability described in Hammer and Heymann [1983]. Let $f: \Lambda U \to \Lambda Y$ be a rational ΛK-linear map. We say that f is *θ-irreducible* if it has the minimal MacMillan degree in $\Sigma(f)$. From Theorem 7.9 we directly obtain the following.

Corollary 7.11. *Let $f: \Lambda U \to \Lambda Y$ be an injective rational ΛK-linear map with stability indices $\sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_m$ and reachability indices $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m$. Then, f is θ-irreducible if and only if $\lambda_i = \sigma_i$ for all $i = 1, \ldots, m$.*

We recall from Hammer and Heymann [1983] that a ΛK-linear map $f: \Lambda U \to \Lambda Y$ is called *strictly observable* whenever $\operatorname{Ker} \pi f \subset \Omega U$. That there is a close connection between strictly observable maps and θ-irreducible ones is indicated by the following

Proposition 7.12. *Let $f: \Lambda U \to \Lambda Y$ be an injective linear input/output map. Then, f is θ-irreducible if and only if $\Delta^\theta(f) \subset \Omega U$.*

Proof. If f is θ-irreducible, then by Corollary 7.11, $\operatorname{Ker} \tilde{f}$ and $\Delta^\theta(f)$ have equal order indices. But then, since $\operatorname{Ker} \tilde{f} \subset \Delta^\theta(f)$, it follows, by Lemma 6.9, that $\operatorname{Ker} \tilde{f} = \Delta^\theta(f)$, and thus $\Delta^\theta(f) \subset \Omega U$. Conversely, if $\Delta^\theta(f) \subset \Omega U$, then, since $\operatorname{Ker} \tilde{f} = \Delta^\theta(f) \cap \Omega U$, we have $\operatorname{Ker} \tilde{f} = \Delta^\theta(f)$, so that the canonical reachability indices of f are equal to its stability indices, and f is θ-irreducible by Corollary 7.11. □

Now, let $f = N_0 D_0^{-1}$ be a zero representation of f, and define $g := D_0^{-1}$. Then, $\operatorname{Ker} \pi g = D_0[\operatorname{Ker} \pi] = D_0[\Omega U] = \Delta^\theta(f)$, where the last equality is by the definition of the zero representation. Combining this with Proposition 7.12, we obtain

Corollary 7.13. *Let $f: \Lambda U \to \Lambda Y$ be a rational ΛK-linear map, let $f = N_0 D_0^{-1}$ be a zero representation of f, and denote $g := D_0^{-1}$. Then, f is θ-irreducible if and only if g is strictly observable (or, equivalently, if and only if D_0 is a polynomial map).*

We conclude this section with a brief indication of how our discussion can be generalized to the noninjective case, and thereafter we give an example for the computation of the stability indices.

The noninjective case: Let $f\colon \Lambda U \to \Lambda Y$ be a rational ΛK-linear map, let $q := \dim_{\Lambda K}\operatorname{Ker} f$, and assume that $q \neq 0$. By Lemma 5.10, there exists an $\Omega_\theta^- K$-unimodular map $\ell\colon \Lambda U \to \Lambda U$ such that $f\ell = (f_0, 0)$, where $f_0\colon \Lambda K^{(m-q)} \to \Lambda Y$ is injective. Let $\sigma_1^0 \leq \sigma_2^0 \leq \cdots \leq \sigma_{(m-q)}^0$ be the stability indices of f_0.

Lemma 7.14. *The stability indices $\sigma_1^0, \ldots, \sigma_{m-q}^0$ are uniquely determined by f.*

Proof. Let $\ell'\colon \Lambda U \to \Lambda U$ be any $\Omega_\theta^- K$-unimodular map for which $f\ell'$ is of the form $f\ell' = (f', 0)$, where $f'\colon \Lambda K^{m-q} \to \Lambda Y$ is injective, and let $\sigma_1', \ldots, \sigma_{m-q}'$ be the stability indices of f'. We have to show that $\sigma_i' = \sigma_i^0$ for all $i = 1, \ldots, m-q$. Substituting $f = (f_0, 0)\ell^{-1}$, we obtain that $(f_0, 0)\ell^{-1}\ell' = (f', 0)$. Whence, since f_0 is injective, it follows that

$$\ell^{-1}\ell' = \begin{pmatrix} \ell_1 & 0 \\ \ell_* & \ell_2 \end{pmatrix},$$

where $\ell_1\colon \Lambda K^{m-q} \to \Lambda K^{m-q}$ and $\ell_2\colon \Lambda K^q \to \Lambda K^q$ are $\Omega_\theta^- K$-unimodular, and $\ell_*\colon \Lambda K^{m-q} \to \Lambda K^q$ is causal and input/output stable. But then, $f' = f_0\ell_1$, and, since ℓ_1 is $\Omega_\theta^- K$-unimodular, it follows by Corollary 7.7 that $\sigma_i' = \sigma_i^0$ for all $i = 1, \ldots, m$. □

We generalize now the definition of the stability indices to the noninjective case as follows. Adhering to our above notation, the *stability indices* $\sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_m$ of f are: $\sigma_i = 0$ for all $i = 1, \ldots, q$, and $\sigma_i := \sigma_{i-q}^0$ for $i = q+1, \ldots, m$. In view of Lemma 7.14, $\sigma_1, \ldots, \sigma_m$ are uniquely determined by f.

Using the representation $f\ell = (f_0, 0)$ and the fact that both of ℓ and ℓ^{-1} are causal as well as input/output stable, it can be shown that, under the above definition of stability indices, Theorems 7.4, 7.6, 7.7, 7.8, 7.9, 7.10, and 7.11 continue to hold when the injectivity assumption of f therein is released. Also, Corollary 7.13 holds in the noninjective case when "strictly observable" therein is replaced by "extended strictly observable" (see Hammer and Heymann [1983]).

Example. Computation of stability indices: For the sake of simplicity, we choose the stability set θ as the set of all powers of $(z+1)$, and we consider the noninjective map

$$f := \left(\frac{(z+1)^3(z+2)}{(z+3)^5}, \frac{(z+2)(z+5)}{(z+3)^5} \right) : \Lambda K^2 \to \Lambda K.$$

Defining

$$\ell := \begin{pmatrix} 1 & -\dfrac{(z+5)}{(z+1)^3} \\ 0 & 1 \end{pmatrix},$$

we have that $\mathscr{L}$ is $\Omega_\theta^- K$-unimodular, and $f\mathscr{L} = (f_0, 0)$, where $f_0 := (z+1)^3(z+2)/(z+3)^5$. Further, denoting $N_0 := (z+2)$, and $D_0 := (z+3)^5/(z+1)^3$, we have that $f_0 = N_0 D_0^{-1}$ is a zero representation of f_0, and the stability index of f_0 is $\sigma_0 = -\operatorname{ord}(z+3)^5/(z+1)^3 = 2$. The stability indices of f are then $\sigma_1 = 0$ and $\sigma_2 = 2$. □

References

1. F. M. Brasch and J. B. Pearson. Pole placement using dynamic compensators, *IEEE Transactions on Automatic Control*, 15:34–43 (1970).
2. P. Brunovski. A classification of linear controllable systems, Kybernetika, 3:173–187 (1970).
3. C. A. Desoer, R. W. Liu, J. Murray, and R. Saeks. Feedback system design: the fractional representation approach to analysis and synthesis, *Trans IEEE*, 25:399–412 (1980).
4. P. A. Furhmann. Algebraic system theory: an analyst's point of view, Journal of the Franklin Institute, 301:521–540 (1976).
5. I. C. Gokhberg and M. G. Krein. Systems of integral equations on a half line with kernels depending on the difference of arguments, Am. Math. Soc., Translations, 14: 2:217–287 (1960).
6. J. Hammer. Linear dynamic output feedback: invariants and stability, to appear in *IEEE Trans. on Automatic Control*, April 1983 (1981).
7. J. Hammer and M. Heymann. Pole assignment and minimal feedback design, *International Journal on Control*, 37:63–88 (1983).
8. J. Hammer and M. Heymann. Causal factorization and linear feedback, *SIAM J. on Control and Optimization*, 19:445–468 (1981).
9. J. Hammer and M. Heymann. Strictly observable rational linear systems, *SIAM J. on Control and Optimization*, 21:1–16 (1983).
10. M. L. J. Hautus and M. Heymann. Linear feedback—an algebraic approach, *SIAM Journal on Control and Optimization* 16:83–105 (1978).
11. R. E. Kalman. Algebraic structure of linear dynamical systems. I: The module of Σ, *Proceedings of the National Academy of Science* (USA), 54:1503–1508 (1965).
12. R. E. Kalman. *Lectures on Controllability and Observability*, CIME (1968).
13. R. E. Kalman. Kronecker invariants and feedback, in *Ordinary Differential Equations, 1971 NRL-MRC Conference*, L. Weiss ed., pp. 459–471, Academic Press, New York (1971).
14. R. E. Kalman, P. L. Falb and M. A. Arbib. *Topics in Mathematical System Theory*, McGraw-Hill, New York (1969).
15. P. P. Khargonekar and E. Emre. Further results on polynomial characterizations of (F, G)-invariant subspaces, preprint, Center for Mathematical System Theory, University of Florida, Gainesville, FL 32611, U.S.A. (1980).
16. H. Kwakernaak and R. Sivan. *Linear Optimal Control Systems*, Wiley-Interscience, New York (1972).
17. C. C. MacDuffee. *The Theory of Matrices*, Chelsea, New York (1934).
18. A. S. Morse. System invariants under feedback and cascade control, Proceedings of the conference on Mathematical System Theory, Udine, Italy, pages 61–74. Appeared in *Lecture Notes in Economics and Mathematical Systems*, 131, (G. Marchesini and S. Mitter, editors), Springer Verlag, Berlin (1975).
19. H. F. Münzner and D. Prätzel-Wolters. Minimal bases of polynomial modules, structural indices and Brunovsky-transformations, *Int. J. Control* 30:291–318 (1979).
20. L. Pernebo. An algebraic theory for the design of controllers for linear multivariable systems—part I: structure matrices and feedforward design, *Trans. IEEE*, AC 26:171–183 (1981).
21. H. H. Rosenbrock. *State Space and Multivariable Theory*, Nelson, London (1970).
22. J. H. M. Wedderburn. *Lectures on Matrices*, Colloquium publication of the AMS, *XVII*. Also, Dover publication, New York, 1964 (1934).
23. W. A. Wolovich. *Linear Multivariable Systems*, Applied Mathematical Sciences Series, No. 11, Springer Verlag, New York (1974).

24. W. A. Wolovich and P. L. Falb. Invariants and canonical forms under dynamic compensation, *SIAM Journal on Control*, 14:996–1008 (1976).
25. W. M. Wonham. *Linear Multivariable Control: A Geometric Approach*, Lecture Notes in Economics and Mathematical Systems, No. 101, Springer Verlag, New York (1974).
26. B. F. Wyman. *Linear Systems Over Commutative Rings*, Lecture Notes, Stanford University, Stanford, CA., U.S.A. (1972).
27. D. C. Youla. On the factorization of rational matrices, *IRE Transactions on Information Theory*, pp. 172–189 (1961).
28. O. Zariski and P. Samuel. *Commutative Algebra*, D. Van Nostrand Co., New York (1958).

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