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Robust stabilization of non-linear systems

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The problem of stabilizing a non-linear system whose parameters are not accurately known is discussed. Specifically, it is assumed that the non-linear system Σ that needs to be stabilized has a recursive representation of the form $x_{k+1} = f(x_k, u_k)$, where only a nominal description of the recursion function f is given, and where the actual recursion function may deviate from the nominal one. An explicit procedure for the stabilization of the system Σ is derived. The procedure consists of the construction of a pair of dynamic compensators—a precompensator and a feedback compensator—which, when connected in a closed loop around the system Σ, yield an internally stable control configuration. This configuration maintains its internal stability as long as the deviation of the recursion function f from nominality is within certain bounds. The compensators consist of recursive systems that can be readily implemented, and which are derived in an explicit form in terms of the nominal recursion function of the system Σ.

1. Introduction

Ever since the inception of classical linear feedback theory more than half a century ago, it has been intuitively clear that a closed-loop feedback system, when properly designed, tends to be insensitive to variations in the parameters of the system around which it was constructed. Indeed, the need to counterbalance uncertainties in systems is one of the main reasons for the widespread use of feedback in practical engineering. In this paper we study the effect of non-linear feedback on uncertainties in the parameters of non-linear systems. More specifically, we discuss the problem of stabilizing a non-linear system Σ when only a nominal description of it is given. We denote the nominal description by Σ_n. The actual system Σ that is placed in the stabilizing control loop may differ from the nominal system. Now, assume that a control configuration that stabilizes the nominal system Σ_n is designed. The basic question in which we are interested is the following: is it possible to design the control configuration in such a way that it will remain stable when the actual system Σ is inserted in it instead of the nominal system Σ_n; and, if such a design is possible, how is it performed? In other words, our main interest is in the preservation of internal stability under variations in the parameters of the given system. We restrict our attention to the case of discrete-time systems.

In order to make our discussion as transparent as possible, we do not discuss non-linear systems in an abstract form. Rather, we limit our attention to discrete-time non-linear systems that are recursive, and have a recursive representation of the form

$$x_{k+1} = f(x_k, u_k), \quad k = 0, 1, 2, \ldots$$

where $\{x_k\}$ is the output sequence of the system, $\{u_k\}$ is its input sequence, and the initial condition x_0 is specified. The function f is called a 'recursion function' of the

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system. We focus our discussion on the case where the recursion function of the system that needs to be stabilized is not accurately known. We denote by f_n the recursion function of the nominal system Σ_n, and we assume that the actual system Σ has a recursion function f of the form

$$f(x_k, u_k) = f_n(x_k, u_k) + v(x_k, u_k)$$

where the function v describes the deviation of the actual recursion function from the nominal one. In these terms, we can state, in a more precise form, the basic question investigated in this paper:

Assume that the nominal recursion function f_n of the system is given. Is it possible to design an internally stable control configuration that will stabilize the actual system Σ, irrespective of the deviation function v, as long as the latter is continuous and its norm does not exceed a prespecified bound? If such a design is possible, how is it performed?

The above question is, in fact, the underlying theme of the theory of robust non-linear control. This question is answered in the subsequent sections of the paper, and we provide explicit and applicable design procedures for robust stabilization of non-linear systems. The designs derived through these procedures maintain their internal stability as long as the deviation v of the recursion function f from nominality does not exceed, in its norm, a predetermined magnitude. The basic control configuration that we use is the one that has also been used in Hammer (1988). We show that this control configuration can be designed to allow deviations in the recursion function of the system that needs to be stabilized, without destroying the (internal) stability of the configuration. The description of our methodology is fairly simple on the level of principles.

As mentioned, our present discussion is based on the stabilization theory for non-linear systems developed in Hammer (1986, 1987, 1988), and so we start with a brief description of some fundamental facts relating to this theory. Let Σ be a discrete-time system, accepting sequences of m-dimensional vectors as input, and generating sequences of p-dimensional vectors as output. We study the control of the system Σ through the classical configuration given in Fig. 1, where Σ is the given system that needs to be stabilized, π is a dynamic precompensator, ϕ is a dynamic feedback compensator, and $\Sigma_{(\pi,\phi)}$ denotes the closed-loop system. As we have repeatedly concluded in our previous studies of the non-linear stabilization problem, it is of particular advantage to choose the precompensator π and the feedback compensator ϕ in the form

$$\left.\begin{aligned} \pi &= B^{-1} \\ \phi &= A \end{aligned}\right\} \qquad (1)$$

where A and B are stable systems with B being invertible, and where A and B^{-1} are causal. As we show, an additional significant advantage of this configuration is that it allows a rather straightforward design procedure yielding a feedback compensator ϕ and a precompensator π which stabilize the system Σ, and which preserve that stabilization under variations in the parameters of Σ. In other words, the configuration yields robust stabilization of the given system.

Of fundamental importance to our discussion is the theory of fraction representations of non-linear systems developed by Hammer (1985 a, 1987). A right fraction

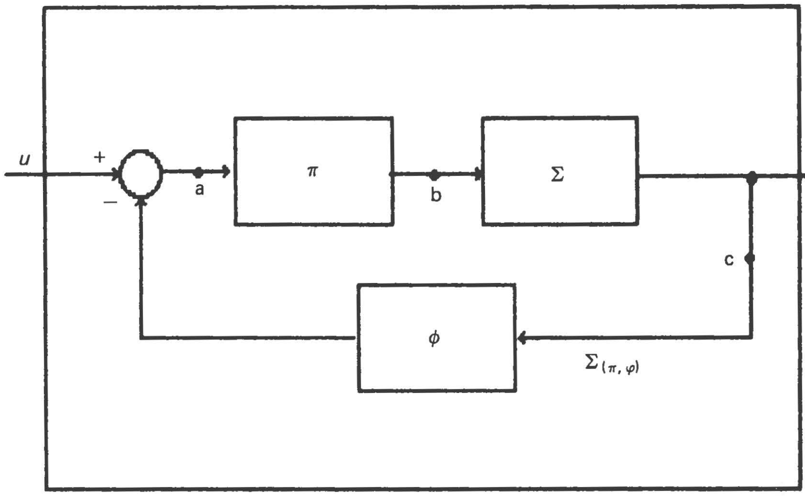

Figure 1.

representation of the non-linear system Σ is a representation of the form $\Sigma = PQ^{-1}$, where P and Q are stable systems, with Q being invertible. Now, assume that the system Σ has a right fraction representation $\Sigma = PQ^{-1}$, and that the compensators π and ϕ are given by (1). Then, it can readily be seen (e.g. Hammer 1984 a) that, under some mild assumptions, the input–output relationship induced by $\Sigma_{(\pi,\phi)}$ is given by

$$\Sigma_{(\pi,\phi)} = \Sigma\pi[I + \phi\Sigma\pi]^{-1} = PQ^{-1}B^{-1}[I + APQ^{-1}B^{-1}]^{-1} = P[AP + BQ]^{-1} \tag{2}$$

Denoting

$$M := AP + BQ \tag{3}$$

we obtain

$$\Sigma_{(\pi,\phi)} = PM^{-1} \tag{4}$$

and it follows that, whenever the stable systems A and B are selected so that the stable system M also has a stable inverse M^{-1}, the closed-loop system $\Sigma_{(\pi,\phi)}$ becomes input–output stable. In fact, $\Sigma_{(\pi,\phi)}$ is internally stable under these circumstances if the systems A and B satisfy some additional mild requirements (Hammer 1986 and 1987). A stable system M which is invertible and whose inverse M^{-1} is also stable is called a 'unimodular' system.

We now describe briefly, in a very qualitative and inaccurate way, the basic idea on which our robust stabilization theory rests. Suppose we have one appropriate pair of systems A and B for which $M := AP + BQ$ is unimodular. The systems P and Q, which arise from a right fraction representation of the given system Σ depend, of course, on Σ. Consequently, deviations of Σ from its nominal value Σ_n cause deviations of P and Q from their nominal values. Let $\Sigma_n = P_nQ_n^{-1}$ be a fraction representation of the nominal system. Let Σ be the actual system with the deviation, and suppose we can construct for it a fraction representation $\Sigma = PQ^{-1}$ in which the numerator P satisfies $P = P_n$, where P_n is the numerator of the fraction representation of the nominal system Σ_n. Namely, assume that the effect of the deviation can be completely described by a deviation of the denominator system Q from its nominal value Q_n. Denote $\omega := Q - Q_n$, and note that ω is a stable system and that $Q = Q_n + \omega$. Furthermore, suppose that, for every real number $\varepsilon > 0$, there is a causal and stable system A_ε satisfying the equation $A_\varepsilon P_n + \varepsilon Q_n = M$. Notice that when the latter holds and (1) is used, the system Σ may be stabilized using $B = \varepsilon I$ (and $A = A_\varepsilon$), in which case the precompensator $\pi = (1/\varepsilon)I$ is a simple amplifier. By taking ε arbitrarily small, we may arbitrarily increase the gain of this amplifier, and thus arbitrarily increase the forward path gain. Finally, suppose there is a real number $\delta > 0$ such that the system $M' := M + \mathscr{M}$ stays unimodular for every stable system $\mathscr{M}$ with 'magnitude' not exceeding δ, so that a deviation of 'less than δ' does not destroy the unimodularity of M. The existence of δ as well as its value depend, of course, on the nature of the particular unimodular system M.

Now, assume that the nominal system Σ_n is stabilized using the compensators induced by $A = A_\varepsilon$ and $B = \varepsilon I$, via (1). Then, if the system Σ is inserted in the loop instead of the nominal system Σ_n for which the loop was designed, we obtain, recalling the fraction representation $\Sigma = PQ^{-1}$, that

$$AP + BQ = A_\varepsilon P_n + \varepsilon(Q_n + \omega) = A_\varepsilon P_n + \varepsilon Q_n + \varepsilon\omega = M + \varepsilon\omega =: M'$$

Whence, if we choose ε small enough so that the 'magnitude' of $\mathscr{M} := \varepsilon\omega$ is smaller than δ, the system M' is still unimodular, and the input–output relationship $\Sigma_{(\pi,\phi)} = PM'^{-1}$ induced by the closed loop remains stable, despite the deviation in the system Σ. Thus, the deviation does not destroy stability. Basically, this is simply a restatement of the qualitative principle that, in a closed feedback loop, high gain in the forward path can counteract deviations in the parameters of the forward path systems; a principle that has been widely accepted on an intuitive level ever since the classical work of Black (1934) on linear feedback systems. The main advantage of the particular form in which we formulate this principle here is that, in this formulation, the principle can readily be applied to non-linear situations.

Of course, our main interest in this paper is in the design of internally stable control systems, and not in the design of control systems that are merely input–output stable. However, as we shall see in the following sections, the ideas alluded to in the previous paragraph can easily be modified to apply to internal stabilization. We shall also see that, in their accurate form, these ideas do not entail any major restrictions on the class of systems for which our results are valid.

We conclude this section with a brief mention of the background literature regarding the effect of feedback on system uncertainties. The first analytic description of the effect of feedback on system uncertainties is probably due to Black (1934) who analysed the linear scalar case; however, the qualitative idea that feedback is beneficial for situations involving uncertainties is much older, and was most likely already known to Watt in the previous century. Since then, there has been a vast amount of literature published on this subject, and it would be beyond the scope of this paper to review it here. Some insight into the available literature can be obtained from the following publications and their references: Bode (1945), Newton *et al.* (1957), Zames (1966 and 1981), Rosenbrock (1970 and 1974), Desoer and Vidyasagar (1975), and Kimura (1984).

The discussion in this paper is a direct continuation of the work reported in Hammer (1984 a, b, 1985 a, b, 1986, 1987, 1988). We briefly review the pertinent aspects of these papers in § 2. Alternative recent studies on the stabilization of non-linear systems may be found in Vidyasagar (1980), Sontag (1981), Desoer and Lin (1984), Isidori (1985), the references cited in these papers, and others.

2. Background

Our basic objective is to discuss the robustness properties of the stabilization procedure developed by Hammer (1988). We wish to show how that procedure can be utilized to overcome deviations in the parameters of the system Σ that needs to be stabilized. With this objective in mind, we devote this section to a brief review and refinement of the basic aspects of the theory developed by Hammer (1988). The robustness problem itself is discussed in the next section.

Our presentation is for the case of discrete-time systems. The systems we consider accept sequences of m-dimensional real vectors as their input, and generate sequences of p-dimensional real vectors as their output. To be more specific, we denote by $\mathbb{R}$ the set of real numbers, and for an integer $m > 0$, we denote by $\mathbb{R}^m$ the set of all m-dimensional real vectors. By $\mathbb{R}^0$ we simply mean the zero element 0. We denote by $S(\mathbb{R}^m)$ the set of all sequences of the form $u_0, u_1, u_2, \ldots$, with each element u_i belonging to $\mathbb{R}^m$. Given a sequence $u \in S(\mathbb{R}^m)$ and an integer $i \geqslant 0$, we denote by u_i the ith element of the sequence, and interpret the integer i as the time marker. For two integers $i > j \geqslant 0$, we denote by u_j^i the set of elements $u_j, u_{j+1}, \ldots, u_i$. In the set $S(\mathbb{R}^m)$ we induce the usual operation of addition elementwise, so that, given a pair of sequences $u, v \in S(\mathbb{R}^m)$, the sum $w := u + v$ is again a sequence in $S(\mathbb{R}^m)$, with each one of its elements given by $w_i := u_i + v_i$, $i = 0, 1, 2, \ldots$

We regard a system Σ simply as a map $\Sigma : S(\mathbb{R}^m) \to S(\mathbb{R}^p)$, transforming input sequences from $S(\mathbb{R}^m)$ into output sequences from $S(\mathbb{R}^p)$, where m and p are arbitrary positive integers. As mentioned before, the computational results presented in this paper refer to recursive systems that have their state as output, namely, to systems Σ with a recursive representation of the form

$$x_{k+1} = f(x_k, u_k)$$

where the initial condition x_0 is specified, and where $f : \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^p$ is a function, which we usually assume to be continuous. The function f is called a 'recursion function' of the system Σ. Given a subspace $S \subset S(\mathbb{R}^m)$, we denote by $\Sigma[S]$ the image of the set S through Σ, namely, the set of all output sequences that Σ generates from input sequences belonging to S.

Considerations involving causality of systems are rather important to our discussion, so we briefly review here some related definitions. A system $\Sigma : S(\mathbb{R}^m) \to S(\mathbb{R}^p)$ is 'causal' (respectively, 'strictly causal') if the following holds for every pair of input sequences u, $v \in S(\mathbb{R}^m)$: for all integers $i \geqslant 0$ for which $u_0^i = v_0^i$, also $\Sigma u]_0^i = \Sigma v]_0^i$ holds (respectively, $\Sigma u]_0^{i+1} = \Sigma v]_0^{i+1}$). A system $M : S(\mathbb{R}^m) \to S(\mathbb{R}^m)$ is a 'bicausal system' if it is invertible, and if M and M^{-1} are both causal systems.

For the purpose of defining the notion of stability, we introduce some norms on the space of sequences $S(\mathbb{R}^m)$. First, let $w = (w^1, \ldots, w^m)$ be a vector in $\mathbb{R}^m$. Denote $|w| := \max\{|w^i|, i = 1, \ldots, m\}$, the maximal absolute value of the coordinates of w. Then, given an element $u \in S(\mathbb{R}^m)$, define the norm $\rho(u) := \sup\{2^{-i}|u_i|, i = 0, 1, 2, \ldots\}$, which is simply a weighted l^∞-norm. We use this norm to define a metric on $S(\mathbb{R}^m)$, given, for every pair of elements $u, v \in S(\mathbb{R}^m)$, by $\rho(u, v) := \rho(u - v)$. Whenever referring to continuity, we always mean continuity with respect to the topology induced by the metric ρ, unless explicitly stated otherwise. It is also convenient for us to use the notation $|u| := \sup\{|u_i|, i = 0, 1, 2, \ldots\}$ for an element $u \in S(\mathbb{R}^m)$. For a real number $\theta > 0$, we denote by $S(\theta^m)$ the set of all elements $u \in S(\mathbb{R}^m)$ satisfying $|u| \leqslant \theta$, namely, the set of all sequences bounded by θ. A system $\Sigma : S(\mathbb{R}^m) \to S(\mathbb{R}^p)$ is BIBO (bounded-input bounded-output) stable if, for every real number $\theta > 0$, there is a real number $N > 0$ such that $\Sigma[S(\theta^m)] \subset S(N^p)$. Finally, we say that the system $\Sigma: S(\mathbb{R}^m) \to S(\mathbb{R}^p)$ is *stable* if it is BIBO stable, and if, for every real number $\theta > 0$, the restriction $\Sigma: S(\theta^m) \to S(\mathbb{R}^p)$ is a continuous map.

In Hammer (1988) we have developed a procedure for stabilizing recursive nonlinear systems having continuous recursion functions, and this stabilization procedure forms the basis of our discussion. In particular, we show that, when appropriately designed, this procedure stabilizes a system Σ even if the available information about the system Σ is not complete. Specifically, we assume that only a nominal recursion function of the system Σ is known, and that the actual recursion function of the system may deviate from the nominal one. We aim at achieving internal stability despite the deviations in the recursion function. We now briefly review the main results of Hammer (1988).

The stabilization procedure developed in Hammer (1988) applies to recursive systems $\Sigma: S(\mathbb{R}^m) \to S(\mathbb{R}^p)$ having a recursive representation of the form $x_{k+1} = f(x_k, u_k)$, where the recursion function $f: \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^p$ is continuous. A system satisfying these conditions is, in particular, a strictly causal system. Instead of stabilizing the system Σ directly, we stabilize a system of the form $\gamma' + \Sigma$, where γ' is a simple static linear system which is chosen so as to make the system $\Sigma_{\gamma'} := \gamma' + \Sigma$ injective (one to one). There is a basic advantage in dealing with injective systems, since they possess left inverses. Using the fact that the given system Σ is strictly causal, it is easy to see that the system $\Sigma_{\gamma'}$ is injective whenever the static system γ' is injective. In explicit form, we construct γ' as follows (Hammer 1988).

Let $\Sigma: S(\mathbb{R}^m) \to S(\mathbb{R}^q)$ be a strictly causal system. Let $p := \max\{m, q\}$, and define the identity injection maps $\mathscr{I}_1: S(\mathbb{R}^m) \to S(\mathbb{R}^p)$ and $\mathscr{I}_2: S(\mathbb{R}^q) \to S(\mathbb{R}^p)$ as follows. If $q \geqslant m$, write $S(\mathbb{R}^p) = S(\mathbb{R}^q) = S(\mathbb{R}^m) \times S(\mathbb{R}^{q-m})$, let $\mathscr{I}_1: S(\mathbb{R}^m) \to S(\mathbb{R}^p): \mathscr{I}_1[S(\mathbb{R}^m)] = S(\mathbb{R}^m) \times 0$ be the obvious identity injection, and let $\mathscr{I}_2: S(\mathbb{R}^q) \to S(\mathbb{R}^p) (= S(\mathbb{R}^q))$ be the identity map. If $q < m$, write $S(\mathbb{R}^p) = S(\mathbb{R}^m) = S(\mathbb{R}^q) \times S(\mathbb{R}^{m-q})$, let $\mathscr{I}_2: S(\mathbb{R}^q) \to S(\mathbb{R}^p): \mathscr{I}_2[S(\mathbb{R}^q)] = S(\mathbb{R}^q) \times 0$ be the obvious identity injection, and let $\mathscr{I}_1: S(\mathbb{R}^m) \to S(\mathbb{R}^p) (= S(\mathbb{R}^m))$ be the identity map. Then, as we show in a minute, the system

$$\Sigma_\gamma := \gamma \mathscr{I}_1 + \mathscr{I}_2 \Sigma: S(\mathbb{R}^m) \to S(\mathbb{R}^p) \tag{5}$$

where γ is a $p \times p$ constant non-singular matrix, is injective by the strict causality of the system Σ. The implementation of the injections $\mathscr{I}_1$ and $\mathscr{I}_2$ is very simple—it amounts to increasing the dimension of some vectors through augmentation by entries of zeros (see Hammer 1988 for details). To simplify the notation, we usually abbreviate and denote $\mathscr{I}_1 u$ by u and $\mathscr{I}_2 y$ by y. It can be seen that, when stabilizing the system Σ_γ in the configuration (1), we in fact obtain stabilization of the original system Σ in the configuration given in Fig. 2. Note that in Fig. 2 γ is to be interpreted as $\gamma\mathscr{I}_1$, in consistency with our notational convention.

There are several simplifications that result when the system Σ_γ is used instead of the system Σ as the basic system to be stabilized. One of them is the fact that Σ_γ is always injective when Σ is strictly causal, as is shown later. This means that Σ_γ has a left inverse. Moreover, when the original system Σ is recursive, the left inverse of Σ_γ is very easy to compute. Indeed, assume that Σ has a recursive representation $x_{k+1} = f(x_k, u_k)$. Let $u \in S(\mathbb{R}^m)$ be an input sequence, and let $x := \Sigma u$ be the corresponding output sequence. Denoting $z := \Sigma_\gamma u$, and using the abbreviated notation, we obtain $z = x + \gamma u$, so that $z_i = x_i + \gamma u_i$ for all integers $i \geqslant 0$. Therefore

$$z_{k+1} = x_{k+1} + \gamma u_{k+1} = f(x_k, u_k) + \gamma u_{k+1} = f((z - \gamma u)_k, u_k) + \gamma u_{k+1}$$

and, invoking the invertibility of γ, we obtain

$$\left.\begin{aligned} u_{k+1} &= \gamma^{-1}\{z_{k+1} - f((z-\gamma u)_k, u_k)\}, \quad k=0,1,2,\ldots \\ u_0 &= \gamma^{-1}\{z_0 - x_0\} \end{aligned}\right\} \tag{6}$$

where x_0 is the given initial condition of the system Σ, and where the relations are valid for any sequence $z \in \operatorname{Im} \Sigma_\gamma$. Thus, the input sequence u of Σ_γ can readily be computed from the output sequence z of Σ_γ in a recursive manner, using the given recursion function and initial conditions of the system Σ. This evidently amounts to a left inversion of the system Σ_γ, and we use these formulae repeatedly in the sequel. It is also clear from (6) that this left inverse is causal, and we summarize our discussion in the following proposition.

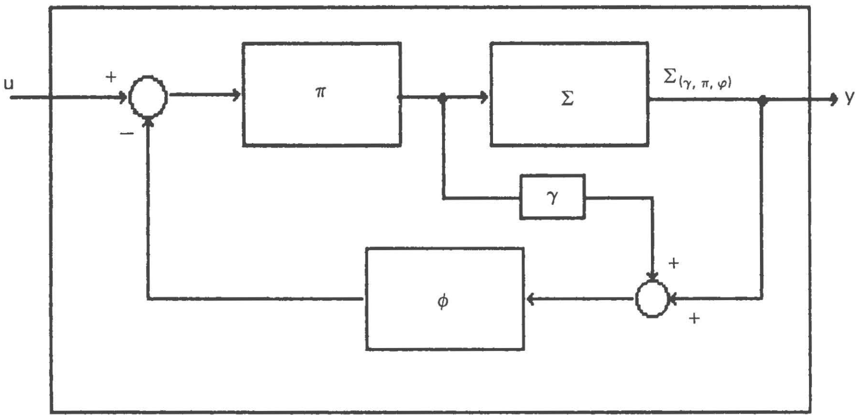

Figure 2.

Proposition 1

Let $\Sigma: S(\mathbb{R}^m) \to S(\mathbb{R}^q)$ be a strictly causal recursive system having a recursive representation $x_{k+1} = f(x_k, u_k)$. Let $p := \max\{m, q\}$, and let γ be a $p \times p$ constant invertible matrix. Then the system $\Sigma_\gamma: S(\mathbb{R}^m) \to \operatorname{Im} \Sigma_\gamma$ defined by (5) is a bicausal system.

The control configurations that we discuss are all internally stable, namely, they are stable, and their stability is not disturbed by small noise signals that might affect the signals at the entry ports of any or all of the subsystems of which the configurations consist. We have seen in Hammer (1988) that if we stabilize the system Σ_γ using the control configuration in Fig. 1 then the configuration in Fig. 2 for Σ is internally stable as well. Thus, we have only to concern ourselves with the Fig. 1 configuration. In order to stabilize this configuration, we use ideas developed in our previous reports, ideas that allow considerable simplification of the stabilization procedure. These ideas involve use of the theory of fraction representations of non-linear systems, which we review shortly. Throughout our discussion we assume that the system Σ is operated by bounded input sequences, namely, that there is a real number $\alpha > 0$ such that $\Sigma: S(\alpha^m) \to S(\mathbb{R}^p)$. The actual value of α is immaterial. Similarly, we assume that the closed-loop system $\Sigma_{(\gamma,\pi,\phi)}$ of Fig. 2 is operated by input sequences bounded by an arbitrary fixed real number $\theta > 0$.

Recall that a right fraction representation of the system $\Sigma: S(\alpha^m) \to S(\mathbb{R}^p)$ is a representation of the form $\Sigma = PQ^{-1}$, where $P: S \to S(\mathbb{R}^p)$ and $Q: S \to S(\alpha^m)$ are stable systems with Q invertible, and where $S \subset S(\mathbb{R}^q)$ is called the factorization space of the fraction representation. Of particular importance to us are coprime right fraction representations, which are fraction representations in which the systems P and Q are right coprime according to the following definition (Hammer 1985 a, 1987). ($P^*[S]$ is the set of all u satisfying $Pu \in S$.)

Definition 1

Let $S \subset S(\mathbb{R}^q)$ be a subspace. A pair of stable systems $P: S \to S(\mathbb{R}^p)$ and $Q: S \to S(\mathbb{R}^m)$ are *right coprime* if the following two conditions are satisfied.

(i) For every real $\tau > 0$ there is a real $\theta > 0$ such that

$$P^*[S(\tau^p)] \cap Q^*[S(\tau^m)] \subset S(\theta^q)$$

(ii) For every real $\tau > 0$, the set $S \cap S(\tau^q)$ is a closed subset of $S(\tau^q)$.

In view of (3), the solution of the stabilization problem involves the search for a pair of stable systems A and B satisfying the equation $AP + BQ = M$, where P and Q arise from a fraction representation $\Sigma = PQ^{-1}$ of the given system Σ, and where M is a unimodular system. The existence of such systems is guaranteed whenever P and Q are right coprime, as given in Theorem 1.

Theorem 1

Let $\Sigma: S(\alpha^m) \to S(\mathbb{R}^p)$ be an injective system, and assume it has a right coprime fraction representation $\Sigma = PQ^{-1}$, where $P: S \to S(\mathbb{R}^p)$ and $Q: S \to S(\alpha^m)$, and where $S \subset S(\mathbb{R}^q)$ for some integer $q > 0$. Then, for every unimodular system $M: S \to S$, there exists a pair of stable systems $A: S(\mathbb{R}^p) \to S(\mathbb{R}^q)$ and $B: S(\alpha^m) \to S(\mathbb{R}^q)$ such that $AP + BQ = M$.

The existence of right coprime fraction representations is related in a fundamental way to the concept of a homogeneous system, which is now defined (Hammer 1985 a, 1987).

Definition 2

A system $\Sigma: S(\mathbb{R}^m) \to S(\mathbb{R}^p)$ is a *homogeneous system* if the following holds for every real number $\alpha > 0$. For every subspace $S \subset S(\alpha^m)$ for which there exists a real number $\tau > 0$ satisfying $\Sigma[S] \subset S(\tau^p)$, the restriction of Σ to the closure $\bar{S}$ of S in $S(\alpha^m)$ is a continuous map $\Sigma: \bar{S} \to S(\tau^p)$.

Homogeneous systems are the only systems possessing right coprime fraction representations, as stated in the following theorem (Hammer 1985 a, 1987).

Theorem 2

An injective system $\Sigma: S(\alpha^m) \to S(\mathbb{R}^p)$ has a right coprime fraction representation if and only if it is a homogeneous system.

The systems that we will consider in this paper are all homogeneous systems, according to the next statement (Hammer 1987).

Proposition 2

Let $\Sigma: S(\mathbb{R}^m) \rightarrow S(\mathbb{R}^p)$ be a recursive system. If Σ has a recursive representation $x_{k+1} = f(x_k, u_k)$ with a continuous recursion function f, then Σ is a homogeneous system.

Furthermore, when the system Σ is homogeneous, the system Σ_γ of (5) is also (Hammer 1987).

Proposition 3

Let $\Sigma: S(\mathbb{R}^m) \rightarrow S(\mathbb{R}^q)$ be a homogeneous system, and let Σ_γ be defined as in (5). Then, $\Sigma_\gamma: S(\mathbb{R}^m) \rightarrow S(\mathbb{R}^p)$ is a homogeneous system.

It is quite easy to construct a right coprime fraction representation for an injective homogeneous system $\Sigma: S(\alpha^m) \rightarrow S(\mathbb{R}^p)$. Indeed, since Σ is injective, its restriction $\Sigma: S(\alpha^m) \rightarrow \operatorname{Im} \Sigma$ is a set isomorphism, and consequently it possesses an inverse $\Sigma^{-1}: \operatorname{Im} \Sigma \rightarrow S(\alpha^m)$. It has been shown (Hammer 1987, § 3) that Σ^{-1} is a stable system, so that, defining the systems

$$\left.\begin{aligned} P &:= I: \operatorname{Im} \Sigma \rightarrow \operatorname{Im} \Sigma \\ Q &:= \Sigma^{-1}: \operatorname{Im} \Sigma \rightarrow S(\alpha^m) \end{aligned}\right\} \tag{7}$$

we obtain a right fraction representation $\Sigma = PQ^{-1}$, which, as can readily be seen, is right coprime. Once we have one right coprime fraction representation $\Sigma = PQ^{-1}$ of the system $\Sigma: S(\alpha^m) \rightarrow S(\mathbb{R}^p)$, any other right coprime fraction representation of Σ is of the form $\Sigma = P_1 Q_1^{-1}$ where $P_1 = PM$ and $Q_1 = QM$, and where M is a unimodular system (Hammer 1985 a, 1987).

Now, let $\Sigma: S(\alpha^m) \rightarrow S(\mathbb{R}^p)$ be the system that needs to be stabilized, and let $\Sigma = PQ^{-1}$ be a right coprime fraction representation of it. Referring to Fig. 1, choose the precompensator π and the feedback compensator ϕ in accordance with (1), so that $\pi = B^{-1}$ and $\phi = A$, where A and B are stable systems, B is invertible and bicausal and A is causal. As indicated in (4), this choice reduces the stabilization problem to the problem of finding an appropriate pair of stable systems A, B that satisfy the equation $AP + BQ = M$, where M is a unimodular system. In Hammer (1988) an implementable solution to this problem has been derived, and our objective in this paper is to study the robustness properties of that solution. For this purpose, we review some further basic facts from Hammer (1988).

First, as seen in Hammer (1988, § 3), there is no impairment of generality when we assume that the dimension of the input space of the systems considered is equal to the dimension of their output space. Thus, we first restrict our attention to systems for which the dimension of the input space is equal to the dimension of the output space, namely, to systems $\Sigma: S(\alpha^m) \rightarrow S(\mathbb{R}^m)$. We also assume that the system Σ has a recursive representation of the form $x_{k+1} = f(x_k, u_k)$, where $f: \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ is a continuous function. In such a case, the system Σ is evidently strictly causal, homogeneous and recursive. Following the earlier discussion, we do not consider the stabilization of the system Σ directly, but rather the stabilization of the system

$$\Sigma_\gamma := \gamma + \Sigma: S(\mathbb{R}^m) \rightarrow S(\mathbb{R}^m) \tag{8}$$

which is the sum of Σ and a non-singular static system represented by the constant $m \times m$ non-singular matrix γ. Clearly, Σ_γ depends on the matrix γ. As seen in

Proposition 1, the system Σ_γ is bicausal whenever the matrix γ is invertible. The $m \times m$ non-singular matrix γ is chosen so that the following condition is satisfied.

Condition 1

There is a real number $\delta > 0$ such that $S(\delta^m) \subset \Sigma_\gamma[S(\alpha^m)]$ for some real number $\alpha > 0$.

As shown by Hammer (1988), a non-singular matrix γ satisfying Condition 1 exists for most cases of practical interest, and its computation is quite straightforward (see Hammer 1988, § 3). Having the matrix γ at our disposition, the stabilization procedure derived by Hammer (1988) proceeds as follows. (Recall that $\Sigma: S(\mathbb{R}^m) \to S(\mathbb{R}^m)$.)

Step 1

First, in view of the fact that the system $\Sigma_\gamma: S(\mathbb{R}^m) \to S(\mathbb{R}^m)$ is bicausal and onto, it has an inverse $\Sigma_\gamma^{-1}: S(\mathbb{R}^m) \to S(\mathbb{R}^m)$. When the non-singular matrix γ is chosen so that Condition 1 is satisfied with the real number $\alpha > 0$, the restriction $Q := \Sigma_\gamma^{-1}: \Sigma_\gamma[S(\alpha^m)] \to S(\alpha^m)$ is a stable system (see Hammer 1988). Given a recursive representation of the system Σ, we can easily obtain a recursive representation of the system Q using (6), and it also follows that Q is bicausal. Furthermore, letting $P := I: \Sigma_\gamma[S(\alpha^m)] \to \Sigma_\gamma[S(\alpha^m)]$ be the restriction of the identity system, it is easy to see that $\Sigma_\gamma = PQ^{-1}$ is a coprime fraction representation, valid over the space $S(\alpha^m)$.

Step 2

Next, let $\delta > 0$ be a real number satisfying Condition 1, so that $S(\delta^m) \subset \Sigma_\gamma[S(\alpha^m)]$. Choose a real number ζ satisfying $0 < \zeta < \delta$, and let $I_\zeta: S(\zeta^m) \to S(\zeta^m)$ be the identity system. We construct a recursive, causal, stable, and uniformly l^∞-continuous system $E: S(\mathbb{R}^m) \to S(\zeta^m)$ which is an extension of I_ζ as follows. We first define a function $e: \mathbb{R}^m \to [-\zeta, \zeta]^m$ given, for every vector $x = (x_1, \dots, x_m) \in \mathbb{R}^m$, by $e(x_1, \dots, x_m) := (\alpha_1, \dots, \alpha_m)$, where $\alpha_i := x_i$ if $|x_i| \leq \zeta$ and $\alpha_i := \zeta \operatorname{sign}(x_i)$ if $|x_i| > \zeta$, and where $\operatorname{sign}(\cdot)$ is ± 1, depending on the sign of the argument. Then, we define the system $E: S(\mathbb{R}^m) \to S(\zeta^m)$ as the (static) recursive system with the representation

$$E: y_k := e(u_k), \quad k = 0, 1, 2, \dots \tag{9}$$

where $y = Eu$.

Step 3

Next, we extend the restriction $Q: S(\delta^m) \to S(\alpha^m)$, which is a stable and causal system, into a stable, bounded, and causal system $Q_*: S(\mathbb{R}^m) \to S(\alpha^m)$, by setting

$$Q_* := QE \tag{10}$$

Step 4

Further, let $\theta > 0$ be the previously chosen bound on the amplitude of the input sequences to the final stabilized system $\Sigma_{(\gamma,\pi,\phi)}$, so that $\Sigma_{(\gamma,\pi,\phi)}$ is operated only by input sequences from $S(\theta^m)$. Choose a recursive, unimodular, bicausal, and uniformly l^∞-continuous system $M: S(\mathbb{R}^m) \to S(\mathbb{R}^m)$, satisfying the condition $M^{-1}[S((5\theta)^m)] \subset S(\zeta^m)$. The unimodular system M controls the dynamical behaviour of the closed-loop system $\Sigma_{(\gamma,\pi,\phi)}$, as shown by Hammer (1988). The condition $M^{-1}[S((5\theta)^m)] \subset S(\zeta^m)$ has been discussed in Hammer (1988), showing that it is simply a scaling condition having no dynamical implications. For instance, a simple choice for M is $M := \beta I : S(\mathbb{R}^m) \to S(\mathbb{R}^m)$, where β is a constant real number satisfying $\beta \geq 5\theta/\zeta$.

Step 5

Finally, let ε be a real number satisfying $0 < \varepsilon < \theta/\alpha$, and define the systems

$$\left.\begin{aligned} A &:= M - \varepsilon Q_* : S(\mathbb{R}^m) \to S(\mathbb{R}^m) \\ B &:= \varepsilon I : S(\mathbb{R}^m) \to S(\mathbb{R}^m) \end{aligned}\right\} \tag{11}$$

where $I : S(\mathbb{R}^m) \to S(\mathbb{R}^m)$ is the identity system. Using (1) and the systems A and B given by (11), we obtain the compensators $\pi = (1/\varepsilon)I$ and $\phi = M - \varepsilon Q_*$. Note that the precompensator $\pi = B^{-1} = (1/\varepsilon)I$ is simply an amplifier with an amplification factor of $1/\varepsilon$. With this choice of compensators, the composite system of Fig. 2 becomes internally stable, and we have the following result, which was proved in Hammer (1988).

Theorem 3

Let $\Sigma : S(\mathbb{R}^m) \to S(\mathbb{R}^m)$ be a strictly causal homogeneous system, let Σ_γ be given by (8), and let $\theta > 0$ be a real number. Assume there is an $m \times m$ constant non-singular matrix γ and a real number $\delta > 0$ such that $S(\delta^m) \subset \Sigma_\gamma[S(\alpha^m)]$ for some real $\alpha > 0$. Let ζ, $\varepsilon > 0$ be real numbers satisfying $\zeta < \delta$ and $\varepsilon < \theta/\alpha$, and let $M : S(\mathbb{R}^m) \to S(\mathbb{R}^m)$ be a recursive, unimodular, bicausal, and uniformly l^∞-continuous system, satisfying the condition $M^{-1}[S((5\theta)^m)] \subset S(\zeta^m)$. Then, for the compensators $\pi := B^{-1}$ and $\phi := A$, where A and B are given by (11) with Q_* given by (10), the closed-loop system $\Sigma_{(\gamma,\pi,\phi)}$ is internally stable for all input sequences $u \in S(\theta^m)$.

We remark that the compensators π and ϕ of Theorem 3 can readily be explicitly computed using the given recursive representations of the systems Σ and M, and they are rather easy to implement (see Hammer 1988). In § 3 we discuss the robustness properties of the stabilizing control configuration described in Theorem 3.

3. Stabilization of systems with uncertainties

In this section we discuss the problem of stabilizing a non-linear recursive system when only a nominal description of the system is known. More accurately, let $\Sigma : S(\mathbb{R}^m) \to S(\mathbb{R}^p)$ be a strictly causal recursive system having a representation of the form $x_{k+1} = f(x_k, u_k)$, where the recursion function f is continuous. Assume that only a nominal recursion function f_n of the system Σ is known, and that the actual recursion function f of Σ is of the form $f(x_k, u_k) = f_n(x_k, u_k) + v(x_k, u_k)$, where v is a continuous function describing the deviation from nominality. We denote by Σ_{n} the system described by the recursion function f_{n}, namely, the nominal system. The problem is as follows. Using the nominal recursion function f_{n}, design a control configuration of the form of Fig. 2 in such a way that the configuration remains internally stable when the actual system Σ is inserted in it. To simplify the treatment, we first assume that the system Σ has an input space of the same dimension as the output space, namely that $\Sigma : S(\mathbb{R}^m) \to S(\mathbb{R}^m)$. Later, we see that this assumption may be removed without complication.

The discussion in this section depends on a result regarding internal stabilization of non-linear systems which has been derived by Hammer (1988), and which we reproduce here. Before doing so, we have to review the following notion. A stable system $A:S(\mathbb{R}^m)\to S(\mathbb{R}^p)$ is *differentially bounded* by a real number $\theta > 0$ if there is a real number $\varepsilon > 0$ such that, for every pair of elements $y, y' \in S(\mathbb{R}^m)$ satisfying $|y - y'| < \varepsilon$, we have $|A(y) - A(y')| < \theta$ (see Hammer 1988 for more details). Also, we say that the two subspaces $S_1 \subset S(\mathbb{R}^q)$ and $S_2 \subset S(\mathbb{R}^m)$ are 'stability-morphic' if there is a bicausal and unimodular isomorphism $M: S_1 \cong S_2$.

Theorem 4

Let $\Sigma: S(\alpha^m) \to S(\mathbb{R}^p)$ be a strictly causal homogeneous system, let $\Sigma_\gamma: S(\alpha^m) \to S(\mathbb{R}^p)$ be given by (5), and let $\theta > 0$ be a real number. Let $\Sigma_\gamma = PQ^{-1}$ be a right coprime fraction representation, and let $S \subset S(\mathbb{R}^q)$ be its factorization space. Assume S contains a subset S' which is stability-morphic to $S((5\theta)^m)$, and let $M: S' \to S((5\theta)^m)$ be a bicausal and unimodular isomorphism. Assume there is a pair of stable systems $A: S(\mathbb{R}^p) \to S(\mathbb{R}^m)$ and $B: S(\mathbb{R}^m) \to S(\mathbb{R}^m)$ satisfying the equation

$$APv + BQv = Mv \quad \text{for all } v \in S'$$

where A is causal and B is bicausal. If A and B are differentially bounded by θ, then the closed-loop system $\Sigma_{(\gamma, B^{-1}, A)}$ is internally stable for all input sequences $u \in S(\theta^m)$.

Now, let $\Sigma_{\mathrm{n}}: S(\mathbb{R}^m) \to S(\mathbb{R}^m)$ be the nominal description of the system that has to be stabilized, and assume there is an $m \times m$ constant non-singular matrix γ satisfying Condition 1. Denoting $\Sigma_{\mathrm{n}\gamma} := \gamma + \Sigma_{\mathrm{n}}$, we then have $S(\delta^m) \subset \Sigma_{\mathrm{n}\gamma}[S(\alpha_1^m)]$ for real numbers δ, $\alpha_1 > 0$. This implies that $(\Sigma_{\mathrm{n}\gamma})^{-1}$ is defined on $S(\delta^m)$ and that the restriction $(\Sigma_{\mathrm{n}\gamma})^{-1}: S(\delta^m) \to S(\mathbb{R}^m)$ is stable. Now, assume that the deviation of Σ from its nominal description Σ_{n} is such that the matrix γ also satisfies Condition 1 for Σ, namely, that $S(\delta^m) \subset \Sigma_\gamma[S(\beta^m)]$, where δ is as given before and $\beta > 0$ is a suitable real number. Then $(\Sigma_\gamma)^{-1}$ is also defined on $S(\delta^m)$, and the restriction $(\Sigma_\gamma)^{-1}: S(\delta^m) \to S(\mathbb{R}^m)$ is a stable system. Denote

$$\alpha := \max\{\alpha_1, \beta\} \tag{12}$$

In view of (7), we can construct for the restrictions of the systems

$$\Sigma_\gamma: (\Sigma_\gamma)^{-1}[S(\delta^m)] \to S(\delta^m)$$

and

$$\Sigma_{\mathrm{n}\gamma}: (\Sigma_{\mathrm{n}\gamma})^{-1}[S(\delta^m)] \to S(\delta^m)$$

the following right coprime fraction representations:

$$\left.\begin{array}{ll} \Sigma_\gamma = PQ^{-1}; & P := I: S(\delta^m) \to S(\delta^m), \quad Q := (\Sigma_\gamma)^{-1}: S(\delta^m) \to (\Sigma_\gamma)^{-1}[S(\delta^m)] \\ \Sigma_{\mathrm{n}\gamma} = P_{\mathrm{n}}Q_{\mathrm{n}}^{-1}; & P_{\mathrm{n}} := I: S(\delta^m) \to S(\delta^m), \quad Q_{\mathrm{n}} := (\Sigma_{\mathrm{n}\gamma})^{-1}: S(\delta^m) \to (\Sigma_{\mathrm{n}\gamma})^{-1}[S(\delta^m)] \end{array}\right\} \tag{13}$$

where I denotes the identity system. As we can see, these fraction representations for Σ_γ and $\Sigma_{\mathrm{n}\gamma}$ both have the same numerators $P = P_{\mathrm{n}}$, so that the effect of the deviation of Σ from nominality is entirely incorporated into the deviation of the denominator Q from the nominal denominator Q_{n}. We have indicated in § 1 the qualitative implications of these facts, and the method by which they simplify the discussion of the robust stabilization problem. We note that the fraction representations of (13) both have the same factorization space $S = S(\delta^m)$. Finally, we comment that a recursive representation of Q_{n} may be obtained in a straightforward manner using (6) with the nominal recursion function f_{n}.

Furthermore, let $\zeta > 0$ be a real number satisfying $\zeta < \delta$, and let $E: S(\mathbb{R}^m) \to S(\zeta^m)$ be the extension of the identity system $I: S(\zeta^m) \to S(\zeta^m)$ defined in (9). As in (10), we define the systems

$$\left.\begin{aligned} Q_{n*} &:= Q_n E: S(\mathbb{R}^m) \to S(\mathbb{R}^m) \\ Q_* &:= QE: S(\mathbb{R}^m) \to S(\mathbb{R}^m) \end{aligned}\right\} \tag{14}$$

Since E is an extension of the identity system, we have $Q_{n*}v = Q_n v$ and $Q_* v = Qv$ for all $v \in S(\zeta^m)$.

We now prepare for the application of Theorem 4. First, from our construction, we have $Q_{n*} = Q_n$ and $Q_* = Q$ over the space $S(\zeta^m)$; we therefore require that the space S' of Theorem 3 be contained in $S(\zeta^m)$. In such a case, the unimodular transformation M of Theorem 3 has to satisfy the condition $M^{-1}[S((5\theta)^m)] \subset S(\zeta^m)$. We remark that the latter condition is purely related to scaling, and has no dynamical implications (see Hammer 1988). Now, assume we have systems A, B and M satisfying all the conditions of Theorem 4 for the nominal system $\Sigma_{n\gamma}$ and its fraction representation $\Sigma_{n\gamma} = P_n Q_n^{-1}$. Then, with $S' := M^{-1}[S((5\theta)^m)] \subset S(\zeta^m)$, we have

$$AP_n v + BQ_n v = Mv \quad \text{for all } v \in S' \tag{15}$$

When we insert the actual system Σ into the configuration, using the same systems A and B, and recalling that $P = P_n$, we obtain the equation

$$APv + BQv = Mv + (BQ - BQ_n)v \quad \text{for all } v \in S' \tag{16}$$

Furthermore, we use the stabilization scheme described by (1) and (11) to stabilize the nominal system Σ_n. In this case, $B = \varepsilon I$ is simply a constant amplifier, and, denoting

$$\left.\begin{aligned} F &:= Q_* - Q_{n*}: S(\mathbb{R}^m) \to S(\mathbb{R}^m) \\ \mathscr{M} &:= \varepsilon F: S(\mathbb{R}^m) \to S(\mathbb{R}^m) \\ M' &:= M + \mathscr{M}: S(\mathbb{R}^m) \to S(\mathbb{R}^m) \end{aligned}\right\} \tag{17}$$

we obtain

$$APv + BQv = M'v \quad \text{for all } v \in S' \tag{18}$$

Now refer to Theorem 4. Suppose we can show that the system M' of (17) is still bicausal and unimodular over the relevant spaces, and that it still satisfies the condition $M'^{-1}[S((5\theta)^m)] \subset S(\zeta^m)$. Then, in view of (18), the conditions of Theorem 4 clearly hold for the actual system Σ with $S' := M'^{-1}[S((5\theta)^m)]$, and therefore the control configuration remains internally stable when the system Σ is substituted in it for the nominal system Σ_n. Thus, the question of whether internal stability is preserved despite the perturbation in the system Σ, simply reduces to the question of whether the perturbation $\mathscr{M}$ allows M' to inherit the unimodularity and bicausality of M. In order to study the latter question, we discuss first some basic properties of $\mathscr{M}$ itself. Recalling that ε is simply a constant factor, it is clear that many interesting properties of $\mathscr{M}$ are determined by the system F, which basically describes the deviation between the denominators Q and Q_n. The system F has the following properties.

Lemma 1

Let $F:S(\mathbb{R}^m)\to S(\mathbb{R}^m)$ be the system given by (17). Then, the following hold true:

(i) F is a strictly causal system;

(ii) F is a stable system;

(iii) $\operatorname{Im} F\subset S((2\alpha)^m)$, where α is given by (12).

Proof

In view of the definition (9) of the system E, it is clearly sufficient to show that (i) holds for the restriction $F':S(\zeta^m)\to S(\mathbb{R}^p)$ of F, which satisfies $F'=Q-Q_{\mathrm{n}}$. Now, we have $Q_{\mathrm{n}}:=(\gamma+\Sigma_{\mathrm{n}})^{-1}$ and $Q:=(\gamma+\Sigma)^{-1}$, and it is easy to directly verify that $Q_{\mathrm{n}}=\gamma^{-1}-\gamma^{-1}\Sigma_{\mathrm{n}}Q_{\mathrm{n}}$ and $Q=\gamma^{-1}-\gamma^{-1}\Sigma Q$, so that $F'=Q-Q_{\mathrm{n}}=\gamma^{-1}(\Sigma_{\mathrm{n}}Q_{\mathrm{n}}-\Sigma Q)$. Recalling that Σ and Σ_{n} are strictly causal by assumption, that Q and Q_{n} are bicausal (Proposition 1), and using the obvious fact that γ is bicausal then $F'=\gamma^{-1}(\Sigma_{\mathrm{n}}Q_{\mathrm{n}}-\Sigma Q)$ implies that F is strictly causal. Part (ii) is a direct consequence of the stability of the systems Q, Q_{n}, and E. To prove (iii), recall that the matrix γ is chosen to satisfy Condition 1 for both Σ and Σ_{n}, so that

$$\operatorname{Im} Q_*=(\Sigma_\gamma)^{-1}E[S(\mathbb{R}^m)]=(\Sigma_\gamma)^{-1}[S(\zeta^m)]\subset(\Sigma_\gamma)^{-1}[S(\delta^m)]\subset S(\alpha_1^m)$$

and, similarly

$$\operatorname{Im} Q_{\mathrm{n}*}=(\Sigma_{\mathrm{n}\gamma})^{-1}[S(\zeta^m)]\subset(\Sigma_{\mathrm{n}\gamma})^{-1}[S(\delta^m)]\subset S(\beta^m)$$

Since $\alpha=\max\{\alpha_1,\beta\}$ and $F=Q_*-Q_{\mathrm{n}*}$, it follows directly that $\operatorname{Im} F\subset S((2\alpha)^m)$. □

Now, let $M:S(\mathbb{R}^m)\to S(\mathbb{R}^m)$ be a unimodular, bicausal, and uniformly l^∞-continuous system satisfying the condition $M^{-1}[S((5\theta)^m)]\subset S(\zeta^m)$, and consider the disturbed system $M':=M+\mathscr{M}$, where $\mathscr{M}:S(\mathbb{R}^m)\to S(\mathbb{R}^m)$ is given by (17). Applying part (i) of Lemma 1 to $\mathscr{M}=\varepsilon F$, we obtain that $\mathscr{M}$ is a strictly causal system. It is quite easy to see that the sum of a bicausal system and a strictly causal system is always a bicausal system (e.g. Hammer 1984 b). Consequently, under our perturbations, M' inherits the bicausality of M. Furthermore, as shown in Lemma 2, under some mild conditions M' inherits all the other properties of M which are relevant to the preservation of internal stability.

Lemma 2

Let $M:S(\mathbb{R}^m)\to S(\mathbb{R}^m)$ be a bicausal unimodular system, and assume there are positive real numbers θ, ξ, and ζ such that the condition $M^{-1}[S((5\theta+\xi)^m)]\subset S(\zeta^m)$ is satisfied. Let $\mathscr{M}:S(\mathbb{R}^m)\to S(\mathbb{R}^m)$ be a stable and strictly causal system satisfying the condition $\mathscr{M}[S(\mathbb{R}^m)]\subset S(\xi^m)$, and denote $M':=M+\mathscr{M}:S(\mathbb{R}^m)\to S(\mathbb{R}^m)$. Then, the following are true:

(i) the system M' is bicausal;

(ii) the restriction $M':S(\zeta^m)\to M'[S(\zeta^m)]$ is unimodular;

(iii) the condition $M'^{-1}[S((5\theta)^m)]\subset S(\zeta^m)$ is satisfied.

Proof

We have discussed the validity of (i) in the paragraph preceding Lemma 2. To prove (ii), we notice that, from the bicausality of M', it follows that the restriction

$M':S(\zeta^m)\to M'[S(\zeta^m)]$ is a set isomorphism. Consequently, since $S(\zeta^m)$ is a compact set and M' is obviously stable, it follows that the restriction $M':S(\zeta^m)\to M'[S(\zeta^m)]$ is in fact a homeomorphism (e.g. Kuratowski 1961), and hence is unimodular. To prove (iii) we let $u\in S((5\theta)^m)$ be an arbitrary element. From (i), the map M' is surjective, so there is an element $v\in S(\mathbb{R}^m)$ such that $u=M'v$. We have $u=Mv+\mathscr{M}v$, or $Mv=u-\mathscr{M}v$, and, in view of the fact that $\mathscr{M}v\in S(\xi^m)$, it follows that $Mv\in S((5\theta+\xi)^m)$, or $v\in M^{-1}[S((5\theta+\xi)^m)]$. But, then, since we have $M^{-1}[S((5\theta+\xi)^m)]\subset S(\zeta^m)$, we obtain that $v=M'^{-1}u\in S(\zeta^m)$. Finally, since the latter is valid for any element $u\in S((5\theta)^m)$, we conclude that $M'^{-1}[S((5\theta)^m)]\subset S(\zeta^m)$, and (iii) holds true. □

Lemma 2 provides us with some rather simple means of accommodating deviations of the system Σ in our stabilization theory. We have to choose the nominal unimodular and bicausal transformation M so that it satisfies the condition

$$M^{-1}[S((5\theta+\xi)^m)]\subset S(\zeta^m) \tag{18}$$

for some real $\xi>0$, instead of just satisfying $M^{-1}[S((5\theta)^m)]\subset S(\zeta^m)$, as required in Theorem 4. Equation (18) is merely an amplitude scaling condition, and has no dynamical implications, as discussed later (see also Hammer 1988, § 3). Assume that (18) is satisfied, and recall that in Lemma 1 we showed that $\mathscr{M}=\varepsilon F$, that F is strictly causal, and that $\operatorname{Im} F\subset S((2\alpha)^m)$. Then, when choosing $\varepsilon\leqslant\xi/(2\alpha)$, we obtain $\mathscr{M}[S(\mathbb{R}^m)]\subset S(\xi^m)$, and it follows that the conditions of Lemma 2 are all satisfied. In such a case, we obtain from Lemma 2 that M' is bicausal and unimodular over the relevant spaces, and that $M'^{-1}[S((5\theta)^m)]\subset S(\zeta^m)$, so that the disturbed system M' satisfies the conditions of Theorem 4. Thus, the configuration $\Sigma_{(\gamma,B^{-1},A)}$ remains internally stable despite the deviations in the system Σ. This discussion completes the proof of Theorem 5 which is stated later. Before this, we summarize the basic steps of our stabilization procedure.

Using the nominal system Σ_n and the right coprime fraction representation $\Sigma_{n\gamma}=P_nQ_n^{-1}$ given by (13), we construct the system Q_{n*} of (14). A recursive representation of Q_n can easily be obtained from the given recursion function f_n of the nominal system Σ_n through (6), and Q_{n*} may be constructed by combining this recursive representation with the recursive representation of the system E of (9). Following (11), we construct the systems

$$\left.\begin{aligned} A &:= M-\varepsilon Q_{n*}:S(\mathbb{R}^m)\to S(\mathbb{R}^m)\\ B &:= \varepsilon I:S(\mathbb{R}^m)\to S(\mathbb{R}^m)\end{aligned}\right\} \tag{19}$$

and, from these, we construct the compensators

$$\left.\begin{aligned} \pi &= (1/\varepsilon)I:S(\mathbb{R}^m)\to S(\mathbb{R}^m)\\ \phi &= M-\varepsilon Q_{n*}:S(\mathbb{R}^m)\to S(\mathbb{R}^m)\end{aligned}\right\} \tag{20}$$

Theorem 5

Let $\Sigma_n:S(\mathbb{R}^m)\to S(\mathbb{R}^m)$ be a strictly causal homogeneous system, and let $\theta>0$ be a real number. Assume there is an $m\times m$ constant non-singular matrix γ and a real number $\delta>0$ such that $S(\delta^m)\subset\Sigma_{n\gamma}[S(\alpha^m)]$ for some real $\alpha>0$, where $\Sigma_{n\gamma}=\gamma+\Sigma_n$. Let ζ, ε, $\xi>0$ be real numbers, where $\zeta<\delta$, $\varepsilon<\min\{\theta/\alpha,\ \xi/(2\alpha)\}$, and let $M:S(\mathbb{R}^m)\to S(\mathbb{R}^m)$ be a bicausal, unimodular, and uniformly l^∞-continuous system satisfying (18). Finally, let π and ϕ be the compensators given by (20). Then, for any strictly causal and homogeneous system $\Sigma : S(\mathbb{R}^m) \to S(\mathbb{R}^m)$ satisfying the condition $S(\delta^m) \subset \Sigma_\gamma[S(\alpha^m)]$, the closed-loop system $\Sigma_{(\gamma,\pi,\phi)}$ is internally stable for all input sequences bounded by θ.

We now provide a brief discussion of condition (18). Consider an arbitrary bicausal, unimodular, and uniformly l^∞-continuous system $M_1 : S(\mathbb{R}^m) \to S(\mathbb{R}^m)$. By the stability of M_1^{-1}, there is, for every choice of the real numbers θ, $\xi > 0$, a real number $\lambda > 0$ satisfying $M_1^{-1}[S((5\theta + \xi)^m)] \subset S(\lambda^m)$. Then, it is obvious that the system $M := M_1(\lambda/\zeta)$ is still unimodular, bicausal, and uniformly l^∞-continuous, and it satisfies $M^{-1}[S((5\theta + \xi)^m)] \subset S(\zeta^m)$. It is also clear that M results from M_1 through a simple scaling transformation, and hence the dynamical properties of M are the same as those of M_1. We may conclude then that (18) is just a requirement for appropriate scaling, and it imposes no adverse restrictions on our systems.

The main qualitative consequence of Theorem 5 is basically the following. The only substantial limitation that the preservation of internal stability imposes on the deviation of the system Σ from the nominal system Σ_n is the requirement that there exists an $m \times m$ constant and non-singular matrix γ such that the conditions

$$S(\delta^m) \subset \Sigma_{n\gamma}[S(\alpha^m)] \quad \text{and} \quad S(\delta^m) \subset \Sigma_\gamma[S(\alpha^m)] \tag{21}$$

are simultaneously satisfied for some real numbers δ, $\alpha > 0$. As we show in the remaining part of this section, this requirement is rather simple to fulfill, and thus our stabilization procedure allows deviations of the parameters of the system Σ from their nominal values.

In the discussion up to this point, we have imposed very few restrictions on the system Σ that needs to be stabilized—we have only required that Σ be a homogeneous and strictly causal system. In order to discuss the explicit implications of condition (21), we restrict ourselves from now on to the consideration of the case when the system Σ has a recursive representation of the form $x_{k+1} = f(x_k, u_k)$, where $f : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ is a continuous function, and where the initial condition x_0 is given. We still assume, as before, that the dimension of the input space of the system Σ is the same as the dimension of its output space, but we show later that the situation in general is very similar.

Assume then that $\Sigma : S(\mathbb{R}^m) \to S(\mathbb{R}^m)$ is the actual system placed in the control loop in Fig. 2, and suppose it has a recursive representation of the form $x_{k+1} = f_n(x_k, u_k) + v(x_k, u_k)$, where f_n is the recursion function of the nominal system $\Sigma_n : S(\mathbb{R}^m) \to S(\mathbb{R}^m)$ for which the loop was designed, and where v is a continuous function describing the deviation from nominality. We wish to study the existence of an $m \times m$ constant non-singular matrix γ satisfying (21) for this particular form of Σ and Σ_n.

Before studying (21) in general, we discuss its implications in the single variable case $m = 1$. The single variable case is, of course, the simplest case here, and an examination of it provides an insight into the general situation. For the sake of clarity, we consider a single variable case in which the non-linearity of the recursion function is bounded. It is convenient to define the following class of systems. Fix a real number $N \geq 0$. Let $\Sigma_n : S(\mathbb{R}) \to S(\mathbb{R})$ be the nominal system, and consider the case where the nominal recursion function is continuous and of the form $f_n(x, u) = ax + bu + \psi(x, u)$, where a and b are real numbers with $b \neq 0$, and where $|\psi(x, u)| \leq N$ for all $x, u \in \mathbb{R}$. For a real number $\Delta > 0$, let $\mathscr{S}(a, b, \Delta)$ be the class of all systems $\Sigma : S(\mathbb{R}) \to S(\mathbb{R})$ having recursive representations of the form $x_{k+1} = f_n(x_k, u_k) + v(x_k, u_k)$, $x_0 = 0$, with v being a continuous function of the form $v(x, u) = \kappa x + \lambda u + \psi_v(x, u)$, where κ and λ are any real numbers satisfying the conditions

$$|\kappa|, |\lambda| \leqslant N \quad \text{and} \quad |\lambda a/b - \kappa| \leqslant \Delta$$

and where $|\psi_v(x, u)| \leqslant N$ for all $x, u \in \mathbb{R}$. Obviously, the class $\mathscr{S}(a, b, \Delta)$ consists of systems whose recursion function $f(x, u) = (a + \kappa)x + (b + \lambda)u + \psi(x, u) + \psi_v(x, u)$ deviates from the nominal recursion function f_n, with κ, λ, and ψ_v determining the deviation. All systems in $\mathscr{S}(a, b, \Delta)$ have, of course, recursion functions with bounded non-linearities, as mentioned.

Proposition 4

Let $\mathscr{S}(a, b, \Delta)$ be the class of systems $\Sigma: S(\mathbb{R}) \to S(\mathbb{R})$ defined in the previous paragraph, and assume that $b \neq 0$ and that $\Delta \leqslant 1 - s$, where $0 < s < 1$. Then, there is a real number $\gamma \neq 0$ for which the following holds. For every real number $\delta > 0$, there is a real number $\alpha > 0$ satisfying $S(\delta^m) \subset \Sigma_\gamma[S(\alpha^m)]$ for all $\Sigma \in \mathscr{S}(a, b, \Delta)$.

Example 1

Before stating the proof of Proposition 4 we consider an example. Assume, for instance, that the nominal system Σ_n is given by the recursive representation

$$\Sigma_n: x_{k+1} = 2x_k + 2u_k + \sin(x_k u_k), \quad x_0 = 0$$

and that the disturbed system Σ has the recursive representation

$$\Sigma: x_{k+1} = (2 + \kappa)x_k + (2 + \lambda)u_k + \sigma \sin(x_k u_k), \quad x_0 = 0$$

where κ, λ, and σ are real numbers. Then, it is easy to see that the class $\mathscr{S}(2, 2, \Delta)$ mentioned in Proposition 4 contains all the systems Σ for which the parameters κ, λ, and σ are in the intervals

$$-\tfrac{1}{3} \leqslant \kappa \leqslant \tfrac{1}{3}, \quad -\tfrac{1}{3} \leqslant \lambda \leqslant \tfrac{1}{3} \quad \text{and} \quad -1 \leqslant \sigma \leqslant 1$$

Recalling Theorem 5, this means that we can design the control configuration in Fig. 2 around the nominal system Σ_n in such a way that it remains internally stable when any of the systems Σ is substituted in it for Σ_n, as long as $-\frac{1}{3} \leqslant \kappa \leqslant \frac{1}{3}$, $-\frac{1}{3} \leqslant \lambda \leqslant \frac{1}{3}$ and $-1 \leqslant \sigma \leqslant 1$. Other types of deviations are also allowed, as long as the conditions of Proposition 4 are satisfied.

Proof

To prove Proposition 4, let $\delta > 0$ and $\gamma \neq 0$ be real numbers, let $\Sigma \in \mathscr{S}(a, b, \Delta)$, let v be an arbitrary sequence in $S(\delta)$, and let $u := \Sigma_\gamma^{-1} v$. Then, using (6) and the fact that $x_0 = 0$, we have

$$\begin{aligned} \gamma u_{k+1} &= v_{k+1} - f(v_k - \gamma u_k), u_k) \\ &= v_{k+1} - [(a + \kappa)(v_k - \gamma u_k) + (b + \lambda)u_k + \psi((v_k - \gamma u_k), u_k) \\ &\quad + \psi_v((v_k - \gamma u_k), u_k)] \\ \gamma u_0 &= v_0 - x_0 = v_0 \end{aligned}$$

Clearly, $|\gamma u_{k+1}| \leqslant |v_{k+1}| + |f((v_k - \gamma u_k), u_k)|$. Denote $n(\gamma) := |(b + \lambda)/\gamma - (a + \kappa)|$ and

$c := (|a| + N)\delta + 2N$. Then, using the assumptions that $|\psi| \leqslant N$, $|\psi_v| \leqslant N$, $|\kappa| \leqslant N$ and $v \in S(\delta)$, so that $|v_k| \leqslant \delta$ and $|v_{k+1}| \leqslant \delta$, we obtain

$$|\gamma u_{k+1}| \leqslant n(\gamma)|\gamma u_k| + (\delta + c)$$

Assume for a moment that $a \neq 0$. Then, we can choose $\gamma := b/a$, which is non-zero since $b \neq 0$. For this value of γ, we obtain $n(\gamma) = |\lambda a/b - \kappa| \leqslant 1 - s$, where, by the assumptions of Proposition 4, s is a fixed number satisfying $0 < s < 1$. Consequently, for every system $\Sigma \in \mathscr{S}(a, b, \Delta)$, we obtain

$$|\gamma u_{k+1}| \leqslant (1 - s)|\gamma u_k| + (\delta + c) \tag{22}$$

and, since $|1 - s| < 1$ and $|\gamma u_0| = |v_0| \leqslant \delta$, it directly follows that there is a real number $\alpha' > 0$ such that $|\gamma u_k| \leqslant \alpha'$ for all integers $k \geqslant 0$. Explicitly, by computing the response of the linear system (22) with the initial condition δ and the constant input $(\delta + c)$, it follows that we can take $\alpha' = \delta + (\delta + c)/s$. Then, letting $\alpha := |1/\gamma|\alpha'$, we obtain that $|u_k| \leqslant \alpha$ for all integers $k \geqslant 0$, or $u \in S(\alpha)$, independently of which system $\Sigma \in \mathscr{S}(a, b, \Delta)$ we use. This shows that there is a real number $\gamma \neq 0$ such that $S(\delta^m) \subset \Sigma_\gamma[S(\alpha^m)]$ for all $\Sigma \in \mathscr{S}(a, b, \Delta)$, which proves our assertion for the case $a \neq 0$. For the case $a = 0$, use $\gamma = 2(|b| + N)/s$, and repeat the above argument. □

The situation where $m > 1$ is completely analogous. We first consider systems having recursion functions in which the non-linear terms are bounded, and we start with some notation. Given an $m \times m$ matrix A with entries a_{ij}, we denote $\|A\| := \max\{|a_{ij}|, i, j = 1, \dots, m\}$. Let $\Sigma_n : S(\mathbb{R}^m) \to S(\mathbb{R}^m)$ be our nominal system, having a recursive representation $x_{k+1} = f_n(x_k, u_k)$ with $x_0 = 0$. Assume the nominal recursion function is of the form $f_n(x_k, u_k) = Fx + Gu + \psi(x, u)$, where F and G are $m \times m$ matrices, and where the function $\psi : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ is continuous and bounded, say $|\psi(x, u)| \leqslant N$ for all $x, u \in \mathbb{R}^m$. Now, for a real number $\Delta > 0$, we define a class $\mathscr{S}(F, G, \Delta)$ of systems that deviate 'by Δ' from the nominal system Σ_n. Specifically, $\mathscr{S}(F, G, \Delta)$ consists of all systems $\Sigma : S(\mathbb{R}^m) \to S(\mathbb{R}^m)$ having recursive representations $x_{k+1} = f_n(x_k, u_k) + v(x_k, u_k)$, $x_0 = 0$, with the deviation function $v : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ being of the form $v(x, u) = \Gamma x + \Lambda u + \psi_v(x, u)$, where Γ and Λ are $m \times m$ matrices satisfying $\|\Gamma\| \leqslant \Delta$ and $\|\Lambda\| \leqslant \Delta$, and where $\psi_v : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ is a bounded continuous function, say $|\psi_v(x, u)| \leqslant N$ for all $x, u \in \mathbb{R}^m$. As usual, we say that a linear system is stabilizable if all its unreachable modes correspond to eigenvalues having absolute value strictly less than one.

Proposition 5

Let $\mathscr{S}(F, G, \Delta)$ be the class of systems $\Sigma : S(\mathbb{R}^m) \to S(\mathbb{R}^m)$ defined in the previous paragraph, and assume that the pair F, G is stabilizable. Then, there is a real number $\Delta > 0$ and an $m \times m$ non-singular matrix γ such that the following holds true. For every real number $\delta > 0$, there is a real number $\alpha > 0$ satisfying $S(\delta^m) \subset \Sigma_\gamma[S(\alpha^m)]$ for all systems $\Sigma \in \mathscr{S}(F, G, \Delta)$.

Proof

Note that, for an $m \times m$ matrix A, we denote by $|A|$ a norm of A such that $|Au| \leqslant |A||u|$ for all elements $u \in \mathbb{R}^m$. Let γ be a non-singular $m \times m$ matrix, let $\delta > 0$ be a real number, and let v be an arbitrary sequence in $S(\delta^m)$. Denote $u := \Sigma_\gamma^{-1} v$, $F' := F + \Gamma$, $G' := G + \Lambda$, $\psi'(x, u) := \psi(x, u) + \psi_v(x, u)$, and note that, by our assumptions, $|\psi'(x, u)| \leqslant 2N$ for all $x, u \in \mathbb{R}^m$. Using the relations (6) for Σ_γ^{-1}, and substituting the particular form of f, we obtain

$$\begin{aligned}\gamma u_{k+1} &= v_{k+1} - f((v_k - \gamma u_k), u_k) = v_{k+1} - F'v_k + F'\gamma u_k - G'u_k \\ &\quad - \psi'((v_k - \gamma u_k), u_k) \\ &= (F' - G'\gamma^{-1})(\gamma u_k) + v_{k+1} - F'v_k - \psi'((v_k - \gamma u_k), u_k)\end{aligned}$$

Denoting

$$z := \gamma u, \quad K := \gamma^{-1} \quad \text{and} \quad w_k := v_{k+1} - F'v_k - \psi'((v_k - \gamma u_k), u_k)$$

we obtain

$$z_{k+1} = (F' - G'K) z_k + w_k \tag{23}$$

which can be regarded as the state representation of a linear system with state z and input w, operating under the static state feedback K. Now, consider the system $z_{k+1} = (F - GK) z_k + w_k$, with the unperturbed parameters F, G and ψ. Then, since the pair (F, G) is stabilizable by our assumption, there is a feedback matrix K (which is $m \times m$ here) such that all the eigenvalues of the matrix $(F - GK)$ have absolute value strictly less than one (Wonham 1967). Moreover, since an arbitrarily small change in the entries of K can transform it into a non-singular matrix in case it is singular, it is easy to see that we can choose the $m \times m$ matrix K here so that it is non-singular, without violating the requirement that all the eigenvalues of the matrix $(F - GK)$ have absolute value strictly less than one; we choose such a matrix K. Then, since the eigenvalues of the matrix $(F - GK)$ depend in a continuous way on the entries of the matrices F and G, there exist real numbers $\Delta > 0$ and $0 < s < 1$ such that the absolute values of the eigenvalues of $(F' - G'K)$ do not exceed $1 - s$ whenever the matrices Γ and Λ satisfy $\|\Gamma\| < \Delta$ and $\|\Lambda\| < \Delta$, namely, whenever $\Sigma \in \mathscr{S}(F, G, \Delta)$.

Now, returning to (23), we have, for all $v \in S(\delta^m)$, that $|w| \leqslant |v| + |F'v| + |\psi'| \leqslant \delta + |F'|\delta + 2N$. Consequently, there is a real number $N' > 0$ such that $|w| \leqslant N'$ whenever $v \in S(\delta^m)$ and $\Sigma \in \mathscr{S}(F, G, \Delta)$. Also, the initial condition for the system (23) satisfies $|z_0| = |\gamma u_0| = |v_0 - x_0| = |v_0| \leqslant \delta$ since $x_0 = 0$. Thus, whenever $v \in S(\delta^m)$ and $\Sigma \in \mathscr{S}(F, G, \Delta)$, the system (23) is operated from initial conditions bounded by δ, with input sequences bounded by N', and all its eigenvalues are with absolute value not exceeding $1 - s$. Through standard properties of asymptotically stable linear discrete-time systems, this implies that there is a real number $N'' > 0$ such that $|z| \leqslant N''$, independently of the quantities v, Γ, Λ, and ψ_v, as long as $v \in S(\delta^m)$ and $\Sigma \in \mathscr{S}(F, G, \Delta)$. Now, recalling that K has been chosen as non-singular, that $\gamma = K^{-1}$, and that $z = \gamma u$, we obtain that $|\gamma u| \leqslant N''$, or $|u| = |Kz| \leqslant |K|N''$ for all sequences $v \in S(\delta^m)$ and for all systems $\Sigma \in \mathscr{S}(F, G, \Delta)$. Taking $\alpha := |K|N''$, we have $|u| \leqslant \alpha$ for all sequences $v \in S(\delta^m)$ and for all systems $\Sigma \in \mathscr{S}(F, G, \Delta)$, which is another way of saying that $S(\delta^m) \subset \Sigma_\gamma[S(\alpha^m)]$ for all systems $\Sigma \in \mathscr{S}(F, G, \Delta)$. □

Remark

We now summarize those aspects of the proof of Proposition 5 that are relevant to implementation. The non-singular matrix γ is chosen so that the eigenvalues of the matrix $(F - G\gamma^{-1})$ are all strictly inside the unit disc in the complex plane. The value of the number Δ appearing is determined by the following requirement. For any pair of matrices F', G' satisfying $\|F' - F\| < \Delta$ and $\|G' - G\| < \Delta$, the eigenvalues of the matrix $(F' - G'\gamma^{-1})$ must all remain strictly inside the unit disc in the complex plane.

□

We now prepare an extension of Proposition 5 for the general class of recursive systems having recursive functions that are continuously differentiable. Let $\Sigma_n : S(\mathbb{R}^m) \to S(\mathbb{R}^m)$ be the nominal system, having a recursive representation $x_{k+1} = f_n(x_k, u_k)$ with $x_0 = 0$. Assume that the nominal recursion function f_n is differentiable at the origin and that $f(0, 0) = 0$, and let (F, G), where F and G are $m \times m$ matrices, be the jacobian matrix of the partial derivatives of f_n at the origin. Now, given a real number $\Delta > 0$, we define a class $\mathcal{S}(f_n, \Delta)$ of systems that deviate 'by Δ' from the nominal system Σ_n. First, we fix a neighbourhood $\mathcal{N}$ of the origin and a real number $N > 0$. Then, $\mathcal{S}(f_n, \Delta)$ consists of all systems $\Sigma : S(\mathbb{R}^m) \to S(\mathbb{R}^m)$ having recursive representations of the form $x_{k+1} = f_n(x_k, u_k) + v(x_k, u_k)$, $x_0 = 0$, where the deviation function $v : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ satisfies the following conditions:

(i) v is twice continuously differentiable over $\mathcal{N}$, and all its second order partial derivatives are bounded in absolute value by N on $\mathcal{N}$;

(ii) $v(0, 0) = 0$; and

(iii) the jacobian matrix (Γ, Λ) of the partial derivatives of v at the origin, partitioned into the $m \times m$ matrices Γ and Λ, satisfies $\|\Gamma\| < \Delta$ and $\|\Lambda\| < \Delta$.

Proposition 6

Let $\mathcal{S}(f_n, \Delta)$ be the class of systems $\Sigma : S(\mathbb{R}^m) \to S(\mathbb{R}^m)$ defined in the previous paragraph. Let (F, G), where F and G are $m \times m$ matrices, be the jacobian matrix of the partial derivatives of the nominal recursion function f_n at the origin, and assume that the pair F, G is stabilizable. Then, there exist real numbers Δ, δ, $\alpha > 0$ and an $m \times m$ non-singular matrix γ such that $S(\delta^m) \subset \Sigma_\gamma[S(\alpha^m)]$ for all systems $\Sigma \in \mathcal{S}(f_n, \Delta)$.

As an example of a class of systems satisfying the conditions of Proposition 6, consider the following. Let the nominal system $\Sigma_n : S(\mathbb{R}) \to S(\mathbb{R})$ be given by the recursive representation

$$x_{k+1} = 2\exp(x_k + u_k) - 2 =: f_n(x_k, u_k)$$

Now, fix some real number $N > 0$. Then, the class of systems $\Sigma : S(\mathbb{R}) \to S(\mathbb{R})$ having recursive representations of the form

$$x_{k+1} = 2\exp(x_k + u_k) - 2 + ax_k + bx_k^2 + cu_k + du_k^2 + gx_k u_k$$

where $|a|$, $|c| < \Delta$ and $|b|$, $|d|$, $|g| < N/2$, is a class of systems contained in $\mathcal{S}(f_n, \Delta)$, and hence Proposition 6 applies to it.

Proof

To prove Proposition 6, we use the fact that f_n is differentiable at the origin. We can write $f_n(x, u) = Fx + Gu + \psi(x, u)$, where

$$\lim_{|(x,u)| \to 0} \frac{\psi(x, u)}{|(x, u)|} = 0$$

Similarly, we have $v(x, u) = \Gamma x + \Lambda u + \psi_v(x, u)$, with

$$\lim_{|(x,u)| \to 0} \frac{\psi_v(x, u)}{|(x, u)|} = 0$$

Also, since we have assumed that the second-order derivatives of v are bounded by N over the neighbourhood $\mathcal{N}$, it follows from the standard properties of the Taylor series that there is a real number $N' > 0$ such that $|\psi_v(x, u)| \leqslant N'|(x, u)|^2$ for all $(x, u) \in \mathcal{N}$. Letting $f(x, u) := f_n(x, u) + v(x, u)$, we obtain that $f(x, u) = F'x + G'u + \psi'(x, u)$, where $F' = F + \Gamma$, $G' = G + \Lambda$, and $\psi'(x, u) = \psi(x, u) + \psi_v(x, u)$. By substitution, it follows then that

$$|\psi'(x, u)| \leqslant |\psi(x, u)| + N'|(x, u)|^2$$

Let $\eta > 0$ be a real number such that the ball $|(x, u)| \leqslant \eta$ is contained in the neighbourhood $\mathcal{N}$.

Now, let γ be a non-singular $m \times m$ matrix, let δ be a real number satisfying $0 < \delta < \eta$, let v be an arbitrary sequence in $S(\delta^m)$, and let $u := \Sigma_\gamma^{-1} v$. Then, as in the proof of Proposition 5, we can write

$$\gamma u_{k+1} = (F' - G'\gamma^{-1})(\gamma u_k) + v_{k+1} - F'v_k - \psi'$$

We select a non-singular $m \times m$ feedback matrix K and a real number $\Delta > 0$ as before, such that the following holds. There is a real number $0 < s < 1$ such that the eigenvalues of the matrix $(F' - G'K)$ do not exceed $1 - s$ in absolute value, for all pairs Γ, Λ satisfying $\|\Gamma\| < \Delta$ and $\|\Gamma\| < \Delta$. We then let $\gamma := K^{-1}$. Denoting $z := \gamma u$, and $w_k := v_{k+1} - F'v_k - \psi'$, we obtain, as in (23),

$$z_{k+1} = (F' - G'K) z_k + w_k \tag{24}$$

which can be regarded as the state representation of a linear system with state z and input w, operating under the static state feedback K. Recalling that

$$\lim_{|(x,u)| \to 0} \frac{\psi(x, u)}{|(x, u)|} = 0$$

and that

$$|\psi'(x, u)| \leqslant |\psi(x, u)| + N'|(x, u)|^2$$

it follows that, for any real number $\chi > 0$, there is a real number τ, with $0 < \tau < \eta$, for which $|\psi'(x, u)| < \chi\tau$ for all pairs (x, u) satisfying $|(x, u)| < \tau$. Assuming that $|(x, u)| < \tau$ and recalling that $|v| \leqslant \delta$, we have

$$|w| \leqslant |v| + |F'v| + |\psi'(x, u)| \leqslant \delta + |F'|\delta + \chi\tau$$

Also, since $\|F' - F\| < \Delta$ and F is a fixed matrix, there is a real number $N_1 > 0$ such that $|F'| < N_1$ for all F' satisfying $\|F' - F\| < \Delta$, and it follows that $|w| \leqslant (1 + N_1)\delta + \chi\tau$.

Further, regarding (24) as a linear system with state z and input w, we can write

$$z = r(z_0) + Z_0(w) \tag{25}$$

where $Z_0(w)$ is the response of the system to the input sequence w from zero initial conditions, and $r(z_0)$ is the response to the zero input sequence from the initial condition z_0. We recall that, from our choice of K and Δ, the matrix $(F' - G'K)$ has all its eigenvalues with absolute values not exceeding $1 - s$ for all admissible F' and G', i.e. for all F' and G' satisfying $\|F' - F\| < \Delta$ and $\|G' - G\| < \Delta$. From standard results on asymptotically stable discrete-time linear systems, it follows that there are real numbers $N_2 > 0$ and $N_3 > 0$ such that $|Z_0(w)| \leqslant N_2|w| \leqslant N_2[(1 + N_1)\delta + \chi\tau]$ and $|r(z_0)| \leqslant N_3|z_0|$ for all admissible F' and G'. We have $v = x + \gamma u$, so that $\gamma u = v - x$, and, since $x_0 = 0$ and $z = \gamma u$, we have $|z_0| = |v_0| \leqslant \delta$, and hence $|r(z_0)| \leqslant N_3\delta$. Substituting this into (25), we obtain $|z| \leqslant N_3\delta + N_2[(1+N_1)\delta + \chi\tau]$. Consequently,

$$|u| = |Kz| \leqslant |K||z| \leqslant |K|\{N_3\delta + N_2[(1+N_1)\delta + \chi\tau]\}$$
$$\leqslant |K|[N_3 + N_2(1+N_1)]\delta + |K|N_2\chi\tau$$

We also have

$$|x| \leqslant |v - \gamma u| \leqslant |v| + |\gamma u| = |v| + |z| \leqslant \delta + N_3\delta + N_2[(1+N_1)\delta + \chi\tau]$$
$$= \{1 + N_3 + N_2(1+N_1)\}\delta + N_2\chi\tau$$

Now, choose τ so that $0 < \tau < \eta$ and $\chi < \min\{1/(2|K|N_2), 1/(2N_2)\}$, and choose δ so that $0 < \delta < \eta$ and $\delta < (\tau/2)\min\{1/\{|K|[N_3 + N_2(1+N_1)]\}, 1/[N_3 + N_2(1+N_1)]\}$. For this choice of δ, we clearly obtain $|u| < \tau$ and $|x| < \tau$ (so that also $|(x, u)| < \tau$) for all $v \in S(\delta^m)$ and for all admissible deviation functions v. Here, an admissible deviation function v is one for which $\Sigma \in \mathscr{S}(f_n, \Delta)$. But, $|u| < \tau$ implies that $\Sigma_\gamma^{-1}[S(\delta^m)] \subset S(\tau^m)$, or $S(\delta^m) \subset \Sigma_\gamma[S(\tau^m)]$, and, upon setting $\alpha := \tau$, we obtain $S(\delta^m) \subset \Sigma_\gamma[S(\alpha^m)]$ for all systems $\Sigma \in \mathscr{S}(f_n, \Delta)$. □

The considerations involved in the choice of the matrix γ and the number Δ of Proposition 6 are stated in detail in its proof, and are briefly summarized in the Remark.

When combining Theorem 5 with Proposition 4, 5 or 6, we obtain an explicit procedure for the robust stabilization of recursive non-linear systems. Before discussing this procedure any further, we wish to indicate how our results up to this point can be made applicable to systems $\Sigma: S(\mathbb{R}^m) \to S(\mathbb{R}^q)$ for which the condition $m = q$ is not satisfied. The simplest way is to add 'dummy' inputs or outputs to the system, as the case may require, to equalize the number of inputs and the number of outputs. Formally, this is carried out as follows. Let $p := \max\{m, q\}$, and let $\Pi: S(\mathbb{R}^p) \to S(\mathbb{R}^m)$ be the standard projection onto the first m coordinates, i.e. for every sequence $u \in S(\mathbb{R}^p)$, the element v_i of the sequence $v := \Pi u$ simply consists of the first m coordinates of the element u_i of the sequence u. Also, let $\mathscr{I}_2: S(\mathbb{R}^q) \to S(\mathbb{R}^p)$ be the injection defined in § 2. Then, the system

$$\Sigma' := \mathscr{I}_2\Sigma\Pi: S(\mathbb{R}^p) \to S(\mathbb{R}^p) \tag{26}$$

has an input space of the same dimension as its output space. A slight reflection shows that internal stabilization of the system Σ' results in internal stabilization of the original system Σ as well. Now, assume that the system Σ has the recursive representation $x_{k+1} = f(x_k, u_k)$. To obtain a recursive representation for the system Σ', we construct for it a recursion function $f': \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^p$ as follows. Let $f_1, \ldots, f_q$ be the coordinate functions of the given recursion function f, and let $f'_1, \ldots, f'_p$ be the coordinate functions of the function f' that we want to construct. Then, for every pair of elements $x = (x_1, \ldots, x_p)$, $u = (u_1, \ldots, u_p) \in \mathbb{R}^p$, we set

$$f'_i(x, u) := f_i((x_1, \ldots, x_q), (u_1, \ldots, u_m)) \quad \text{for } i = 1, \ldots, q$$

and

$$f'_i(x, u) := 0 \quad \text{for } i = q+1, \ldots, p$$

It is easy to see that the system Σ' inherits from Σ all the properties that are relevant to our discussion. For instance, if f has a jacobian matrix (F, G)—where F is $q \times q$ and G

is $q \times m$—with the pair F, G being stabilizable, then f' has a jacobian matrix (F_1, G_1)—where F_1 and G_1 are $p \times p$—with the pair F_1, G_1 being stabilizable. Thus, using the elementary transformation taking Σ into Σ', we can easily transform every system into one having an input space of the same dimension as its output space.

We can now provide a step-by-step description of the robust stabilization procedure that we have developed. Let $\Sigma_0 : S(\mathbb{R}^m) \to S(\mathbb{R}^q)$ be the given nominal system, and let $x_{k+1} = f_0(x_k, u_k)$ be its recursive representation. Using the procedure described in the previous paragraph, we replace Σ_0 by the system

$$\Sigma_n := \mathscr{I}_2 \Sigma_0 \Pi : S(\mathbb{R}^p) \to S(\mathbb{R}^p) \tag{27}$$

and we denote by f_n the recursion function $\mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^p$ of Σ_n. We now construct a control configuration that robustly stabilizes the system Σ_n, and hence also robustly stabilizes the original system Σ_0. The nominal recursive representation on which the following computations are based is given by $x_{k+1} = f_n(x_k, u_k)$, and the initial condition is $x_0 = 0$. It is convenient to introduce the notation $\mathscr{S}(*, \Delta)$ for a 'neighbourhood' of 'radius' Δ of the system Σ, by which we simply mean a generic notation, referring to one of the sets $\mathscr{S}(a, b, \Delta)$, $\mathscr{S}(F, G, \Delta)$, or $\mathscr{S}(f_n, \Delta)$ mentioned in Proposition 4, 5, or 6. We assume that the given recursion function f_0 (and hence the recursion function f_n) satisfies all the conditions involved in the use of these sets of systems, so that, when $\mathscr{S}(*, \Delta)$ is $\mathscr{S}(a, b, \Delta)$, we have $b \neq 0$; when $\mathscr{S}(*, \Delta)$ is $\mathscr{S}(F, G, \Delta)$, the pair F, G is stabilizable; and when $\mathscr{S}(*, \Delta)$ is $\mathscr{S}(f_n, \Delta)$, the jacobian matrix (F, G) of f_n at the origin, when partitioned into the pair of $p \times p$ matrices F and G, yields a stabilizable pair.

The stabilization procedure described is based on the particular form of Theorem 3, due to its simplicity. We comment however that in Hammer (1988) some more general forms of compensators have also been described, and these too could be employed in the robust stabilization scheme, with only obvious minor changes in the details.

Our stabilization procedure consists of the following steps.

Step 1

Choose a real number $\theta > 0$. This number serves as the bound on the amplitude of the input sequences of the final stabilized closed-loop system.

Step 2

Find a constant $p \times p$ non-singular matrix γ for which there are three real numbers $\Delta, \delta, \alpha > 0$ such that the condition $S(\delta^p) \subset \Sigma_\gamma[S(\alpha^p)]$ holds for all systems $\Sigma \in \mathscr{S}(*, \Delta)$. Some methods through which such a matrix γ can be computed are outlined in the proofs of Propositions 4, 5, and 6. Notice that $\Sigma_{n\gamma}$ is bicausal.

Step 3

Choose a positive number ξ, and, using the numbers θ, δ, and α of the previous steps, choose constant positive numbers $\zeta < \delta$ and $\varepsilon < \min\{\theta/\alpha, \xi/(2\alpha)\}$. Choose a recursive, unimodular, bicausal, and uniformly l^∞-continuous system $M : S(\mathbb{R}^p) \to S(\mathbb{R}^p)$ satisfying $M^{-1}[S((5\theta + \xi)^p)] \subset S(\zeta^p)$. The system M determines the dynamical behaviour of the closed-loop system (see Hammer 1988). An elementary possible choice for M is $M := \beta I$, where $I : S(\mathbb{R}^p) \to S(\mathbb{R}^p)$ is the identity system, and where β is any real number satisfying $\beta \geqslant (5\theta + \xi)/\zeta$.

Step 4

Let $E : S(\mathbb{R}^p) \to S(\zeta^p)$ be the static system constructed in (9). Define the system

$Q_{n*} := \Sigma_{n\gamma}^{-1} E : S(\mathbb{R}^p) \to S(\alpha^p)$. This system is a combination of the two recursive systems $\Sigma_{n\gamma}^{-1}$ (whose recursive representation is given by (6)) and E (whose recursive representation is given by (9)), and therefore can readily be implemented on a digital computer.

Step 5

Construct the systems $A := M - \varepsilon Q_{n*} : S(\mathbb{R}^p) \to S(\mathbb{R}^p)$ and $B := \varepsilon I : S(\mathbb{R}^p) \to S(\mathbb{R}^p)$, where $I : S(\mathbb{R}^p) \to S(\mathbb{R}^p)$ is the identity system and ε from Step 3, is given as in (19). Using these systems and (20), we construct the precompensator $\pi = B^{-1} = (1/\varepsilon) I$, which is simply an amplifier in this case, and the feedback compensator $\phi := A = M - \varepsilon Q_{n*}$. Then, for any system $\Sigma \,\varepsilon\, \mathscr{S}(*, \Delta)$, the closed loop $\Sigma_{(\gamma,\pi,\phi)}$ around Σ is internally stable for all input sequences from $S(\theta^p)$, and we obtain robust stabilization of the system Σ_0. The exact input–output relationship induced by this configuration depends, of course, on Σ, and its form is as derived by Hammer (1988).

We conclude our discussion with an example regarding the computation of the compensators π and ϕ that yield robust stabilization of a given nominal system. In order not to clutter the presentation unnecessarily, we provide here a rather simple example. The situation in general is similar.

Example 2

We consider the design of a robust stabilization scheme for the class of systems described in Example 1 and follow the steps of the design procedure.

Step 1

We choose $\theta = 2$ as the bound on the amplitude of our input sequences to the closed-loop system.

Step 2

Following the proof of Proposition 4, we take $\gamma = b/a = 2/2 = 1$, and we choose $\delta = 1$. From the same proof, we have $\alpha = [\delta + (\delta + c)/s]/\gamma$, where $c = (|a| + N)\delta + 2N$, and here we can take $N = 1$, $s = 1/3$, so that $\alpha = 19$.

Step 3

We choose $\xi = 2$. Then, we take $\zeta = 1/2$ and $\varepsilon = 1/20$. We also take the simple choice for the unimodular system M as an amplifier $M = 25$.

Step 4

Now, we construct the system Q_{n*}. We denote by $\{x_k\}$ the input sequence of Q_{n*}, and denote by $\{z_k\}$ its output sequence. Then, we have the recursive relations

$$Q_{n*} : \begin{cases} y_k := e(x_k) \\ z_{k+1} = y_{k+1} - 2y_k - \sin[(y_k - z_k) z_k] \\ z_0 = y_0 \end{cases}$$

$k = 0, 1, 2, \ldots$, where the function $e : \mathbb{R} \to \mathbb{R}$ is given by $e(x) = x$ if $|x| \leqslant 1/2$ and $e(x) = (1/2)\operatorname{sign}(x)$ if $|x| > 1/2$, and where $\operatorname{sign}(x) = \pm 1$, depending on the sign of x.

Step 5

We let

$$\phi = 25 - (1/20) Q_{n*}$$

$$\pi = 20$$

Then, for any system $\Sigma: S(\mathbb{R}) \to S(\mathbb{R})$ having a recursive representation of the form $x_{k+1} = (2+\kappa)x_k + (2+\lambda)u_k + \sigma \sin(x_k u_k)$, where $-\frac{1}{3} \leqslant \kappa \leqslant \frac{1}{3}$, $-\frac{1}{3} \leqslant \lambda \leqslant \frac{1}{3}$ and $-1 \leqslant \sigma \leqslant 1$, the closed loop $\Sigma_{(1,\pi,\phi)}$ around Σ is internally stable for all input sequences from $S(2)$. As we see, the compensators π and ϕ obtained can readily be implemented on a digital computer.

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References

Black, H. S., 1934, *Bell Systems Tech. J.*, **13,** 1.

Bode, H. W., 1945, *Network Analysis and Feedback Amplifier Design* (New York: Van Nostrand).

Desoer, C. A., and Lin, C. A., 1984, *Int. J. Control*, **40,** 37.

Desoer, C. A., and Vidyasagar, M., 1975, *Feedback Systems: Input–Output Properties* (New York: Academic Press).

Hammer, J., 1984 a, *Int. J. Control*, **40,** 1; 1984 b, *Ibid.*, **40,** 953; 1985 a, *Ibid.*, **42,** 1; 1985 b, The concept of rationality and stabilization of nonlinear systems. *Proc. Symp. on Mathematical Theory of Networks and Systems*, Stockholm, Sweden, edited by C. Byrnes and A. Lindquist (Amsterdam: North-Holland); 1986, *Int. J. Control*, **44,** 1349; 1987, *Ibid.*, **46,** 455; 1988, *Ibid.*, **48,** 1183.

Isidori, A., 1985, *Nonlinear Control Systems: an Introduction*, Lecture Notes in Control and Information Sciences, Vol. 72 (Berlin: Springer-Verlag).

Kimura, H., 1984, *I.E.E.E. Trans autom. Control*, **29,** 788.

Kuratowski, K., 1961, *Introduction to set theory and topology* (New York: Pergamon Press).

Newton, G. C., Jr., Gould, L. A., and Kaiser, J. F., 1957, *Analytical Design of Linear Feedback Controls* (New York: Wiley).

Rosenbrock, H. H., 1970, *State Space and Multivariable Theory* (London: Nelson); 1974, *Computer-aided Control System Design* (London: Academic Press).

Sontag, E. D., 1981, *Inf. Control*, **51,** 105.

Vidyasagar, M., 1980, *I.E.E.E. Trans. autom. Control*, **25,** 504.

Zames, G., 1966, *I.E.E.E. Trans. autom. Control*, **11,** 228; 1981, *Ibid.*, **26,** 301.