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Pole assignment and minimal feedback design†

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The problem of pole assignment is considered for three types of output feedback configurations: a combination of dynamic precompensation and dynamic output feedback; dynamic precompensation and unity output feedback; and pure dynamic output feedback. In all three cases, the conditions for pole assignment depend on certain integer invariants which are determined, roughly speaking, by the unstable poles, by the unstable zeros, and by the zeros at infinity of the transfer matrix of the given system.

1. Introduction

Let f be the transfer matrix of a given linear time-invariant system, and consider the classical control configuration shown in Fig. 1, where v is a causal precompensator and r is a causal output feedback, and where $f_{(v,r)}$ denotes the resulting system. Throughout our discussion we assume that this configuration is internally stable; that is, that all its modes, including the unobservable and the unreachable ones, are stable.

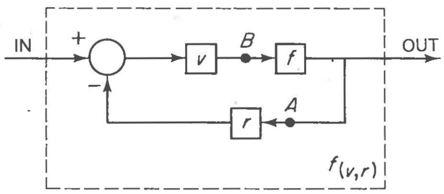

Figure 1.

In the present paper we consider the problem of pole assignment in the following three versions of this configuration, all of which have been extensively employed in various contexts in the control theoretic literature.

(a) Pole shifting with both *precompensation and feedback*, where both of v and r are allowed.

(b) Pole shifting with *unity feedback*, where the feedback r is required to be the identity matrix.

(c) Pole shifting with *pure output feedback*, where the precompensator v is required to be the identity matrix.

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In all cases we are particularly interested in the strong version of the pole shifting problem, namely, in the assignment of invariant factors for a realization of the final transfer matrix $f_{(v,r)}$. The study of the effect of feedback on the invariant factors was initiated by Rosenbrock (1970) in his study of state feedback. Later, Rosenbrock and Hayton (1978) obtained sufficient conditions for the assignment of invariant factors by pure output feedback.

As was noted by Rosenbrock (1970) and by Brunovski (1970), a major role in the study of state feedback phenomena is played by integer invariants, namely by the reachability indices—or the Kronecker invariants (Kalman 1971)—of the given system f. One of the main themes of the present paper is to show that the situation for the internally stable configuration (Fig. 1) is similar. Its (input/output) structural properties are also determined by certain integer invariants. These integer invariants, however, are different from the reachability or the observability indices. They depend, roughly speaking, on the unstable poles, on the unstable zeros, and on the zeros at infinity of the given system f. We shall review these integer invariants in § 2. Using them, we shall study the problems (a), (b) and (c) in §§ 3, 5 and 6, respectively.

In order to give a qualitative description of the nature of our present results, we describe them now for the case of a non-zero single-input single-output transfer matrix f. Let ρ be the number of unstable poles, let ζ be the number of unstable zeros, and let η be the number of zeros at infinity of f. We denote $\theta := \zeta+\eta$, and $[\rho-1]^+ := \rho-1$, if $\rho \geqslant 1$, and $[\rho-1]^+ := 0$ if $\rho<1$. Now, let ϕ be a monic polynomial having all its roots in the left side of the complex plane (i.e. ϕ has stable roots). Then, under the requirement of internal stability, we show that the following hold.

(a') Precompensation and feedback: There exists a causal pair v, r (where v is non-singular) such that $f_{(v,r)}$ has a coprime polynomial fraction representation with denominator ϕ if and only if $\deg \phi \geqslant \theta$.

(b') Unity feedback: If $\deg \phi \geqslant \theta+[\rho-1]^+$, then there exists a causal non-singular v such that $f_{(v,I)}$ has a polynomial fraction representation with denominator ϕ.

(c') Pure output feedback: If $\deg \phi \geqslant 2\theta-1$, then there exists a causal r such that $f_{(I,r)}$ has a polynomial fraction representation with denominator ϕ.

In order to compare with previous results in the literature, let λ be the reachability index of f. Then, Brasch and Pearson (1970) and Rosenbrock and Hayton (1978) obtained the sufficient condition $\deg \phi \geqslant 2\lambda-1$ for both (b') and (c'). Since always $\theta+[\rho-1]^+ \leqslant 2\lambda-1$ and $2\theta-1 \leqslant 2\lambda-1$, the present sufficient conditions are sharper. Moreover, the present conditions show, roughly speaking, that certain 'stable components' of the given system f have no effect on pole shifting. For actual numerical examples comparing these conditions, see §§ 5 and 6.

We remark that there is a difference between our present point of view and the point of view adopted by the above references, in the following sense. In the above references, full realizations of the final system are considered, whereas in our present discussion we consider input–output properties, and we disregard the hidden modes of the final system after ensuring their stability. It is interesting to note that the condition (b') for unity feedback also depends on the number of unstable poles of f, whereas the condition (c') for pure output feedback depends only on the unstable and on the infinite zeros of f.

Our discussion of pole assignment by unity feedback involves a study of the following problem, which is of independent interest (Desoer, personal communication).

Minimal feedback design problem

Minimize the dynamical order of the feedback compensator r in Fig. 1, without affecting the input–output transfer matrix $f' := f_{(v,r)}$.

Intuitively speaking, this calls for the inclusion of as much as possible of the compensation dynamics in the precompensator v. The independent interest in this problem comes from classical sensitivity considerations. Qualitatively, under conditions of high forward gain, parameters of $f_{(v,r)}$ are more sensitive to variations in the parameters of r than they are to variations in the parameters of v (and f). Therefore, if r is reduced, the design will contain fewer critical parameters. We consider the problem of minimal feedback design in § 4. We show that the minimal dynamical order for r is related to the number of unstable poles of f.

Questions related to pole shifting have been extensively investigated in the control theoretic literature. Thus, pole assignment by multivariable state feedback was considered by Wonham (1967), by Heymann (1968), by Simon and Mitter (1968), and by many others. The problem of assigning invariant factors by state feedback was studied by Rosenbrock (1970), by Dickinson (1974), and by Munzner and Pratzel-Wolters (1979). Pole assignment by dynamic output feedback was examined by Brasch and Pearson (1970), and by Rosenbrock and Hayton (1978). Pole shifting by static output feedback was investigated by Kimura (1975), by Davison and Wang (1975), and by Brockett and Byrnes (1981). Internal stabilization of feedback systems was considered by Wonham and Pearson (1974), by Wonham (1974), by Desoer and Chan (1975), by Desoer *et al.* (1980), by Pernebo (1981), by Francis and Vidyasagar (1980), and by the references cited in these works.

2. Integer invariants

In the present section we review certain integer invariants from Hammer (1981, 1983). These integer invariants play a central role in our discussion in the present paper. We start with a brief review of our setup.

Let K be a field, and let S be a K-linear space. We denote by ΛS the set of all formal Laurent series with coefficients in S, of the form

$$s = \sum_{t=t_0}^{\infty} s_t z^{-t} \tag{1}$$

where, for all t, $s_t \in S$. Then, under the operations of coefficient-wise addition and convolution as scalar multiplication, the set ΛK is endowed with a field structure, and the set ΛS forms a linear space over ΛK. Moreover, if the

K-linear space S is finite-dimensional, then so also is ΛS as a ΛK-linear space, and $\dim_{\Lambda K} \Lambda S = \dim_K S$.

Further, let U and Y be finite-dimensional K-linear spaces, and let Σ be a K-linear time-invariant system, admitting input values from U and having its output values in Y. Assume also that Σ possesses a transfer matrix T. Then, clearly, T has its entries in the field ΛK, and thus induces in a natural way a ΛK-linear map $f_T : \Lambda U \to \Lambda Y$. Conversely, let $f: \Lambda U \to \Lambda Y$ be a ΛK-linear map. Then f can, of course, be represented as a matrix, relative to specified bases $u_1, \ldots, u_m$ in ΛU and $y_1, \ldots, y_p$ in ΛY. Of particular importance is the case when $u_1, \ldots, u_m$ belong to U and $y_1, \ldots, y_p$ belong to Y, where U and Y are regarded as subsets of ΛU and ΛY, respectively. In this case the matrix representation Z_f of f is called a *transfer matrix*, and, if $f = f_T$, we clearly have that Z_f coincides with T. Thus, a transfer matrix and a ΛK-linear map are equivalent quantities. (For a more abstract interpretation in the discrete-time case, see Kalman *et al.* (1969) and Wyman (1972).) Throughout our discussion, *all matrix representations will tacitly be assumed to be transfer matrices.* No sharp distinction between a map and its transfer matrix will be made.

The field ΛK contains as subsets the set $\Omega^+ K$ of all (polynomial) elements of the form $k = \sum_{t=t_0}^{0} k_t z^{-t}$, $t_0 \leqslant 0$, and the set $\Omega^- K$ of all (power series) elements of the form $k = \sum_{t=0}^{\infty} k_t z^{-t}$. Both of these sets form principal ideal domains under the operations defined in ΛK. A ΛK-linear map $f: \Lambda U \to \Lambda Y$ is *polynomial* if all the entries in its transfer matrix are in $\Omega^+ K$. The map f is *causal* (respectively, *strictly causal*) if all the entries in its transfer matrix belong to $\Omega^- K$ (respectively, $z^{-1}\Omega^- K$). A ΛK-linear map $f: \Lambda U \to \Lambda Y$ is *rational* if there exists a non-zero polynomial $\psi \in \Omega^+ K$ such that ψf is a polynomial map. If f is both rational and strictly causal, then it is called an *i/o (input/output) map*.

In discussion of causality it is sometimes convenient to employ the following classical notion of order. Let $s = \sum s_t z^{-t}$ be an element in ΛS. The *order* of s is defined as $\operatorname{ord} s := \min_t \{s_t \neq 0\}$ if $s \neq 0$, and $\operatorname{ord} s := \infty$ if $s = 0$. The *leading coefficient* $\hat{s}$ of s is then defined as $\hat{s} := s_{\operatorname{ord} s}$ if $s \neq 0$, and $\hat{s} := 0$ if $s = 0$. In this terminology, a ΛK-linear map $f: \Lambda U \to \Lambda Y$ is causal if and only if $\operatorname{ord} fu \geqslant \operatorname{ord} u$ for all elements $u \in \Lambda U$, and f is *strictly causal* if and only if $\operatorname{ord} fu > \operatorname{ord} u$ for all $u \in \Lambda U$. Finally, a ΛK-linear map $l: \Lambda U \to \Lambda U$ is *bicausal* if it is causal and if it possesses an inverse which is causal as well (Hautus and Heymann 1978).

We turn next to proper bases. Let $s_1, \ldots, s_n \in \Lambda S$ be a set of elements. Then $s_1, \ldots, s_n$ are *properly independent* if their leading coefficients $\hat{s}_1, \ldots, \hat{s}_n (\in S)$ are K-linearly independent. A basis consisting of properly independent elements is called a *proper basis.* It can be shown that every ΛK-linear subspace $R \subset \Lambda U$ has a proper basis (Hammer and Heymann 1981). Let $u_1, \ldots, u_m$ be a proper basis of ΛU. Then, a ΛK-linear map $f: \Lambda U \to \Lambda Y$ is causal if and only if $\operatorname{ord} fu_i \geqslant \operatorname{ord} u_i$ for all $i = 1, \ldots, m$ (Wolovich 1974, Hammer and Heymann 1983). Also, a ΛK-linear map $l: \Lambda U \to \Lambda U$ is bicausal if and only if $lu_1, \ldots, lu_m$ are properly independent, and $\operatorname{ord} lu_i =$

ord u_i for all $i=1, \ldots, m$. A proper basis $u_1, \ldots, u_m$ is *ordered* if ord $u_i \leqslant$ ord u_{i+1} for all $i=1, \ldots, m-1$.

To discuss stability, we let σ be a multiplicatively closed set of polynomials in Ω^+K (i.e. for every pair of elements $k_1, k_2 \in \sigma$ also $k_1 k_2 \in \sigma$). We say that σ is a *stability set* if it satisfies (i) $0 \notin \sigma$, and (ii) σ contains a first-degree polynomial, that is, there is an element $\alpha \in K$ such that $z+\alpha \in \sigma$ (Morse 1975). Now let σ be a stability set. We denote by $\Omega_\sigma K$ the set of all elements k in ΛK which can be expressed as a polynomial fraction $k=\alpha/\beta$, where $\beta \in \sigma$. It can be shown that $\Omega_\sigma K$ forms a principal ideal domain under the operations defined in ΛK (e.g. Hammer 1981). A ΛK-linear map $f: \Lambda U \rightarrow \Lambda Y$ is *i/o* (*input/output*) *stable* (in the sense of σ) if all entries in its transfer matrix belong to $\Omega_\sigma K$. It can readily be seen that this notion of stability includes the classical notion of stability in linear control theory, where all the roots of the characteristic polynomial are located within a prescribed region of the complex plane (intersecting the real line).

When both stability and causality are of interest, one defines the intersection $\Omega_\sigma^- K := \Omega_\sigma K \cap \Omega^- K$. Then, clearly, a ΛK-linear map $f: \Lambda U \rightarrow \Lambda Y$ is both causal and i/o stable if and only if all entries in its transfer matrix belong to $\Omega_\sigma^- K$. It was shown by Morse (1975) that $\Omega_\sigma^- K$ forms a principal ideal domain under the operations of addition and multiplication defined in ΛK.

Several types of unimodular maps appear in our discussion, and we review now the terminology. We say that a ΛK-linear map $l: \Lambda S \rightarrow \Lambda S$ is Ω^+K- (respectively Ω^-K-, $\Omega_\sigma K$-, $\Omega_\sigma^- K$-) *unimodular* if l has an inverse l^{-1} and if both of l and l^{-1} are polynomial (respectively causal, i/o stable, both causal and i/o stable). In particular, an Ω^+K-unimodular map is the usual polynomial unimodular map, and an Ω^-K-unimodular map is the bicausal map.

We next turn to certain canonical representations of systems in the stability sense, following Hammer (1981). Let $N: \Lambda U \rightarrow \Lambda Y$ and $D: \Lambda U \rightarrow \Lambda Y'$ be i/o stable ΛK-linear maps. We say that N and D are *right σ^+-coprime* if there exist i/o stable ΛK-linear maps $A: \Lambda Y \rightarrow \Lambda U$ and $B: \Lambda Y' \rightarrow \Lambda Y$ such that $AN+BD=I$ (the identity map). Now let $f: \Lambda U \rightarrow \Lambda Y$ be a rational ΛK-linear map. A (matrix fraction) representation of the form $f=ND^{-1}$, where $N: \Lambda U \rightarrow \Lambda Y$ and $D: \Lambda U \rightarrow \Lambda U$ are i/o stable, is called a *right stability representation* of f. In case N and D are right σ^+-coprime, we say that the stability representation is *canonical*. Left stability representations are defined in a dual way. It can be shown that every rational ΛK-linear map has both right and left canonical stability representations.

If $f=ND^{-1}$ is a right canonical stability representation, then we say that D is a *right σ^+-denominator* of f. It is worthwhile to note that f is i/o stable if and only if its right σ^+-denominators are $\Omega_\sigma K$-unimodular.

Two particular types of canonical stability representations are distinguished by their minimality properties. One of these representations characterizes the unstable poles of the system, and the other one the unstable zeros. Let $f: \Lambda U \rightarrow \Lambda Y$ be a rational ΛK-linear map. A right stability representation NP^{-1} of f is called a *right pole representation* whenever the following hold : (i) P is a polynomial map, and (ii) if $f=RQ^{-1}$ is any right stability representation with Q polynomial, then P is a polynomial left divisor of Q. The matrix

P is then called a *right pole matrix* of f. Further, a right stability representation $f = ZD^{-1}$ is called a *right zero representation* whenever (i) Z is a polynomial map, and (ii) if RQ^{-1} is any right stability representation with R polynomial, then Z is a polynomial left divisor of R. The matrix Z is then called a *right zeros matrix* of f (compare also with Pernebo (1981)). Left pole and zero representations are defined in a dual way. It can be shown that pole and zero representations exist, and are canonical stability representations (Hammer 1981).

A right pole representation is constructed as follows. Let $f = RT^{-1}$ be a right coprime polynomial matrix fraction representation. One factors $T = PT_1$ into a multiple of polynomial matrices, where T_1^{-1} is i/o stable, and where $\det P$ is polynomially coprime with every element in the stability set σ. Then, letting $N := RT_1^{-1}$, it can be shown that $f = NP^{-1}$ is a right pole representation of f. A right zero representation is constructed dually—one factors $R = ZR_1$ into a multiple of polynomial matrices, where R_1 is square non-singular and R_1^{-1} is i/o stable, and where the invariant factors of Z are polynomially coprime with every element in σ. Then, denoting $D := TR_1^{-1}$, it can be shown that $f = ZD^{-1}$ is a right zero representation of f.

When considering pole and zero representations, it is convenient to employ the following type of matrices. Let $P : \Lambda U \to \Lambda U$ be a polynomial matrix. We say that P is *completely unstable* (in the sense of σ) if the invariant factors of P are (polynomially) coprime with every element in σ. It can then be seen that a canonical right stability representation $f = NP^{-1}$ is a pole representation if and only if P is a completely unstable polynomial map. The situation for zero representations is, of course, analogous. The following is a useful technical property of completely unstable maps, which can be easily verified (Hammer and Khargonekar 1981).

Lemma 2.1

Let $R : \Lambda U \to \Lambda Y$ and $Q : \Lambda U \to \Lambda U$ be polynomial maps and assume that Q is non-singular and completely unstable. If the map $Q^{-1}R$ is i/o stable, then it is a polynomial map.

Pole and zero representations induce certain types of integer invariants on which much of our discussion in the present paper depends. These integers arise in a way which is similar to the way in which the reachability indices arise from coprime polynomial matrix fraction representations. In order to emphasize this analogy, we start with a review of the reachability indices. Let $f : \Lambda U \to \Lambda Y$ be a rational ΛK-linear map and let $f = RQ^{-1}$ be a right coprime polynomial matrix fraction representation. Let $M : \Lambda U \to \Lambda U$ be a polynomial unimodular matrix such that the matrix QM has properly independent and ordered columns $q_1, \ldots, q_m$. The integers $\lambda_i := -\operatorname{ord} q_i$, $i = 1, \ldots, m$, are then the reachability indices of f (Wolovich 1974, Forney 1975).

Similarly, let $f = NP^{-1}$ be a right pole representation of f, and let $M : \Lambda U \to \Lambda U$ be a polynomial unimodular matrix such that the matrix PM has properly independent and ordered columns $p_1, \ldots, p_m$. The integers $\rho_i := -\operatorname{ord} p_i$, $i = 1, \ldots, m$, are called the *right pole indices* of f (Hammer 1981). The *left pole indices* $\rho'_1, \ldots, \rho'_p$ of f are defined (dually) as the right pole indices of the transpose of f. The integer $\rho(f) := -\operatorname{ord} \det P$ is called the *pole degree* of f, and it is equal to the number of unstable poles of f. These integers are related through the following equality (Hammer 1982)

$$\sum_{i=1}^{m} \rho_i = \sum_{i=1}^{p} \rho'_i = \rho(f) \tag{2}$$

We next define an additional set of integer invariants. Let $f : \Lambda U \to \Lambda Y$ be an injective ΛK-linear map, and let $f = ZD^{-1}$ be a right zero representation of f. As before, we let $M : \Lambda U \to \Lambda U$ be a polynomial unimodular matrix such that the matrix DM has properly independent and ordered columns $d_1, \ldots, d_m$. The integers $\theta_i := -\operatorname{ord} d_i$, $i = 1, \ldots, m$, are called the *right stability indices* of f (Hammer 1981). When f is non-injective, the stability indices are defined as follows. One lets $l : \Lambda U \to \Lambda U$ be an $\Omega_\sigma^- K$-unimodular map such that $fl = (f_0, 0)$, where f_0 is injective. Then, let $\theta_1{}^0 \geqslant \theta_2{}^0 \geqslant \ldots \geqslant \theta_k{}^0$ be the stability indices of f_0. The (right) stability indices $\theta_1, \ldots, \theta_m$ of f are then defined as $\theta_i := \theta_i{}^0$ for $i = 1, \ldots, k$, and $\theta_i := 0$ for $i = k+1, \ldots, m$. It can be shown that $\theta_1, \ldots, \theta_m$ are uniquely determined by f (Hammer 1981). In the present paper we use only *right* stability indices, therefore the qualifier 'right' will be omitted. For a detailed study of the pole and stability indices the reader is referred to Hammer (1981).

An additional type of integer-invariants that we shall need is related to the inversion of ΛK-linear maps. Let $f : \Lambda U \to \Lambda Y$ be a ΛK-linear map. We say that f is (left) *σ^--invertible* if there exists a causal and i/o stable map $h : \Lambda Y \to \Lambda U$ such that $hf = I$, the identity. Evidently, every σ^--invertible map is injective. Conversely, an injective rational map can be made σ^--invertible by premultiplying it with a suitable matrix. The 'minimal' such matrix plays a central role in our discussion, and it is defined as follows. Let $f : \Lambda U \to \Lambda Y$ be an injective rational ΛK-linear map, and let $D_\sigma : \Lambda U \to \Lambda U$ be a non-singular and i/o stable matrix. We say that D_σ is a *σ-annihilator* of f if (i) $fD_\sigma{}^{-1}$ is (left) σ^--invertible, and (ii) for every non-singular and i/o stable matrix $D : \Lambda U \to \Lambda U$ for which fD^{-1} is σ^--invertible, D_σ is a right σ^--divisor of D (i.e. $DD_\sigma{}^{-1}$ is both causal and i/o stable) (Hammer 1983). Roughly speaking, the σ-annihilator exactly cancels the unstable and the infinite zeros of f. We shall need an explicit expression for a σ-annihilator of f, which we obtain as follows.

Construction of σ-annihilators and σ-latency indices

Let $f : \Lambda U \to \Lambda Y$ be an injective rational ΛK-linear map, and let $f = D^{-1}N$ be a left coprime polynomial matrix fraction representation of f. We factor $N = N_S N_U$ into a multiple of polynomial matrices, where N_U is square non-singular and completely unstable, and where N_S has invariant factors $\phi_1, \ldots, \phi_m$ which satisfy $1/\phi_i \in \Omega_\sigma K$ for all $i = 1, \ldots, m$ (i.e. are stable). Then, we denote $g := D^{-1}N_S$ and we let $M : \Lambda U \to \Lambda U$ be a polynomial unimodular matrix such that gM has properly independent and ordered columns $g_1, \ldots, g_m$. The integers $\nu_i := \operatorname{ord} g_i$, $i = 1, \ldots, m$, are called the *σ-latency indices* of f (Hammer 1983). Qualitatively, these integers are determined by the unstable and by the infinite zeros of f. Now let $h := M^{-1}N_U$, so that $f = gh$, and let $(z + \alpha)$ be a first-degree polynomial in σ. Then the matrix

$$D_\sigma := [\operatorname{diag}((z+\alpha)^{-\nu_1}, \ldots, (z+\alpha)^{-\nu_m})]h : \Lambda U \to \Lambda U \tag{3}$$

is a σ-annihilator of f (Hammer 1983). Any other σ-annihilator D'_σ of f is then, by definition, of the form $D'_\sigma = lD_\sigma$, where $l: \Lambda U \to \Lambda U$ is an arbitrary Ω_σ^- K-unimodular map. □

Let $f: \Lambda U \to \Lambda Y$ be a ΛK-linear map. We recall that $\rho(f)$ is the pole degree of f, which is equal to the number of unstable poles of f. It is also convenient to use the number of unstable zeros of f, which we call the *zero degree* $\zeta(f)$, and which we define as follows. Let $f = ZD^{-1}$ be a zero representation of f, and let $\psi_1, \ldots, \psi_k$ be the invariant factors of the polynomial matrix Z. Then, $\zeta(f) := \sum_{i=1}^{k} \deg \psi_i$. Pole and zero degrees can be used to identify coprimeness through the following property, which can be directly verified.

Lemma 2.2

Let $f: \Lambda U \to \Lambda Y$ be a ΛK-linear map, and let $f = ND^{-1}$ be a stability representation. Then, $\rho(f) \leqslant \zeta(D)$ and $\zeta(f) \leqslant \zeta(N)$. Moreover, N and D are right σ^+-coprime if and only if $\rho(f) = \zeta(D)$, or, equivalently, if and only if $\zeta(f) = \zeta(N)$.

We conclude this section with the following technical property which we shall need below.

Lemma 2.3

Let $f: \Lambda U \to \Lambda Y$ be an injective linear i/o map, and let D be a right σ^+-denominator of f. Let D_σ be a σ-annihilator of f, and let D_0 be a right σ^+-denominator of fD_σ^{-1}. Denote $R := D_0^{-1}(D_\sigma D)$. Then, (i) *$R$ is i/o stable,* (ii) *D_0 and D_σ are left σ^+-coprime, and* (iii) *R and D are right σ^+-coprime.*

Proof of Lemma 2.3

(i) Follows from the fact that $fD = (fD_\sigma^{-1})(D_\sigma D)$ is i/o stable. (ii) Let $A := fD_\sigma^{-1} D_0$. Then, by definition $\zeta(f) = \zeta(D_\sigma)$; $\zeta(A) = 0$; and $\rho(A) = 0$. Thus, since $f = AD_0^{-1} D_\sigma$, $\zeta(f) = \zeta(D_0^{-1} D_\sigma)$. But then, $\zeta(D_0^{-1} D_\sigma) = \zeta(D_\sigma)$, and D_0, D_σ are left σ^+-coprime by (dual of) Lemma 2.2. The latter also implies that $\rho(f) = \rho(D_0^{-1} D_\sigma) = \zeta(D_0)$. (iii) By definition, $\rho(f) = \zeta(D)$. Also, since $f = ARD^{-1}$, we have $\rho(f) = \rho(RD^{-1})$. Whence, $\rho(RD^{-1}) = \zeta(D)$, so that R, D are right σ^+-coprime by Lemma 2.2. □

3. Pole assignment with precompensation and feedback

In the present section we consider the problem of pole assignment in configurations of the form of Fig. 1, where both of v and r are allowed. We pay particular attention to the internal stability of this configuration; namely, we ensure the stability of all its modes, including the unreachable and the unobservable ones. The main result of this section is the following. (We say that a non-zero polynomial ϕ is stable if $1/\phi$ is i/o stable.)

Theorem 3.1

Let $f: \Lambda U \to \Lambda Y$ be an i/o map with stability indices $\theta_1 \geqslant \theta_2 \geqslant \ldots \geqslant \theta_m$, and let $k :=$ rank f. Let $\phi_1, \ldots, \phi_k$ be a set of monic stable polynomials, where ϕ_{i+1}

divides ϕ_i for all $i=1, \ldots, k-1$. Then, the following are equivalent.

(i) $\sum_{i=1}^{j} \deg \phi_i \geqslant \sum_{i=1}^{j} \theta_i$, *for all $j=1, \ldots, k$.*

(ii) *There exists a pair of causal matrices $v : \Lambda U \to \Lambda U$ and $r : \Lambda Y \to \Lambda U$, where v is non-singular, such that $f_{(v,r)}$ is internally stable and has a coprime polynomial representation $f_{(v,r)} = G^{-1}H$, where G has $\phi_1, \ldots, \phi_k$ as its (non-trivial) invariant factors.*

Theorem 3.1 is of a nature similar to the classical result obtained by Rosenbrock (1970) (see also Dickinson (1974)) for the case of state feedback, which can be summarized as follows. Let $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_m$ be the reachability indices of f, and let $\phi_1, \ldots, \phi_m$, where ϕ_{i+1} divides ϕ_i, be a set of monic polynomials. Denote by f_F the system resulting when a state feedback F is applied in a canonical realization of f. Then, there exists an F such that $\phi_1, \ldots, \phi_m$ are the invariant factors of a reachable realization of f_F if and only if $\sum_{i=1}^{j} \deg \phi_i \geqslant \sum_{i=1}^{j} \lambda_i$ for all $j=1, \ldots, m$ (with equality holding for $j=m$). The main difference between the two results is that in the case of Theorem 3.1 the stability indices play the major role. One can show that $\theta_i \leqslant \lambda_i$ for all $i=1, \ldots, m$ (Hammer 1981). Qualitatively, in our present case, the stable zeros of the system do not affect pole shifting.

Before proving Theorem 3.1, we have to review some aspects from Hammer (1982), on which most of our discussion in the present paper is based. Let $f : \Lambda U \to \Lambda Y$ be an injective linear i/o map, and let $l : \Lambda U \to \Lambda U$ be a non-singular, causal and i/o stable precompensator for which fl is i/o stable. In Hammer (1982) we considered the problem of constructing (if possible) compensators $v : \Lambda U \to \Lambda U$ and $r : \Lambda Y \to \Lambda U$ satisfying

$$fl \overset{\sigma}{=} f_{(v,r)}$$

where the symbol $\overset{\sigma}{=}$ indicates that the transfer matrices fl and $f_{(v,r)}$ are equal, and that (the right-hand side) $f_{(v,r)}$ is internally stable. In order to review the construction of v and r, let D_σ be a σ-annihilator of f, and let D_0 be a right σ^+-denominator of fD_σ^{-1}. The construction depends then on a matrix partial-fraction decomposition

$$(D_\sigma l)^{-1} = PA^{-1} + QB^{-1} \tag{4}$$

where $P, A, Q, B : \Lambda U \to \Lambda U$ are i/o stable matrices. We say that this partial fraction decomposition is *reduced* if the pairs P, A and Q, B are right σ^+-coprime. For notational convenience, we abbreviate by σ^+-LCRM the *least common right multiple* of two matrices over the principal ideal domain $\Omega_\sigma K$. In this notation, the following holds (Hammer 1983).

Theorem 3.2

Let $f : \Lambda U \to \Lambda Y$ be an injective linear i/o map, and let $l : \Lambda U \to \Lambda U$ be a non-singular, causal and i/o stable precompensator. Let D_σ be a σ-annihilator of f, and let D_0 be a right σ^+-denominator of fD_σ^{-1}. Then, the following (i) *and*

(ii) *are equivalent.*

(i) *There exist causal maps* $v : \Lambda U \to \Lambda U$ *and* $r : \Lambda Y \to \Lambda U$ *such that* $fl \overset{\sigma}{=} f_{(v,r)}$.

(ii) *There exists a reduced partial-fraction decomposition* (4) *satisfying :*

(α) A *and* B *are left* σ^+*-coprime and have* $D_\sigma l$ *as a* σ^+*-LCRM.*

(β) D_σ *is a left* σ^+*-divisor of* A.

(γ) D_0 *is a left* σ^+*-divisor of* B.

A detailed discussion of partial-fraction decompositions satisfying condition (ii) of Theorem 3.2 and of their construction is given in Hammer (1983). Below, we shall explicitly construct these decompositions whenever we shall encounter them. In case condition (ii) of Theorem 3.2 is satisfied, the compensators v and r are obtained as follows (Hammer 1983).

Construction of v and r

Assume that there exists a partial-fraction decomposition (4) satisfying Theorem 3.2(ii). Let $\mathbf{L}^+ : \Lambda U \to \Omega^+ U : \sum u_t z^{-t} \mapsto \sum_{t<0} u_t z^{-t}$ be the polynomial truncation operator, and define the matrices

$$\left.\begin{aligned} g &:= PA^{-1} + \mathbf{L}^+(QB^{-1}) \\ h &:= QB^{-1} - \mathbf{L}^+(QB^{-1}) \end{aligned}\right\} \tag{5}$$

(It can be shown then that g is non-singular.) Further, recalling that f is injective, let $l_0 : \Lambda Y \to \Lambda Y$ be an $\Omega_\sigma^- K$-unimodular map such that $l_0 f = \begin{pmatrix} f_0 \\ 0 \end{pmatrix}$, where f_0 is square non-singular. Define

$$\left.\begin{aligned} v &:= (gD_\sigma)^{-1} : \Lambda U \to \Lambda U \\ r &:= [h(f_0 D_\sigma^{-1})^{-1}, 0]l_0 : \Lambda Y \to \Lambda U \end{aligned}\right\} \tag{6}$$

Then, it is shown in Hammer (1983) that both v and r are causal, and that $fl \overset{\sigma}{=} f_{(v,r)}$. □

The general case where f is possibly non-injective can be reduced to the injective case as follows (see Hammer (1983) for a detailed discussion).

The non-injective case

Let $g : \Lambda U \to \Lambda Y$ be a non-zero ΛK-linear map. We say that g has a *static kernel* if there exists a non-singular (static) matrix $V : U \to U$ such that $gV = (g_0, 0)$ where g_0 is injective. Now let $f : \Lambda U \to \Lambda Y$ be a non-zero i/o map of rank k, and let $l : \Lambda U \to \Lambda U$ be a non-singular, causal and i/o stable precompensator. In the present paper we shall need to consider only the particular case when fl has a static kernel. Assume then that fl has a static kernel, and let $V : U \to U$ be a non-singular static matrix such that $flV = (f'_0, 0)$, where f'_0 is injective. Denote $l' := lV$.

Further, let $l_U : \Lambda U \to \Lambda U$ be an $\Omega_\sigma^- K$-unimodular matrix such that $f_S := fl_U = (f_0, 0)$, where f_0 is injective (the existence of l_U follows by the Hermite normal form theorem). Denote $l_S := l_U^{-1} l'$, so that $f_S l_S = fl'$.

Next we partition $l_S = \begin{pmatrix} l_1 & l_2 \\ l_3 & l_4 \end{pmatrix}$, where l_1 is $k \times k$, and we define the causal and i/o stable matrix

$$l'' := \begin{pmatrix} l_1 & 0 \\ 0 & I \end{pmatrix} : \Lambda U \rightarrow \Lambda U \tag{7}$$

which clearly satisfies $f_S l'' = f_S l_S$. The last equation implies that l_1 is non-singular, and that $f'_0 = f_0 l_1$.

We are now faced with the injective map f_0 and the precompensator l_1, to which Theorem 3.2 and the construction (5) apply. If there exists a pair of causal maps v_1 and r_1 satisfying $f_0 l_1 \overset{\sigma}{=} f_{0(v_1, r_1)}$, then, defining

$$\left.\begin{aligned} v &:= l_U \begin{pmatrix} v_1 & 0 \\ 0 & I \end{pmatrix} : \Lambda U \rightarrow \Lambda U \\ r &:= \begin{pmatrix} r_1 \\ 0 \end{pmatrix} : \Lambda Y \rightarrow \Lambda U \end{aligned}\right\} \tag{8}$$

it can be seen that $fl \overset{\sigma}{=} f_{(v,r)} V^{-1}$ (Hammer 1983). Thus, the non-injective case reduces to the injective case. □

Before turning to the proof of Theorem 3.1, we recall the following fundamental result due to Rosenbrock (1970) (see also Münzner and Prätzel-Wolters (1979)).

Theorem 3.3

Let $\phi_1, \ldots, \phi_m$ be a set of monic polynomials, where ϕ_{i+1} divides ϕ_i for all $i = 1, \ldots, m-1$, and let $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_m \geqslant 0$ be a set of integers. Then, the following are equivalent.

(i) $\sum_{j=1}^{i} \deg \phi_j \geqslant \sum_{j=1}^{i} \lambda_j$ *for all* $i = 1, \ldots, m$.

(ii) *There exists an $m \times m$ polynomial matrix with properly independent and ordered columns of respective degrees $\xi_1 \geqslant \xi_2 \geqslant \ldots \geqslant \xi_m$, where $\xi_i \geqslant \lambda_i$ for all $i = 1, \ldots, m$, which has $\phi_1, \ldots, \phi_m$ as its invariant factors.*

Remark

Theorem 3.3 is usually stated under the requirement that equality holds for $i = m$ in (i) $\left(\text{i.e. } \sum_{j=1}^{m} \deg \phi_j = \sum_{j=1}^{m} \lambda_j\right)$, in which case one has in (ii) that $\xi_i = \lambda_i$ for all $i = 1, \ldots, m$. From this statement, the present one is obtained through the following elementary lemma.

Lemma 3.1

Let $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_m \geqslant 0$ and $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_m \geqslant 0$ be integers satisfying $\sum_{i=1}^{j} a_i \geqslant \sum_{i=1}^{j} \lambda_i$ for all $j = 1, \ldots, m$. Then there exist integers $\xi_1 \geqslant \xi_2 \geqslant \ldots \geqslant \xi_m$ such that $\xi_j \geqslant \lambda_j$ and $\sum_{i=1}^{j} a_i \geqslant \sum_{i=1}^{j} \xi_i$ for all $i = 1, \ldots, m$, where $\sum_{i=1}^{m} a_i = \sum_{i=1}^{m} \xi_i$.

Proof of Lemma 3.1

We give an algorithm that constructs $\xi_1, \ldots, \xi_m$. Define $\xi'_i := a_i$, $i=1, \ldots, m$. (*) If $\xi'_i \geqslant \lambda_i$ for all $i=1, \ldots, m$, then set $\xi_i := \xi'_i$, $i=1, \ldots, m$. Otherwise, let n be the first integer for which $\xi'_n < \lambda_n$. Then, clearly, $n>1$, and, since $\sum_{i=1}^{n} \xi'_i \geqslant \sum_{i=1}^{n} \lambda_i$, there is a $j<n$ such that $\xi'_j > \lambda_j$. Let $k<n$ be the maximal integer for which $\xi'_k > \lambda_k$ and define $\xi''_i := \xi'_i$ for all $i \neq k, n, i \leqslant m$, and $\xi''_k := \xi'_k - 1$, $\xi''_n := \xi'_n + 1$. Now set $\xi'_i := \xi''_i$, $i=1, \ldots, m$, and repeat from (*). □

Proof of Theorem 3.1

(i)⇒(ii) : Let $l_0 : \Lambda U \to \Lambda U$ be an $\Omega_\sigma^- K$-unimodular map such that $fl_0 = (f_0, 0)$, where f_0 is injective, and let $f_0 = ZD^{-1}$ be a zero representation of f_0. We can assume that the columns $d_1, \ldots, d_k$ of D are properly independent and ordered, in which case, by definition, $\operatorname{ord} d_i = -\theta_i$, $i=1, \ldots, k$. By Theorem 3.3 there exists a $k \times k$ polynomial matrix S with properly independent columns $q_1, \ldots, q_k$ such that (a) $\mu_i := -\operatorname{ord} q_i \geqslant \theta_i$, $i=1, \ldots, k$, and (b) $\phi_1, \ldots, \phi_k$ are the invariant factors of S. Since the $\{\phi_i\}$ are stable, S^{-1} is i/o stable.

We define now $l_1 := DS^{-1}$, and note that, by the proper independence of the columns and the fact that $\mu_i \geqslant \theta_i$ for all $i=1, \ldots, k$, the matrix l_1 is causal. It is evidently also non-singular and i/o stable, and satisfies $f'_0 := f_0 l_1 = ZS^{-1}$. Defining $l := l_0 \begin{pmatrix} l_1 & 0 \\ 0 & I \end{pmatrix}$, we have that $f' := fl = (f'_0, 0)$, so that f' has a canonical polynomial representation with invariant factors $\phi_1, \ldots, \phi_k$. Thus, our proof will conclude upon showing that there exist causal v and r such that $fl \overset{\sigma}{=} f_{(v,r)}$. In view of (8), this will follow if we show that there exist causal v_1 and r_1 such that $f_0 l_1 \overset{\sigma}{=} f_{0(v_1, r_1)}$, which we next do.

Let D_σ be a σ-annihilator of f_0, and let D_0 be a right σ^+-denominator of $f_0 D_\sigma^{-1}$. Note also that, since S is $\Omega_\sigma K$-unimodular, l_1 is actually a right σ^+-denominator of f_0. Then, letting $R := D_0^{-1}(D_\sigma l_1)$, it follows by Lemma 2.3 that R is i/o stable ; D_0 and D_σ are left-, whereas R and l_1 are right-σ^+-coprime. This also implies that $(D_\sigma l_1)$ is a σ^+-LCRM of D_0 and D_σ. Let P and Q be i/o stable matrices such that $Pl_1 + QR = I$. Then, $(D_\sigma l_1)^{-1} = (Pl_1 + QR)(D_\sigma l_1)^{-1} = PD_\sigma^{-1} + QD_0^{-1}$ is a partial-fraction decomposition satisfying Theorem 3.2(ii). Whence, v_1 and r_1 can be constructed through (5) and (6). This proves (i)⇒(ii).

The converse direction (ii)⇒(i) is a consequence of the following facts : (a) the equivalent precompensator $l_{(v,r)} := v[I + rfv]^{-1}$ is both causal and i/o stable, and (b) $f_{(v,r)} = fl_{(v,r)}$ is i/o stable (Hammer 1981 §7). □

4. Minimal feedback design

In the present section we consider the problem of minimal feedback design, where we wish to minimize the dynamical order of the feedback compensator. For our purposes in this paper, we shall need to consider only the case when the given system f is surjective (i.e. onto). The extension to the more general case will be considered in a separate report.

Let $f : \Lambda U \to \Lambda Y$ be an i/o map, and consider a configuration $f_{(v,r)}$ as depicted in Fig. 1. For the minimal feedback design problem, the transfer matrices f and $f' := f_{(v,r)}$ are regarded as fixed, whereas the compensators v and r are variable, and we wish to choose them in the desired way. It is therefore convenient to state our discussion in terms of f and f', and to use as an auxiliary underlying quantity an equivalent non-singular and causal precompensator $l : \Lambda U \to \Lambda U$ satisfying $f' = fl$. Explicitly, in terms of v and r, l is given by

$$l = v[I + rfv]^{-1} \tag{9}$$

In Hammer (1983) we showed that if $f_{(v,r)}$ is internally stable then l is i/o stable, and, or course, so also is fl. Before stating the main result of the present section, we distinguish between several quantities.

Let $f : \Lambda U \to \Lambda Y$ be an i/o map, and let $l : \Lambda U \to \Lambda U$ be a non-singular, causal and i/o stable precompensator for which $f' := fl$ is i/o stable. Let P be a right pole matrix of f, let Z be a right zeros matrix of f, and let Z' be a right zero matrix of f'. Let $\mathbf{P}_f$ (respectively, $\mathbf{Z}_f$, $\mathbf{Z}_{f'}$) denote the set of polynominal prime divisors of the invariant factors of P (respectively, Z, Z'), including multiplicity. Since l is i/o stable, we have that $\mathbf{Z}_f \subset \mathbf{Z}_{f'}$. Let $\mathbf{Z}_f{}^l$ denote the difference set $\mathbf{Z}_f{}^l := \mathbf{Z}_{f'} \backslash \mathbf{Z}_f$ (i.e. the unstable zeros added by l). When the field K is infinite (e.g. real), we evidently have that, for almost every l, the following intersection is empty

$$\mathbf{P}_f \cap \mathbf{Z}_f{}^l = \emptyset \tag{10}$$

We shall also assume below that the final system $f' := fl$ has a static kernel. This assumption does not imply any restriction on the original system f. Actually, all the systems $f_{(v,r)}$ constructed in the proof of Theorem 3.1 have static kernels (and they also satisfy condition (10)). We can now state the main result of the present section, which shows that the minimal dynamical order of the feedback compensator r does not exceed the number of unstable poles of the given system f.

Theorem 4.1

Let $f : \Lambda U \to \Lambda Y$ be a surjective i/o map, and let $l : \Lambda U \to \Lambda U$ be a non-singular, causal and i/o stable precompensator for which fl is i/o stable. Assume that (10) *is satisfied, and that fl has a static kernel. Let $\rho(f)$ be the pole degree of f, and let $n(f)$ be the number of non-zero left pole indices of f. Then there exists a pair of causal maps $v : \Lambda U \to \Lambda U$ and $r : \Lambda Y \to \Lambda U$ such that $fl \overset{\sigma}{=} f_{(v,r)}$, where the Macmillan degree $\mu(r) \leqslant \rho(f) - n(f)$.*

Before stating the proof of Theorem 4.1, we define certain truncation operators which we shall need for it. Let $f : \Lambda U \to \Lambda Y$ be a rational ΛK-linear map, and let $(z + \alpha)$ be a first-degree polynomial in the stability set σ. We define now a particular type of polynomial truncation operator $\mathbf{L}_\alpha{}^+$ as follows: (i) The matrix $N := \mathbf{L}_\alpha{}^+(f)$ is polynomial and divisible by $(z + \alpha)$ (i.e. $\{1/(z + \alpha)\}N$ also is a polynomial matrix), and (ii) $f - \mathbf{L}_\alpha{}^+(f)$ is causal.

Construction of $\mathbf{L}_\alpha{}^+(f)$: Assume first that $f \in \Lambda K$ (is a scalar), and let $n' := \mathbf{L}^+(f)$, where $\mathbf{L}^+ := \Lambda K \to \Omega^+ K : \sum_t k_t z^{-t} \mapsto \sum_{t<0} k_t z^{-t}$ is the usual polynomial truncation operator. Now, let $a \in K$ be such that the polynomial

$n := n' + a$ is divisible by $(z+\alpha)$. Then, $\mathbf{L}_\alpha^{+}(f) := n$ satisfies our requirements. When f is a matrix, apply the same procedure separately to each entry. □

Next, we define the truncation operator $\mathbf{L}_\alpha^{-}$ which extracts the causal and stable part from a given transfer matrix, as follows: (i) $\mathbf{L}_\alpha^{-}(f)$ is both causal and stable, and (ii) the matrix $f^{+} := f - \mathbf{L}_\alpha^{-}(f)$ has a right coprime polynomial matrix fraction representation $f^{+} = NP^{-1}$, where P is completely unstable and N is divisible by $(z+\alpha)$ (i.e. f^{+} has no stable poles, and $\{1/(z+\alpha)\}N$ is a polynomial matrix). Note that P is then necessarily a right pole matrix of f.

Construction of $\mathbf{L}_\alpha^{-}(f)$: We start with the scalar case. Let $f \in \Lambda K$ be a non-zero rational scalar, and let $f = \beta(z)/\gamma(z)$ be a coprime polynomial fraction representation of f (where $\beta(z), \gamma(z) \in \Omega^{+}K$). We factor $\gamma(z) = \delta(z)\epsilon(z)$ into a multiple of polynomials, where $\delta(z)$ is coprime with every element in the stability set σ, and where $1/\epsilon(z)$ is i/o stable. (Clearly $\delta(z)$ is a 'pole matrix' of f.) Then, since $\delta(z)$ and $\epsilon(z)$ are evidently coprime, there exist polynomials $\beta'(z)$ and $\beta''(z)$ satisfying $f = \beta'(z)/\delta(z) + \beta''(z)/\epsilon(z)$, where $\deg \beta''(z) \leqslant \deg \epsilon(z)$. Further, since $(z+\alpha) \in \sigma$, we have that $\delta(-\alpha) \neq 0$, and whence there exists an element $a \in K$ such that $\beta'(-\alpha) + a\delta(-\alpha) = 0$. Then, $(z+\alpha)$ is a divisor of the polynomial $\beta'(z) + a\delta(z)$, and

$$\mathbf{L}_\alpha^{-}(f) := \beta''(z)/\epsilon(z) - a$$

satisfies our requirements.

When f is a matrix with entries $f_{i,j}$, then the (i, j) entry of the matrix $\mathbf{L}_\alpha^{-}(f)$ is simply $\mathbf{L}_\alpha^{-}(f_{i,j})$. □

The following technical result can easily be verified (using (3)).

Lemma 4.1

Let $f: \Lambda U \to \Lambda U$ be a square non-singular ΛK-linear map with left pole indices $\rho_1 \geqslant \rho_2 \geqslant \ldots \geqslant \rho_m$. Let $f = P^{-1}Q$ be a left pole representation of f, where P has properly independent and ordered rows. Finally, let $(z+\alpha) \in \sigma$. Then

$$D_\sigma := [\operatorname{diag}((z+\alpha)^{\rho_1}, \ldots, (z+\alpha)^{\rho_m})]^{-1}Q$$

is a σ-annihilator of f.

It will be convenient to use the following notion. Let $A, B: \Lambda U \to \Lambda U$ be non-singular i/o stable matrices. We say that the multiple AB is *interchangeable* if there exist i/o stable matrices $A', B': \Lambda U \to \Lambda U$ such that $AB = B'A'$, where (i) A and B' are left σ^{+}-coprime, and (ii) B and A' are right σ^{+}-coprime (see also Wolovich (1978)). If (i) and (ii) hold, then $AB = B'A'$ is called an *interchange equation.*

Lemma 4.2

Let $A, B: \Lambda U \to \Lambda U$ be non-singular i/o stable matrices. If $\det A$ *and* $\det B$ *are σ^{+}-coprime, then the multiple AB is interchangeable.*

Proof of Lemma 4.2

Let $M_1, M_2 : \Lambda U \to \Lambda U$ be $\Omega_\sigma K$-unimodular matrices such that $\delta := M_1(AB)M_2$ is in Smith canonical form, say $\delta = \mathrm{diag}\,(\delta_1, \ldots, \delta_m)$. For each $i = 1, \ldots, m$, we factor $\delta_i = \delta'_i\, \delta''_i$, where $\delta'_i, \delta''_i \in \Omega_\sigma K$; δ'_i is a σ^+-divisor of det B ; and δ''_i is a σ^+-divisor of det A. Now let $B' := M_1^{-1}[\mathrm{diag}\,(\delta'_1, \ldots, \delta'_m)]$ and $A' := [\mathrm{diag}\,(\delta''_1, \ldots, \delta''_m)]M_2^{-1}$, so that $AB = B'A'$. The fact that det A and det B are σ^+-coprime implies then that the latter is an interchange equation. □

Proof of Theorem 4.1

Let $l_U : \Lambda U \to \Lambda U$ be an $\Omega_\sigma^- K$-unimodular map such that $f_S := fl_U = (f_0, 0)$, where f_0 is injective. Then, since f is surjective, f_0 is square non-singular, say $p \times p$. Also, since fl has a static kernel, we can assume that $fl = (f'_0, 0)$, where, again, f'_0 is $p \times p$ and non-singular. Let $l_S := l_U^{-1}\, l$, so that $fl = f_S l_S$, and partition $l_S = \begin{pmatrix} l_1 & l_2 \\ l_3 & l_4 \end{pmatrix}$, where l_1 is $p \times p$. Then, defining $l' := \begin{pmatrix} l_1 & 0 \\ 0 & I \end{pmatrix}$, we clearly have that $f_S l_S = f_S l'$ and $f'_0 = f_0 l_1$. In view of (8), our proof will conclude upon showing that the theorem holds for the pair f_0, l_1.

To this end, let D_σ be the σ-annihilator of f_0 given by Lemma 4.1, and let ND_0^{-1} be a right coprime polynomial matrix fraction representation of $f_0 D_\sigma^{-1}$. Then D_0 is a right pole matrix of $f_0 D_\sigma^{-1}$, and, by the construction of D_σ, the matrix N has invariant factors $(z+\alpha)^{\rho_1}, \ldots, (z+\alpha)^{\rho_m}$. Next let D be a right σ^+-denominator of f_0, and let $D'_\sigma := D_0^{-1}\, D_\sigma D$. By Lemma 2.3, the equation $D_0 D'_\sigma = D_\sigma D$ is an interchange equation. Also, since $f_0 l_1$ is i/o stable, so also is $G := D^{-1} l_1$, and we have $D_\sigma l_1 = D_\sigma DG = D_0 D'_\sigma\, G$. By (10), it follows that det D and det G are σ^+-coprime, so that, by Lemma 4.2, we can construct an interchange equation $DG = G'D'$. Then, letting $E := D_\sigma G'$ and $E' := D'_\sigma\, G$, we obtain $D_\sigma l_1 = ED' = D_0 E'$, where the last equality is an interchange equation. This also implies that $(D_\sigma l_1)$ is a σ^+-LCRM of E and D_0.

Finally, let P, Q be i/o stable matrices such that $PD' + QE' = I$. Then $(D_\sigma l_1)^{-1} = (PD' + QE')(D_\sigma l_1)^{-1} = PE^{-1} + QD_0^{-1}$, and this partial-fraction decomposition satisfies Theorem 3.2(ii). We define

$$g := PE^{-1} + \mathbf{L}_\alpha^-(QD_0^{-1}) + \mathbf{L}_\alpha^+(QD_0^{-1})$$

$$h := QD_0^{-1} - \mathbf{L}_\alpha^-(QD_0^{-1}) - \mathbf{L}_\alpha^+(QD_0^{-1})$$

Then, h is causal, and, since D_0 is a completely unstable polynomial matrix, the matrix $R := hD_0$ is polynomial and divisible by $(z+\alpha)$. Letting $P' := gE$, we have that P' is i/o stable, and $(D_\sigma l)^{-1} = P'E^{-1} + RD_0^{-1}$ still satisfies Theorem 3.2(ii). We now use (6) to obtain the following causal pair v_0, r_0 satisfying $f_0 l_1 \overset{\sigma}{=} f_{0(v_0, r_0)}$. (Note that since f_0 is an isomorphism, we have $l_0 = I$ in (6).)

$$v_0 := (gD_\sigma)^{-1} = G'P'^{-1}$$

$$r_0 := h(f_0 D_\sigma^{-1})^{-1} = RN^{-1}$$

To check the Macmillan degree $\mu(r_0)$, we recall that R is divisible by $(z+\alpha)$ and that the invariant factors of N are $(z+\alpha)^{\rho_1}, \ldots, (z+\alpha)^{\rho_m}$. Whence, $\mu(r_0) \leqslant \left(\sum_{i=1}^{m} \rho_i \right) - n(f) = \rho(f) - n(f)$, concluding our proof. □

We remark that the proof of Theorem 4.1 contains an explicit construction of the required compensators v and r. We conclude this section with a numerical example for the application of Theorem 4.1.

Example

We let K be the field of real numbers, and we let the stability set σ be the set of all non-zero polynomials having their roots in the open left half of the complex plane. Assume that the given system is $f = (z-1)/[(z-2)(z+1)]$, and that the desired final system is $f' = (z-1)/(z+1)^2$. (Note that $\rho(f) = 1$.) The equivalent precompensator is then given by $l = f'/f = (z-2)/(z+1)$, and it satisfies the requirements of Theorem 4.1. We construct now the compensators v and r following the proof of Theorem 4.1, and choosing $\alpha = 1$. We then have

$$D_\sigma = (z-1)/(z+1)^2, \quad D_0 = (z-2)$$

Whence $(D_\sigma l) = (z-1)(z-2)/(z+1)^3$, and we have the partial-fraction decomposition

$$(D_\sigma l)^{-1} = \frac{z^2 - 4z - 5}{z-1} + 9\left(\frac{z+1}{z-2}\right)$$

Thus $g = (z^2 - 4z - 5)/(z-1)$, and $h = 9(z+1)/(z-2)$. Consequently

$$v = (gD_\sigma)^{-1} = \frac{(z+1)^2}{z^2 - 4z - 5}$$

$$r = h(fD_\sigma^{-1})^{-1} = 9$$

We see that indeed $\mu(r) = 0 = \rho(f) - 1$. □

5. Pole assignment by unity feedback

In the present section we consider the problem of pole assignment by unity output feedback configurations of the form of Fig. 2, where f is the given transfer matrix, v is a causal precompensator, and $f_{(v,I)}$ denotes the resulting system. We require, of course, that the configuration be internally stable. As before, our attention is directed toward the input–output poles of the final system, and, after ensuring internal stability, we disregard its

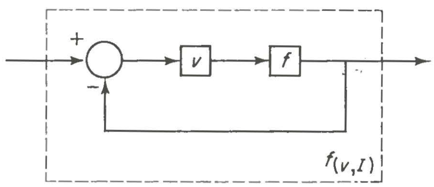

Figure 2.

hidden (i.e. unreachable or unobservable) modes. The main result of the present section considers the assignment of invariant factors by unity feedback configurations, as follows. (For an integer n, we denote $[n]^+ := n$ if $n \geqslant 0$, and $[n]^+ := 0$ otherwise. We also denote $p := \dim Y$.)

Theorem 5.1

Let $f: \Lambda U \to \Lambda Y$ be a surjective (onto) linear i/o map with stability indices $\theta_1 \geqslant \theta_2 \geqslant \ldots \geqslant \theta_m$ and with left pole indices $\rho_1 \geqslant \rho_2 \geqslant \ldots \geqslant \rho_p$. Let $\phi_1, \ldots, \phi_p$ be a set of monic and stable polynomials, where ϕ_{i+1} divides ϕ_i for all $i = 1, \ldots, p-1$. Assume that

$$\sum_{j=1}^{i} \deg \phi_j \geqslant \sum_{j=1}^{i} (\theta_j + [\rho_1 - 1]^+) \quad \text{for all } i = 1, \ldots, p$$

Then there exists a causal precompensator $v: \Lambda Y \to \Lambda U$ such that $f_{(v,I)}$ is internally stable and has a polynomial representation $f_{(v,I)} = G^{-1}H$, where G has $\phi_1, \ldots, \phi_p$ as its invariant factors.

We shall consider the generalization of the above theorem to the case of non-surjective maps later in this section. Before stating the proof of Theorem 5.1, we wish to compare it with existing results in the literature. For the situation described in the theorem, the result of Rosenbrock and Hayton (1978) is as follows. Let $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_m$ be the reachability indices of f, and let $\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_p$ be its observability indices. If

$$\sum_{i=1}^{j} \deg \phi_i \geqslant \sum_{i=1}^{j} (\lambda_i + \mu_1 - 1) \quad \text{for all } i = 1, \ldots, m \tag{11}$$

then there exists a causal v such that $\phi_1, \ldots, \phi_m$ are the invariant factors of the closed-loop system. Thus, our present conditions are of the same general form, but the integers that determine them are of a different nature. In Hammer (1981) it was shown that $\theta_i \leqslant \lambda_i$ for all $i = 1, \ldots, m$ and $\rho_i \leqslant \mu_i$ for all $i = 1, \ldots, p$, so that the present condition allows lower degrees for the $\{\phi_i\}$. We next compare the two conditions for an actual numerical example.

Example

Let K be the field of real numbers, and let the stability set σ be the set of all polynomials having their roots in the open left half of the complex plane. Consider the transfer function

$$f = \frac{(z-1)(z+1)^4}{(z-2)^2(z+2)^6}$$

Then, we have $\theta = 4$, $\rho = 2$ and $\lambda = \mu = 8$. Whence, Theorem 5.1 leads to $\deg \phi \geqslant 5$, whereas the condition (11) leads to $\deg \phi \geqslant 15$. As we see, what actually happens is that certain 'stable components' of the transfer matrix f have no effect on input–output pole shifting. □

Proof of Theorem 5.1

Since f is surjective, there exists an Ω_σ^- K-unimodular map $l_U : \Lambda U \to \Lambda U$ such that $fl_U = (f_0, 0)$, where f_0 is $p \times p$ and non-singular. Let $f_0 = ZD^{-1}$ be a right zero representation of f_0, where D has properly independent and ordered columns $d_1, \ldots, d_p$. Then by definition, ord $d_i = -\theta_i$, $i = 1, \ldots, p$. By assumption and Theorem 3.3, there is a $p \times p$ polynomial matrix Q with properly independent and ordered columns $q_1, \ldots, q_p$ satisfying (i) $\mu_i := -\text{ord}\, q_i \geqslant \theta_i + [\rho_1 - 1]^+$, $i = 1, \ldots, p$, and (ii) $\phi_1, \ldots, \phi_p$ are the invariant factors of Q.

Now let $(z+\alpha) \in \sigma$, and define the non-singular and i/o stable matrix

$$l_1 := D(z+\alpha)^{[\rho_1 - 1]^+} Q^{-1}$$

Then, since $\mu_i \geqslant \theta_i + [\rho_1 - 1]^+$ for all $i = 1, \ldots, p$, l_1 is also causal, and $f_0 l_1 = ZQ^{-1}(z+\alpha)^\beta$, where we abbreviated $\beta := [\rho_1 - 1]^+$. Denote $l_S := \begin{pmatrix} l_1 & 0 \\ 0 & I \end{pmatrix}$ and $l := l_U l_S$, and let v, r be the causal pair satisfying $fl \overset{\sigma}{=} f_{(v,r)}$ constructed in the proof of Theorem 4.1. We make now the following change in Fig. 1. We remove the input from IN and apply it instead by subtracting it at point A in the diagram (and we reduce the number of independent inputs to dim Y). Let f_A denote the resulting transfer matrix. Then denoting

$$v_A := vr \tag{12 a}$$

we evidently have

$$f_A = f_{(v_A, I)} \tag{12 b}$$

Since $f_{(v,r)}$ is internally stable, so also is $f_{(v_A, I)}$. An explicit block-diagram computation shows that $f_A = f_{(v,r)} r$.

Further, we recall from the proof of Theorem 4.1 that $r = \begin{pmatrix} r_0 \\ 0 \end{pmatrix}$, where r_0 is $p \times p$. Thus, $f_{(v_A, I)} = f_{(v,r)} r = (ZQ^{-1}(z+\alpha)^\beta, 0) \begin{pmatrix} r_0 \\ 0 \end{pmatrix}$. By the construction of r_0 in the proof of Theorem 4.1 (see in particular the end of that proof), the matrix $P := (z+\alpha)^\beta r_0$ is polynomial. Also, since Z is completely unstable and Q^{-1} is stable, the pair Z, Q is polynomially right coprime. Whence, the transfer matrix $f_{(v_A, I)} = (ZQ^{-1}P, 0)$ has a left polynomial representation $G^{-1}H$, where G has the invariant factors of Q, that is $\phi_1, \ldots, \phi_p$. Thus, the compensator v_A satisfies the theorem, and our proof concludes. □

Remark

It can be shown that the feedback r_0 in the above proof can be chosen to have full rank. In such a case, the final system $f_{(v_A, I)}$ will have the same rank as the original one f. We omit the proof of this possibility. □

As we have already remarked, the assumption that the given transfer matrix f in Theorem 5.1 is surjective can be removed. We next discuss this point.

The non-surjective case

In order to avoid complications, we shall assume a certain assumption which is generically (i.e. in almost all cases) valid. Let $f: \Lambda U \to \Lambda Y$ be a non-zero i/o map of rank k with stability indices $\theta_1 \geqslant \theta_2 \geqslant \ldots \geqslant \theta_m$. Let $f = AB^{-1}$ be a canonical stability representation of f. Assume that there exists a static matrix $V: Y \to K^k$ satisfying (i) the i/o map $f' := Vf$ is surjective, and (ii) the matrices (VA) and B are still σ^+-coprime. (We discuss this assumption below.) Denote by $\rho'_1 \geqslant \rho'_2 \geqslant \ldots \geqslant \rho'_k$ the left pole indices of f'.

We now show that the following extension of Theorem 5.1 holds: *If*

$$\sum_{j=1}^{i} \deg \phi_j \geqslant \sum_{j=1}^{i} (\theta_j + [\rho'_1 - 1]^+), \quad \textit{for all } i = 1, \ldots, k$$

then there exists a causal v *such that* $f_{(v,V)}$ *is internally stable and has a polynomial representation* $f_{(v,V)} = G^{-1}H$, *where* G *has* $\phi_1, \ldots, \phi_k$ *as its invariant factors.* (We note that the unity feedback has been replaced by the static (constant gain) feedback V.)

The proof of this statement is by a slight modification of the proof of Theorem 5.1. Let $l_U : \Lambda U \to \Lambda U$ be an Ω_σ^- K-unimodular map for which $fl_U = (f_1, 0)$, where $f_1 : \Lambda K^k \to \Lambda Y$ is injective, and note that $f_0 := Vf_1$ is square non-singular. Let $f_1 = ZD^{-1}$ be a zero representation of f_1, where D has properly independent and ordered columns $d_1, \ldots, d_k$. (We remark that $f_0 = (VZ)D^{-1}$ is a canonical stability representation by our above assumption, but is not necessarily a zero representation.) Then, by definition, ord $d_i = -\theta_i$, $i = 1, \ldots, k$. Denote $\beta' := [\rho'_1 - 1]^+$, and let $l_1 := D(z+\alpha)^{\beta'}Q^{-1}$, where Q is defined as in the proof of Theorem 5.1 under the substitution $\rho_1 \mapsto \rho'_1$. Following the proof of Theorem 5.1, we construct from l_1 (around the system $f' = Vf$) the compensators v, $r = \begin{pmatrix} r_0 \\ 0 \end{pmatrix}$, and $v_A = vr$. An explicit computation shows then that $f_{(v_A, V)} = fv_A[I + f'v_A]^{-1} = (ZQ^{-1}P, 0)$, where $P := (z+\alpha)^{\beta'}r_0$ is a polynomial matrix. By the definition of Q this proves our assertion.

We turn finally to the existence of V. When K in the field of real numbers and σ is the set of all polynomials having their roots in the left side of the complex plane, it is easy to see that the following holds for almost every f: (*)Z has a non-singular $k \times k$ submatrix C such that det C and det D are σ^+-coprime (i.e. have no unstable zeros in common). When (*) holds, the matrix $V : Y \to K^k$ for which $C = VZ$ clearly satisfies (i) and (ii) above. □

Assignment of characteristic polynomials

In some cases one may be interested only in the assignment of the characteristic polynomial, disregarding the detailed structure of the invariant factors. In such cases, a sufficient condition somewhat stronger than the one of Theorem 5.1 can be obtained as follows. Let $f : \Lambda U \to \Lambda Y$ be a non-zero i/o map with stability indices $\theta_1 \geqslant \theta_2 \geqslant \ldots \geqslant \theta_m$. By Brasch and Pearson (1970) (who use Wonham (1967)), there exists a static output feedback $F : Y \to U$ and a vector $V : K \to U$ such that the single input system $f' := f_F V : \Lambda K \to \Lambda Y$ (where f_F is obtained by applying the output feedback F to f) has Macmillan degree $\mu(f') = \mu(f)$. Let θ be the stability index of f', and let $\rho_1 \geqslant \rho_2 \geqslant \ldots \geqslant \rho_p$ be its left pole indices. Now, let ϕ be any monic stable polynomial with

$$\deg \phi \geqslant \theta + [\rho_1 - 1]^+ \tag{13}$$

We next construct an internally stable unity feedback configuration around f' for which ϕ is a characteristic polynomial. We note that (13) allows lower degrees for ϕ than the condition of Brasch and Pearson (1970), which reads $\deg \phi \geqslant \mu(f) + \mu_1 - 1$ (where μ_1 is the maximal observability index of f'), since clearly $\theta \leqslant \mu(f)$ and $\rho_1 \leqslant \mu_1$. A numerical example comparing the two conditions was given earlier in this section.

Construction

Let $u_2, \ldots, u_p \in \Omega_\sigma^- K^p$ be strictly causal and stable column vectors for which the augmented matrix $f'' := [f', u_2, \ldots, u_p]$ is square and non-singular. Since $u_2, \ldots, u_p$ are stable, f'' has the same left pole indices as f'. Now, let $f' = N'd^{-1}$ be a right zero representation of f', and note that d is scalar and ord $d = -\theta$. We define $\psi := d(z+\alpha)^{[\rho_1 - 1]^+}/\phi$, where $(z+\alpha) \in \sigma$, and consider the $p \times p$ non-singular, causal and i/o stable matrix

$$l := \begin{pmatrix} \psi & 0 & \ldots & 0 \\ \hdashline 0 & & & \\ \vdots & & I_{(p-1)\times(p-1)} & \\ 0 & & & \end{pmatrix}$$

Then $f''l$ is i/o stable and the conditions of Theorem 4.1 are satisfied. Let v and r be the causal compensators constructed in the proof of Theorem 4.1 satisfying $f''l \stackrel{\sigma}{=} f''_{(v,r)}$. We recall (see the proof of Theorem 5.1) that $(z+\alpha)^{[\rho_1-1]^+} r$ is a polynomial matrix. Further, let a and b be the first row of v^{-1} and r, respectively, and define the $p \times p$ matrices

$$v_1^{-1} := \begin{pmatrix} & a & \\ \hdashline 0 & & \\ \vdots & I_{(p-1)\times(p-1)} & \\ 0 & & \end{pmatrix}, \quad r_1 := \begin{pmatrix} b \\ \hdashline 0 \\ \vdots \\ 0 \end{pmatrix} : \Lambda K^p \to \Lambda K^p$$

Then, noting that $l^{-1} = v_1^{-1} + r_1 f$, it follows by the particular form of l that we still have $f''l \stackrel{\sigma}{=} f''_{(v_1, r_1)}$. Denote now by $\gamma := (z+\alpha)^{[\rho_1-1]^+} b$ the respective polynomial row vector, and let $Q^{-1}N''$ be a left coprime polynomial matrix fraction representation of $N'\phi^{-1}$. (Note that, since N' is completely unstable and ϕ is stable, det $Q = \phi$.) Returning now to (12 *a*) and (12 *b*), and letting $v_A := v_1 r_1$, we have

$$f''_{(v_A, I)} = f''_{(v_1, r_1)} r_1 = f'' l r_1 = Q^{-1} N'' \gamma$$

Whence, since det $Q = \phi$, we obtained that ϕ is a characteristic polynomial of $f''_{(v_A, I)}$. This completes our construction. We observe that our construction involves two static output feedback loops—an internal one by F, and an external one by unity feedback. □

6. Pole assignment by pure output feedback

In this section we consider the problem of pole assignment in configurations of the form of Fig. 3, where f is a given transfer matrix, r is a causal feedback compensator, and f_r is the resulting system. We assume, of course, that the configuration is internally stable. The following is the main result of the present section. (We recall that the σ-latency indices were defined in § 2. We also note that, by strict causality, the σ-latency indices of an injective i/o map $f: \Lambda U \to \Lambda Y$ satisfy $\nu_i \geqslant 1$ for all $i = 1, \ldots, m$.)

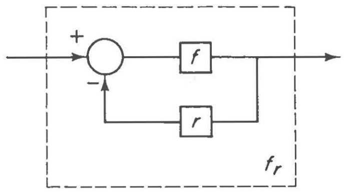

Figure 3.

Theorem 6.1

Let $f: \Lambda U \to \Lambda Y$ be an injective i/o map with stability indices $\theta_1 \geqslant \theta_2 \geqslant \ldots \geqslant \theta_m$ and with σ-latency indices $\nu_1 \geqslant \nu_2 \geqslant \ldots \geqslant \nu_m$. Let $\phi_1, \ldots, \phi_m$, where ϕ_{i+1} divides ϕ_i for all $i=1, \ldots, m-1$, be a set of monic and stable polynomials. If

$$\sum_{i=1}^{j} \deg \phi_i \geqslant \sum_{i=1}^{j} (\theta_i + \nu_1 - 1), \quad \textit{for all } j = 1, \ldots, m$$

then there exists a causal feedback r such that f_r is internally stable and has a polynomial representation $f_r = G^{-1}H$, where G has $\phi_1, \ldots, \phi_m$ as its (non-trivial) invariant factors.

The proof of Theorem 6.1 will be stated later in this section, following a preliminary discussion. We remark that the injectivity assumption in this theorem can be released through a method dual to the one employed in § 5 for the non-surjective case, and we shall discuss this point later in the section. Now, we wish to consider a numerical example.

Example

We return to the transfer matrix f of the example in § 5. Then, $\theta = 4$ and also $\nu = 4$. Whence, the condition of Theorem 6.1 becomes $\deg \phi \geqslant 7$. We recall that the condition (11) of Rosenbrock and Hayton (1978) was in this case $\deg \phi \geqslant 15$, whereas the condition for unity feedback (Theorem 5.2) was in this case $\deg \phi \geqslant 5$. □

Returning to the condition (11) of Rosenbrock and Hayton (1978), we remark that, since $\theta_i \leqslant \lambda_i$, $i = 1, \ldots, m$, and $\nu_1 \leqslant \mu_1$, the present conditions allow lower degree for the $\{\phi_i\}$. As we see in the above example, the present conditions can be considerably sharper, depending on the given transfer matrix f.

The proof of Theorem 6.1 is, in a way, similar to the proof of Theorem 5.1, but, instead of depending on the minimization of the feedback r in Fig. 1, it depends on the minimization of the precompensator v. The basic idea is as follows. Let $f: \Lambda U \to \Lambda Y$ be an i/o map, and let $v: \Lambda U \to \Lambda U$ and $r: \Lambda Y \to \Lambda U$, where v is non-singular, be causal compensators such that $f_{(v,r)}$ is internally stable. Denoting $f' := f_{(v,r)}$ and $r' := vr$, we obtain

$$f_{r'} = f' v^{-1} \tag{14}$$

Actually $f_{r'}$ is the transfer matrix obtained when the input in Fig. 1 is removed from point IN and applied by adding it at point B. Whence, since $f_{(v,r)}$ is internally stable, so also is $f_{r'}$. Now, from (14) we see that the dynamical properties of $f_{r'}$ are determined by those of f' and of v^{-1}. In order to obtain a low dynamical order for $f_{r'}$ we shall minimize the dynamical orders of f' and of v^{-1}. This will then lead to the desired pole-shifting theorem. The minimization of the dynamical order of f' was considered in § 3. We consider next the reduction of the dynamical order of v. We start with some preliminary considerations.

Let $f: \Lambda U \to \Lambda Y$ be an injective linear i/o map with σ-latency indices $\nu_1 \geqslant \ldots \geqslant \nu_m$. We define the *$\sigma$-latency degree* $\nu(f) := \sum_{i=1}^{m} \nu_i$. Next, let $v: \Lambda U \to \Lambda U$ be a non-singular ΛK-linear map. The *latency degree* of v is simply $\eta(v) :=$ ord (det v), that is, the number of zeros at infinity. The connection between the latency degree and the Macmillan degree $\mu(\cdot)$ is given by

$$\mu(v) = \mu(v^{-1}) + \eta(v) \tag{15}$$

For an injective map $f: \Lambda U \to \Lambda Y$ we define the latency degree as follows. Let $l: \Lambda Y \to \Lambda Y$ be a bicausal ΛK-linear map such that $lf = \begin{pmatrix} f_0 \\ 0 \end{pmatrix}$, where f_0 is square non-singular (the existence of l follows by the Hermite normal form theorem). Then, the latency degree is $\eta(f) :=$ ord (det f_0), and it is uniquely determined by f (Hammer and Heymann 1981). Recalling the zero degree $\zeta(\cdot)$ from § 2, it follows by construction (see (3)) that, for an injective map $f: \Lambda U \to \Lambda Y$

$$\nu(f) = \eta(f) + \zeta(f), \tag{16}$$

that is, the number of zeros of f which are unstable or infinite. Using this notation (and letting $m := \dim U$), we have the following result, which shows that the dynamical order of the precompensator v in Fig. 1 can always be kept below the σ-latency degree of the desired system $f' := f_{(v,r)}$.

Theorem 6.2

Let $f: \Lambda U \to \Lambda Y$ be an injective i/o map, and let $l: \Lambda U \to \Lambda U$ be a non-singular, causal and i/o stable precompensator. Assume that condition (ii) *of Theorem* (3.2) *is satisfied, and let $\nu(fl)$ be the σ-latency degree of fl. Then, there exists a pair of causal maps $v: \Lambda U \to \Lambda U$ and $r: \Lambda Y \to \Lambda U$ such that $fl \overset{\sigma}{=} f_{(v,r)}$, where the Macmillan degree $\mu(v) \leqslant \nu(fl) - m$.*

The proof of Theorem 6.2 depends on the following auxiliary results (see also Hammer (1983)).

Lemma 6.1

Let $g: \Lambda U \to \Lambda U$ be a non-singular causal ΛK-linear map, and let $g^{-1} = g_1 + g_2$ be a decomposition into a sum, where $g_2: \Lambda U \to \Lambda U$ is strictly causal. Then, g_1 is non-singular, ${g_1}^{-1}$ is causal, and the latency degrees $\eta({g_1}^{-1}) = \eta(g)$.

Proof of Lemma 6.1

We have $g_1 = g^{-1} - g_2 = g^{-1}(I - gg_2)$. Now, since gg_2 is strictly causal, the map $l := I - gg_2$ is bicausal. Whence, $g_1 = g^{-1}l$ is non-singular, ${g_1}^{-1} = l^{-1}g$ is causal, and, since $\eta(l) = \eta(l^{-1}) = 0$, also $\eta({g_1}^{-1}) = \eta(l^{-1}) + \eta(g) = \eta(g)$. □

The following are two technical results which were proved in Hammer (1983).

Lemma 6.2

Let $D_\sigma, l: \Lambda U \to \Lambda U$ be non-singular i/o stable maps, and let $(D_\sigma l)^{-1} = PA^{-1}+QB^{-1}$ be a reduced partial-fraction decomposition. Then the following are equivalent:

(i) *A and B are left σ^+-coprime, and have $(D_\sigma l)$ as a σ^+-LCRM.*
(ii) *The zero degrees $\zeta(D_\sigma l) = \zeta(A)+\zeta(B)$.*

Lemma 6.3

Let $f: \Lambda U \to \Lambda Y$ be a rational ΛK-linear map, and let $l: \Lambda U \to \Lambda U$ be a non-singular, causal and i/o stable precompensator. If fl is i/o stable, then $\zeta(fl) = \zeta(f)+\zeta(l)-\rho(f)$, where $\zeta(\cdot)$ is the zero degree, and $\rho(\cdot)$ is the pole degree.

Proof of Theorem 6.2

Let D_σ be the σ-annihilator of f given by (3). By assumption, there exists a reduced partial-fraction decomposition

$$(D_\sigma l)^{-1} = PA^{-1}+QB^{-1}$$

satisfying the conditions of Theorem 3.2. We define the matrices

$$\left.\begin{aligned} g &:= PA^{-1} - \mathbf{L}_\alpha^{-}(PA^{-1}) + \mathbf{L}_\alpha^{+}(QB^{-1}) \\ h &:= QB^{-1} + \mathbf{L}_\alpha^{-}(PA^{-1}) - \mathbf{L}_\alpha^{+}(QB^{-1}) \end{aligned}\right\} \tag{17}$$

Now, by definition of $\mathbf{L}_\alpha^{-}$ and $\mathbf{L}_\alpha^{+}$, it follows that g has a right coprime polynomial matrix fraction representation ND^{-1}, where D is completely unstable and N is divisible by $(z+\alpha)$. Also, A is still a right σ^+-denominator of g, so that $\zeta(A) = \zeta(D) = \deg(\det D)$.

Further, let $D_\sigma = RS^{-1}$ be a right coprime polynomial matrix fraction representation, and note that, by (3), R is completely unstable and S is divisible by $(z+\alpha)$. Then, since D_σ is a left σ^+-divisor of A, and since R and D are both completely unstable polynomial matrices, it follows by Lemma 2.1 that R is a polynomial left divisor of D. Whence, $D = RD_1$ for some polynomial matrix D_1. We also note (see (3)) that $\deg(\det S) = \nu(f)$, and that $\zeta(f) = \zeta(R) = \deg(\det R)$. Consequently, $\zeta(A) = \zeta(D) = \zeta(D_1)+\zeta(R) = \zeta(D_1)+\zeta(f)$. Finally, we note that B is still a right σ^+-denominator of h.

Now, applying the construction (6) to our present g and h, we obtain a causal pair v, r satisfying $fl \overset{\sigma}{=} f_{(v,r)}$. We have

$$v^{-1} = gD_\sigma = ND_1^{-1}S^{-1} \tag{18}$$

Since both N and S are divisible by $(z+\alpha)$, it follows that the Macmillan degree $\mu(v^{-1}) \leqslant \deg(\det D_1)+\deg(\det S)-m$. Substituting on the right-hand side some previous equalities, we obtain

$$\mu(v^{-1}) \leqslant \zeta(A)-\zeta(f)+\nu(f)-m = \eta(f)+\zeta(A)-m$$

where the last step is by (16).

Further, since $l^{-1}=v^{-1}+rf$, we have by Lemma 6.1 that $\eta(l)=\eta(v)$. Applying (15), we obtain $\mu(v)=\mu(v^{-1})+\eta(v)\leqslant\zeta(A)+\eta(f)+\eta(l)-m=\zeta(A)+\eta(fl)-m$. Finally, by Lemma 6.2, $\zeta(D_\sigma l)=\zeta(A)+\zeta(B)$, and by Theorem 3.2(ii)(γ), $\zeta(B)\geqslant\rho(f)$. Whence,

$$\zeta(A)=\zeta(D_\sigma l)-\zeta(B)=\zeta(D_\sigma)+\zeta(l)-\zeta(B)\leqslant\zeta(f)+\zeta(l)-\rho(f)=\zeta(fl)$$

by Lemma 6.3. Thus, using (16), we obtain $\mu(v)\leqslant\eta(fl)+\zeta(fl)-m=\nu(fl)-m$, concluding our proof. □

We are now in a position to state the proof of the main theorem of this section.

Proof of Theorem 6.1

Assume that $\phi_1, \ldots, \phi_m$ satisfy $\sum_{i=1}^{j} \deg\phi_i \geqslant \sum_{i=1}^{j} (\theta_i+\nu_1-1)$ for all $j=1, \ldots, m$. By Theorem 3.3 there exists then a polynomial matrix T having properly independent and ordered columns $t_1, \ldots, t_m$ satisfying

(*a*) $-\operatorname{ord} t_i\geqslant\theta_i+\nu_1-1$, $i=1, \ldots, m$, and
(*b*) $\phi_1, \ldots, \phi_m$ are the invariant factors of T.

Now let $f=ZD^{-1}$ be a right zero representation of f, where D has properly independent and ordered columns $d_1, \ldots, d_m$. We have then, by definition, that $\operatorname{ord} d_i=-\theta_i$, $i=1, \ldots, m$. Defining $l:=[(z+\alpha)^{\nu_1-1}D]T^{-1}$, where $(z+\alpha)\in\sigma$, we clearly obtain that l is non-singular, causal and i/o stable, and

$$fl=ZT^{-1}(z+\alpha)^{\nu_1-1}$$

It is also clear that $\zeta(l)=\zeta(D)=\rho(f)$. Moreover, by an argument similar to the one used in the proof of Theorem 3.1, it follows that condition (ii) of Theorem 3.2 holds for l.

We apply now to l the construction described in the proof of Theorem 6.2, using the same notation. We then obtain the compensators v, r satisfying $fl\overset{\sigma}{=}f_{(v,r)}$. In view of Theorem 3.2(ii) and Lemma 6.2, we have $\zeta(D_\sigma l)=\zeta(A)+\zeta(B)$; $\zeta(A)\geqslant\zeta(D_\sigma)=\zeta(f)$; and $\zeta(B)\geqslant\zeta(D_0)=\rho(f)$. Since in our present case $\zeta(l)=\rho(f)$, we have $\zeta(D_\sigma l)=\zeta(D_\sigma)+\zeta(l)=\zeta(f)+\rho(f)$. Whence, the previous inequalities imply $\zeta(A)=\zeta(f)$ and $\zeta(B)=\rho(f)$, and it follows that the matrix D_1 in (18) is polynomial unimodular, say $D_1=M$. Consequently, still referring to (18), and recalling that the invariant factors of S are $(z+\alpha)^{\nu_1}, \ldots, (z+\alpha)^{\nu_m}$ and that N is divisible by $(z+\alpha)$, it follows that the matrix $C:=(z+\alpha)^{\nu_1-1}v^{-1}$ is a polynomial matrix.

Finally, returning to (14), we recall that $f_{r'}$ is internally stable, and we obtain $f_{r'}=flv^{-1}=ZT^{-1}C$. Now, Z is a completely unstable polynomial matrix whereas T^{-1} is i/o stable, so that Z, T are polynomially right coprime. Thus, since C is also polynomial, $f_{r'}$ has a left polynomial representation $G^{-1}H$, where G has the invariant factors of T, that is, $\phi_1, \ldots, \phi_m$. This concludes our proof. □

In case one is interested only in the characteristic polynomial of the final system, then the condition of Theorem 6.1 can be sharpened. This is done similarly to § 5 of Rosenbrock and Hayton (1978). Let $f: \Lambda U\to\Lambda Y$ be a non-zero linear i/o map with stability indices $\theta_1\geqslant\theta_2\geqslant\ldots\geqslant\theta_m$. It was shown by Brasch and Pearson (1970) (using a result by Wonham (1967)) that there exists a static output feedback $F: Y \to U$ and a static matrix $V: K \to U$ such that (the single-input system) $f' := f_F V$ has Macmillan degree $\mu(f') = \mu(f)$. The i/o map f' is then clearly injective, and it has a single stability index θ and a single σ-latency index ν. Now let ϕ be a monic stable polynomial satisfying

$$\deg \phi \geqslant \theta + \nu - 1 \tag{19}$$

Then, by Theorem 6.1, there exists a causal $r_1: Y \to K$ such that f'_{r_1} is internally stable and has ϕ as a characteristic polynomial (the single non-trivial invariant factor). But then, defining

$$r := Vr_1 + F \tag{20}$$

we obtain that $f_r V$ is internally stable and has ϕ as a characteristic polynomial of an observable realization. Thus, (19) is a sufficient condition for the assignment of characteristic polynomials by pure output feedback (and a static (constant-gain) precompensator V).

We finally remark that (19) allows lower degrees for ϕ than the condition obtained for a similar problem by Brasch and Pearson (1970), which reads $\deg \phi \geqslant \mu(f) + \mu_1 - 1$, where μ_1 is the maximal observability index of f.

We conclude with a brief consideration of the extension of Theorem 6.1 to the non-injective case.

The non-injective case

In order to avoid complications, we use a method which is only generically valid. It is dual to the one employed for the non-surjective case in § 5. Let $f: \Lambda U \to \Lambda Y$ be a non-zero i/o map of rank k, and let $f = D^{-1}N$ be a left canonical stability representation of f. We can choose this representation so that $N = \begin{pmatrix} N_0 \\ 0 \end{pmatrix}$, where N_0 is a $k \times m$ matrix (having full rank k). Assume now that N_0 contains a $k \times k$ non-singular submatrix Q for which $\det Q$ is σ^+-coprime with $\det D$. When K is the field of real numbers, and when σ is the set of all polynomials having their roots in the left half of the complex plane, then a suitable submatrix Q exists for almost every f (i.e. generically).

Now, let $V: K^k \to U$ be a static (constant gain) matrix such that $Q = N_0 V$, and let $N' := NV$. Then, the i/o map $f' := fV$ is clearly injective, and $f' = D^{-1}N'$ is a left canonical stability representation of f'. Apply now Theorem 6.1 to f', and let $r': \Lambda Y \to \Lambda K^k$ be a feedback which makes $f'_{r'}$ internally stable, and which assigns the desired invariant factors $\phi_1, \ldots, \phi_k$. Then, defining $r := Vr'$, it follows that $f_r V$ is internally stable and has a (observable) realization with the same invariant factors $\phi_1, \ldots, \phi_k$. This extends Theorem 6.1 to the non-injective case (adding the static (constant-gain) precompensator V).

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