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Preservation of stability under system perturbations in nonlinear control: A simple characterization *

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Abstract: A simple characterization of the perturbations a nonlinear closed-loop control system can tolerate before losing its stability is derived. The characterization is given in purely algebraic terms, with the topological aspects being automatically incorporated through the theory of fraction representations of nonlinear systems. The implications of system perturbations on internal stability are also discussed. The presentation is for the case of discrete-time nonlinear systems.

Keywords: Nonlinear control; robust control; fraction representation.

1. Introduction

Perhaps one of the most fascinating properties of closed loop control systems is their prowess to preserve stability despite relatively large perturbations in the parameters of the forward path of the loop. The purpose of the present paper is to provide some simple insight into this phenomenon for the case of nonlinear additive feedback systems. Specifically, we show that the class of system perturbations under which stability is preserved can be characterized in simple algebraic terms, with only indirect reference to stability properties. This is achieved through the use of the theory of fraction representations of nonlinear systems. In addition to shedding light on one of the most fundamental aspects of control theory, this result also permits the derivation of some very simple sufficient conditions for the preservation of internal stability under system perturbations. These conditions consist merely of certain amplitude bounds on the perturbations; Their simplicity makes them particularly apt for use in practical design applications, where they provide the design engineer with a clear insight into the class of perturbations the closed loop system can tolerate before losing its internal stability. An indication of the effect of tolerable variations on some basic performance characteristics of the closed loop system is also provided. The presentation is for the case of discrete-time nonlinear systems.

Of critical importance to the present discussion is the theory of fraction representations of nonlinear systems. Recall that a right fraction representation of a nonlinear system Σ is a factorization of the system into a composition of two systems, one of which is stable and the other is the inverse of a stable system, in the form

$$\Sigma = PQ^{-1}, \tag{1.1}$$

where P and Q are stable systems with Q being invertible. In order to be specific, the discussion is centered around the control configuration shown in Figure 1. Here, Σ is the system that needs to be controlled and stabilized; π is a causal dynamic precompensator; and ϕ is a causal dynamic feedback compensator. The closed loop system is denoted by $\Sigma_{(\pi,\phi)}$. As seen in [8,9], it is particularly

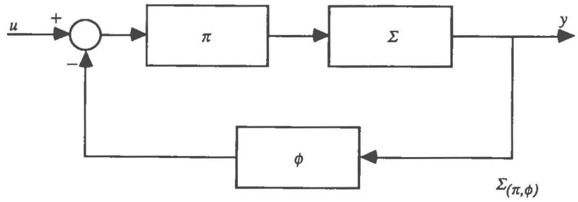

Fig. 1.

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convenient to choose the compensators π and ϕ in the form

$$\pi = B^{-1}, \qquad \phi = A, \tag{1.2}$$

where A and B are stable systems, B is invertible, and A and B^{-1} are causal systems. Assume now that the system Σ has a right coprime fraction representation $\Sigma = PQ^{-1}$. Then, a direct computation shows that the input/output relation induced by the closed loop system is given by

$$\begin{aligned}\Sigma_{(\pi,\phi)} &= \Sigma\pi[I+\phi\Sigma\pi]^{-1}\\ &= PQ^{-1}B^{-1}[I+APQ^{-1}B^{-1}]^{-1}\\ &= P[AP+BQ]^{-1}.\end{aligned} \tag{1.3}$$

Denoting

$$M := AP + BQ, \tag{1.4}$$

we obtain that

$$\Sigma_{(\pi,\phi)} = PM^{-1}. \tag{1.5}$$

The basic design objective is to choose A and B in such a way as to make the system M unimodular, where a unimodular system is a stable system possessing a stable inverse. Furthermore, since the image of M becomes the domain of the closed loop system $\Sigma_{(\pi,\phi)} = PM^{-1}$, the chosen systems A and B must yield an M with an appropriate image (see Section 3 for details). Then, (1.5) shows that the closed loop system is input/output stable, and a few further mild restrictions on A and B guaranty that the closed loop is in fact internally stable [8]. We can now state the basic question considered in the present paper.

1.6. Question. Let Σ_n be a given nominal system, and let π and ϕ be a fixed pair of compensators (of the form (1.2)) for which the closed loop system $\Sigma_{n(\pi,\phi)}$ is stable. Find the class of all systems Σ for which the closed loop system $\Sigma_{(\pi,\phi)}$ remains stable (for the fixed π and ϕ).

As it turns out, the formalism developed in [8,9,10] facilitates the derivation of a particularly simple answer to this question. Specifically, the class of all systems Σ for which $\Sigma_{(\pi,\phi)}$ is stable is characterized in terms of a purely algebraic condition, with the stability aspects of the problem being taken care of automatically through the framework of the theory of fraction representations of nonlinear systems. Qualitatively and somewhat inaccurately stated, these conditions consist of certain amplitude bounds on the deviation of Σ from the nominal system Σ_n. The exact statement is provided in Section 3.

Studies of the effect of system uncertainties on the performance of control systems have a long history and extensive literature, which is beyond our scope to survey here. Some insight into the available literature in this area, which is mostly confined to the case of linear systems, can be gained from [1,2,14,21,22,16,17,20,5,13], the references cited in these papers, and others. The present discussion depends heavily on the theory of fraction representations of nonlinear systems, various aspects of which are discussed in [6–12,4,18,19,3,15], the references cited in these papers, and others.

2. Notation and background

The present section is devoted to a brief outline of the framework of [6–12], which forms the basis of the present discussion. Let $S(R^m)$ be the set of all sequences $u = \{u_0, u_1, u_2, \ldots\}$ of m-dimensional real vectors $u_j \in R^m$, $j = 0, 1, 2, \ldots$. Adopting the standard input/output point of view, a system is regarded simply as a map $\Sigma : S(R^m) \to S(R^p)$ transforming input sequences of m-dimensional real vectors into output sequences of p-dimensional real vectors. For a subset $S \subseteq S(R^m)$, let $\Sigma[S]$ be the image of S through Σ, namely, the set of all output sequences generated by Σ from input sequences belonging to S.

In the current investigation we are mostly interested in nonlinear systems Σ that can be described by equations of the form

$$x_{k+1} = f(x_k, u_k), \tag{2.1a}$$

$$y_k = h(x_k), \tag{2.1b}$$

where

$$u = \{u_0, u_1, u_2, \ldots\} \in S(R^m)$$

is the input sequence;

$$y = \{y_0, y_1, y_2, \ldots\} \in S(R^p)$$

is the output sequence; and

$$x = \{x_0, x_1, x_2, \dots\} \in S(R^q)$$

is an intermediate sequence in state space. The initial condition x_0 is specified, and the functions $f: R^q \times R^m \to R^q$ and $h: R^q \to R^p$ are assumed to be continuous. A system Σ possesing a representation of the form (2.1) with continuous f and h is said to have a *continuous realization*. By the *input/state part* Σ_s of Σ we refer to the system described by the recursion $x_{k+1} = f(x_k, u_k)$.

To deal with bounded sequences of vectors, let $\theta > 0$ be a real number, and denote by $S(\theta^m)$ the set of all sequences $u \in S(R^m)$ whose elements satisfy $u_i \in [-\theta, \theta]^m$ for all integers $i \geq 0$. Then, a system $\Sigma: S(R^m) \to S(R^p)$ is *BIBO* (*Bounded-Input Bounded-Output*)*-stable* if for every real number $\theta > 0$ there is a real number $N > 0$ such that $\Sigma[S(\theta^m)] \subset S(N^p)$.

Denote by $|\cdot|$ the l^∞-norm, so that, for a vector $a = (a_1, \dots, a_m) \in R^m$, one has

$$|a| = \max\{|a_i|, i = 1, \dots, m\}.$$

For a sequence $u \in S(R^m)$, set $|u| = \sup_{i \geq 0} |u_i|$. An important place in our framework is occupied by the norm ρ defined on the space $S(R^m)$ by

$$\rho(u) := \sup_{i \geq 0} 2^{-i} |u_i|$$

for all $u \in S(R^m)$, which is a weighted l^∞-norm. In the sequel, all notions of continuity over spaces of sequences are with respect to the norm ρ.

A system $\Sigma: S(R^m) \to S(R^p)$ is *stable* if it is BIBO-stable, and if, for every real number $\theta > 0$, the restriction $\Sigma: S(\theta^m) \to S(R^p)$ is a continuous map. A system $M: S_1 \to S_2$, where $S_1 \subset S(R^m)$ and $S_2 \subset S(R^p)$, is said to be *unimodular* if it is a set isomorphism, and if M and M^{-1} are both stable systems.

The notion of causality is important to the present discussion. A system $\Sigma: S(R^m) \to S(R^p)$ is *causal* (respectively, *strictly causal*) if the following is satisfied for all integers $i \geq 0$ and for all input sequences $u, v \in S(R^m)$: whenever $u_j = v_j$ for all $j = 0, \dots, i$, also $(\Sigma u)_j = (\Sigma v)_j$ for all $j = 0, \dots, i$ (respectively $j = 0, \dots, i+1$). A system $\Sigma: S_1 \to S_2$, where $S_1 \subset S(R^m)$ and $S_2 \subset S(R^p)$, is *bicausal* if it is causal and if it has an inverse Σ^{-1} which is also causal.

The sum of two systems $\Sigma_1, \Sigma_2: S_1 \to S_2$ is defined, as usual, by

$$(\Sigma_1 + \Sigma_2)u := \Sigma_1 u + \Sigma_2 u$$

for all $u \in S_1$. The following is a simple but useful consequence of causality considerations (e.g., [7]).

2.2. Proposition. *Let $\Sigma_1, \Sigma_2: S \to S(R^p)$ be two causal systems, where $S \subset S(R^m)$, and let $\Sigma := \Sigma_1 + \Sigma_2$. Assume that the restriction $\Sigma_1: S \to \Sigma_1[S]$ is bicausal, and that Σ_2 is strictly causal. Then, Σ is an injective (one to one) system, and the restriction $\Sigma: S \to \Sigma[S]$ is bicausal.*

We turn now to a brief review of some notions from the theory of fraction representations of nonlinear systems. A right fraction representation of a system $\Sigma: S(R^m) \to S(R^p)$ is determined by three quantities: a subset $S \subset S(R^q)$, where $q > 0$ is some integer, and a pair of stable systems $P: S \to S(R^p)$ and $Q: S \to S(R^m)$, where Q is invertible and $\Sigma = PQ^{-1}$. The subset S is called the *factorization space* of the fraction representation. The fraction representation $\Sigma = PQ^{-1}$ is said to be *right coprime* if the stable systems P and Q are right coprime according to the following definition [8]. (For a system $P: S_1 \to S_2$ and a subset $S \subset S_2$, denote by $P^*[S]$ the inverse image of the set S through P, namely, the set of all input sequences $u \in S_1$ for which $Pu \in S$.)

2.3. Definition. Let $S \subset S(R^q)$ be a subset. Two stable systems $P: S \to S(R^p)$ and $Q: S \to S(R^m)$ are *right coprime* if the following conditions hold:

(i) For every real number $\tau > 0$ there exists a real number $\theta > 0$ such that

$$P^*[S(\tau^p)] \cap Q^*[S(\tau^m)] \subset S(\theta^q).$$

(ii) For every number $\tau > 0$, the set $S \cap S(\tau^q)$ is a closed subset of $S(\tau^q)$ (with respect to the topology induced by ρ).

Of particular importance to our present discussion are right coprime fraction representations $\Sigma = PQ^{-1}$ in which the denominator system Q is a bicausal system. In [12] it was shown that such fraction representations exist for all stabilizable systems that possess a continuous realization. The explicit construction of such fraction representations was also described there, and it depends on the theory of reversible state feedback. Briefly, let $\Sigma_s : S(R^m) \to S(R^q)$ be the input/state part of the system Σ described by (2.1), and let $\sigma : R^q \times R^m \to R^m$ be a continuous function. When σ is used as a state feedback function for the system Σ_s, it yields a closed loop system $\Sigma_{s\sigma}$ whose recursive representation is given by

$$x_{k+1} = f(x_k, \sigma(x_k, u_k)).$$

The feedback function $\sigma(x, u)$ is called *reversible* if it is injective in u for any possible state x. The feedback operation induced by a reversible feedback function can be 'undone' by another feedback function to retrieve the original system Σ_s from the closed loop system $\Sigma_{s\sigma}$. We say that the feedback function σ stabilizes the system Σ_s over the input space $S(\theta^m)$ if the restriction $\Sigma_{s\sigma} : S(\theta^m) \to S(R^q)$ is a stable system. For the sake of convenience, we reproduce here the following result from [12], where all relevant constructions are described in detail.

2.4. Theorem. *Let $\Sigma : S(R^m) \to S(R^p)$ be a system having a continuous realization, and let $\Sigma_s : S(R^m) \to S(R^q)$ be the input/state part of Σ. Assume there is a reversible feedback function $\sigma : R^q \times R^m \to R^m$ that stabilizes the system Σ_s over the input space $S(\theta^m)$. Then, the system Σ has a right coprime fraction representation $\Sigma = PQ^{-1}$, where $P : S(\theta^m) \to S(R^p)$ and $Q : S(\theta^m) \to Q[S(\theta^m)]$ are (stable) systems with continuous realizations, and Q is bicausal. (The factorization space of this fraction representation is $S(\theta^m)$.)*

The discussion of the present paper depends on some results on internal stabilization of nonlinear systems derived in [8], which we now review briefly. First, the control configuration of Figure 1 is said to be *internally stable* if (i) the input/output system $\Sigma_{(\pi,\phi)}$ is stable, and if (ii) this stability is not destroyed by small noises added to the inputs of the systems π, Σ, and ϕ within the closed loop. In mathematical terms, statement (ii) is to mean that the output sequence of the closed loop system depends in a continuous way on the aforementioned noise signals (and all internal signals of the loop remain bounded), as long as the noise amplitude is sufficiently small (see [8] for a formal definition). Our main interest here is in stabilization over bounded input spaces, and we shall asume that the input sequences of the closed loop system are restricted to the space $S(\theta^m)$ for some real number $\theta > 0$.

It is convenient to introduce the following notion. A stable system $A : S(R^m) \to S(R^p)$ is *differentially bounded* by a real number $\theta > 0$ if there is a real number $\varepsilon > 0$ such that, for every pair of elements y, $y' \in S(R^m)$ satisfying $|y - y'| < \varepsilon$, one has $|A(y) - A(y')| < \theta$ (see [8] for more details). Two subsets $S_1 \subset S(R^q)$ and $S_2 \subset S(R^m)$ are said to be *stability morphic* if there is a bicausal and unimodular isomorphism $M : S_1 \cong S_2$. The following result was derived in [8]. (We say that a fraction representation $\Sigma = PQ^{-1}$ is valid over a subset S_I of input sequences if the fraction representation is valid (at least) for the restriction of Σ to S_I.)

2.5. Theorem. *Let $\Sigma : S(R^m) \to S(R^p)$ be a causal system, and assume it has a right coprime fraction representation $\Sigma = PQ^{-1}$ valid over a subset $S_I \subset S(R^m)$ of input sequences, and having the factorization space $S \subset S(R^q)$. Let $\theta > 0$ be a real number. Assume S contains a subset S' which is stability morphic to $S((5\theta)^m)$, and let M: $S' \cong S((5\theta)^m)$ be a unimodular and bicausal system. Assume further there is a pair of stable systems $A : S(R^p) \to S(R^m)$ and $B : S(R^m) \to S(R^m)$ satisfying the equation $APv + BQv = Mv$ for all $v \in S'$, where A is causal and B is bicausal. If A and B are differentially bounded by θ, then the closed loop system $\Sigma_{(B^{-1}, A)}$ is internally stable over the input space $S(\theta^m)$.*

For our present application, it is convenient to restate Theorem 2.5 in the following somewhat refined form, whose validity is easily confirmed by a reexamination of the proof of Theorem 3.9 in [8].

2.6. Theorem. *Let $\Sigma : S(R^m) \to S(R^p)$ be a causal system, and assume it has a right coprime fraction representation $\Sigma = PQ^{-1}$ valid over a subset $S_I \subset S(R^m)$ of input sequences, and having the factorization space $S \subset S(R^q)$. Let $\theta, \gamma > 0$ be real numbers satisfying $\gamma > 4\theta$, and denote $\alpha := \gamma - 4\theta$. Assume S contains a subset S' which is stability morphic to $S(\gamma^m)$, and let M: $S' \cong S(\gamma^m)$ be a unimodular and bicausal system. Assume further there is a pair of stable systems $A : S(R^p) \to S(R^m)$ and $B : S(R^m) \to S(R^m)$ satisfying the equation $APv + BQv = Mv$ for all $v \in S'$, where A is causal and B*

is bicausal. If A and B are differentially bounded by θ, then the closed loop system $\Sigma_{(B^{-1},A)}$ is internally stable over the input space $S(\alpha^m)$.

3. The effect of system perturbations

We restrict our attention to the case where the system Σ that needs to be controlled is strictly causal, and consider the stabilization of Σ using the control configuration of Figure 1 with compensators π and ϕ of the form (1.2). Note that the class of strictly causal systems includes many systems of practical interest; For instance, every system possessing a continuous realization of the form (2.1) is strictly causal.

In order to discuss the effect of system perturbations, assume that only a nominal description Σ_{n} of the given system is provided, and let Σ be the actual system inserted into the closed loop of Figure 1. We require Σ_{n} and Σ to have right coprime fraction representations $\Sigma_{\mathrm{n}} = P_{\mathrm{n}}Q_{\mathrm{n}}^{-1}$ and $\Sigma = PQ^{-1}$, where the denominator systems Q_{n} and Q are both bicausal, and where the factorization space for both fraction representations is $S(\beta^m)$ for some real number $\beta > 0$. Such fraction representations can be constructed under quite general conditions, as demonstrated by Theorem 2.4. The strict causality of the systems Σ_{n} and Σ, combined with the bicausality of Q_{n} and Q, implies that the numerator systems P_{n} and P are both strictly causal systems.

Examine now the configuration of Figure 1 with the nominal system Σ_{n}, using compensators π and ϕ of the form (1.2). The critical equation is (1.4), which we rewrite here in the form

$$AP_n + BQ_n = M_n, \tag{3.1}$$

and require that it be valid over the factorization space $S(\beta^m)$. Here, the term AP_n is strictly causal due to the strict causality of P_n and the causality of A; the term BQ_n is bicausal, since B and Q_n are both bicausal. Proposition 2.2 then implies that the system

$$M_{\mathrm{n}} : S(\beta^m) \to M_{\mathrm{n}}[S(\beta^m)]$$

is bicausal, which, in particular, implies that $M_{\mathrm{n}} : S(\beta^m) \to M_{\mathrm{n}}[S(\beta^m)]$ is a set isomorphism, and M_{n}^{-1} exists. Consequently, following the path leading to (1.5), the input/output relation of the closed loop system can be expressed in the form

$$\Sigma_{\mathrm{n}(\pi,\phi)} = P_{\mathrm{n}}M_{\mathrm{n}}^{-1}. \tag{3.2}$$

Note that we did not assume that M_{n} is a unimodular system. However, the system M_{n} is clearly stable, since so are the systems A, P_{n}, B, and Q_{n} of which it consists; Furthermore, the domain $S(\beta^m)$ is compact in the topology induced by the metric ρ and, since M_{n} is a set isomorphism, it follows by a standard result in basic topology that $M_{\mathrm{n}} : S(\beta^m) \to M_{\mathrm{n}}[S(\beta^m)]$ is actually a unimodular system. This completes the proof of the next statement.

3.3. Lemma. *The system*

$$M_{\mathrm{n}} : S(\beta^m) \to M_{\mathrm{n}}[S(\beta^m)]$$

of (3.1) *is a bicausal and unimodular system.*

The lemma directly yields that the inverse

$$M_{\mathrm{n}}^{-1} : M_{\mathrm{n}}[S(\beta^m)] \to S(\beta^m)$$

is stable, and, combining this with (3.2), we obtain that the closed loop system $\Sigma_{\mathrm{n}(\pi,\phi)}$ is (input/output) stable over the domain of input sequences $M_{\mathrm{n}}[S(\beta^m)]$.

From the control theoretic point of view, stability of a system is meaningful only if it is valid over a domain of input sequences of the form $S(\gamma^m)$ for some real number $\gamma > 0$. This guarantees that all input sequences of amplitude not exceeding γ are permitted; Otherwise, only very peculiar input sequences are allowed. Imposing this requirement on the closed loop system $\Sigma_{\mathrm{n}(\pi,\phi)}$, it follows that in order for $\Sigma_{\mathrm{n}(\pi,\phi)}$ to be (meaningfully) stable, there must be a real number $\gamma > 0$ such that $S(\gamma^m) \subset M_{\mathrm{n}}[S(\beta^m)]$. We have then the following.

3.4. Theorem. *Let $\Sigma_{\mathrm{n}} : S(R^m) \to S(R^p)$ be a strictly causal system possessing a right coprime fraction representation $\Sigma_{\mathrm{n}} = P_{\mathrm{n}}Q_{\mathrm{n}}^{-1}$ with the factorization space $S(\beta^m)$, where $Q_n : S(\beta^m) \to Q_{\mathrm{n}}[S(\beta^m)]$ is bicausal and $\beta > 0$. Let $A : S(R^p) \to S(R^m)$ and $B : S(R^m) \to S(R^m)$ be a pair of stable systems, with A causal and B bicausal. Set $\pi := B^{-1}$ and $\phi := A$. Then, the closed loop system $\Sigma_{\mathrm{n}(\pi,\phi)}$ is (input/output) stable over the domain of* *input sequences $S(\gamma^m)$ for some real number $\gamma > 0$ if and only if*

$$S(\gamma^m) \subset (AP_n + BQ_n)[S(\beta^m)].$$

Thus, we have obtained a very simple (and purely algebraic) characterization of the input/output stability of a strictly causal nonlinear system over a bounded domain of input sequences. As we can see, all the topological considerations related to the notion of stability are automatically incorporated through the theory of fraction representations of nonlinear systems.

We turn now to an investigation of the effect of system perturbations on the stability of the closed loop system of Figure 1. In explicit terms, we substitute the system Σ for the nominal system Σ_n, and we would like to find out under what conditions the closed loop $\Sigma_{(\pi,\phi)}$ remains stable. This is in fact quite simple. Using the fraction representation $\Sigma = PQ^{-1}$, we know from Theorem 3.4 that the closed loop system $\Sigma_{(\pi,\phi)}$ will remain stable with the same π and ϕ if and only if there is a real number $\delta > 0$ such that

$$S(\delta^m) \subset (AP + BQ)[S(\beta^m)];$$

the stability will then be valid over the input domain $S(\delta^m)$. From this fact, we can derive some simple conditions on the permissible deviation of Σ from Σ_n, as follows.

For Σ we have the right coprime fraction representation $\Sigma = PQ^{-1}$, with factorization space $S(\beta^m)$. Any other right coprime fraction representation of Σ with the factorization space $S(\beta^m)$ and bicausal denominator is of the form $\Sigma = P_1Q_1^{-1}$, where $P_1 := PM$ and $Q_1 := QM$ with $M: S(\beta^m) \to S(\beta^m)$ being a unimodular and bicausal system [7,8]. Apriori, we do not know, of course, which one of these fraction representations of Σ should be used in the analysis, but, as we show below (and as one might intuitively expect), all are equally suitable. Denote

$$\Delta_a(M) := AP_1 - AP_n = APM - AP_n, \tag{3.5a}$$

$$\Delta_b(M) := BQ_1 - BQ_n = BQM - BQ_n. \tag{3.5b}$$

Then, equation (1.4) for the closed loop system around Σ with the fraction representation $\Sigma = P_1Q_1^{-1}$ becomes

$$\begin{aligned} AP_1 + BQ_1 &= AP_n + \Delta_a(M) + BQ_n + \Delta_b(M) \\ &= M_n + \Delta_a(M) + \Delta_b(M). \end{aligned} \tag{3.6}$$

Letting

$$\Delta(M) := \Delta_a(M) + \Delta_b(M), \tag{3.7a}$$

$$\Delta := \Delta(I), \tag{3.7b}$$

$$\mathscr{M} := M_n + \Delta, \tag{3.7c}$$

where $I: S(\beta^m) \to S(\beta^m)$ is the identity system, it directly follows that

$$AP + BQ = \mathscr{M}, \tag{3.8}$$

and

$$\begin{aligned} AP_1 + BQ_1 &= APM + BQM \\ &= (AP + BQ)M = \mathscr{M}M, \end{aligned} \tag{3.9}$$

where (3.8) and (3.9) hold over the fractorization space $S(\beta^m)$ of the fraction representation of Σ. In complete analogy with Lemma 3.3, it follows that $\mathscr{M}: S(\beta^m) \to \mathscr{M}[S(\beta^m)]$ is a bicausal and unimodular sytem. The input/output relation of the closed loop system is then given by $\Sigma_{(\pi,\phi)} = P_1(\mathscr{M}M)^{-1}$, and it is input/output stable over the input space $\mathscr{M}M[S(\beta^m)]$, the domain of $(\mathscr{M}M)^{-1}$.

In view of the discussion of the paragraph preceding Theorem 3.4, meaningful stability of the closed loop system $\Sigma_{(\pi,\phi)}$ is obtained only if there is a real number $\delta > 0$ such that $S(\delta^m) \subset \mathscr{M}M[S(\beta^m)]$. But, since $M[S(\beta^m)] = S(\beta^m)$ by definition of M, the latter is equivalent to

$$S(\delta^m) \subset \mathscr{M}[S(\beta^m)] = (M_n + \Delta)[S(\beta^m)]. \tag{3.10}$$

The input/output relation of the perturbed closed loop system over the space $S(\delta^m)$ of input sequences is then given by

$$\Sigma_{(\pi,\phi)} = P_1(\mathscr{M}M)^{-1} = P\mathscr{M}^{-1} : S(\delta^m) \to S(R^p). \tag{3.11}$$

In view of the discussion leading to Theorem 3.4, the existence of δ is a necessary and sufficient condition for the (meaningful) stability of the closed loop around the perturbed system Σ. We shall heretofore regard δ as the *largest* real number for which (3.10) is valid (a maximum for δ exists here due to the compactness of the domains). Clearly, depending on whether $\delta \le \gamma$ or $\delta \ge \gamma$, the domain of inputs over which the stability of the perturbed closed loop system is valid may be either smaller or larger than the input domain over which stability holds in the nominal case. Of course, if there is no strictly positive number δ for which (3.10) is valid, the closed loop system $\Sigma_{(\pi,\phi)}$ is not stable over any useful input space. Another direct consequence of (3.9), (3.10), and (3.11) is that M is of no consequence here, and any appropriate fraction representation $\Sigma = PQ^{-1}$ of the perturbed system can be used, as one would intuitively expect. We summarize now our discussion.

3.12. Summary. (i) The strictly causal nominal system $\Sigma_n : S(R^m) \to S(R^p)$ is assumed to have a right coprime fraction representation $\Sigma_n = P_n Q_n^{-1}$ with the factorization space $S(\beta^m)$ and a bicausal denominator $Q_n : S(\beta^m) \to Q_n[S(\beta^m)]$. In view of Theorem 2.4, this is basically a stabilizability assumption on the nominal system Σ_n.

(ii) A stable closed loop configuration of the form depicted in Figure 1 with compensators of the form (1.2) is designed for the nominal system Σ_n. Then, the system

$$M_n := AP_n + BQ_n : S(\beta^m) \to M_n[S(\beta^m)]$$

is a unimodular and bicausal system, and there is a real number $\gamma > 0$ such that $S(\gamma^m) \subset M_n[s(\beta^m)]$.

(iii) The perturbed system $\Sigma : S(R^m) \to S(R^p)$ is inserted into the closed loop instead of the nominal system Σ_n. The system Σ is assumed to possess a right coprime fraction representation $\Sigma = PQ^{-1}$ having the factorization space $S(\beta^m)$ and a bicausal denominator $Q : S(\beta^m) \to Q[S(\beta^m)]$. In view of Theorem 2.4, this assumption is basically a stabilizability assumption on the perturbed system Σ.

(iv) Define the deviation

$$\Delta = (AP - AP_n) + (BQ - BQ_n)$$

as in (3.7). Then, the class of perturbed systems Σ for which the closed loop system remains stable with the given compensators $\phi = A$ and $\pi = B^{-1}$ is characterized as follows.

3.13. Theorem. *The closed loop system $\Sigma_{(\pi,\phi)}$ is input/output stable if and only if the deviation Δ satisfies the following condition: there is a real number $\delta > 0$ such that*

$$S(\delta^m) \subset (M_n + \Delta)[S(\beta^m)].$$

When the latter holds, the closed loop system $\Sigma_{(\pi,\phi)}$ is input/output stable over the space of input sequences $S(\delta^m)$.

Interpreting the theorem in intuitive terms, we can view the effect of the disturbance Δ as a 'shift' of the image of the nominal unimodular and bicausal system M_n. As seen from the above discussion, this 'shift' always maintains the unimodularity and the bicausality of the disturbed system $M_n + \Delta$. However, in order to preserve the stability of the closed loop system, the 'shift' has to be such that a subset of the form $S(\delta^m)$, $\delta > 0$, remains contained within the image of $M_n + \Delta$. Recall that the original unimodular system M_n contains the subset $S(\gamma^m)$ in its image, by 3.12 (ii).

Note that the source causing the deviation Δ does not need to be restricted to deviations of the system Σ, but may also include deviations of the compensators π and ϕ, as long as those deviations do not destroy the causality of A and the bicausality of B.

As a final comment on Theorem 3.13, we observe that the necessary and sufficient condition for the preservation of stability under system variations is very simple in nature, and involves only the verification of the amplitude condition

$$S(\delta^m) \subset (M_n + \Delta)[S(\beta^m)],$$

which is purely algebraic in nature. This simplicity provides yet another manifestation of the power of the fraction representation approach to nonlinear control. Somewhat philosophically, we may say that the fraction representation approach has the advantages of automatically incorporating the topological considerations of the theory of stabilization of nonlinear systems, leaving us to verify only relatively simple algebraic conditions.

By somewhat restricting the permissible the deviation $\Delta(M)$, we can obtain a particularly simple sufficient condition for the preservation of stability under system perturbations. Recall that $\Delta(M) = \Delta_a(M) + \Delta_b(M)$, where $\Delta_a(M)$ and $\Delta_b(M)$ are given by (3.5). Due to the strict causality of P_1 and P_n it follows that $\Delta_a(M)$ is always a strictly causal system. On the other hand, $\Delta_b(M)$ is in general only causal. A very simple condition for the preservation of stability is obtained when $\Delta_b(M)$ is restricted to be strictly causal. In order to provide some motivation for this restriction, ignore for a moment the requirement that $M[S(\beta^m)] = S(\beta^m)$, and consider the linear case. Then, M can always be chosen so as to make $\Delta_b(M)$ strictly causal. Indeed, since BQ and BQ_n are both bicausal, they can be expressed in the linear case in the form $BQ = V + C$ and $BQ_n = W + D$, where V and W are nonsingular static linear transformations, and C and D are strictly causal. Taking $M := V^{-1}W$, it follows that

$$\Delta_b(M) = BQM - BQ_n$$
$$= VM + CM - W - D = CM - D,$$

which is a strictly causal system. However, in the general nonlinear case, the situation in this regard may not be as simple.

Nevertheless, assume there is a bicausal unimodular system $M: S(\beta^m) \to S(\beta^m)$ such that $\Delta_b(M)$ is strictly causal. Then, $\Delta(M) = \Delta_a(M) + \Delta_b(M)$ is also strictly causal. Furthermore, from (3.7) and (3.5) it follows directly that $\Delta(M)$ is a stable system, and thus, there always is a real number $\eta > 0$ such that

$$\Delta(M)[S(\beta^m)] \subset S(\eta^m).$$

Recall that $S(\gamma^m) \subset M_n[S(\beta^m)]$ by 3.12 (ii), and define $\mu := \gamma - \eta$.

3.14. Lemma. *Assume there is a bicausal and unimodular $M: S(\beta^m) \to S(\beta^m)$ for which $\Delta(M)$ is strictly causal, and let μ, γ, and η be as above. If $\mu > 0$, then*

$$S(\mu^m) \subset (M_n + \Delta)[S(\beta^m)].$$

Proof. We first show that whenever $\mu > 0$, then

$$S(\mu^m) \subset (M_n + \Delta(M))[S(\beta^m)].$$

Let $s = \{s_0, s_1, s_2, \dots\}$ be any sequence in $S(\mu^m)$. We construct (element by element) a sequence $w \in S(\beta^m)$ satisfying $s = (M_n + \Delta(M))w$. Note that by the strict causality of $\Delta(M)$, the first output element $(\Delta(M)v)_0 =: \Delta(M)_0$ is independent of v for all $v \in S(\beta^m)$. Consequently, the element $s_0 - (\Delta(M)v)_0 \in R^m$ is independent of v, and satisfies

$$|s_0 - (\Delta(M)v)_0| \le |s_0| + |(\Delta(M)v)_0|$$
$$\le \mu + \eta = \gamma.$$

Now, fix a sequence v. Since $S(\gamma^m) \subset M_n[S(\beta^m)]$, there is an input sequence

$$w^0 = \{u_0, w_1, w_2, \dots\} \in S(\beta^m)$$

satisfying $M_n w^0 = s - \Delta(M)v$. Then,

$$(M_n w^0)_0 = s_0 - (\Delta(M)v)_0$$
$$= s_0 - \Delta(M)_0 = s_0 - (\Delta(M)w^0)_0.$$

This implies that the input sequence w^0 satisfies

$$[(M_n + \Delta(M))w^0]_0 = s_0,$$

and we managed to match the first element of s.

In preparation for a recursion, assume there is an integer $i \ge 0$ and an input sequence

$$w^i = \{u_0, u_1, \dots, u_i, a_{i+1}, a_{i+2}, \dots\} \in S(\beta^m)$$

such that

$$[(M_n + \Delta(M))w^i]_j = s_j$$

for all $j = 0, \dots, i$. Invoking our bounds, we obtain

$$|s - \Delta(M)w^i| \le |s| + |\Delta(M)w^i| \le \mu + \eta = \gamma,$$

and, since $S(\gamma^m) \subset M_n[S(\beta^m)]$, there is an input sequence $w^{i+1} \in S(\beta^m)$ such that

$$M_n w^{i+1} = s - \Delta(M)w^i.$$

Combining with the recursion assumption, we have $(M_n w^{i+1})_j = (M_n w^i)_j$ for all $j = 0, \dots, i$, and using the bicausality of M_n, it follows that $(w^{i+1})_j = (w^i)_j = u_j$ for all $j = 0, \dots, i$. Hence, the input sequence w^{i+1} is of the form

$$w^{i+1} = \{u_0, u_1, \dots, u_i, u_{i+1}, b_{i+2}, b_{i+3}, \dots\}.$$

By the strict causality of $\Delta(M)$, the element $(\Delta(M)w^i)_{i+1}$ is uniquely determined by the input elements $(w^i)_0, \dots, (w^i)_i$ and, since those are the same as the elements number $0, \dots, i$ of w^{i+1}, we get

$$(\Delta(M)w^{i+1})_{i+1} = (\Delta(M)w^i)_{i+1}.$$

This yields that

$$(M_n w^{i+1})_j = s_j - (\Delta(M)w^{i+1})_j$$

for all $j = 0, \dots, i+1$, or

$$[(M_n + \Delta(M))w^{i+1}]_j = s_j$$

for all $j = 0, \ldots, i+1$. By recursion, there is then an input sequence $w \in S(\beta^m)$ satisfying

$$(M_n + \Delta(M))w = s.$$

Since the argument is valid for any sequence $s \in S(\mu^m)$, we obtain

$$S(\mu^m) \subset (M_n + \Delta(M))[S(\beta^m)].$$

Finally, by (3.6), (3.9), and (3.7), we have

$$M_n + \Delta(M) = (M_n + \Delta)M,$$

and, since $M[S(\beta^m)] = S(\beta^m)$, it follows that

$$\begin{aligned} S(\mu^m) &\subset (M_n + \Delta(M))[S(\beta^m)] \\ &= (M_n + \Delta)M[S(\beta^m)] \\ &= (M_n + \Delta)[S(\beta^m)], \end{aligned}$$

and the assertion holds. □

The lemma can be rephrased in the following terms. Note that

$$M_n + \Delta = (I + \Delta M_n^{-1})M_n;$$

thus, the lemma is concerned with the containment

$$S(\mu^m) \subset (I + \Delta M_n^{-1})M_n[S(\beta^m)].$$

In other words, we need to find whether for every element $s \in S(\mu^m)$ there is an element $v \in \operatorname{Im} M_n$ satisfying $v + \Delta M_n^{-1}v = s$. Rewriting this in the form $v = s - \Delta M_n^{-1}v$, it basically amounts to the existence of a fixed point. The lemma can be proved under a variety of assumptions on Δ, the simplest of which is being used here.

When Lemma 3.14 is combined with Theorem 3.13, it follows directly that whenever $\mu > 0$, the closed loop system around Σ is stable for input sequences bounded by μ (at least). This yields the foillowing simple sufficient condition for the preservation of stability under system perturbations. (Recall from 3.12 (ii) that $S(\gamma^m)$ was the input/output stability domain for the nominal closed loop system.)

3.15. Corollary. *Under the conditions of Theorem* 3.13 *and Lemma* 3.14, *let* $\eta > 0$ *be such that*

$$\Delta(M)[S(\beta^m)] \subset S(\eta^m),$$

and assume that $\eta < \gamma$. *Then, there is a real number* $\delta > 0$ *such that the closed loop system* $\Sigma_{(\pi,\phi)}$ *is (input/output) stable for input sequences bounded by* δ.

As we can see, the corollary provides a simple sufficient condition on the deviations under which stability of the closed loop system is preserved. Of course, when this condition is not met, the necessary and sufficient condition of Theorem 3.13 has to be checked.

Another interesting point related to Corollary 3.15 is the following. High forward gain in the loop, i.e., high gain for the precompensator π, is obtained for low gain of B, since $\pi = B^{-1}$. Thus, when π is a high gain device, relatively large variations in the denominator Q of Σ are permitted, since the influence of such deviations on $\Delta(M)$ comes through the term $BQM - BQ_n$, and the low gain of B will have an attenuating effect.

Internal stability

We turn now to an examination of the preservation of internal stability under system perturbations. Using Theorem 2.6 as the starting point, let A and B be a pair of stable systems satisfying the conditions of the theorem for the nominal system Σ_n, so that the closed loop $\Sigma_{n(B^{-1},A)}$ is internally stable. Since A and B satisfy the conditions of Theorem 2.6, we have that A and B are stable and differentially bounded by $\theta > 0$; A is causal and B is bicausal; and

$$AP_n + BQ_n = M_n, \tag{3.16}$$

where M_n is a unimodular and bicausal system. By our construction of the fraction representation $\Sigma_n = P_n Q_n^{-1}$, its factorization space is $S = S(\beta^m)$. For the sake of simplicity, we assume that M_n was taken with $S' = S(\beta^m)$ in Theorem 2.6, so that M_n: $S(\beta^m) \cong S(\gamma^m)$ with $\gamma > 4\theta$. Under these circumstances, (1.5) shows that the input/output relation induced by the closed loop system of Figure 1 around the nominal system Σ_n, with the compensators $\pi = B^{-1}$ and $\phi = A$, is given by

$$\Sigma_{n(\pi,\phi)} = P_n M_n^{-1}, \tag{3.17}$$

and the closed loop system is internally stable over the input domain $S(\alpha^m)$, where $\alpha = \gamma - 4\theta$.

Assume now that the strictly causal system Σ is inserted into the closed loop of Figure 1 instead of the nominal system Σ_n, using the same compensators π and ϕ. Recall that Σ has the right coprime fraction representation $\Sigma = PQ^{-1}$, with the factorization space $S(\beta^m)$. Setting $\mathscr{M} := AP + BQ$, it follows by our discussion of (3.8) that $\mathscr{M}: S(\beta^m) \to \mathscr{M}[S(\beta^m)]$ is a unimodular and bicausal system. Just as in (3.7), we have that $\mathscr{M} = M_n + \Delta$. Suppose now there is a real number $\delta > 4\theta$ such that

$$S(\delta^m) \subset \mathscr{M}[S(\beta^m)]. \tag{3.18}$$

Then, since A and B satisfy the conditions of Theorem 2.6, it follows by the same theorem that the closed loop system $\Sigma_{(\pi,\phi)}$ is internally stable over the input space $S(\alpha^m)$, where $\alpha = \delta - 4\theta$. The input/output relation of the closed loop system is then given by

$$\Sigma_{(\pi,\phi)} = P\mathscr{M}^{-1} : S(\alpha^m) \to S(R^p). \tag{3.19}$$

Thus, the existence of a real number $\delta > 4\theta$ satisfying $S(\delta^m) \subset \mathscr{M}[S(\beta^m)]$ is a sufficient condition for the internal stability of the closed loop around the perturbed system Σ. We shall heretofore regard δ as the *largest* real number for which (3.18) is satisfied (a maximum exists here due to the compactness of the domain). The domain of inputs over which the internal stability of the perturbed closed loop system is valid depends on the size of δ, and may be either smaller or larger than the domain over which internal stability holds for the nominal system Σ_n, depending on whether $\delta \le \gamma$ or $\delta \ge \gamma$. We can summarize our discussion as follows.

3.20. Summary. (i) The strictly causal nomimal system $\Sigma_n: S(R^m) \to S(R^p)$ is assumed to have a right coprime fraction representation $\Sigma_n = P_n Q_n^{-1}$ with the factorization space $S(\beta^m)$ and a bicausal denominator $Q_n: S(\beta^m) \to Q_n[S(\beta^m)]$. In view of Theorem 2.4, this is basically a stabilizability assumption on the nominal system Σ_n.

(ii) An internally stable closed loop configuration of the form depicted in Figure 1 with compensators of the form (1.2) is designed for the nominal system Σ_n. In the spirit of Theorem (2.6), the stable systems $A: S(R^p) \to S(R^m)$ and $B: S(R^m) \to S(R^m)$ of (1.2) are chosen differentially bounded by the real number $\theta > 0$; A is causal; B is bicausal; the system

$$AP_n + BQ_n = M_n:\ S(\beta^m) \cong S(\gamma^m)$$

is unimodular and bicausal; and $\gamma > 4\theta$. This guarantees the internal stability of the closed loop around the nominal system Σ_n for input sequences of amplitude not exceeding $\alpha = \gamma - 4\theta$.

(iii) The perturbed system $\Sigma: S(R^m) \to S(R^p)$ is inserted into the closed loop instead of the nominal system Σ_n. It is assumed that there is a right coprime fraction representation $\Sigma = PQ^{-1}$ having the factorization space $S(\beta^m)$ and bicausal denominator $Q: S(\beta^m) \to Q[S(\beta^m)]$. Again, in view of Theorem 2.4, this is basically a stabilizability assumption on the perturbed system Σ.

(iv) Define the deviation

$$\Delta = (AP - AP_n) + (BQ - BQ_n)$$

as in (3.7). Then, the following is true.

3.21. Theorem. *If there is a real number $\delta > 4\theta$ for which*

$$S(\delta^m) \subset (M_n + \Delta)[S(\beta^m)],$$

then the closed loop system $\Sigma_{(\pi,\phi)}$ is internally stable over the input domain $S(a^m)$, where $a := \delta - 4\theta$.

In the spirit of Corollary 3.15, we can derive here too a simple sufficient condition for the preservation of internal stability under system perturbations. Under the conditions and the notation of Lemma 3.14, recall that

$$S(\mu^m) \subset (M_n + \Delta)[S(\beta^m)].$$

If also $\mu > 4\theta$, Theorem 3.21 implies that the closed loop system around Σ is internally stable for input sequences bounded by $\alpha = \mu - 4\theta$ (at least). In other words, internal stabilization over the input space $S(\alpha^m)$ is then assured. This yields the following simple sufficient condition for the preservation of internal stability under system perturbations. (Recall that $S(\gamma^m)$ is the domain of input/output stability of the closed loop with the nominal system Σ_n (from 3.20 (ii)), and that A and B are differentially bounded by θ.)

3.22. Corollary. *Under the conditions of Lemma 3.14 and Theorem 3.21, let $\eta > 0$ be such that*

$$\Delta(M)[S(\beta^m)] \subset S(\eta^m).$$

If $\eta < \gamma - 4\theta$, then there is a real number $\alpha > 0$ such that the closed loop system $\Sigma_{(\pi,\phi)}$ is internally stable for input sequences bounded by α.

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