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Robust Stabilization of Nonlinear Systems, the Separation Theorem, and Asymptotic Observers

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***Abstract*—The problem of robustly stabilizing a nonlinear system is revisited through a re-examination of the separation theorem, whereby a nonlinear system can be stabilized by a combination of an asymptotic observer and a state feedback controller. The main emphasis is on the notion of *strict asymptotic observers* – asymptotic observers designed to tolerate uncertainty. It is shown that strict asymptotic observers can be constructed directly from *strict observer functions* – functions that yield an asymptotically stable differential equation when subtracted from the differential equation of the observed system.**

I. Introduction

The observer-controller configuration of Figure 1 has played an important role in linear control theory for decades. The present note examines its use in nonlinear control.

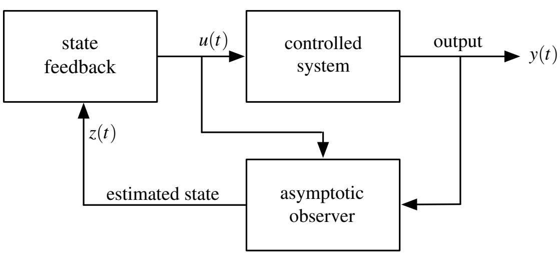

Fig. 1. The observer-controller configuration

Consider a nonlinear time-invariant system Σ given by

$$\Sigma : \begin{array}{l} \dot{x}(t) = f(x(t), u(t)),\ x(0) = x_0, \\ y(t) = h(x(t)), \end{array} \tag{1}$$

where $x(t)$ is the state, $u(t)$ is the input, and $y(t)$ is the output of Σ. Denoting by R the real numbers, and letting n, m, and p be positive integers, we set $x(t) \in R^n$, $u(t) \in R^m$, and $y(t) \in R^p$. The *recursion function* $f : R^n \times R^m \to R^n$ and the *output function* $h : R^n \to R^p$ of Σ are continuous functions. The initial state x_0 of Σ is unknown. We assume that Σ has a (possibly unstable) stationary point at the origin:

$$f(0,0) = 0,\ h(0) = 0. \tag{2}$$

The first line of (1) is the *input/state part* Σ_s of Σ, namely,

$$\Sigma_s : \dot{x}(t) = f(x(t), u(t)),\ x(0) = x_0. \tag{3}$$

As usual, the $\ell^\infty-$ norm $|\cdot|$ of a scalar a is the absolute value $|a|$; for a vector $a = (a_1, ..., a_n) \in R^n$, it is $|a| = \max_{i=1,2,...,n} |a_i|$; and, for a function $u(t)$, it is $|u| = \sup_{t \geq 0} |u(t)|$. A function $u(t)$ is *bounded* if $|u| < \infty$. All input signals of Σ are bounded and piecewise continuous functions.

A static state feedback controller for the input/state part Σ_s of Σ is formed by a function $\varphi : R^n \to R^m$ according to

$$u(t) = \varphi(x(t)).$$

Combining Σ_s and φ yields the system of Figure 2:

$$\Sigma_{s\varphi} : \dot{x}(t) = f(x(t), \varphi(x(t))),\ x(0) = x_0.$$

Static state feedback can asymptotically stabilizes all asymptotically stabilizable input/state systems (HAMMER[2012]).

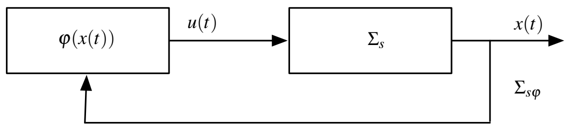

Fig. 2. Static state feedback

The observer-controller configuration uses an observer to generate an estimate $z(t)$ of the state $x(t)$ of Σ; then, it feeds $z(t)$ into a static state feedback controller. How accurate must the estimate $z(t)$ be? As the initial state of Σ is unknown, the estimate $z(t)$ may be quite inaccurate near the initial time. An asymptotic observer creates an asymptotic estimate $z(t)$ of $x(t)$ satisfying

$$\lim_{t \to \infty} [z(t) - x(t)] = 0. \tag{4}$$

We are faced then with two questions: *(i)* When does an asymptotic observer exist and how is it built? *(ii)* Can the asymptotic estimate of an observer replace the real state as input to state feedback?

An asymptotic observer $\mathscr{O}$ is a differential equation

$$\mathscr{O} : \dot{z}(t) = s(z(t), u(t), y(t)),\ t \geq 0,\ z(0) = z_0, \tag{5}$$

where $u(t)$ and $y(t)$ are the input and output signals of the observed system Σ; see Figure 3. Recalling the state $x(t)$ of Σ, the observer signal $z(t)$ must satisfy (4).

The initial states of $\mathscr{O}$ and of Σ are unrelated. In Section II we show that the recursion function s of $\mathscr{O}$ is determined by a 'strict observer function' ω defined by the feature:

The difference $\kappa(x,u) := f(x,u) - \omega(h(x),u)$ is the recursion function of an asymptotically stable system.

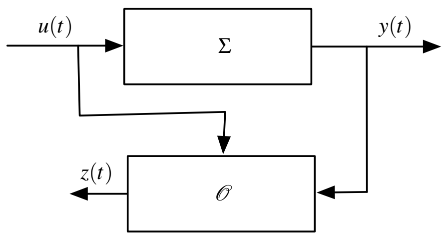

Fig. 3. An asymptotic observer $\mathscr{O}$

Once ω is found, s can be assembled with no further ado; ω can be derived from the functions f and h given in (1) by Liapunov's second method (section VII).

Combining an asymptotic observer $\mathscr{O}$ with a state feedback function φ yields the system $\Sigma_\varphi^{\mathscr{O}}$ of Figure 4. Here, the estimate $z(t)$ created by $\mathscr{O}$ is fed into φ instead of the true state of Σ. For the design of state feedback functions, see HAMMER [2012] and the references cited there.

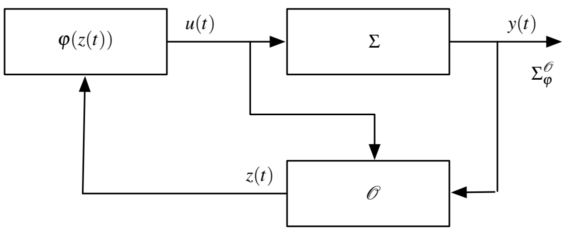

Fig. 4. The observer-controller configuration

We show in section VI that, like in the linear case, an asymptotic estimate $z(t)$ is adequate for asymptotic stabilization, when fed into a state feedback function that asymptotically and robustly stabilizes the input/state part Σ_s of Σ. Intuitively, this is expected: in time, $z(t)$ gets closer and closer to the state $x(t)$; after some time, $z(t)$ becomes sufficiently close to $x(t)$ for the robustness feature of φ to take over. From then on, φ reacts to $z(t)$ approximately as it would react to $x(t)$, driving Σ asymptotically to the origin to achieve asymptotic stabilization.

This leads to the *separation theorem* (section VI): a combination of an asymptotic observer and a stabilizing static state feedback stabilizes a nonlinear system under appropriate conditions. Thus, the observer-controller configuration is an effective tool in nonlinear stabilization. When this is combined with the fraction representation theory of nonlinear systems, more general controllers can be obtained (HAMMER [1988]).

There is an extensive literature on asymptotic observers (KALMAN and BUCY [1961], LUENDBERGER [1966], SONG and GRIZZLE [1995], FRIDMAN, SHTESSEL, EDWARDS, and YAN [2007], the references cited in these papers, and others); this note does not intend to provide a comprehensive literature survey on asymptotic observers. Similarly, there is an extensive literature on nonlinear system stabilization (LASALLE and LEFSCHETZ [1961], LEFSCHETZ [1965], HAMMER [1984, 1985, 1988, 1989, and 1994], DESOER and KABULI [1988], CHEN and FIGUEIREDO [1990], PAICE and MOORE [1990], SANDBERG [1993], PAICE and van der SCHAFT [1994], BARAMOV and KIMURA [1995], GEORGIOU and SMITH [1997], LOGEMANN, RYAN, and TOWNLEY [1999], the references cited in these papers, and many others. This note does not intend to provide a comprehensive literature survey on the stabilization of nonlinear systems.

The paper is organized as follows. Section II explores basic features of nonlinear asymptotic observers. Section III introduces strict Liapunov stability – a strong notion of stability underlying asymptotic observers. Nonlinear asymptotic observers are constructed in section IV, and their robustness is examined in Section V. Section VI states the separation theorem for nonlinear systems, while section VII uses Liapunov's second method to derive nonlinear asymptotic observers. We conclude in section VIII with an example.

II. The structure of asymptotic observers

An asymptotic observer fulfills the convergence requirement (4). To examine this requirement, we write a differential equation for the difference $z(t)-x(t)$ by combining the observer equation (5) with the system equation (1):

$$\dot{z}(t)-\dot{x}(t)=s(z(t),u(t),y(t))-f(x(t),u(t)). \tag{6}$$

The requirement $\lim_{t\to\infty}[z(t)-x(t)]=0$ means that, for every $\varepsilon>0$, there is a time $T(\varepsilon)>0$ such that $|z(t)-x(t)|<\varepsilon$ for all $t\geq T(\varepsilon)$. Let us start the system Σ from the initial condition $x_0:=x(T(\varepsilon))$; start the observer $\mathscr{O}$ from the initial condition $z_0:=z(T(\varepsilon))$; and apply the input signal $u'(t):=u(t+T(\varepsilon))$, $t\geq 0$. By time invariance, this shifts the original behavior over the time interval $[T(\varepsilon),\infty)$ to the time interval $[0,\infty)$. After this shift, denote by $x'(t)$ and $z'(t)$ the states at the time t of Σ and of $\mathscr{O}$, respectively; then, $|z'(t)-x'(t)|<\varepsilon$ for all $t\geq 0$. As this process can be done for any input signal $u(t)$, we can say the following: if Σ and $\mathscr{O}$ start from certain initial conditions that are close to each other, then the trajectories of the two systems remain close at all times.

From this, it is just a small further step to imposing the following general requirement: if $\mathscr{O}$ and Σ start from identical initial conditions, then the state trajectories of $\mathscr{O}$ and of Σ should remain identical at all times. In fact, this requirement is part of the intuitive notion of an asymptotic observer.

Definition 2.1: Let Σ be a system of the form (1) with input signal $u(t)$, state $x(t)$, and initial condition $x(0)=x_0$, and let $\mathscr{O}$ be a system of the form (5) with state $z(t)$ and initial condition $z(0)=z_0$. Then, $\mathscr{O}$ is an *asymptotic observer* of Σ if it satisfies the following requirements:

(i) $\lim_{t\to\infty}[z(t)-x(t)]=0$ for any input signal $u(t)$ and for any initial conditions x_0 and z_0;

(ii) If $z_0=x_0$, then $z(t)=x(t)$ for all $t\geq 0$ and for all input signals $u(t)$.

The *observer error* is the difference $\xi(t):=z(t)-x(t)$. □

Using the observer error $\xi(t)$, we have $z(t)=\xi(t)+x(t)$; substituting into (6), and recalling that $y(t)=h(x(t))$, we obtain a differential equation for the observer error $\xi(t)$, with $u(t)$ and $x(t)$ formally serving as input signals:

$$\begin{aligned}\dot{\xi}(t) &= s(\xi(t)+x(t),u(t),h(x(t))) - f(x(t),u(t)),\\ \xi_0 &= \xi(0) = z_0 - x_0.\end{aligned} \tag{7}$$

When $\mathscr{O}$ and Σ start from the same initial condition $z_0 = x_0$, we have $\xi_0 = 0$. By Definition 2.1*(ii)*, this implies that $z(t) = x(t)$, or $\xi(t) = 0$, for all $t \geq 0$ and for all input signals $u(t)$. But then, also $\dot{\xi}(t) = 0$ for all $t \geq 0$, and (7) yields $s(x(t),u(t),h(x(t))) - f(x(t),u(t)) = 0$ for all $t \geq 0$, or

$$s(x,u,h(x)) = f(x,u). \tag{8}$$

Defining the function

$$\sigma(z,u,y) := s(z,u,y)) - f(z,u), \tag{9}$$

where z is the state of $\mathscr{O}$ and $y = h(x)$ is the output of Σ, it follows from (8) that $\sigma(x,u,h(x)) = 0$. Changing the name of the variable, we get

$$\sigma(z,u,h(z)) = 0. \tag{10}$$

Rewriting (9) in the form $s(z,u,y) = f(z,u) + \sigma(z,u,y)$, our asymptotic observer equation (5) becomes

$$\mathscr{O} : \dot{z}(t) = f(z(t),u(t)) + \sigma(z(t),u(t),y(t)),\ z(0) = z_0. \tag{11}$$

An important fact about (11) is that the dependence of σ on z factors over the output function h of Σ, as follows (Im h denotes the image of the function h).

Lemma 2.2: Let $h : R^n \to R^p$ be the output function of the system Σ of (1), let $y = h(x)$ be the output of Σ, let $\mathscr{O}$ be an asymptotic observer for Σ, and let z be the state of $\mathscr{O}$. Then, referring to (11), there is a function $\mu : R^p \times R^m \times R^p \to R^n$ such that $\sigma(z,u,y) = \mu(h(z),u,y)$ for all z, u, and $y \in$ Im h.

Proof: (sketch) This follows from the fact that σ is constant over all values of z over which h is constant. ■

This yields the general form of an asymptotic observer:

Corollary 2.3: Let Σ be a system of the form (1) with the recursion function f and the output function h, and let $\mathscr{O}$ be an asymptotic observer for Σ. Then, there is a function μ such that $\mathscr{O}$ is described by the equation

$$\mathscr{O} : \dot{z}(t) = f(z(t),u(t)) + \mu(h(z(t)),u(t),y(t)).\square \tag{12}$$

Substituting $y = h(x)$ into (12), we obtain

$$\dot{z}(t) = f(z(t),u(t)) + \mu(h(z(t)),u(t),h(x(t))). \tag{13}$$

Inserting (13) into (6) and using $z(t) = \xi(t) + x(t)$, we get

$$\begin{aligned}\dot{\xi}(t) =& f(\xi(t)+x(t),u(t)) - f(x(t),u(t)) +\\ &+ \mu(h(\xi(t)+x(t)),u(t),h(x(t))).\end{aligned} \tag{14}$$

As it describes the observer error, the solution of (14) must always tend to zero. This implies a particularly strong notion of stability introduced next.

III. Strict Liapunov Stability

Definition 3.1: Let

$$\begin{aligned}&\dot{\theta}(t) = g(\theta(t),w(t)),\ t \geq 0,\ \theta(0) = \theta_0,\\ &\Lambda(t) = \phi(\theta(t)),\end{aligned} \tag{15}$$

be a system, where $\theta(t) \in R^n$, $w(t) \in R^m$, and $\Lambda(t) \in R^p$ for all $t \geq 0$, and where $g : R^n \times R^m \to R^n$ and $\phi : R^n \to R^p$ are continuous functions satisfying $g(0,0) = 0$ and $\phi(0) = 0$. Assume that (15) has a unique solution $\theta(t)$, $t \geq 0$, for every initial condition θ_0 and for every piecewise continuous and bounded input function $w(t)$. A *strict Liapunov function* for (15) is a function $V : R^n \to R$ that satisfies the following conditions for all initial conditions θ_0 and for all piecewise continuous and bounded input signals $w(t)$:

(i) $V(\theta) > 0$ for all $\theta \neq 0$ and $V(0) = 0$;

(ii) $\partial V/\partial\theta$ exists and is a continuous function;

(iii) The set $\{\theta : V(\theta) \leq A\}$ is a bounded subset of R^n for every real number $A \geq 0$;

(iv) $\dot{V}(\theta(t)) < 0$ when $\theta(t) \neq 0$, and $\dot{V}(0) = 0$.

The system (15) is *strictly Liapunov stable* if there is a strict Liapunov function for $\theta(t)$. □

When no input signal appears in (15), then strict Liapunov stability reduces to the standard notion of Liapunov stability. However, with input signal, strict Liapunov stability is a rather strong notion of stability: the solution always decays to zero, irrespective of the input signal, as follows.

Proposition 3.2: Let $\theta(t)$ be the state of a strictly Liapunov stable differential equation of the form (15) with a piecewise continuous and bounded input function $w(t)$. Then, $\theta(t)$ is a bounded function and $\lim_{t\to\infty}\theta(t) = 0$.

Proof: (sketch) Denote $\eta(t) := V(\theta(t))$. Then, according to Definition 3.1*(i)* and *(iv)*, we have $\eta(t) \geq 0$ and $\dot{\eta}(t) < 0$ as long as $\eta(t) \neq 0$, so that $\eta(t)$ is a bounded monotone decreasing function, and $\eta(0) \geq \eta(t) \geq 0$. By Definition 3.1*(iii)*, this implies that $\theta(t)$ is bounded as well, proving the first part of the proposition.

Next, monotonicity, continuity, and boundedness imply that $\lim_{t\to\infty}\eta(t)$ exists, say $\lim_{t\to\infty}\eta(t) =: \eta_\infty$ and, for every $\varepsilon > 0$, there is a time $\tau(\varepsilon) \geq 0$ such that $\mu_\infty \leq \eta(t) \leq \eta_\infty + \varepsilon$ for all $t \geq \tau(\varepsilon)$. By Definition 3.1*(i)*, we have that $\eta_\infty \geq 0$. To show that $\eta_\infty = 0$, assume, by contradiction, that $\eta_\infty > 0$.

Now, induce the shifts $\eta_{\tau(\varepsilon)}(t) := \eta(t+\tau(\varepsilon))$ and $\theta_{\tau(\varepsilon)}(t) := \theta(t+\tau(\varepsilon))$, so that $\theta_{\tau(\varepsilon)}(t) \in \{\theta : V(\theta) \leq \eta_\infty + \varepsilon\}$ for all $t \geq 0$. By Definition 3.1*(iii)* and the continuity of V, the set $\{\theta : V(\theta) \leq \eta_\infty + \varepsilon\}$ is compact. Consequently, there is a sequence of times $t_1, t_2, \ldots \to \infty$ for which $\lim_{i\to\infty}\theta_{\tau(\varepsilon)}(t_i)$ exists, say $\lim_{i\to\infty}\theta_{\tau(\varepsilon)}(t_i) = \alpha$. Then, also $\lim_{i\to\infty}V(\theta(t_i)) = \eta_\infty$, and, by continuity, $\eta_\infty = \lim_{i\to\infty}V(\theta_{\tau(\varepsilon)}(t_i)) = V(\lim_{i\to\infty}\theta_{\tau(\varepsilon)}(t_i)) = V(\alpha)$. As $\eta_\infty > 0$ by assumption, $\alpha \neq 0$ by Definition 3.1*(i)*.

Next, as $\lim_{t\to\infty}\eta(t)$ exists, there is, for every $\Delta > 0$, a time $t(\Delta^2) \geq 0$ such that $|\eta(t_1) - \eta(t_2)| < \Delta^2$ for all $t_1, t_2 \geq t(\Delta^2)$. Consequently, $|\eta(t+\Delta) - \eta(t))/\Delta| \leq \Delta$ for all $t \geq t(\Delta^2) + |\Delta|$. Selecting a sequence $\{\Delta_i\}_{i=0}^{\infty} \to 0$ and choosing a sequence of times $\tau_0, \tau_1, \tau_2, \ldots \to \infty$ such that $\tau_i \geq t(\Delta_i^2) + |\Delta_i|$ for all $i = 0, 1, 2, \ldots$, leads to the conclusion

$$\lim_{t\to\infty} \dot{\eta}(t) = 0. \tag{16}$$

Further, we have $\dot{\eta}(t) = (\partial V(\theta)/\partial\theta) g(\theta(t), w(t))$. Using the sequence $\{t_i\}_{i=1}^{\infty}$ and recalling that $w(t)$ is bounded, it follows that there is a subsequence $\{t_{i_j}\}_{j=1}^{\infty}$ such $\{w(t_{i_j})\}_{j=1}^{\infty}$ converges, say $\lim_{j\to\infty} w(t_{i_j}) = \beta$. By continuity, Definition 3.1*(iv)*, and the fact that $\alpha \neq 0$, we have $\lim_{j\to\infty} \dot{\eta}(t_{i_j}) = \lim_{j\to\infty} \dot{V}(\theta_{\tau(\varepsilon)}(t_{i_j})) = \lim_{j\to\infty} (\partial V(\theta_{\tau(\varepsilon)}(t_{i_j}))/\partial\theta) g(\theta_{\tau(\varepsilon)}(t_{i_j}), w(t_{i_j})) = \partial V(\alpha)/\partial\theta\, g(\alpha, \beta) = \dot{V}(\alpha) < 0$, contradicting (16). As this contradiction originates from the assumption that $\eta_\infty > 0$, we obtain $\eta_\infty = 0$, or $\lim_{t\to\infty} V(\theta(t)) = 0$. Then, $\limsup_{t\to\infty} |\theta(t)| \leq |\bigcap_{\delta\to 0}\{\theta : V(\theta) \leq \delta\}| = 0$. Hence, $\lim_{t\to\infty} |\theta(t)| = 0$. ■

The following notion is the main subject of our discussion.

Definition 3.3: A *strict asymptotic observer* for the system Σ of (1) is an asymptotic observer $\mathscr{O}$ of the form (12) whose observer error (14) is strictly Liapunov stable, with $w(t) := (u(t), x(t))$ being regarded as the input signal. □

IV. Building strict asymptotic observers

Given a strict asymptotic observer $\mathscr{O}$ for Σ, there is a strict Liapunov function V so that the observer error $\xi(t)$ satisfies $\dot{V}(\xi(t)) = \frac{\partial V}{\partial \xi}\dot{\xi}(t) < 0$ for $\xi(t) \neq 0$. Using (14), this yields

$$\begin{aligned}\frac{\partial V}{\partial \xi}[f(\xi(t) + x(t), u(t)) - f(x(t), u(t)) +\\ \mu(h(\xi(t) + x(t)), u(t), h(x(t)))] < 0\end{aligned} \tag{17}$$

for all $\xi(t) \neq 0$. Due to uncertainties about the recursion function f of Σ, we can assume no strict relation between the values of $x(t)$ and $u(t)$ at a time $t > 0$. Furthermore, as $u(t)$ is only piecewise continuous, we can change the value $u(t)$ arbitrarily at any time t, irrespective of the value of $x(t)$. These facts imply that inequality (17) must be valid for any pair of values $(x(t), u(t))$ at any time $t > 0$. In particular, we can take $x(t) = 0$ and set $u(t)$ at an arbitrary value, without violating the inequality. Substituting this into (14) and using the fact that $h(0) = 0$, we obtain the strictly Liapunov stable differential equation $\dot{\xi}(t) = f(\xi(t), u(t)) + [\mu(h(\xi(t)), u(t), 0) - f(0, u(t))]$. Defining the function $\omega(h(\xi), u) := -[\mu(h(\xi), u, 0) - f(0, u)]$, we can rewrite this equation in the form

$$\dot{\xi}(t) = f(\xi(t), u(t)) - \omega(h(\xi(t)), u(t)). \tag{18}$$

Now, rewrite (18) twice with different variables:

$$\begin{aligned}\dot{\zeta}(t) &= f(\zeta(t), u(t)) - \omega(h(\zeta(t)), u(t)),\\ \dot{\chi}(t) &= f(\chi(t), u(t)) - \omega(h(\chi(t)), u(t)).\end{aligned} \tag{19}$$

As these equations are both strictly Liapunov stable, Proposition 3.2 implies that $\lim_{t\to\infty} \zeta(t) = 0$ and $\lim_{t\to\infty} \chi(t) = 0$ for all input functions $u(t)$ and for all initial conditions. Then, the difference $\vartheta(t) := \zeta(t) - \chi(t)$ satisfies

$$\begin{aligned}\dot{\vartheta}(t) = \dot{\zeta}(t) - \dot{\chi}(t) = [f(\zeta(t), u(t)) - \omega(h(\zeta(t)), u(t))] -\\ [f(\chi(t), u(t)) - \omega(h(\chi(t)), u(t))]\end{aligned} \tag{20}$$

and $\lim_{t\to\infty} \vartheta(t) = \lim_{t\to\infty} \zeta(t) - \lim_{t\to\infty} \chi(t) = 0$ for all input functions $u(t)$ and for all initial conditions. These facts allow us to assemble an asymptotic observer for Σ by setting

$$\mathscr{O} : \begin{aligned}&\dot{z}(t) = f(z(t), u(t)) - [\omega(h(z(t)), u(t)) - \omega(y(t), u(t))],\\ &z(0) = z_0.\end{aligned} \tag{21}$$

For this observer, the error $\xi(t) = z(t) - x(t)$ satisfies

$$\begin{aligned}\dot{\xi}(t) = \dot{z}(t) - \dot{x}(t) = &[f(z(t), u(t)) - \omega(h(z(t)), u(t))] -\\ &[f(x(t), u(t)) - \omega(h(x(t)), u(t))].\end{aligned}$$

Taking $\zeta(0) := z(0)$ and $\chi(0) := x(0)$, this equation becomes the same as (20). As a result, the observer error $\xi(t)$ decays asymptotically to zero for all input functions $u(t)$ and initial conditions. This leads to the following important conclusion.

Theorem 4.1: Let Σ be a system of the form (1) with recursion function f and output function h. The following two statements are equivalent.

(i) There is a strict asymptotic observer for Σ.

(ii) There is a continuous function $\omega : R^p \times R^m \to R^n$ for which the differential equation $\dot{\zeta}(t) = f(\zeta, u) - \omega(h(\zeta), u)$ is strictly Liapunov stable.

Proof: (i) implies *(ii)* by the preceding discussion. Conversely, if *(ii)* is valid, let V be a strict Liapunov function for the equation in *(ii)*, and let the observer $\mathscr{O}$ be given by (21). By the discussion preceding the theorem's statement, the observer error satisfies $\xi(t) = z(t) - x(t) = \zeta(t) - \chi(t)$. As V is a strict Liapunov function, a slight reflection shows that so is $V_1(\xi(t), \chi(t)) := V(\xi(t) + \chi(t)) + V(\chi(t))$. Hence, the observer error is strictly Liapunov stable. ■

This yields the general form of strict asymptotic observers:

Corollary 4.2: With the function ω of Theorem 4.1, a strict asymptotic observer for Σ is given by (21). □

Thus, to find a strict asymptotic observer, we need only find a function ω for which $f(x, u) - \omega(h(x), u)$ is the recursion function of a strictly Liapunov stable system. This provides a simple recipe for finding strict asymptotic observers.

Definition 4.3: A continuous function ω that satisfies Theorem 4.1*(ii)* is a *strict observer function* for Σ. □

V. Accounting for the effects of disturbances

In practice, disturbances and noises may affect an observer's inputs, as depicted in Figure 5, where $\upsilon(t)$ and $\upsilon'(t)$ the disturbance signals described by piecewise continuous functions of time. The following statement, whose proof is provided in HAMMER [2013], shows that a strict asymptotic observer can tolerate such disturbances and noises.

Theorem 5.1: Let Σ be a system given by (1) with input $u(t)$, state $x(t)$, and output $y(t)$, and assume that there is a strict asymptotic observer $\mathscr{O}$ for Σ. Let $z(t)$ be the estimate of $x(t)$ produced by $\mathscr{O}$ in the presence of the disturbance signals $\upsilon(t)$ and $\upsilon'(t)$ depicted in Figure 5. Then, for every pair of real numbers $A, \varepsilon > 0$, there is a real number $\delta > 0$ such that $\limsup_{t\to\infty} |z(t) - x(t)| < \varepsilon$ as long as $|\upsilon|, |\upsilon'| < \delta$ and $|u| \leq A$. Further, if $x(t)$ is bounded, then so is $z(t)$. □

Thus, strict asymptotic observers can tolerate real settings.

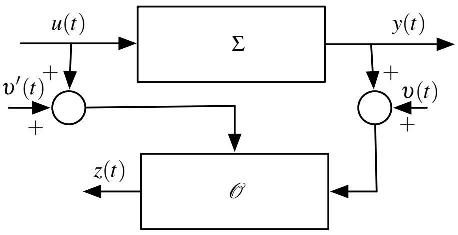

Fig. 5. Observer with disturbances

VI. The Separation Theorem

In the observer-controller configuration $\Sigma_\varphi^{\mathscr{O}}$ of Figure 4, the state feedback function φ receives as input the estimate $z(t)$ produced by the observer $\mathscr{O}$, instead of receiving the true state $x(t)$ of Σ. As we have here a composition of systems, we must discuss internal stability (e.g., HAMMER [1984]).

A. Internal stability

The observer-controller configuration of Figure 6 is affected by disturbances $\upsilon_1, \upsilon_2, \upsilon_3$, and υ_4. Let $x(\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4, t)$ be the state of Σ at time t as a function of the disturbance signals. We use the following terminology.

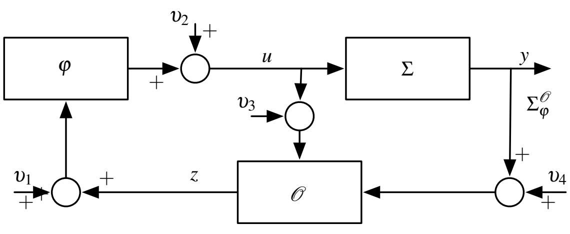

Fig. 6. Internal stability of the observer-controller configuration

Definition 6.1: Let Σ be a system of the form (1), let φ be a state feedback function that asymptotically stabilizes the input/state part Σ_s of Σ, and let $\mathscr{O}$ be a strict asymptotic observer for Σ. Referring to Figure 6, let $x(\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4, t)$ be the state of Σ and let $z(\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4, t)$ be the state of $\mathscr{O}$ at a time $t \geq 0$. Denote by x_0 and z_0 the initial conditions of Σ and of $\mathscr{O}$, respectively. Then, the observer-controller configuration $\Sigma_\varphi^{\mathscr{O}}$ of Figure 6 is *internally and asymptotically stable* if, for every pair of real numbers $\varepsilon, A > 0$, there is a real number $\delta > 0$ such that the following are true whenever $|x_0|, |z_0| \leq A$ and $|\upsilon_i| < \delta$ for all $i = 1, 2, 3, 4$:

(i) $x(\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4, t)$ and $z(\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4, t)$ are both bounded signals; and

(ii) $\limsup_{t\to\infty} |x(\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4, t)| < \varepsilon$. □

Internal asymptotic stability guaranties that, for small disturbances, all signals in the observer-controller configuration remain bounded and the state of the controlled system asymptotically approaches a close vicinity of the origin.

B. State feedback

To be practically applicable, state feedback controllers must be able to tolerate small noises and disturbances like those represented by υ and υ' in Figure 7. Here, φ serves as feedback around the input/state part Σ_s of the controlled system Σ. The closed loop system is denoted by $\Sigma_{s\varphi}$.

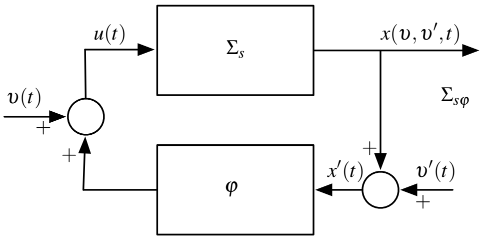

Fig. 7. Robust state feedback

Definition 6.2: Let Σ be a system of the form (1) with input/state part Σ_s and initial state x_0. Referring to the closed loop configuration of Figure 7, let $\varphi: R^n \to R^m$ be a state feedback function, let $\upsilon(t)$ and $\upsilon'(t)$ be disturbance signals, and denote by $x(\upsilon, \upsilon', t)$ the state of the closed loop system at the time t. Then, the feedback function φ *internally and asymptotically stabilizes* Σ_s if the following are valid:

(i) φ is a piecewise continuous function; and

(ii) For every pair of real numbers $\varepsilon, A > 0$, there is a real number $\delta > 0$ such that *(a)* and *(b)* are valid whenever $|x_0| \leq A$ and $|\upsilon|, |\upsilon'| < \delta$:

(a) $x(\upsilon, \upsilon', t)$ is a bounded function, and

(b) $\limsup_{t\to\infty} |x(\upsilon, \upsilon', t)| < \varepsilon$. □

The following is a consequence of time invariance.

Proposition 6.3: Let Σ be a system described by (1), let $\varphi: R^n \to R^m$ be a state feedback function that internally and asymptotically stabilizes the input/state part Σ_s of Σ, and let $\upsilon(t)$ and $\upsilon'(t)$ be the disturbance signals of Figure 7. Then, for every pair of real numbers $A, \varepsilon > 0$, there is a real number $\delta > 0$ such that the following are valid whenever $\limsup_{t\to\infty} |\upsilon(t)| < \delta$ and $\limsup_{t\to\infty} |\upsilon'(t)| < \delta$:

(i) $x(\upsilon, \upsilon', t)$ is a bounded function, and

(ii) $\limsup_{t\to\infty} |x(\upsilon, \upsilon' t)| < \varepsilon$. □

C. The separation theorem

The separation theorem allows us to separate the design of the feedback from the design of the observer, as follows.

Theorem 6.4: Let Σ be a system of the form (1). Assume that there are a state feedback function $\varphi: R^n \to R^m$ that internally and asymptotically stabilizes the input/state part Σ_s of Σ and a strict asymptotic observer $\mathscr{O}$ for Σ. Then, the observer-controller configuration $\Sigma_\varphi^{\mathscr{O}}$ is internally and asymptotically stable.

Proof: (sketch) By Theorem 5.1, $\limsup_{t\to\infty} |z(\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4, t) - x(\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4, t)|$ is as small as desired for sufficiently small noises υ_3 and υ_4. Referring to Figure 7, denote $\upsilon''(t) := z(\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4, t) - x(\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4, t)$, $\upsilon(t) := \upsilon_2(t)$, and $\upsilon'(t) := \upsilon''(t) + \upsilon_1(t)$. Proposition 6.3 states that $x(\upsilon, \upsilon', t)$ is bounded and $\limsup_{t\to\infty} |x(\upsilon, \upsilon', t)|$

can be made arbitrarily small as long as $\limsup_{t\to\infty}|\upsilon(t)|$ and $\limsup_{t\to\infty}|\upsilon'(t)|$ are sufficiently small. The proof arises from these facts. ■

Consequently, the observer-controller configuration is a viable tool for stabilizing nonlinear systems.

VII. Finding Strict Observer Functions

By Corollary 4.2, a strict asymptotic observer is determined by a strict observer function ω. Such functions can be found via Liapunov's second method. Indeed, referring to Theorem 4.1, we seek a continuous function ω for which

$$\dot{\zeta}(t) = f(\zeta,u) - \omega(h(\zeta),u) \tag{22}$$

is strictly Liapunov stable. We must find a strict Liapunov function V which, in particular, satisfies $dV(\zeta(t))/dt = (\partial V/\partial\zeta)\dot{\zeta} = (\partial V/\partial\zeta)[f(\zeta,u) - \omega(h(\zeta),u)] < 0$, or

$$(\partial V/\partial\zeta)\rho(\zeta,u) < (\partial V/\partial\zeta)\psi(h(\zeta),u) \tag{23}$$

for all $\zeta \neq 0$ and all u. We state this point formally.

Theorem 7.1: Let Σ be a system of the form (1). Then, there is a strict asymptotic observer for Σ if and only if there is a strict Liapunov function $V(\zeta)$ and a continuous function ω satisfying (23). □

VIII. Example

Consider a system Σ with recursion and output functions

$$f(x_1,x_2,u) = \begin{pmatrix} x_2^3 + x_1^2 + u \\ x_1^3 + ux_1 \end{pmatrix},\ h(x_1,x_2) = x_1.$$

We can use here

$$\begin{aligned} \omega(h(\zeta),u) &= \omega(\zeta_1,u) = \begin{pmatrix} \zeta_1 + \zeta_1^2 + u \\ \zeta_1 + \zeta_1^3 + u\zeta_1 \end{pmatrix}, \\ V &= \zeta_1^2/2 + \zeta_2^4/4. \end{aligned}$$

Then, (22) becomes

$$\begin{pmatrix} \dot{\zeta}_1 \\ \dot{\zeta}_2 \end{pmatrix} = \begin{pmatrix} \zeta_2^3 - \zeta_1 \\ -\zeta_1 \end{pmatrix},$$

and (23) takes the form

$$\begin{aligned} (\partial V/\partial\zeta)\rho &= \zeta_1\zeta_2^3, \\ (\partial V/\partial\zeta_1)\psi_1 + (\partial V/\partial\zeta_2)\psi_2 &= \zeta_1^2 + \zeta_2^3\zeta_1, \end{aligned}$$

so that $(\partial V/\partial\zeta)\rho < (\partial V/\partial\zeta)\psi$ for $\zeta_1 \neq 0$. Thus, ω is a strict observer function, and, by (21), the resulting strict asymptotic observer is

$$\dot{z}(t) = \begin{pmatrix} z_2^3(t) - z_1(t) + y_1(t) + y_1^2(t) + u(t) \\ -z_1(t) + y_1(t) + y_1^3(t) + u(t)y_1(t) \end{pmatrix},\ z(0) = z_0.$$

References

[1] L. BARAMOV and H. KIMURA [1995] "Nonlinear coprime factorizations and parametrization of a class of stabilizing controllers", Proceedings of the 1995 IEEE conference on Decision and Control, pp. 981-986.

[2] G. CHEN and R.J.P. de FIGUEIREDO [1990] "Construction of the left coprime fractional representations for a class of nonlinear control systems", Sys. & Cont. Let., vol. 14, pp. 353-361.

[3] C.A. DESOER and M.G. KABULI [1988] "Right factorization of a class of nonlinear systems", Trans. IEEE, vol. AC-33, pp. 755-756.

[4] L. FRIDMAN, Y. SHTESSEL, C. EDWARDS, and X.-G. YAN [2007] "Higher-order sliding-mode observer for state estimation and input reconstruction in nonlinear systems." Int. J. Robust and Nonlinear Cont. vol. 18 no. 4-5, pp. 399-412.

[5] T.T. GEORGIOU and M.C. SMITH [1997] "Robustness analysis of nonlinear feedback systems: an input-output approach", Trans. IEEE, vol. AC-42, pp. 1200-1221.

[6] J. HAMMER [1984] "Nonlinear systems: stability and rationality", Int. J. Control, vol. 40, pp. 1-35.

[7] J. HAMMER [1985] "Nonlinear systems, stabilization, and coprimeness", Int. J. Control, vol. 42, pp. 1-20.

[8] J. HAMMER [1988] "Assignment of dynamics for nonlinear recursive feedback systems", Int. J. Control, vol. 48 No. 3, pp. 1183-1212.

[9] J. HAMMER [1989] "Robust stabilization of nonlinear systems", Int. J. Control, vol. 49 No. 2, pp. 629-653.

[10] J. HAMMER [1994] "Internally stable nonlinear systems with disturbances: a parametrization", IEEE Transactions on Automatic Control, vol. AC-39, pp. 300 - 314.

[11] J. HAMMER [2012a] "State Feedback Control of Nonlinear Systems: A Simple Approach", Int. J. Cont. (to appear).

[12] J. HAMMER [2012b] "A Simple Approach to Nonlinear State Feedback Design", Proc. Asian Cont. Conf., June 2013.

[13] R.E. KALMAN and R.S. BUCY [1961] "New results in linear filtering and prediction theory" J. Basic Eng., vol. 83 no. 3, pp. 95-108.

[14] J.F. LASALLE and S. LEFSCHETZ [1961] "Stability by Liapunov's direct method with applications", Academic Press, NY.

[15] S. LEFSCHETZ [1965] "Stability of nonlinear control systems", Academic Press, N.Y.

[16] LOGEMANN, E.P. RYAN, and S. TOWNLEY [1999] "Integral control of linear systems with actuator nonlinearities: lower bounds for the maximal regulating gain", Trans. IEEE, vol. AC-44, pp. 1315-1319.

[17] D. G. LUENBERGER [1966] "Observers for Multivariable Systems", IEEE Trans. Aut. Cont., vol. 11, no. 2, pp. 190-197.

[18] A.D.B. PAICE and J.B. MOORE [1990] "Robust stabilization of nonlinear plants via left coprime factorizations", Systems and Control Letters, vol. 15, pp. 121-129.

[19] A.D.B. PAICE and A.J. van der SCHAFT [1994] "Stable kernel representation and the Youla parametrization for nonlinear systems", Proc. of the 33rd IEEE Conference on Decision and Control, Lake Buena Vista, FL, pp. 781-786.

[20] I.W. SANDBERG [1993] "Uniform approximation and the circle criterion", Trans. IEEE, vol. AC-38, pp. 1450-1458.

[21] Y. SONG and J.W. GRIZZLE [1995] "The extended Kalman filter as a local asymptotic observer for discrete-time nonlinear systems." J. Math. Syst., Estim. and Cont., vol. 5.1, pp. 59-78.