2018 Annual American Control Conference (ACC)
June 27–29, 2018. Wisconsin Center, Milwaukee, USA

Recovering in Minimal Time from Feedback Loss: Bounded Overshoots

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***Abstract*— When feedback is restored after a period of feedback loss, the problem is to reduce in minimal time operating errors that may have accumulated during feedback absence. This problem is examined here under a constraint on the maximal overshoot of the controlled system. It is shown that robust optimal controllers that satisfy this constraint exist under rather general conditions. It is also shown that optimal performance can be approximated as closely as desired by bang-bang controllers that are relatively easy to design and implement.**

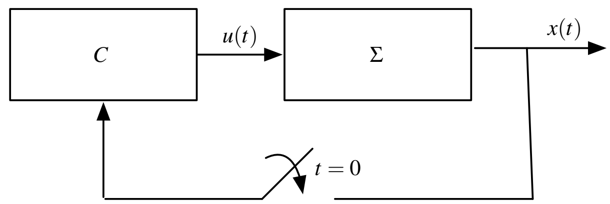

Fig. 1: The control configuration

I. Introduction

Control systems often impose restrictions on the maximal amplitude of their output signal. Such restrictions come to protect the system and ensure safe operation. In the present note, we consider optimal recovery from a period of feedback loss for systems that impose a constraints on the maximal amplitude of their output signal. Of course, recovery starts once feedback has been restored. The objective is to reduce in minimal time operating errors that may have accumulated during a period of feedback loss, without violating a specified bound on the amplitude of the controlled system's response.

Loss of feedback is not an uncommon event in control engineering practice; feedback loss may result from component malfunctions, poor operating conditions, demand for stealthy operation, or from restrictions on data flow in the feedback channel ([1]–[3]). Another important application is sampled-data systems, where no feedback is available between sample times.

The control configuration we consider is depicted in Figure 1. Here, Σ is the controlled system, and C is a controller. The feedback connection to the controller C was restored at the time $t = 0$, after having been disrupted for some time. During the period of feedback absence, increased operating errors may have accumulated. Feedback is restored momentarily at $t = 0$, transmitting the state $x(0) = x_0$ of Σ to the controller C. Only a single sample is transmitted; feedback is disconnected again after transmitting the sample x_0. Based on this sample, the objective of the controller C is to reduce in minimal time operating errors that may have accumulated during feedback absence, without violating a specified bound on the maximal amplitude of the controlled system's response. In this note, we show that such optimal feedback controllers exist under rather broad conditions, and that optimal performance can be approximated as closely as desired by controllers that generate bang-bang signals. Considering that bang-bang signals are characterized by a finite string of scalars – their switching times, controllers that generate bang-bang signals are relatively easy to design and implement.

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Our objectives can be summarized as follows.

Without loss of generality, we can assume that the desired nominal operating point of the closed loop system is at the zero state $x = 0$ of Σ, since the state coordinates of Σ can be shifted appropriately if necessary. Then, the objective of the controller C is to bring Σ from the initial state x_0 to the nominal operating point in minimal time.

To accommodate modeling errors and other uncertainties, a deviation of $\ell > 0$ from the nominal operating point is acceptable. Accordingly, the requirement is to reach in minimal time a state x of Σ that satisfies

$$|x|_2^2 := x^\top x \leq \ell;$$

this must be accomplished without causing the state of Σ to exceed a specified magnitude bound of $A > 0$.

Problem 1. *(i)* Under what conditions is there an optimal controller that guides Σ in minimal time from its initial state x_0 to the domain

$$\rho(\ell) := \left\{x : |x|_2^2 \leq \ell\right\},$$

without violating the state amplitude bound A of Σ.
(ii) When optimal controllers exist, find simple-to-calculate-and-implement controllers that approximate optimal performance. □

This note expands the work of [4]–[10] and is based on material from [11]–[25], the references cited in these papers, and many other publications. The note is organized as follows. The mathematical framework is described in Section II and basic facts are discussed in Section III. In Section IV we prove the existence of optimal controllers, and in Section V we show that optimal performance can be approximated as closely as desired by bang-bang controllers. An example is provided in Section VI, and concluding observations can be found Section VII.

II. Notation and setup

A. Basics

We denote by R the compactified set of real numbers (including $-\infty$ and ∞); R^+ denotes the set of non-negative real numbers. The absolute value of a real number r is denoted by $|r|$. The $L^\infty-$norm of a constant $n \times m$ matrix $G = (G_{ij}) \in R^{n\times m}$ is $|G| := \max_{i,j} |G_{ij}|$, while the $L^\infty-$norm of a matrix function of time $g : R^+ \to R^{n\times m} : t \mapsto g(t)$ is $|g|_\infty := \sup_{t\geq 0} |g(t)|$. We refer to $|g|_\infty$ as the *amplitude* of g. The L^2-norm of a vector $x \in R^n$ is denoted by $|x|_2$, so that $|x|_2^2 = x^\top x$.

B. The controlled system's model

The system Σ of Figure 1 is an input-affine time-varying nonlinear system of the form

$$\Sigma : \begin{array}{l} \dot{x}(t) = a(t, x(t)) + b(t, x(t))u(t), \\ x(0) = x_0; \end{array} \tag{1}$$

here, $x(t) \in R^n$ is the state, $u(t) \in R^m$ is the input signal, and $a : R^+ \times R^n \to R^n : (t, x) \mapsto a(t, x)$ and $b : R^+ \times R^n \to R^{n\times m} : (t, x) \mapsto b(t, x)$ are continuous functions that satisfy the Lipchitz conditions

$$\begin{aligned} |a(t, x') - a(t, x)| &\leq \alpha^+|x' - x|, \\ |b(t, x') - b(t, x)| &\leq \alpha^+|x' - x|, \\ a(t, 0) = 0,\ |b(t, 0)| &\leq \alpha^+, \end{aligned} \tag{2}$$

where $\alpha^+ > 0$ is a specified constant.

Uncertainties in the model of Σ are represented by decomposing a and b into nominal (a_0 and b_0) and uncertain (a_γ and b_γ) parts:

$$\begin{aligned} a(t, x) &= a_0(t, x) + a_\gamma(t, x), \\ b(t, x) &= b_0(t, x) + b_\gamma(t, x). \end{aligned} \tag{3}$$

Here, $a_0 : R^+ \times R^n \to R^n$ and $b_0 : R^+ \times R^n \to R^{n\times m}$ are nominal specified continuous functions that satisfy the Lipschitz conditions

$$\begin{aligned} |a_0(t, x') - a_0(t, x)| &\leq \alpha|x' - x|, \\ |b_0(t, x') - b_0(t, x)| &\leq \alpha|x' - x|, \\ a_0(t, 0) = 0,\ |b_0(t, 0)| &\leq \alpha, \end{aligned} \tag{4}$$

where $\alpha \geq 0$ is a specified constant. The nominal system Σ_0 is then

$$\Sigma_0 : \begin{array}{l} \dot{x}(t) = a_0(t, x(t)) + b_0(t, x(t))u(t),\ t \geq 0, \\ x(0) = x_0. \end{array}$$

The functions $a_\gamma : R^+ \times R^n \to R^n$ and $b_\gamma : R^+ \times R^n \to R^{n\times m}$ of (3) are unspecified continuous functions that represent modeling uncertainties; they satisfy the Lipschitz conditions

$$\begin{aligned} |a_\gamma(t, x') - a_\gamma(t, x)| &\leq \gamma|x' - x|, \\ |b_\gamma(t, x') - b_\gamma(t, x)| &\leq \gamma|x' - x|, \\ a_\gamma(t, 0) = 0,\ |b_\gamma(t, 0)| &\leq \gamma, \end{aligned} \tag{5}$$

where $\gamma > 0$ is a specified constant characterizing the uncertainty level, and $\alpha^+ = \alpha + \gamma$.

C. Signals

Our space of input signals is the Hilbert space $L_2^{\omega,m}$ of all Lebesgue measurable functions $f, g : R^+ \to R^m$ with the inner product

$$\langle f, g\rangle := \int_0^\infty e^{-\omega s} f^\top(s)g(s)ds,$$

where $\omega > 0$ is a real constant ([23], [24]).

The system Σ of Figure 1, like most practical systems, allows only bounded input signals. Denoting by $K > 0$ the input bound of Σ, the set of all permissible input signals is

$$U(K) := \left\{u \in L_2^{\omega,m} : |u|_\infty \leq K\right\}.$$

D. Formal problem statement

In addition to the input signal bound K, the system Σ also imposes a state amplitude bound $A > 0$ to avoid undesirable overshoots. The state $x(t)$ of Σ must satisfy at all times

$$|x(t)| \leq A.$$

Notation 2. Denote by $\mathcal{F}_\gamma(\Sigma_0)$ the family of all systems of the form (1), subject to the requirements (2), (3), (4) and (5). All systems $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ have the same initial state $x(0) = x_0$; the same permissible set of input signals $U(K)$; the same state amplitude bound A; and the initial state satisfies $|x_0| \leq A$. The state $x(t)$ of Σ that results from an input signal u is denoted by $\Sigma(x_0, u, t) := x(t)$. □

For a member $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$, the set of all input signals for which the state amplitude remains bounded by A until the time t is given by

$$U(K, A, \Sigma, t) := \{u \in U(K) : |\Sigma(x_0, u, \theta)| \leq A \text{ for all } \theta \in [0, t]\}.$$

Due to the modeling uncertainty described in (3), it is not known which member of $\mathcal{F}_\gamma(\Sigma_0)$ is the active member. Therefore, we must make sure that the states of all members of $\mathcal{F}_\gamma(\Sigma_0)$ remain bounded by A. The set of all input signals for which the state amplitudes of all members of $\mathcal{F}_\gamma(\Sigma_0)$ remain bounded by A during the time interval $[0, t]$ is

$$U(K, A, \gamma, t) := \bigcap_{\Sigma\in\mathcal{F}_\gamma(\Sigma_0)} U(K, A, \Sigma, t). \tag{6}$$

Returning to a single member $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$, the shortest time during which an input signal $u \in U(K)$ can bring the state of Σ from x_0 into $\rho(\ell)$, while complying with the state amplitude bound A, is

$$t(x_0, \ell, A, \Sigma, u) := \inf_{t\geq 0} \left\{|\Sigma(x_0, u, t)|_2^2 \leq \ell, u \in U(K, A, \Sigma, t)\right\},$$

where $t(x_0, \ell, A, \Sigma, u) := \infty$ if the infimum does not exist, i.e., if there is no time $t \geq 0$ at which the two conditions $|\Sigma(x_0, u, t)|_2^2 \leq \ell$ and $u \in U(K, A, \Sigma, t)$ are valid.

The minimal time at which an input signal $u \in U(K)$ can bring the states of all members of $\mathcal{F}_\gamma(\Sigma_0)$ from x_0 into $\rho(\ell)$, while complying with the state amplitude bound A, is

$$t(x_0, \ell, A, \gamma, u) := \inf_{t\geq 0} \left\{ \begin{array}{l} \left(\sup_{\Sigma\in\mathcal{F}_\gamma(\Sigma_0)} |\Sigma(x_0, u, t)|_2^2\right) \leq \ell, \\ u \in U(K, A, \gamma, t), \end{array} \right\} \tag{7}$$

where $t(x_0,\ell,A,\gamma,u) := \infty$ if there no time $t \geq 0$ at which the two conditions $\sup_{\Sigma\in\mathcal{F}_\gamma(\Sigma_0)} |\Sigma(x_0,u,t)|_2^2 \leq \ell$ and $u \in U(K,A,\gamma,t)$ are valid.

Finally, the shortest time during which any input signal $u \in U(K)$ can bring the state of every member of $\mathcal{F}_\gamma(\Sigma_0)$ from x_0 into $\rho(\ell)$, while complying with the state amplitude bound A, is

$$t^*(x_0,\ell,A,\gamma) = \inf_{u\in U(K)} t(x_0,\ell,A,\gamma,u), \tag{8}$$

where $t^*(x_0,\ell,A,\gamma) := \infty$ if there is no infimum.

In Section IV, we show under rather broad conditions that $t^*(x_0,\ell,A,\gamma) < \infty$ and that there is an optimal input signal $u^*(x_0,\ell,A,\gamma) \in U(K)$ that achieves this minimal time, satisfying $t^*(x_0,\ell,A,\gamma) = t(x_0,\ell,A,\gamma,u^*(x_0,\ell,A,\gamma))$. The main requirement for an optimal solution to exist is controllability of the nominal system Σ_0; an additional requirement is that the uncertainty parameter γ not be excessively large.

An optimal input signal $u^*(x_0,\ell,A,\gamma)$, being a vector valued function of time, is often difficult to calculate and implement. Section V shows that $u^*(x_0,\ell,A,\gamma)$ can be replaced by a bang-bang input signal, without significantly deviating from optimal performance. As bang-bang signals are easier to calculate and implement, this fact makes it possible to achieve close to optimal performance with relatively little complication.

Here is a summary of our objectives.

Problem 3. *(i)* Find conditions under which there is an optimal input signal $u^*(x_0,\ell,A,\gamma) \in U(K)$ satisfying $t^*(x_0,\ell,A,\gamma) = t(x_0,\ell,A,\gamma,u^*(x_0,\ell,A,\gamma))$.
(ii) If $u^*(x_0,\ell,A,\gamma)$ exists, find a simple-to-calculate-and-implement input signal that can replace $u^*(x_0,\ell,A,\gamma)$ without significant departure from optimal performance. □

III. Preliminary facts

A. Bounded minimal time

We start by reproducing a statement from [4] and [5], showing that the response of our systems is bounded at all finite times.

Proposition 4. *For every finite time $T \geq 0$, there is a real number $M(T) \geq 0$ such that $|\Sigma(x_0,u,t)| \leq M(T)$ for all times $t \in [0,T]$, for all input signals $u \in U(K)$, and for all members $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$.* □

The following variant of the notion of controllability is essential to our discussion.

Definition 5. A system $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ is (K,A)*–controllable* from the initial state x_0 if there is an input signal $u \in U(K)$ and a finite time $t_A \geq 0$ such that $\Sigma(x_0,u,t_A) = 0$ and $|\Sigma(x_0,u,t)| \leq A$ for all $t \in [0,t_A]$. □

We show next that (K,A)–controllability of the nominal system Σ_0 guarantees that the minimal time $t^*(x_0,\ell,A,\gamma)$ is finite, provided the uncertainty parameter γ is not excessively large. As will become clear in the course of our discussion, this fact guarantees that only one system – the nominal system – has to be checked in order to make sure that Problem 3 has a solution.

Proposition 6. *Assume that the nominal system Σ_0 is (K,A_0)–controllable from the initial state x_0. Then, for every pair of real numbers $\ell > 0$ and $A > A_0$, there is an uncertainty parameter $\gamma > 0$ for which the minimal time $t^*(x_0,\ell,A,\gamma)$ is finite.*

Proof (sketch). (K,A_0)–controllability of Σ_0 implies the existence of a time $t_{A_0} \geq 0$ and an input signal $u_{A_0} \in U(K)$ for which $\Sigma_0(x_0,u_{A_0},t_{A_0}) = 0$ and $|\Sigma_0(x_0,u_{A_0},t)| \leq A_0$ for all $t \in [0,t_{A_0}]$. Set $x(t) := \Sigma_0(x_0,u_{A_0},t)$ and, for $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$, set $x'(t) := \Sigma(x_0,u_{A_0},t)$ and $\xi(t) = x'(t) - x(t)$. Since Σ_0 and Σ have the same initial state, $\xi(0) = 0$.

Let $t_1,t_2 \in [0,t_{A_0}]$, $t_1 < t_2$, be two times, and consider a time $t \in [t_1,t_2]$. Using (1), (3), (4), and (5) yields

$$|\xi(t)| \leq |\xi(t_1)| + \int_{t_1}^{t} (\alpha|\xi(s)| + \gamma|x'(s)|)\,ds + \int_{t_1}^{t} (\alpha|\xi(s)| + \gamma|x'(s)| + \gamma)\,|u_{A_0}(s)|ds.$$

By Proposition 4, this leads to the inequality

$$\begin{aligned}\sup_{s\in[t_1,t_2]} |\xi(s)| \leq |\xi(t_1)| + \alpha(1+K)(t_2-t_1) \sup_{s\in[t_1,t_2]} |\xi(s)| \\ + \gamma[M(t_{A_0})(1+K)+K](t_2-t_1).\end{aligned} \tag{9}$$

Next, choose $\mu > 0$ such that $\alpha(1+K)\mu < 1$, set $t_2 := t_1 + \mu$, and denote $\eta := [1-\alpha(1+K)\mu]^{-1}$ and $\eta_1 := [M(t_{A_0})(1+K)+K]\mu\eta$. Then, from (9), we obtain

$$\sup_{s\in[t_1,t_1+\mu]} |\xi(s)| \leq \eta|\xi(t_1)| + \gamma\eta_1. \tag{10}$$

Further, for an integer $q \geq t_{A_0}/\mu$, construct the partition

$$[0,t_{A_0}] \subseteq \{[0,\mu],[\mu,2\mu],\ldots,[(q-1)\mu,q\mu]\}.$$

Then, for an integer $k \in [0,q-1]$, set $t_1 := k\mu$ in (10) to obtain

$$\sup_{s\in[k\mu,(k+1)\mu]} |\xi(s)| \leq \eta|\xi(k\mu)| + \gamma\eta_1, k = 0,1,2,\ldots,q-1.$$

By properties of linear recursions, this yields

$$\sup_{s\in[0,t_{A_0}]} |\xi(s)| \leq \gamma\eta_1\eta^{q-1}.$$

Thus, setting $\delta := \min\{(A-A_0),\ell\}$, the proposition is valid for

$$0 < \gamma < \frac{\delta}{\eta_1\eta^{q-1}}.$$

□

By Proposition 6, only one system – the nominal system – has to be checked to guarantee that the minimal time of Problem 3*(i)* is finite.

B. Compactness and continuity

We utilize the following mathematical notions (e.g., [26], [27]).

Definition 7. Let H be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$.
(i) A sequence $\{v_n\}_{n=1}^{\infty}$ of members of H *converges weakly* to a member $v \in H$ if $\lim_{i\to\infty}\langle v_i, y\rangle = \langle v, y\rangle$ for every $y \in H$.
(ii) A subset W of H is *weakly compact* if every sequence of members of W has a subsequence that converges weakly to a member of W. □

The next statement is reproduced here from [23] and [24].

Lemma 8. *The set of signals $U(K)$ is weakly compact in $L_2^{\omega,m}$.* □

We need a few more mathematical notions (e.g., [26], [27]).

Definition 9. Let S be a subset of a Hilbert space H, and let z be a member of S. A functional $F : S \to R$ is *weakly lower semi-continuous at z* if the following is true for every sequence $\{z_i\}_{i=1}^{\infty} \subseteq S$ that converges weakly to z: whenever $F(z)$ is bounded, there is, for every real number $\varepsilon > 0$, an integer $N > 0$ such that $F(z) - F(z_i) < \varepsilon$ for all $i \geq N$.

A function $G : S \times R \to R^n : (s,t) \mapsto G(s,t)$ is *weakly continuous* at z at a time t if the following is true for every sequence $\{z_i\}_{i=1}^{\infty} \subseteq S$ that converges weakly to z: for every real number $\varepsilon > 0$, there is an integer $N > 0$ such that $|G(z,t) - G(z_i,t)| < \varepsilon$ for all $i \geq N$.

Given two times $t_1 < t_2$, the function G is *uniformly weakly continuous* over the interval $[t_1,t_2]$ if the following is true for every sequence $\{z_i\}_{i=1}^{\infty} \subseteq S$ that converges weakly to z: for every real number $\varepsilon > 0$, there is an integer $N > 0$ such that $\sup_{t\in[t_1,t_2]}|G(z,t) - G(z_i,t)| < \varepsilon$ for all integers $i \geq N$. □

The following statement, quoted here from [4] and [5], shows that our systems have a certain continuity property.

Lemma 10. *For a member $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$, the function $\Sigma(x_0,\cdot,\cdot) : U(K) \times R^+ \to R^n : (u,t) \mapsto \Sigma(x_0,u,t)$ is uniformly weakly continuous over every finite interval of time.* □

A slight reflection shows that (6) can be rewritten in the form

$$U(K,A,\gamma,t) = \left\{ u \in U(K) : \sup_{\substack{\Sigma\in\mathcal{F}_\gamma(\Sigma_0)\\ 0\leq s\leq t}} |\Sigma(x_0,u,s)| \leq A \right\}.$$

Clearly, if the amplitude bound A is maintained up to a time $t_2 > 0$, then it is also maintained up to any time $t_1 \leq t_2$. In other words, $U(K,A,\gamma,t)$ is monotone decreasing as a function of the time t, i.e.,

$$U(K,A,\gamma,t_2) \subseteq U(K,A,\gamma,t_1) \text{ for all } t_2 \geq t_1.$$

IV. Existence of optimal solutions

To prove the existence of optimal solutions of Problem 3, we need the following mathematical facts (e.g., [26], [27]).

Theorem 11. (i) *A weakly continuous functional is weakly lower semi-continuous.*
(ii) *Let S and A be topological spaces and assume that, for every member $a \in A$, there is a weakly lower semi-continuous functional $f_a : S \to R$. If $\sup_{a\in A} f_a(s)$ exists at each point $s \in S$, then the functional $f(s) := \sup_{a\in A} f_a(s)$ is weakly lower semi-continuous on S.* □

Now, paraphrasing a proof used in [4] and [5], define the functional

$$\psi(t,u) := \begin{cases} \sup_{\Sigma\in\mathcal{F}_\gamma(\Sigma_0)} |\Sigma(x_0,u,t)|_2^2 & \text{if } u \in U(K,A,\gamma,t), \\ \infty & \text{if } u \notin U(K,A,\gamma,t). \end{cases} \tag{11}$$

Then, the next statement is a consequence of Theorem 11 (see [10] for details).

Lemma 12. *The functional $\psi(t,\cdot) : U(K) \to R$ of (11) is weakly lower semi-continuous over U(K) at every time $t \geq 0$.* □

Considering that $t(x_0,\ell,A,\gamma,u) = \inf\{t \geq 0 : \psi(t,u) \leq \ell\}$ and using Proposition 6, Lemma 12, and a method similar to the one employed to prove an analogous statement in [4] and [5], the following can be verified (see [10] for details).

Proposition 13. *Let $A_0, A, \ell, \gamma > 0$ be real numbers, where $A > A_0$. Assume that the nominal system Σ_0 is (K,A_0)-controllable from the initial state x_0, and that the uncertainty parameter γ is compatible with Proposition 6 for the current A, A_0, and ℓ. Then, the functional $t(x_0,\ell,A,\gamma,u)$ of (7) is weakly lower semi-continuous as a function of u over $U(K)$.* □

We can prove now the main result of this section, namely, that Problem 3 has a solution under rather general conditions:

Theorem 14. *Let $A_0, A, \ell, \gamma > 0$ be real numbers, where $A > A_0$. Assume that the nominal system Σ_0 is (K,A_0)-controllable from the initial state x_0, and that the uncertainty parameter γ is compatible with Proposition 6 for the current A, A_0, and ℓ. Then, referring to the notation of Problem 3, the following hold.*
(i) There is a finite minimal time $t^*(x_0,\ell,A,\gamma)$.
(ii) There is an optimal input signal $u^*(x_0,\ell,A,\gamma) \in U(K)$ satisfying $t^*(x_0,\ell,A,\gamma) = t(x_0,\ell,A,\gamma,u^*(x_0,\ell,A,\gamma))$, while abiding by the state amplitude bound A.

Proof (sketch). According to the Generalized Weierstrass Theorem, a weakly lower semi-continuous functional attains a minimum in a weakly compact set (e.g., [27]). Thus, the present theorem is a consequence of Proposition 13 and Lemma 8. □

V. Bang-bang approximation

Optimal input signals that meet the requirements of Problem 3*(i)* are, in general, vector valued functions of time; as such, they may be hard to calculate and implement. We show in this section that optimal performance can be approximated by using bang-bang input signals – signals that are relatively easy to calculate and implement.

Specifically, we show that a bang-bang input signal $u^\pm$ can reduce operating errors to a slightly higher bound $\ell' > \ell$ at least as quickly as the minimal time $t^*(x_0, \ell, A, \gamma)$ for the error bound ℓ:

Theorem 15. *Let $A_0, A, \ell, \ell', \gamma > 0$ be real numbers, where $A > A_0$ and $\ell' > \ell$. Assume that the nominal system Σ_0 is (K, A_0)−controllable from the initial state x_0. Then, there are an uncertainty parameter γ and a bang-bang control input signal $u^\pm \in U(K)$ such that $t(x_0, \ell', A, \gamma, u^\pm) \le t^*(x_0, \ell, A, \gamma)$, where $u^\pm$ has a finite number of switchings.* □

The proof of Theorem 15 depends on the following statement, according to which the response to any input signal can be approximated by the response to a bang-bang input signal. The proof of this statement is along lines similar to the proofs of related statements in [8], [4], [5], [23] and [24], and is omitted here (see [10] for details).

Theorem 16. *Let Σ be a system of the form (1) with the initial state x_0. Let $u \in U(K)$ be an input signal of Σ, and let $t' > 0$ be a finite time. Then, for every real number $\varepsilon > 0$, there is a bang-bang input signal $u^\pm \in U(K)$ (with a finite number of switchings) and an uncertainty parameter $\gamma > 0$ for which the following is true. The difference between the response $x(t) := \Sigma(x_0, u, t)$ of Σ to u and the response $x^\pm(t) := \Sigma(x_0, u^\pm, t)$ of Σ to $u^\pm$ satisfies the inequality $|x(t) - x^\pm(t)| < \varepsilon$ at all times $0 \le t \le t'$ and for all members $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$.* □

We also need the following feature of the optimal time $t^*(x_0, \ell, A, \gamma)$.

Proposition 17. *The minimal time $t^*(x_0, \ell, A, \gamma)$ of (8) is a monotone decreasing function of the state amplitude bound A.*

Proof (sketch). Let $A' > A > 0$; then, since $U(K, A, \gamma, t) \subseteq U(K, A', \gamma, t)$, it follows by minimality that $t^*(x_0, \ell, A, \gamma) = t(x_0, \ell, A, \gamma, u^*(x_0, \ell, A, \gamma)) = t(x_0, \ell, A', \gamma, u^*(x_0, \ell, A, \gamma)) \ge t^*(x_0, \ell, A', \gamma)$. □

We can prove now the main result of this section, namely, that optimal performance can be approximated as closely as desired by using bang-bang input signals.

Proof of Theorem 15 (sketch). Select a number A' such that $A > A' > A_0$. By Theorem 14, there is a $\gamma > 0$ and an optimal input signal $u^* := u^*(x_0, \ell, A', \gamma)$ achieving the minimal time $t^* := t^*(x_0, \ell, A', \gamma)$.

Let $\varepsilon > 0$ be a real number. Invoking Theorem 16, let $u^\pm \in U(K)$ be a bang-bang input signal such that $|\Sigma(x_0, u^*, t) - \Sigma(x_0, u^\pm, t)| < \varepsilon$ for all $t \in [0, t^*]$ and all $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$. Choosing $0 < \varepsilon \le A - A'$, we get $|\Sigma(x_0, u^\pm, t)| \le A$ for all $t \in [0, t^*]$ and all $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$. Next, using the vector relation $z^\top z = y^\top y - 2y^\top(y - z) + (y - z)^\top(y - z) \le y^\top y + 2n|y||y - z| + n|y - z|^2$, we get

$$\begin{aligned}
&\Sigma^\top(x_0, u^\pm, t^*)\Sigma(x_0, u^\pm, t^*) \\
&\le \Sigma^\top(x_0, u^*, t^*)\Sigma(x_0, u^*, t^*) \\
&+ 2n\,|\Sigma(x_0, u^*, t^*)|\,\left|\Sigma(x_0, u^\pm, t^*) - \Sigma(x_0, u^*, t^*)\right| \\
&+ n\left|\Sigma(x_0, u^\pm, t^*) - \Sigma(x_0, u^*, t^*)\right|^2 \\
&\le \ell + 2n\sqrt{\ell}\varepsilon + n\varepsilon^2.
\end{aligned}$$

Finally, choose $\varepsilon > 0$ sufficiently small to satisfy $\ell + 2n\sqrt{\ell}\varepsilon + n\varepsilon^2 \le \ell'$ and $\varepsilon \le A - A'$. This yields $\Sigma(x_0, u^\pm, t^*) \in \rho(\ell')$ and $|\Sigma(x_0, u^\pm, t| \le A$ for all $t \in [0, t^*]$ and all $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$. The theorem follows then by Proposition 17. □

Theorem 15 provides a relatively simple way to design and implement controllers whose performance is as close as desired to optimal performance.

VI. Example

Example 18. Consider a slightly modified version of the inverted pendulum of [28]:

$$\Sigma : \begin{aligned} &\dot{x}_1(t) = x_2(t), \\ &\dot{x}_2(t) = d_1 \sin x_1(t) + d_2 x_2(t) + d_3 \tanh u(t), \end{aligned}$$

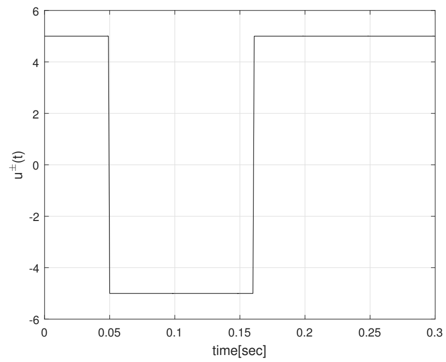
where $x_1(t)$ represents the pendulum's angle to the perpendicular axis, and d_1, d_2, and d_3 are constant parameters. The nominal values are $d_1^0 = 24.527$, $d_2^0 = -0.107$, and $d_3^0 = 12.5$; the initial state is $x_0 = [-\pi/8, -2]^\top$; the input amplitude bound is $K = 5$; the state amplitude bound is $A = 2$; and the operating error bound is $\ell = 0.1$. The values of d_1, d_2, and d_3 are unspecified in the ranges $d_1 \in [21, 27]$, $d_2 \in [-0.3, -0.1]$, and $d_3 \in [10, 14]$.

A numerical search process (see, e.g., [6]) yields the minimal time of 0.246 seconds for reducing the operating error to the specified level: this is the minimal time required to bring the system from its initial state into the domain $\rho(0.1)$. A bang-bang input signal that achieves almost the same time is shown in Figure 2(a); as can be seen in the figure, this bang-bang signal has only two switching times. The system's response is shown in Figures 2(b) and 2(c) for the following parameter values:

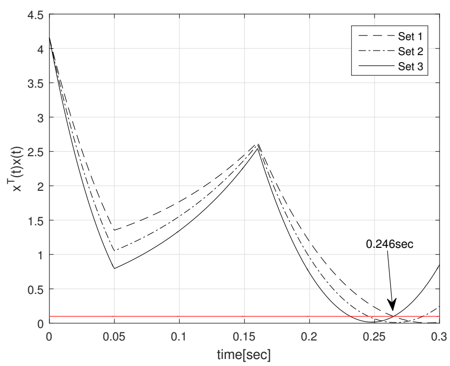
Set 1: $d_1 = 21$, $d_2 = -0.3$, $d_3 = 10$;
Set 2: $d_1 = 24$, $d_2 = -0.2$, $d_3 = 12$;
Set 3: $d_1 = 27$, $d_2 = -0.1$, $d_3 = 14$.

VII. Conclusion

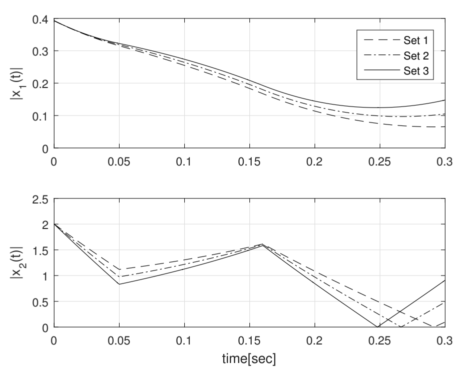
The problem of reducing operating errors in minimal time during recovery from an interruption in feedback service, was examined under a constraint on the maximal overshoot of the controlled system. It was shown that optimal controllers that solve this problem exist under broad conditions, and that optimal performance can be approximated as closely as desired by bang-bang controllers that are relatively easy to design and implement. These results have many practical applications, including the quick reduction of inter-sample errors in sampled data control systems, after the arrival of the next sample.



(a) Bang-bang input signal



(b) $x^\top(t)x(t)$



(c) Individual state trajectories

Fig. 2: Control with state amplitude bound

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