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Recovering from Feedback Failure in Minimal Time ★

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Abstract: It is well known that feedback failure increases operating errors in control systems. The objective here is to develop controllers that reduce such operating errors in minimal time, once feedback has been restored. It is shown that there are robust optimal feedback controllers that attain this objective, and that the performance of these controllers can be approximated as closely as desired by bang-bang controllers that are easy to derive and implement.

Keywords: nonlinear systems, optimal control, feedback failure.

1. INTRODUCTION

Feedback failures, or disruptions in feedback service, are not uncommon in engineering practice. Component failures, inauspicious operating conditions, or deliberate policies that restrict feedback data flow (as in networked control systems, see Nair et al. (2007), Zhivogyladov and Middleton (2003), Montestruque and Antsaklis (2004), and others), create periods of feedback unavailability. Inevitably, disruptions in feedback service cause increased operating errors in control systems. An important need is to develop controllers that reduce such errors in minimal time, once feedback has been restored.

Potential application of such controllers abound. Such controllers improve the performance of sampled control systems by reducing inter-sample errors as quickly as possible, once a new sample has arrived. In biomedicine, for instance, such controllers give rise to optimal treatment protocols for diabetes, helping correct glucose levels as quickly as possible, once a deviation from normal has been detected.

The control configuration is depicted in Figure 1, where the switch closes momentarily at $t = 0$, after having been open for some time. Here, Σ is the controlled system and C is the controller. The input signal $u(t)$ of Σ is generated by C, and $x(t)$ is the state of Σ. Closure of the switch provides C with access to the state $x(0) = x_0$, prompting C to create an input signal $u(t)$ that guides Σ to reduce in minimal time operating errors that have accumulated during open loop operation. After shifting the state of Σ as appropriate, we assume that error-free operation of Σ is at the zero state $x(t) = 0$. Thus, the objective of C is to bring Σ from x_0 to the zero state in minimal time.

We take into consideration two practical issues: *(i)* there is an uncertainty about the model of Σ; and *(ii)* There is a bound $K > 0$ on the input amplitude of Σ.

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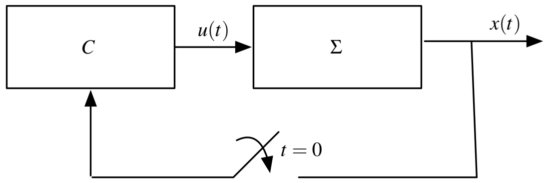

Fig. 1. Feedback is restored momentarily at $t = 0$.

As a result of uncertainties, we cannot expect Σ to reach precisely the zero state. Instead, using $(\cdot)^T$ to denote transpose, we require the controller C to bring Σ close to the zero state, to a state $x(t)$ that satisfies

$$|x(t)|_2^2 := x^T(t)x(t) \leq \delta, \quad (1)$$

where $\delta > 0$ is specified. We refer to (1) as the $\delta-$*vicinity* of the origin. In these terms, we concentrate on the following.

Problem 1. Let Σ be a system whose model is not precisely known, and assume that Σ is at the state x_0 at $t = 0$. Given $\delta > 0$, find an optimal input signal that takes Σ to the $\delta-$vicinity of the origin in minimal time. If such an optimal signal exists, derive an easy-to-implement input signal that approximates optimal performance. □

In section 3, we show that Problem 1 has an optimal solution, and, in section 4, we show that optimal performance can be approximated by bang-bang signals. Bang-bang signals are convenient for calculation and implementation, since they are determined by their switching times.

This note relies on earlier results in optimization theory, including Kelendzheridze (1961), Pontryagin et al. (1962), Neustadt (1966, 1967), Gamkrelidze (1965), Luenberger (1969), Young (1969), Warga (1972), Chakraborty and Hammer (2009), Chakraborty and Shaikshavali (2009), the references cited in these works, and many others. Yet, to the best of our knowledge, Problem 1 has not been addressed so far in published literature.

The note is organized as follows. The formal framework is introduced in section 2, while section 3 includes a proof of the existence of an optimal solution of Problem 1. Section 4 shows that bang-bang signals can approximate optimal performance, and section 5 provides an example.

2. BASICS

2.1 Systems and uncertainty

First, some notation. Denoting by R the real numbers and by $|r|$ the absolute value of $r \in R$, the $L^\infty-$norm of a matrix $v = (v_{ij}) \in R^{n\times m}$ is

$$|v| = \max_{\substack{i=1,2,\ldots,n\\ j=1,2,\ldots,m}} |v_{ij}|.$$

For a function $v(t) \in R^{n\times m}$, $t \geq 0$, the $L^\infty-$norm is

$$|v(t)|_\infty := \sup_{t\geq 0} |v(t)|,$$

where $|v(t)|_\infty := \infty$ if the supremum does not exist.

We consider a class of input-affine nonlinear systems of the form

$$\Sigma : \quad \dot{x}(t) = a(t, x(t)) + b(t, x(t))u(t),\ t \geq 0, \tag{2}$$

where $x(t) \in R^n$ is the state and $u(t) \in R^m$ is the input signal; $a(t) \in R^n$ and $b(t) \in R^{n\times m}$ are continuous functions. The initial time is $t = 0$, and the initial state is $x(0) = x_0$. Considering that practical systems usually impose an input amplitude bound, we restrict input signals $u(t)$ to those satisfying $|u(t)|_\infty \leq K$, where $K > 0$ is specified. The class of systems (2) includes all linear time-varying systems as well as certain non-linear systems used in engineering practice to model flexible joints, special electrical motors, and other devices (e.g., Modeling et al. (2006)).

To allow for uncertainties in Σ, we regard the functions $a(t, x)$ and $b(t, x)$ of (2) as sums of nominal and error terms

$$\begin{aligned} a(t, x) &= a_0(t, x) + a_\gamma(t, x),\\ b(t, x) &= b_0(t, x) + b_\gamma(t, x), \end{aligned} \tag{3}$$

where $a_0(t, x)$ and $b_0(t, x)$ are specified nominal continuous functions, and $a_\gamma(t, x)$ and $b_\gamma(t, x)$ are unspecified continuous functions describing uncertainties. The following Lipschitz inequalities hold for all $t \geq 0$ and all $x_1, x_2 \in R^n$:

$$\begin{aligned} |a_0(t, x_2) - a_0(t, x_1)| &\leq M|x_2 - x_1|,\\ |b_0(t, x_2) - b_0(t, x_1)| &\leq M|x_2 - x_1|,\\ a_0(t, 0) := 0,\ b_0(t, 0) &\leq M, \end{aligned} \tag{4}$$

$$\begin{aligned} |a_\gamma(t, x_2) - a_\gamma(t, x_1)| &\leq \gamma|x_2 - x_1|,\\ |b_\gamma(t, x_2) - b_\gamma(t, x_1)| &\leq \gamma|x_2 - x_1|,\\ a_\gamma(t, 0) := 0,\ b_\gamma(t, 0) &\leq \gamma; \end{aligned} \tag{5}$$

here, $M > 0$ and $\gamma > 0$ are specified bounds, with γ describing uncertainty. The nominal model is Σ_0:

$$\Sigma_0 : \quad \dot{x}(t) = a_0(t, x(t)) + b_0(t, x(t))u(t),\ t \geq 0,\ x(0) = x_0. \tag{6}$$

Definition 2. The family $\mathcal{F}_\gamma(\Sigma_0)$ consists of all systems Σ of the form (2), where $a(t, x)$ and $b(t, x)$ are continuous functions given by (3), (4), and (5), and where $a_\gamma(t, x)$ and $b_\gamma(t, x)$ are unspecified continuous functions. □

Recall that a system with no finite escape time has a response that is bounded at all finite times but may diverge as $t \to \infty$. The following can be verified (see Yu and Hammer (2015)).

Proposition 3. Members of the family of systems $\mathcal{F}_\gamma(\Sigma_0)$ have no finite escape times. □

2.2 Spaces and Reachability

We use the mathematical framework of Chakraborty and Hammer (2009). Given a number $\alpha > 0$ and an integer $m > 0$, the space $L_2^{\alpha,m}$ consists of all Lebesgue measurable functions $f, g : R^+ \to R^m$ with the inner product

$$\langle f, g\rangle := \int_0^\infty e^{-\alpha t} f^T(s)g(s)ds. \tag{7}$$

The set $U(K)$ of input signals of the family $\mathcal{F}_\gamma(\Sigma_0)$ consists of all members of $L_2^{\alpha,m}$ that are bounded by $K > 0$, namely,

$$U(K) := \left\{u \in L_2^{\alpha,m} : |u|_\infty \leq K\right\}. \tag{8}$$

For a member $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ with initial condition x_0 and an input signal $u \in U(K)$, we denote the response at the time t by

$$x(t) := \Sigma(x_0, u, t).$$

The following notion is critical to our discussion.

Definition 4. Given $K > 0$, a system Σ_0 of the form (6) is $K-$*reachable* at the initial state x_0 if there is an input signal $u \in U(K)$ that takes Σ_0 from x_0 to the zero state in finite time.□

Reachability of the nominal system Σ_0 implies a related form of reachability for the entire family $\mathcal{F}_\gamma(\Sigma_0)$, as follows.

Proposition 5. Let Σ_0 be a system of the form (6) that is $K-$reachable at the initial state x_0. Then, for every $\delta > 0$, there is a $\gamma > 0$ for which the following is true: there is an input signal $u \in U(K)$ and a time $\tau \geq 0$ such that $\Sigma^T(x_0, u, \tau)\Sigma(x_0, u, \tau) < \delta$ for all members $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$.

Proof. *(sketch)* By assumption, there is an input function $u \in U(K)$ and a time $\tau \geq 0$ for which $x(\tau) := \Sigma_0(x_0, u, \tau) = 0$. For a member $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ set $x'(t) := \Sigma(x_0, u, t)$ and $\xi(t) := x'(t) - x(t)$. Then, using (5), we have for $0 \leq t' < t \leq \tau$ that

$$\begin{aligned} \xi(t) = \xi(t') + \int_{t'}^{t} \Big\{&[a_0(s, x'(s)) - a_0(s, x(s))]\\ +[a_\gamma(s, x'(s)) - a_\gamma(s, 0)] &+ [b_0(s, x'(s)) - b_0(s, x(s))]u(s)\\ &+[b_\gamma(s, x'(s)) - b_\gamma(s, 0) + b_\gamma(s, 0)]u(s)\Big\}ds. \end{aligned}$$

Proposition 3 indicates that there is a number $N > 0$ such that $|x(t)| \leq N$ and $|x'(t)| \leq N$ for all $t \in [0, \tau]$. Using (4), (5), and the bound $|u|_\infty \leq K$, we obtain

$$\sup_{t'\leq\theta\leq t} |\xi(\theta)| \leq |\xi(t')|$$

$$+\left(\sup_{t'\leq\theta\leq t} |\xi(\theta)|\right) M(t - t')(1 + K) + \gamma(t - t')(N(1 + K) + K).$$

Now, choose a $\Delta > 0$ such that $M\Delta(1 + K) \leq 1/2$ and $p := \tau/\Delta$ is an integer. Then, at $t' = i\Delta$, $t = (i + 1)\Delta$, $i \in \{0, 1, ..., p - 1\}$, we get

$$\sup_{i\Delta\leq\theta\leq(i+1)\Delta} |\xi(\theta)| \leq 2|\xi(i\Delta)| + 2\gamma\Delta(N(1 + K) + K),$$

$i = 0, 1, ..., p - 1$. This yields $\sup_{0\leq\theta\leq\tau} |\xi(\theta)| \leq q_{p-1}\gamma\Delta(N(1 + K) + K)$, where q_k is the solution of the recursion $q_{k+1} = 2q_k + 2 = 2(q_k + 1)$, $q_0 = 0$, $k = 0, 1, 2, ..., p - 1$. Thus, the proposition holds for $\gamma < \delta/\left[q_{p-1}\Delta(N(1 + K) + K)\right]$. □

2.3 Statement of the Problem

Problem 6. For the family $\mathcal{F}_\gamma(\Sigma_0)$, let x_0 be the initial state, let $U(K)$, $K > 0$, be the set of input signals, let M and γ be as in (4) and (5), and assume that $\gamma, \delta > 0$ satisfy Proposition 5. Denote

$$t_f(x_0,u) := \inf_{t\geq 0}\left\{\sup_{\Sigma\in\mathcal{F}_\gamma(\Sigma_0)} |\Sigma(x_0,u,t)|_2^2 \leq \delta\right\},\ u\in U(K), \text{ and}$$

$$t_f^*(x_0) := \inf_{u\in U(K)} t_f(x_0,u).$$

(i) Find the minimal time $t_f^*(x_0)$.

(ii) When $t_f^*(x_0) < \infty$, show that there is an optimal input signal $u^*(x_0) \in U(K)$ satisfying $t_f^*(x_0) = t_f(x_0, u^*(x_0))$.

(iii) If $u^*(x_0)$ exists, find an easy-to-implement signal that approximates optimal performance. □

In section 3, we show that $u^*(x_0)$ exists and in section 4 we show that optimal performance remains almost unaffected, when $u^*(x_0)$ is replaced by an appropriate bang-bang signal.

3. OPTIMAL SOLUTIONS

Our proof of the existence of an optimal solution of Problem 6 depends on two important facts validated in this section:

(i) $U(K)$ – the set of input functions – is 'compact'.

(ii) $t_f(x_0,u)$ is a 'continuous function' of u.

Considering that a continuous function attains a minimum over a compact domain, these facts imply that optimal quantities $t_f^*(x_0)$ and $u^*(x_0)$ exist. We review a few notions.

Definition 7. Let H be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$.

(i) A sequence $\{v_n\}_{n=1}^\infty \subseteq H$ *converges weakly* to a member $v\in H$ if $\lim_{n\to\infty}\langle v_n, y\rangle = \langle v, y\rangle$ for every $y \in H$.

(ii) A subset W of H is *weakly compact* if every sequence of members of W has a subsequence that converges weakly to a member of W.

(iii) Let S be a subset of a Hilbert space H, and let z be a point of S. A functional $F: S\to R$ is *weakly lower semi-continuous at* z if the following is true for every sequence $\{z_n\}_{n=1}^\infty \subseteq S$ that converges weakly to z: whenever $F(z)$ is bounded, there is, for every $\varepsilon > 0$, an integer $N > 0$ such that $F(z) - F(z_n) < \varepsilon$ for all $n \geq N$. If F is weakly lower semi-continuous at every point $z\in S$, then F is *weakly lower semi-continuous* on S. The function F is *weakly continuous* at z if there is, for every $\varepsilon > 0$, an integer $N > 0$ such that $|F(z) - F(z_n)| < \varepsilon$ for all $n \geq N$. □

The following is taken from Chakraborty and Hammer (2009).

Lemma 8. (Lemma 3.2 of Chakraborty and Hammer (2009)) The set $U(K)$ of (8) is weakly compact in the topology of the Hilbert space $L_2^{\alpha,m}$. □

We turn now to a continuity feature of the time $t(x_0,u)$ of Problem 6.

Proposition 9. In the notation of Problem 6, the function $t(x_0,u)$ is a weakly lower semi-continuous function of u over $U(K)$. □

The proof of Proposition 9 requires several auxiliary results.

Lemma 10. In the notation of Problem 6, let $\{u_i\}_{i=1}^\infty \subseteq U(K)$ be a sequence that converges weakly to $u\in U(K)$. Then, $\lim_{i\to\infty}\Sigma(x_0,u_i,t) = \Sigma(x_0,u,t)$ for every system $\Sigma\in\mathcal{F}_\gamma(\Sigma_0)$ at every time $t\geq 0$.

Proof. *(sketch)* Set $x(t,u_i) := \Sigma(x_0,u_i,t)$, $x(t,u) := \Sigma(x_0,u,t)$, and $x(t,i) := x(t,u) - x(t,u_i)$. We show that $\lim_{i\to\infty} x(t,i) = 0$. As the initial condition of Σ is always x_0, we get from (2) and (3) that

$$\begin{aligned} x(t,i) &= \int_0^t [a(\theta,x(\theta,u)) - a(\theta,x(\theta,u_i))]d\theta \\ &+ \int_0^t [b(\theta,x(\theta,u))u(\theta) - b(\theta,x(\theta,u_i))u_i(\theta)]d\theta \\ &= \int_0^t [a(\theta,x(\theta,u)) - a(\theta,x(\theta,u_i))]d\theta \\ &+ \int_0^t [b(\theta,x(\theta,u)) - b(\theta,x(\theta,u_i))]u_i(\theta) \\ &+ \int_0^t b(\theta,x(\theta,u))[u(\theta)-u_i(\theta)]d\theta. \end{aligned}$$

Using the bounds (4), (5), $|u|_\infty \leq K$ and $|u_i|_\infty \leq K$, yields

$$\sup_{0\leq\tau\leq t}|x(\tau,i)| \leq (M+\gamma)\big[\sup_{0\leq\tau\leq t}|x(\tau,i)|t + \sup_{0\leq\tau\leq t}|x(\tau,i)|Kt\big] + \sup_{0\leq\tau\leq t}\left|\int_0^\tau b(\theta,x(\theta,u))[u(\theta)-u_i(\theta)]d\theta\right|.$$

For a value $\zeta > 0$ of t at which $\{1-\zeta[(M+\gamma)(1+K)\} > 0$, define $\mu := \{1-\zeta[(M+\gamma)(1+K)\}$. Then,

$$\sup_{0\leq\tau\leq\zeta}|x(\tau,i)| \leq \frac{1}{\mu}\left\{\sup_{0\leq\tau\leq\zeta}\left|\int_0^\tau b(\theta,x(\theta,u))[u(\theta)-u_i(\theta)]d\theta\right|\right\}. \tag{9}$$

Next, define the function

$$y_\tau(\theta) := \begin{cases} e^{\alpha\theta}b(\theta,x(\theta,u)) & 0\leq\theta\leq\tau, \\ 0 & \text{else,}\end{cases}$$

so that

$$\int_0^\tau b(\theta,x(\theta,u))[u(\theta)-u_i(\theta)]d\theta = \langle u-u_i, y_\tau\rangle.$$

As the sequence $\{u_i\}_{i=1}^\infty$ converges weakly to u, there is, for every $\beta > 0$, an integer $N_\tau \geq 0$ satisfying $|\langle u-u_i, y_\tau\rangle| < \beta$ for all $i\geq N_\tau$. We prove by contradiction that there is an integer $N\geq 0$ such that $\sup_{0\leq\tau\leq\zeta}|\langle u-u_i,y_\tau\rangle| < \beta$ for all $i\geq N$.

Indeed, if there is no such N, then there is a sequence of times $\{\tau_j\}_{j=0}^\infty \subseteq [0,\zeta]$ such that

$$\left|\langle u-u_j, y_{\tau_j}\rangle\right| \geq \beta \tag{10}$$

for all $j = 0,1,\ldots$; this sequence must include a convergent subsequence $\lim_{k\to\infty}\tau_{j_k} = \tau' \in [0,\zeta]$. By weak convergence of $\{u_i\}$, there is an integer $N' \geq 0$ such that $\left|\langle u-u_{j_k}, y_{\tau'}\rangle\right| < \beta/2$ for all $k\geq N'$. By Proposition 3, there is a number $A > 0$ such that $|x(\theta,u)| \leq A$ for all $\theta\in[0,\zeta]$. Combining with (4) and (5), and recalling that $0\leq\tau_{j_k},\tau'\leq\zeta$ for all integers $k\geq 0$, we get

$$\left|\langle u-u_{j_k}, y_{\tau'}\rangle - \langle u-u_{j_k}, y_{\tau_{j_k}}\rangle\right| \leq 2(M+\gamma)AK|\tau'-\tau_{j_k}|.$$

Let $N^* \geq N'$ be an integer satisfying

$$|\tau'-\tau_{j_k}| < \frac{\beta}{4(M+\gamma)AK}$$

for all $k\geq N^*$. Then,

$$\begin{aligned}\left|\langle u-u_{j_k}, y_{\tau_{j_k}}\rangle\right| &\leq \left|\langle u-u_{j_k}, y_{\tau_{j_k}}\rangle - \langle u-u_{j_k}, y_{\tau'}\rangle\right| \\ &+ \left|\langle u-u_{j_k}, y_{\tau'}\rangle\right| < \beta/2+\beta/2 = \beta\end{aligned}$$

for all $k\geq N^*$, contradicting (10). Consequently, for every $\beta > 0$, there is an $N\geq 0$ such that

$$\sup_{0\leq\tau\leq\zeta}|\langle u-u_i, y_\tau\rangle| < \beta \text{ for all } i\geq N. \tag{11}$$

Now, given $\xi > 0$, select $0 < \beta < \mu\xi$. Then, (11) and (9) imply that there is an integer $N_\xi \geq 0$ satisfying

$$\sup_{0\leq\tau\leq\zeta} |x(\tau,i)| < \xi \tag{12}$$

for all $i \geq N_\xi$; this proves that $\lim_{i\to\infty} x(\tau,i) = 0$ for all $\tau \in [0,\zeta]$. Employing a construction similar to the one used in the proof of Proposition 5, we conclude that $\lim_{i\to\infty} x(t,i) = 0$ at any finite time $t \geq 0$. □ Definition 7 and Lemma 10 imply the following.

Corollary 11. In the notation of Problem 6, the function $\Sigma(x_0,v,t)$ is weakly continuous over $U(K)$ for every $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ at every time $t \geq 0$. □

Actually, the proof of Lemma 10 implies the following stronger result (see Yu and Hammer (2015) for details).

Corollary 12. In the notation of Problem 6, let $\{u_i\}_{i=1}^{\infty} \subseteq U(K)$ be a sequence converging weakly to $u \in U(K)$. Then, for every $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ and for every $\tau,\varepsilon > 0$, there is an integer $N(\varepsilon) > 0$ such that $\sup_{t\in[0,\tau]} |\Sigma(x_0,u,t) - \Sigma(x_0,u_i,t)| < \varepsilon$ for all integers $i \geq N(\varepsilon)$. □

Next, a few mathematical facts (see Willard (1970)).

Theorem 13.

(i) A weakly continuous function is weakly lower semi-continuous.

(ii) Let S and A be topological spaces and assume that, for every member $a \in A$, there is a weakly lower semi-continuous function $f_a : S \to R$. If $\sup_{a\in A} f_a(s)$ exists at each point $s \in S$, then the function $f(s) := \sup_{a\in A} f_a(s)$ is weakly lower semi-continuous on S. □

We can prove now the following.

Lemma 14. In the notation of Problem 6, let $v \in U(K)$ and set $\psi(t,v) := \sup_{\Sigma\in\mathcal{F}_\gamma(\Sigma_0)} |\Sigma(x_0,v,t)|_2^2$. Then, $\psi(t,v)$ is weakly lower semi-continuous over U(K) at every $t \geq 0$.

Proof. *(sketch)* A continuous function of a weakly continuous function is weakly continuous; hence, by Corollary 11 and Theorem 13*(i)*, the function $|\Sigma(x_0,v,t)|_2^2$ is weakly lower semi-continuous on $U(K)$ at every $t \geq 0$. Then, the lemma follows from Theorem 13*(ii)*. □

Proof. *(of Proposition 9; sketch)* By Lemma 14, we have $t_f(x_0,u) := \inf_t \{t \geq 0 : \psi(t,u(t)) \leq \delta\}$. Denote

$$\theta(u) := \inf_t \{t \geq 0 : \psi(t,u(t)) \leq \delta\}. \tag{13}$$

Let $u \in U(K)$ be such that $\theta(u) < \infty$ (see Proposition 5); let $\{u_i\}_{i=1}^{\infty} \subseteq U(K)$ be a sequence converging weakly to u, and denote $\psi_i(t) := \psi(t,u_i(t))$, $i = 1,2,...$, and $\psi_0(t) := \psi(t,u(t))$. Then, $\theta(u_i) := \inf\{t \geq 0 : \psi_i(t) \leq \delta\}$ and $\theta(u) := \inf\{t \geq 0 : \psi_0(t) \leq \delta\}$. We show next that $\theta(u)$ is a weakly lower semi-continuous function over $U(K)$, namely, that for every $\varepsilon > 0$, there is an integer $N > 0$ such that

$$\theta(u_i) > \theta(u) - \varepsilon \text{ for all } i \geq N. \tag{14}$$

Indeed, given $\varepsilon > 0$, there are two options:

Case 1: There is an integer $N > 0$ such that $\theta(u_i) \geq \theta(u)$ for all $i \geq N$; then (14) clearly holds.

Case 2: Case 1 is not valid; then, there is a sequence of integers $j_1, j_2, \ldots$ such that $\theta(u_{j_k}) < \theta(u)$ for all integers $k \geq 1$.

In Case 2, the inequalities $\theta(u) < \infty$ and $\theta(u_{j_k}) < \theta(u)$ imply that $\theta(u_{j_k}) < \infty$. By (13), there is a $\bar{t} \in [\theta(u) - \varepsilon, \theta(u))$ at which $\psi_0(\bar{t}) > \delta$, or

$$\psi_0(\bar{t}) - \delta > 0. \tag{15}$$

By Lemma 14 there is, for every $\mu > 0$, an integer $N > 0$ such that $\psi_0(\bar{t}) - \psi_{j_k}(\bar{t}) < \mu$ for all $k \geq N$. Taking $\mu := (\psi_0(\bar{t}) - \delta)/2$ yields $\psi_0(\bar{t}) - \psi_{j_k}(\bar{t}) < (\psi_0(\bar{t}) - \delta)/2$ for all $k \geq N$, or $\psi_{j_k}(\bar{t}) > (\psi_0(\bar{t}) + \delta)/2$ for all $k \geq N$. Consequently, by (15), we have $\psi_{j_k}(\bar{t}) > \delta$ for all $k \geq N$, which implies $\theta(u_{j_k}) > \bar{t}$. Recalling that $\bar{t} \in [\theta(u) - \varepsilon, \theta(u))$, it follows that $\theta(u_{j_k}) > \theta(u) - \varepsilon$ for all $k \geq N$. As $t_f(x_0,u) = \theta(u)$, the proposition holds. □

Lemma 8 and Proposition 9 allow us to invoke the generalized Weierstrass Theorem (e.g. Zeidler (1985)), according to which a weakly lower semi-continuous function attains a minimum in a weakly compact set. This proves that Problem 6 has an optimal solution, as follows.

Theorem 15. In the notation of Problem 6,

(i) There is a finite minimal time $t_f^*(x_0)$, and

(ii) There is an optimal input function $u^*(x_0) \in U(K)$ satisfying $t_f^*(x_0) = t_f(x_0,u^*(x_0))$. □

Thus, our optimization problem has an optimal solution under rather general conditions. In the next section, we show that optimal performance can be approximated by using bang-bang input signals – signals that are easy to calculate and implement.

4. BANG-BANG APPROXIMATION OF OPTIMAL PERFORMANCE

The optimal input signal $u^*(x_0)$ may be hard to compute and implement; instead, we show that optimal performance can be closely approximated by a bang-bang input signal that is easy to compute and implement. Formally, a *bang-bang* member of $U(K)$ has components that switch between $-K$ or $+K$ as necessary. We need the following auxiliary result.

Lemma 16. In the notation of (2), (3), and Problem 6, let $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ be a system with initial state x_0 and input $u \in U(K)$. Setting $x(t,u) := \Sigma(x_0,u,t)$, the following hold for all $u \in U(K)$:

(i) There are bounds $B_0(x_0) \geq 0$ and $B_\gamma(x_0) \geq 0$ for which

$$\begin{cases} \sup\limits_{0\leq\tau\leq t_f^*(x_0)} |b_0(\tau,x(\tau,u))| \leq B_0(x_0), & \text{and} \\ \sup\limits_{0\leq\tau\leq t_f^*(x_0)} \left|b_\gamma(\tau,x(\tau,u))\right| \leq B_\gamma(x_0). \end{cases}$$

(ii) For every $\rho > 0$, there is a $\beta(x_0,\rho) > 0$ such that

$$\begin{cases} \left|b_0(t',x(t',u)) - b_0(t'',x(t'',u))\right| < \rho, & \text{and} \\ \left|b_\gamma(t',x(t',u)) - b_\gamma(t'',x(t'',u))\right| < \rho, \end{cases} \tag{16}$$

for all $t', t'' \in \left[0, t_f^*(x_0)\right]$ for which $|t' - t''| < \beta(x_0,\rho)$.

Proof. *(sketch)* Part *(i)* of the lemma is valid since $b_0(t,x(t))$ and $b_\gamma(t,x(t))$ are continuous functions of time and there is a bound $|x(\tau)| \leq N(x_0)$ for all $0 \leq \tau \leq t_f^*(x_0)$ and all $u \in U(K)$ (see Proposition 3). Part *(ii)* of the lemma is a consequence of (a) $b_0(t,x(t))$, $b_\gamma(t,x(t))$, and $\Sigma(x_0,u,t)$ are all uniformly continuous over $[0,t_f^*(x_0)]$; and (b) $U(K)$ is weakly compact by Lemma 8 (see Yu and Hammer (2015) for details). □ The feasibility of using bang-bang signals to approximate optimal performance stems from the following.

Lemma 17. In the notation of Problem 6, let $\theta \in \left[0, t_f^*(x_0)\right)$ and $\sigma_0 > 0$ be numbers, let $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ be a system, and let $u^*(x_0)$ be the optimal input. Then, there are numbers $\eta \in \left(0, t_f^*(x_0) - \theta\right]$ and $\mu(\eta) > 0$, and a bang-bang input signal $u^\pm(x_0) \in U(K)$ such that

(i) $u^\pm(x_0)$ has a finite number of switchings during the time $[\theta, \theta + \eta]$.

(ii) The difference in the responses $x^*(t) := \Sigma(x_0, u^*(x_0), t)$ and $x^\pm(t) := \Sigma(x_0, u^\pm(x_0), t)$ satisfies

$$\sup_{t \in [\theta, \theta+\eta]} |x^*(t) - x^\pm(t)| < \mu(\eta) |x^*(\theta) - x^\pm(\theta)| + \sigma_0.$$

(iii) η and $\mu(\eta)$ depend only on M, γ, and K.

Proof. *(sketch)* Given two numbers $\theta \in \left[0, t_f^*(x_0)\right)$ and $\rho > 0$, let $\beta(x_0, \rho) > 0$ be as in Lemma 16. We select later two numbers $\eta \in \left(0, t_f^*(x_0) - \theta\right]$ and $\lambda > 0$ satisfying

$$0 < \lambda < \beta(x_0, \rho) \text{ and } \eta/\lambda \text{ is an integer.} \tag{17}$$

Denote $r := \eta/\lambda - 1$, and consider the partition

$$[\theta, \theta+\eta] = \{[\theta, \theta+\lambda], [\theta+\lambda, \theta+2\lambda], \ldots, [\theta + r\lambda, \theta + (r+1)\lambda]\}. \tag{18}$$

Construct a bang-bang signal $u^\pm(x_0) = (u_1^\pm(x_0,t), u_2^\pm(x_0,t), \ldots, u_m^\pm(x_0,t))^T \in U(K)$ as follows: Letting $u^*(x_0,t) = (u_1^*(x_0,t), u(x_0,t), \ldots, u_m^*(x_0,t))^T$ be the optimal input signal, choose in each interval of (18) a point $\omega_{\ell_i} \in [\theta + \ell\lambda, \theta + (\ell+1)\lambda]$ satisfying

$$K[2(\omega_{\ell_i} - (\theta + \ell\lambda)) - \lambda] = \int_{\theta+\ell\lambda}^{\theta+(\ell+1)\lambda} u_i^*(s) ds, \tag{19}$$

and set

$$u_i^\pm(x_0,t) := \begin{cases} K & \text{for } t \in [\theta + \ell\lambda, \omega_{\ell_i}), \text{ and} \\ -K & \text{for } t \in [\omega_{\ell_i}, \theta + (\ell+1)\lambda) \text{ if } \omega_{\ell_i} \neq (\ell+1)\lambda, \end{cases} \tag{20}$$

$\ell = 0, 1, 2, \ldots, r$, $i = 1, 2, \ldots, m$. Then,

$$\int_{\theta+\ell\lambda}^{\theta+(\ell+1)\lambda} [u_i^*(x_0,s) - u_i^\pm(x_0,s)] ds = 0, \; i = 1, \ldots, m, \; \ell = 0, \ldots, r. \tag{21}$$

Denote

$$\xi(t) := x^*(t) - x^\pm(t). \tag{22}$$

Then, for $t \in [\theta, t_f^*]$, we have

$$\begin{aligned}\xi(t) = \xi(\theta) + \int_\theta^t \Big[& \left(a_0(s, x^*(s)) + a_\gamma(s, x^*(s))\right) \\ & - \left(a_0(s, x^\pm(s)) + a_\gamma(s, x^\pm(s))\right) \\ & + \left(b_0(s, x^*(s)) + b_\gamma(s, x^*(s))\right) u^*(x_0, s) \\ & - \left(b_0(s, x^\pm(s)) + b_\gamma(s, x^\pm(s))\right) u^\pm(x_0, s)\Big] ds.\end{aligned}$$

From (4), (5), and the fact that $|u^*(x_0)|_\infty \leq K$ and $|u^\pm(x_0)|_\infty \leq K$, we get

$$\begin{aligned}\sup_{t \in [\theta, \theta+\eta]} |\xi(t)| \leq & |\xi(\theta)| + (M+\gamma)(1+K)\eta \left(\sup_{t \in [\theta, \theta+\eta]} |\xi(t)|\right) \\ & + \sup_{t \in [\theta, \theta+\eta]} \left| \int_\theta^t \Big[b_0(s, x^*(s)) \right. \\ & \left. + b_\gamma(s, x^*(s)) \Big] \left(u^*(x_0,s) - u^\pm(x_0,s)\right) ds \right|,\end{aligned}$$

or

$$\begin{aligned}(1 - (M+\gamma)(1+K)\eta) \sup_{t \in [\theta, \theta+\eta]} |\xi(t)| \leq & |\xi(\theta)| \\ & + \sup_{t \in [\theta, \theta+\eta]} \left| \int_\theta^t \Big[b_0(s, x^*(s)) \right. \\ & \left. + b_\gamma(s, x^*(s)) \Big] \left(u^*(x_0,s) - u^\pm(x_0,s)\right) ds \right|.\end{aligned}$$

Choose now a number $\eta \in \left(0, t_f^*(x_0) - \theta\right]$ for which $(M+\gamma)(1+K)\eta < 1$, and set

$$\mu(\eta) := \frac{1}{1 - (M+\gamma)(1+K)\eta}. \tag{23}$$

Then, part *(iii)* of the lemma holds, and

$$\begin{aligned}\sup_{t \in [\theta, \theta+\eta]} |\xi(t)| \leq & \mu(\eta) |\xi(\theta)| + \mu(\eta) \sup_{t \in [\theta, \theta+\eta]} \left| \int_\theta^t \Big[b_0(s, x^*(s)) \right. \\ & \left. + b_\gamma(s, x^*(s)) \Big] \left(u^*(x_0,s) - u^\pm(x_0,s)\right) ds \right|.\end{aligned} \tag{24}$$

To examine the last integral, use the partition (18) with (20), (19), and (21), and let $q(t) \in \{0, 1, 2, \ldots, r\}$ be the integer satisfying $t \in [q(t)\lambda, (q(t)+1)\lambda]$. Then,

$$\begin{aligned}& \sup_{t \in [\theta, \theta+\eta]} \left| \int_\theta^t \left(b_0(s, x^*(s)) + b_\gamma(s, x^*(s))\right) \left(u^*(s) - u^\pm(x_0,s)\right) ds \right| \\ & = \sup_{t \in [\theta, \theta+\eta]} \left| \sum_{i=0}^{q(t)-1} \int_{\theta+i\lambda}^{\theta+(i+1)\lambda} \Big\{ b_0(\theta + i\lambda, x^*(\theta+i\lambda)) \right. \\ & \qquad - b_0(\theta + i\lambda, x^*(\theta+i\lambda)) \\ & \qquad + b_\gamma(\theta + i\lambda, x^*(\theta+i\lambda)) - b_\gamma(\theta + i\lambda, x^*(\theta+i\lambda)) \\ & \qquad + b_0(s, x^*(s)) + b_\gamma(s, x^*(s)) \Big\} \left(u^*(x_0,s) - u^\pm(x_0,s)\right) ds + \\ & \left. \int_{\theta+q(t)\lambda}^t \Big[b_0(s, x^*(s)) + b_\gamma(s, x^*(s)) \Big] \left(u^*(x_0,s) - u^\pm(x_0,s)\right) ds \right|.\end{aligned}$$

Using (21), Lemma 16, (16) and (17), we get $\sup_{t \in [\theta, \theta+\eta]}$

$$\begin{aligned}& \left| \int_\theta^t \Big[b_0(s, x^*(s)) + b_\gamma(s, x^*(s)) \Big] \left(u^*(x_0,s) - u^\pm(x_0,s)\right) ds \right| \\ & \leq q(t)\lambda\rho 2K + q(t)\lambda\rho 2K + \Big[B_0(x_0) + B_\gamma(x_0) \Big] 2K\lambda,\end{aligned}$$

so that $\sup_{t \in [\theta, \theta+\eta]}$

$$\begin{aligned}& \left| \int_\theta^t \Big[b_0(s, x^*(s)) + b_\gamma(s, x^*(s)) \Big] \left(u^*(x_0,s) - u^\pm(x_0,s)\right) ds \right| \\ & \leq 4K\rho\eta + 2K \Big[B_0(x_0) + B_\gamma(x_0) \Big] \lambda.\end{aligned}$$

Now, choose ρ so that $0 < \rho < \frac{\sigma_0}{8\mu(\eta)K\eta}$; then, choose $\lambda > 0$ so that η/λ is an integer and

$$0 < \lambda < \min\left\{ \beta(x_0, \rho), \frac{\sigma_0}{4\mu(\eta)K \Big[B_0(x_0) + B_\gamma(x_0) \Big]} \right\}.$$

This yields $\sup_{t \in [\theta, \theta+\eta]} |\xi(t)| < \mu(\eta) |\xi(\theta)| + \sigma_0$, as required. □

We have reached the main result of this section, which shows that the optimal input signal can be replaced by a bang-bang signal without a significant impact on performance.

Theorem 18. In the notation of Problem 6, let $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$. Then, for every $\sigma > 0$, there is a bang-bang signal $v^\pm(x_0, t) \in U(K)$ with a finite number of switchings in the interval $[0, t_f^*(x_0)]$ for which $\sup_{t \in [0, t_f^*(x_0)]} |x^*(t) - x_v^\pm(t)| < \sigma$, where $x^*(t) = \Sigma(x_0, u^*(x_0), t)$ and $x_v^\pm(t) = \Sigma(x_0, v^\pm(x_0), t)$.

Proof. *(sketch)* By Lemma 17 with $\theta = 0$ and $\sigma_0 > 0$, there is a $\eta \in \left(0, t_f^*(x_0)\right]$ and a bang-bang signal $u^{\pm,1}(x_0, t)$ with a finite number of switchings over $[0, \eta]$ such that $\sup_{t \in [0,\eta]} |x^*(t) - x_v^{\pm}(t)| < \mu(\eta)|x^*(0) - x_v^{\pm}(0)| + \sigma_0$, where $x_v^{\pm}(t) := \Sigma(x_0, v^{\pm}(x_0), t)$ and $v^{\pm}(x_0, t) := u^{\pm,1}(x_0, t)$, $t \in [0, \eta]$. Considering that $x^*(0) = x^{\pm,1}(0) = x_0$, we have $\sup_{t \in [0,\eta]} |x^*(t) - x^{\pm,1}(t)| < \sigma_0$. Create the partition

$$[0, t_f^*(x_0)] = \left\{[0, \eta], [\eta, 2\eta], ..., [(k-1)\eta, k\eta], [k\eta, t_f^*(x_0)]\right\},$$

where k is the integer part of $t_f^*(x_0)/\eta$. For an integer $i \in \{2, ..., k\}$, assume, by recursion, that the signal $v^{\pm}(x_0)$ has been constructed over the interval $[0, (i-1)\eta]$. Using Lemma 17 with $\theta = i\eta$ yields a bang-bang signal $u^{\pm,(i+1)}(x_0, t)$ with a finite number of switchings in $[i\eta, \min\{(i+1)\eta, t_f^*(x_0)\})$. Set $v^{\pm}(x_0) := u^{\pm,(i+1)}(x_0, t)$, $t \in (i\eta, (i+1)\eta]$. Then, by Lemma 17,

$$\sup_{t \in \left[i\eta, \min\left\{(i+1)\eta, t_f^*(x_0)\right\}\right]} |x^*(t) - x_v^{\pm}(t)| < \chi_i,$$

where $\chi_i := \mu(\eta)\left|x^*(i\eta) - x^{\pm,1}(i\eta)\right| + \sigma_0 = \mu(\eta)\chi_{i-1} + \sigma_0$, and χ_i is determined by the recursion

$$\chi_{i+1} = \mu(\eta)\chi_i + \sigma_0, \; i = 0, 1, 2, ..., \quad \chi_0 = 0.$$

This implies that σ_0 can be chosen so that $\chi_i < \sigma$ for all $i \in \{0, 1, ..., k\}$. For such σ_0, we have $\sup_{t \in \left[0, t_f^*(x_0)\right]} |x^*(t) - x_v^{\pm}(t)| < \sigma$, as required. □

Thus, bang-bang signals can achieve performance that is almost optimal. Appropriate bang-bang signals can be constructed by numerical optimization.

5. EXAMPLE

Consider the family of systems

$$\mathcal{F}: \begin{array}{l} \dot{x}_1(t) = -c(1 + 0.5\cos(t))x_1(t) + (1-t)u(t), \\ \dot{x}_2(t) = d(1 - 0.5\sin(t))x_2(t) + (1-t)u(t), \end{array} \; t \geq 0,$$

where $1.4 \leq c \leq 1.6$, $0.9 \leq d \leq 1.1$, the input bound is $|u(t)|_\infty \leq 5$, and the initial state is $x_0 = (3.5, -1)^T$. The goal is to construct an optimal input signal $u^*(x_0)$ that takes all members of $\mathcal{F}$ in minimal time from x_0 to a state x satisfying $x^T x \leq 1.25$. Numerical optimization yields that the minimal time is $t_f^*(x_0) = 0.73$. Figure 3 shows that a similar time can be achieved by the bang-bang signal with 2 switchings depicted in Figure 2.

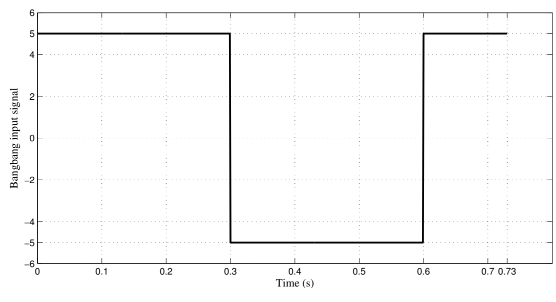

Fig. 2. A bang-bang input signal $v^{\pm}(x_0)$ that approximates optimal performance.

6. CONCLUSION

We have seen that there exist optimal controllers that reduce open-loop operating errors in minimal time, once feedback is re-instated. We have also seen that optimal controllers can be

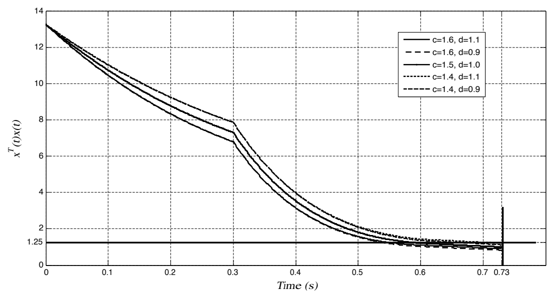

Fig. 3. $x^T(t)x(t)$ for the family $\mathcal{F}$ with the bang-bang input signal $v^{\pm}(x_0)$ of Figure 2.

replaced by bang-bang controllers without significantly spoiling optimal performance. Bang-bang controllers are relatively easy to calculate and implement.

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