INTERNATIONAL JOURNAL OF CONTROL
2020, VOL. 93, NO. 6, 1303–1316
https://doi.org/10.1080/00207179.2018.1504234

Periodic sampling: maximising the sampling period

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ABSTRACT
The process of periodic sampling is investigated for a class of nonlinear systems. The objective is to achieve the longest sampling period that is compatible with a specified error bound. It is shown that there are robust optimal controllers that achieve this objective. It is also shown that the performance of such optimal controllers can be approximated by bang-bang controllers – controllers that are relatively easy to design and implement.

ARTICLE HISTORY
Received 16 January 2018
Accepted 20 July 2018

KEYWORDS
Nonlinear systems; periodic sampling; optimal control; maximal sampling period

1. Introduction

Periodic sampling is among the most common operations performed in modern control engineering. It underpins the functionality of sampled-data control systems – systems that employ digital controllers to control continuous-time systems. In many applications, it is desirable to increase the sampling period without increasing associated operating errors. Increasing the sampling period allows more time to process control data between successive data samples, making it possible to utilise more sophisticated control algorithms. A longer sampling period also reduces data load in feedback communication channels, a central consideration in the design of networked control systems (Montestruque & Antsaklis, 2004; Nair, Fagnani, Zampieri, & Evans, 2007; Zhivogyladov & Middleton, 2003). In addition, longer sampling periods contribute to a system's concealment and stealth, a material advantage in certain applications.

There are many additional enterprises that may benefit from longer sampling periods. For example, consider a biotechnology manufacturing facility. Here, workers must inspect periodically the status of organisms used in the manufacture of biological products. Maximizing the time interval between these inspections may reduce manufacturing costs. Applications in medicine also abound. For instance, in the treatment of diabetes, patients would benefit from a longer time interval between consecutive samplings of their blood glucose level. Many other potential applications come to mind.

It goes without saying that, between samples, a sampled-data control system operates without feedback. As classical control theory reminds us, the lack of feedback may increase operating errors. For a particular controlled system, the magnitude of such operating errors depends on two main factors: (i) the length of the time span between samples, namely, the length of the sampling period; and (ii) the nature of the input signal the controlled system receives between samples, namely, the nature of the controller controlling the system. The present paper concentrates on the existence, the design, and the implementation of robust optimal controllers that make it possible to utilise the maximal sampling period, without violating specified bounds on operating errors and other specifications. We show in Section 3 that such optimal controllers do exist for a broad family of nonlinear input-affine systems. The main requirement for the existence of such optimal controllers is a certain controllability property the controlled system must possess.

The design and construction of optimal controllers is often an arduous engineering task. It may require the calculation and the implementation of intricate vector-valued functions of time – the signals an optimal controller must generate as input to the controlled system. In Section 4, we show that the performance of optimal controllers can be approximated by bang-bang controllers, namely, by controllers that generate bang-bang signals as input to the controlled system. Bang-bang controllers are relatively easy to design and implement, since a bang-bang signal is characterised by its switching times – a list of scalars.

The control configuration we consider is depicted in Figure 1. Here, the system Σ is controlled by the controller C, which generates the input signal $u(t)$ of Σ. The state of Σ at the time t is $x(t)$. As seen in the figure, the controller's feedback channel closes momentarily every T seconds, delivering periodic samples of $x(t)$ with a sampling period of T. These samples are used by C to control Σ.

As mentioned earlier, the magnitude of inter-sample operating errors experienced by Σ depends on the length of the sampling period T and on the design of the controller C. Our objective is to design the controller C so as to achieve the longest possible sampling period T, without violating a specified operating error bound $\ell > 0$ and without overloading the controlled system Σ. In Section 3, we characterise the maximal sampling period T that is consistent with this objective. We also show in that section that robust optimal controllers C that achieve the maximal sampling period exist under rather broad conditions. The main condition that the controlled system Σ must satisfy in order for such optimal

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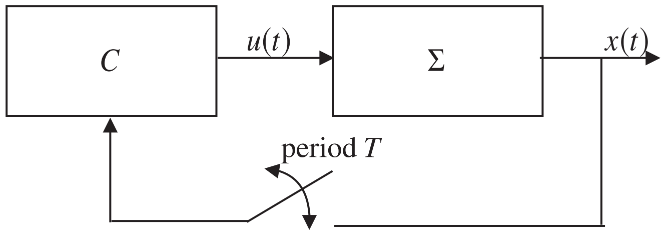


Figure 1. The control configuration.

controllers to exist is a certain controllability condition. Later on, in Section 4, we develop a relatively simple methodology for practical design and implementation of controllers that approximate optimal performance. We show that optimal performance can be approximated as closely as desired by controllers that generate bang-bang signals as input to the controlled system Σ.

To make our discussion specific, we assume that the requirement is to keep the state of Σ in the vicinity of a specified nominal target state x_{target}. By appropriately shifting the state coordinates of Σ, we can assume that the nominal target state is the zero state $x = 0$. Due to errors and uncertainties affecting Σ and its environment, it is not possible to guide Σ to stay exactly at the zero state. Instead, a specified maximal deviation of $\ell > 0$ from the zero state is acceptable. This deviation is specified in terms of the L^2−norm of the state; using $^\top$ to indicate transpose, the requirement is to control Σ so that its state $x(t)$ satisfies the inequality

$$x^\top(t)x(t) \le \ell \tag{1}$$

at all times t. We refer to ℓ as the *operating error bound*.

To conform with structural limitations of the controlled system Σ and avoid overloading Σ, input signals to Σ cannot exceed a specified amplitude bound of $K > 0$. Our objectives can now be summarised in the following form.

Problem 1.1: Let Σ be a system with input amplitude bound $K > 0$ and operating error bound $\ell > 0$.

(i) Find conditions under which there is a robust optimal sampled-data controller C with periodic sampling period T that controls Σ so as to fulfil the following goals:
  (a) The amplitude of the signal that C creates as input for Σ does not exceed K.
  (b) The state $x(t)$ of Σ satisfies $x^\top(t)x(t) \le \ell$ at all times t.
  (c) The periodic sampling period T is the maximal one consistent with (a) and (b).

(ii) Find controllers that approximate optimal performance and are relatively easy to design and implement.

In this paper, Problem 1.1 is investigated for a class of nonlinear time-invariant and input-affine systems. In Section 3, we show that robust optimal controllers C that fulfil the requirements of Problem 1.1(i) exist under rather broad conditions. The main requirement for the existence of such controllers is a certain controllability property the controlled system Σ must possess. This controllability property requires that it be possible to drive Σ from a given initial state to the origin, without violating specified input signal amplitude bounds. In fact, it can be seen from the statement of Problem 1.1(i) that such a controllability property is also necessary for the existence of an appropriate controller.

As mentioned earlier, the effort required to compute, design, and implement optimal controllers is often considerable. To overcome this potential hindrance and address Problem 1.1(ii), we show in Section 4 that optimal performance can be approximated by controllers that generate bang-bang signals as input for the controlled system Σ. In our case, where input signals to Σ must be bounded by K, bang-bang signals are piecewise-constant signals, whose components switch between the values of $+K$ and $-K$ a finite number of times during every finite time interval. As bang-bang signals are determined by their switching times, they can often be derived by relatively simple numerical search and optimisation algorithms (see Section 5). Implementation of bang-bang signals is also relatively easy. The observation of the current paragraph further establishes a principle espoused in Chakraborty and Hammer (2009, 2010), Yu and Hammer (2016a), and Choi and Hammer (2017, 2018b), according to which bang-bang controllers can approximate optimal performance in a wide range of optimisation problems.

The most common technique currently utilised in sampled-data control systems is the so-called sample-and-hold technique. In this technique, a constant input signal is delivered to the controlled system Σ between samples, instead of the optimal input signal developed in the present paper. In Section 5, we use an example of a single-link manipulator to compare the outcome of the optimal input signal derived in this paper to the outcome of the sample-and-hold technique. In this example, optimal input signals permit a substantially longer sampling period than the longest sampling period achievable via sample-and-hold. Naturally, other examples may yield greater or lesser improvements. In any case, by its nature of optimality, the optimal technique developed in this paper always yields a maximal sampling period.

The discussion of this paper relies on earlier work by the authors (Chakraborty & Hammer, 2009, 2010; Choi & Hammer, 2017, 2018b; Yu & Hammer, 2016a) as well as on the foundations of the theory of constrained optimisation. The latter include Kelendzheridze (1961), Pontryagin, Boltyansky, Gamkrelidze, and Mishchenko (1962), Gamkrelidze (1965), Neustadt (1966), Neustadt (1967), Luenberger (1969), Young (1969), Warga (1972), the references cited in these studies, and many others. It seems, however, that there are no earlier published reports on the existence and the implementation of robust optimal controllers that achieve maximal sampling periods, while complying with specified bounds on operating errors and control signal amplitudes.

The paper is organised as follows. Section 2 formulates Problem 1.1 in more precise mathematical terms. Section 3 proves the existence of robust optimal controllers that fulfil the objectives of Problem 1.1(i). Problem 1.1(ii) is examined in Section 4, where we show that, without significantly degrading performance, optimal controllers can be replaced by controllers that generate bang-bang input signals for the controlled system Σ.

Section 5 presents an example, and the paper concludes in Section 6 with a brief summary.

2. Background and problem formulation

In this section, we introduce the mathematical background and the notation underlying our discussion. We denote by R the compactified set of real numbers, namely, the set of all real numbers augmented by the points $\pm\infty$; by R^+ we denote the set of all non-negative real numbers. The set of all vectors with n real components is denoted by R^n.

The $L^\infty-$norm is used in our discussion. For a real number r, the $L^\infty-$norm is simply the absolute value $|r|$; for an $n \times m$ matrix $V = (V_{ij})$, the $L^\infty-$norm is $|V| := \max_{i,j} |V_{ij}|$, i.e. the largest absolute value of an entry. For a matrix-valued function of time $W : R^+ \to R^{n\times m} : t \mapsto W(t)$, the $L^\infty-$norm is $|W|_\infty := \sup_{t\geq 0} |W(t)|$ and is referred to as the *amplitude* of W.

2.1 System description

We consider nonlinear input-affine time-invariant systems described by differential equations of the form

$$\Sigma : \dot{x}(t) = a(x(t)) + b(x(t))u(t),\ x(0) = x_0, \tag{2}$$

where $x(t) \in R^n$ is the state of the system and $u(t) \in R^m$ is the input signal at the time t. The functions $a : R^n \to R^n$ and $b : R^n \to R^{n\times m}$ are continuous functions satisfying the Lipschitz conditions

$$|a(x) - a(y)| \leq \alpha^+|x - y|, \quad |b(x) - b(y)| \leq \alpha^+|x - y|,$$

for all $x, y \in R^n$; here $\alpha^+ > 0$ is a specified real number.

To accommodate uncertainties and errors affecting the model of the controlled system Σ, we split the functions a and b of (2) into a sum

$$a(x) = a_0(x) + a_\gamma(x), \quad b(x) = b_0(x) + b_\gamma(x), \tag{3}$$

where a_0 and b_0 are specified continuous functions, and a_γ and b_γ are unspecified continuous functions that represent uncertainties. All functions $a_0, a_\gamma : R^n \to R^n$ and $b_0, b_\gamma : R^n \to R^{n\times m}$ satisfy the Lipschitz conditions:

$$|a_0(x) - a_0(y)| \leq \alpha|x - y|, \quad |b_0(x) - b_0(y)| \leq \alpha|x - y|,$$
$$a_0(0) = 0, \quad |b_0(0)| \leq \alpha; \tag{4}$$

$$|a_\gamma(x) - a_\gamma(y)| \leq \gamma|x - y|, \quad |b_\gamma(x) - b_\gamma(y)| \leq \gamma|x - y|,$$
$$|a_\gamma(0)| \leq \gamma, \quad |b_\gamma(0)| \leq \gamma, \tag{5}$$

for all $x, y \in R^n$. Here, $\alpha, \gamma > 0$ are specified real numbers and $\alpha^+ = \alpha + \gamma$. The number γ represents uncertainty; it is usually a small number. We refer to γ as the *uncertainty parameter*. The nominal system is then

$$\Sigma_0 : \dot{x}(t) = a_0(x(t)) + b_0(x(t))u(t), \quad x(0) = x_0. \tag{6}$$

2.2 The sampling process

In the configuration of Figure 1, the feedback channel closes momentarily at the times $\ldots, -T, 0, T, 2T, \ldots$, forming a periodic sampling process with a period of $T > 0$ and sampling intervals $\ldots, [-T, 0], [0, T], [T, 2T], \ldots$ At a sampling time kT, $k = \ldots, -1, 0, 1, 2, \ldots$, the feedback channel delivers to the controller C the sample $x(kT)$ of the state of Σ. As $t = 0$ is one of the sampling times, the state $x(0) = x_0$ is available to C. We refer to x_0 as the *initial state*.

In our ensuing discussion, we concentrate on the sampling interval $[0, T]$, and develop a framework in which all other sampling intervals are regarded as repeats of the sampling interval $[0, T]$. In qualitative terms, this is accomplished by considering x_0 as a set of potential initial states S_0, rather than as a single specified initial state. The set S_0 includes the initial states of all sampling intervals, namely, S_0 includes all states $x(kT)$, $k = \ldots, -1, 0, 1, 2, \ldots$ The specifics of this technique are discussed in Section 2.4.

2.3 Basics

Our discussion takes place in the Hilbert space $L_2^{\omega,m}$ of Lebesgue measurable functions $f, g : R^+ \to R^m$ with the inner product

$$\langle f, g\rangle := \int_0^\infty e^{-\omega t} f^\top(s) g(s)\, ds;$$

here, $\omega > 0$ is a real number (Chakraborty & Hammer, 2009, 2010). Note that with this inner product, all bounded measurable functions produce a bounded inner product.

Many systems encountered in engineering practice impose a bound on the maximal input amplitude they can tolerate. Correspondingly, we enforce a maximal permissible input amplitude on the controlled system Σ, allowing only input signals bounded by a specified bound $K > 0$. Then, the set of permissible input signals of Σ is

$$U(K) = \{u \in L_2^{\omega,m} : |u|_\infty \leq K\}. \tag{7}$$

Notation 2.1: The family of systems $\mathcal{F}_\gamma(\Sigma_0)$ consists of all systems of the form (2), subject to (3), (4), and (5). All members of $\mathcal{F}_\gamma(\Sigma_0)$ share the same initial state x_0 and accept only input signals belonging to $U(K)$. To indicate explicitly the initial state x_0 and the input signal u of a member $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$, we denote the state at the time t by $x(t) = \Sigma(x_0, u, t)$.

The controlled system Σ of Figure 1 is a member of $\mathcal{F}_\gamma(\Sigma_0)$. Due to the uncertainty about the model of Σ expressed by (3)–(5), it is not known specifically which member of $\mathcal{F}_\gamma(\Sigma_0)$ the system Σ is. Notwithstanding, the initial state x_0 of Σ is known, as it is communicated by the feedback channel. In addition to sharing the same initial state, all members of $\mathcal{F}_\gamma(\Sigma_0)$ also share the same input signal u, since, again, it is not known which member of $\mathcal{F}_\gamma(\Sigma_0)$ the controlled system Σ is, so the input signal u cannot be designed individually for each member of $\mathcal{F}_\gamma(\Sigma_0)$.

Recalling the operating error bound $\ell > 0$ of (1), we introduce the ball $\rho(\ell) := \{x \in R^n : x^\top x \le \ell\}$. Then, requirement (1) can be restated in the form

$$x(t) \in \rho(\ell) \quad \text{for all } t, \tag{8}$$

where $x(t)$ is the state of Σ at the time t. The controller C of Figure 1 must guide Σ so as to maintain (8) at all times.

2.4 A periodic framework

The periodic sampling process conducted by the feedback channel of Figure 1 has the sampling period $T > 0$. The initial state $x(0) = x_0$ serves as the starting state for the period $[0, T]$. For the next period, i.e. the period $[T, 2T]$, the starting state is $x_T := x(T)$ – the terminal state of the period $[0, T]$. In general, for an integer $k = \dots, -1, 0, 1, \dots$, the starting state of the period $[kT, (k+1)T]$ is the state x_{kT} – the terminal state of the preceding period.

To make things manageable, we concentrate on the single sampling interval $[0, T]$, and replace the initial state x_0 by the set of all possible starting states $\{x_{kT}\}_{k=\dots-1,0,1,2,\dots}$. As the system Σ is time-invariant, an analysis of this single sampling interval with the collection of all starting states will represent the behaviour for all sampling intervals. To perform this analysis, we must characterise all possible starting states $\{x_{kT}\}_{k=\dots-1,0,1,2,\dots}$.

An exact characterisation of all possible states $\{x_{kT}\}_{k=\dots-1,0,1,2,\dots}$ would be unduly complex, since it must account for the dynamic behaviour of the nonlinear system Σ as well as for the uncertainty about Σ. Instead of engaging in an accurate characterisation of these states, we select a real number $\sigma > 0$ and design the controller C to guide every member $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ from every state $x_{kT} \in \rho(\sigma)$ to a state $x_{(k+1)T} \in \rho(\sigma)$, $k = \dots - 1, 0, 1, 2, \dots$ In addition, C must comply with requirement (8) and assure that the state $x(t)$ of Σ satisfies $x(t) \in \rho(\ell)$ at all times $t \in [kT, (k+1)T]$. In particular, this implies that we must have $\sigma \le \ell$. We refer to σ as the *sample radius*; it bounds the state at the sampling times.

Consider now the sampling interval $[0, T]$, and let C be a time-invariant controller that guides every member $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ from every initial state $x_0 \in \rho(\sigma)$ to a state $x_T \in \rho(\sigma)$, while assuring that $x(t) \in \rho(\ell)$ for all $t \in [0, T]$. By the time-invariance of C and Σ, the same controller action will guide every member $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ from every state $x_{kT} \in \rho(\sigma)$ to a state $x_{(k+1)T} \in \rho(\sigma)$, while keeping $x(t) \in \rho(\ell)$ for all $t \in [kT, (k+1)T], k = \dots, -1, 0, 1, \dots$ We can summarise our discussion as follows.

Conclusion 2.2: Let $\sigma, \ell > 0$ be real numbers, where $\sigma \le \ell$, and refer to the control configuration of Figure 1, where C is a time-invariant controller, $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$, and the sampling period is $T > 0$. Then, the next two statements are equivalent.

(i) The controller C guides every member $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ from every initial state $x_0 \in \rho(\sigma)$ to a state $x_T \in \rho(\sigma)$, while keeping $x(t) \in \rho(\ell)$ for all $t \in [0, T]$.
(ii) The controller C guides every member $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ from every state $x_{kT} \in \rho(\sigma)$ to a state $x_{(k+1)T} \in \rho(\sigma)$, while keeping $x(t) \in \rho(\ell)$ for all $t \in [kT, (k+1)T]$, $k = \dots - 1, 0, 1, 2, \dots$

In view of Conclusion 2.2, it is sufficient to concentrate on the sampling interval $[0, T]$ and derive a controller C that fulfils the requirements of Conclusion 2.2(i). By the Conclusion, such a controller will satisfy the objectives of Problem 1.1(i) at all times. This yields a substantial simplification of Problem 1.1: it allows us to concentrate on the finite interval $[0, T]$, instead of having to work with the entire time axis.

In the sequel, we study sampling periods $T > 0$ for which there is a controller C that fulfils the requirements of Conclusion 2.2(i). To make it easy to refer to such sampling periods, we introduce the following term.

Definition 2.3: A time $T > 0$ is a *feasible sampling period* if there is a controller C that satisfies Conclusion 2.2(i) with the period T.

We are interested in the existence of controllers C that facilitate the longest feasible sampling period T. In more specific terms, our interest is focused on the following.

Problem 2.4: Let $\sigma, \ell > 0$ be real numbers, where $\sigma \le \ell$. Find the longest time $T > 0$ for which the following holds for every initial state $x_0 \in \rho(\sigma)$: there is an input signal $u_{x_0} \in U(K)$ for which every system $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ satisfies

(i) $\Sigma(x_0, u_{x_0}, T) \in \rho(\sigma)$, and
(ii) $\Sigma(x_0, u_{x_0}, t) \in \rho(\ell)$ for all $t \in [0, T]$.

2.5 Formal statement of the problem

2.5.1 Input signals

In this subsection, we rephrase Problem 1.1 in formal terms, using the framework of Conclusion 2.2. Let $\sigma > 0$ be a real number, and consider a system $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ with an initial state $x_0 \in \rho(\sigma)$. In accordance with Conclusion 2.2, we confine our attention to the time interval $[0, T]$ and permit only input signals $u \in U(K)$ that keep the state $x(t)$ of Σ within $\rho(\ell)$ during all times $t \in [0, T]$. The latter restricts input signals to the set

$$U_\ell(x_0, K, \ell, \Sigma, T) := \left\{ u(t) \in U(K) : \sup_{t\in[0,T]} \Sigma^\top(x_0, u, t)\Sigma(x_0, u, t) \le \ell \right\}. \tag{9}$$

As it is not known which member of $\mathcal{F}_\gamma(\Sigma_0)$ the controlled system Σ is, (9) must hold for every member $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$. Therefore, input signals must be confined to the set

$$U_\ell(x_0, K, \ell, \gamma, T) := \bigcap_{\Sigma\in\mathcal{F}_\gamma(\Sigma_0)} U_\ell(x_0, K, \ell, \Sigma, T). \tag{10}$$

Equivalently, (10) can be rewritten in the form

$$U_\ell(x_0, K, \ell, \gamma, T) = \left\{ u \in U(K) : \sup_{\substack{t\in[0,T] \\ \Sigma\in\mathcal{F}_\gamma(\Sigma_0)}} \Sigma^\top(x_0, u, t)\Sigma(x_0, u, t) \le \ell \right\}. \tag{11}$$

By Conclusion 2.2(i), the state of Σ must be taken back into the domain $\rho(\sigma)$ at the end of the time interval $[0, T]$. The set of all input signals that take every member $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ from the initial state x_0 into the domain $\rho(\sigma)$ at the time T is

$$U_\sigma(x_0, K, \sigma, \gamma, T) = \left\{ u \in U(K) : \sup_{\Sigma \in \mathcal{F}_\gamma(\Sigma_0)} \Sigma^\top(x_0, u, T)\Sigma(x_0, u, T) \leq \sigma \right\}. \tag{12}$$

For a member $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$, define the set

$$U_\sigma(x_0, K, \sigma, \Sigma, T) = \{u \in U(K) : \Sigma^\top(x_0, u, T)\Sigma(x_0, u, T) \leq \sigma\}; \tag{13}$$

then, (12) can be rewritten in the form

$$U_\sigma(x_0, K, \sigma, \gamma, T) = \bigcap_{\Sigma \in \mathcal{F}_\gamma(\Sigma_0)} U_\sigma(x_0, K, \sigma, \Sigma, T). \tag{14}$$

According to Conclusion 2.2, we must fulfil two requirements: (i) $x(t) \in \rho(\ell)$ for all $t \in [0, T]$ and (ii) $x(T) \in \rho(\sigma)$. This leads to the intersection of the two sets given by (10) and (14), and yields the set of input signals

$$U'(x_0, K, \ell, \sigma, \gamma, T) := U_\ell(x_0, K, \ell, \gamma, T) \cap U_\sigma(x_0, K, \sigma, \gamma, T). \tag{15}$$

This intersection describes the set of all input signals u that bring every member $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ from the initial state x_0 to a state $x(T) \in \rho(\sigma)$, while complying with the operating error bound ℓ along the way.

Next, by Conclusion 2.2, for T to be a feasible sampling period, it must be possible to take all initial states $x_0 \in \rho(\sigma)$ back into $\rho(\sigma)$ at the time T, without violating the operating error bound ℓ along the way. Thus, sets of input signals $U'(x_0, K, \ell, \sigma, \gamma, T)$ that are empty for some initial states $x_0 \in \rho(\sigma)$ are of no use to us. Eliminating these sets yields the set

$$U(x_0, K, \ell, \sigma, \gamma, T) := \begin{cases} U'(x_0, K, \ell, \sigma, \gamma, T) & \text{if } U'(x_0, K, \ell, \sigma, \gamma, T) \neq \emptyset \text{ for all } x_0 \in \rho(\sigma), \\ \emptyset & \text{otherwise.} \end{cases} \tag{16}$$

We can summarise as follows.

Proposition 2.5: *The following two statements are equivalent.*

(i) *A time $T > 0$ is a feasible sampling period for the family of system $\mathcal{F}_\gamma(\Sigma_0)$ with the sample radius $\sigma > 0$.*
(ii) *$U(x_0, K, \ell, \sigma, \gamma, T) \neq \emptyset, x_0 \in \rho(\sigma)$.*

By Proposition 2.5, the set of all input signals that may be employed in the process of taking Σ from initial states $x_0 \in \rho(\sigma)$ to states $x_T \in \rho(\sigma)$, without violating the operating error bound ℓ along the way, is

$$U(K, \ell, \sigma, \gamma, T) := \bigcup_{x_0 \in \rho(\sigma)} U(x_0, K, \ell, \sigma, \gamma, T).$$

Note that this set of input signals is empty if there is an initial state $x_0 \in \rho(\sigma)$ that cannot be taken to a state $x_T \in \rho(\sigma)$ without violating the operating error bound ℓ along the way. Thus, for a given pair σ and ℓ, a time $T > 0$ is a feasible sampling period if and only if $U(K, \ell, \sigma, \gamma, T) \neq \emptyset$.

The facts that $\sigma \leq \ell$ and $x_0 \in \rho(\sigma)$ imply that

$$U(x_0, K, \ell, \sigma, \gamma, 0) = U(K) \quad \text{for all } x_0 \in \rho(\sigma). \tag{17}$$

(Of course, $T = 0$ is not a valid sampling period.)

2.5.2 Feasible sampling periods

Having discussed potential input signals, we turn to the examination of feasible sampling periods. As seen in (16), potential input signals are all members of the set $U(x_0, K, \ell, \sigma, \gamma, t)$. Assume that the controlled system Σ is at an initial state $x_0 \in \rho(\sigma)$ and is driven by an input signal $u \in U(x_0, K, \ell, \sigma, \gamma, t)$. Then, the longest possible time lapse $T(x_0, \ell, \sigma, \Sigma, u)$ after which Σ returns to the ball $\rho(\sigma)$ without violating the operating error bound ℓ is

$$T(x_0, \ell, \sigma, \Sigma, u) = \{\sup t \geq 0 : \Sigma^\top(x_0, u, t)\Sigma(x_0, u, t) \leq \sigma \text{ and } u \in U(x_0, K, \ell, \sigma, \gamma, t)\}, \tag{18}$$

where $T(x_0, \ell, \sigma, \Sigma, u) := \infty$ if the supremum does not exist. By (17), it follows that $T(x_0, \ell, \sigma, \Sigma, u)$ is defined for all signals $u \in U(K)$ and for all systems $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$, and that $T(x_0, \ell, \sigma, \Sigma, u) \geq 0$.

Recall that the family of systems $\mathcal{F}_\gamma(\Sigma_0)$ represents uncertainty about the controlled system; it is not known which member of $\mathcal{F}_\gamma(\Sigma_0)$ the controlled system Σ of Figure 1 actually is. Consequently, the same input signal u must be used for all members of $\mathcal{F}_\gamma(\Sigma_0)$. For an initial state $x_0 \in \rho(\sigma)$, the longest possible duration $T(x_0, \ell, \sigma, \gamma, u)$ after which u guides every member $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ back to $\rho(\sigma)$ (without violating the operating error bound ℓ along the way) is

$$T(x_0, \ell, \sigma, \gamma, u) = \left\{ \sup t \geq 0 : \sup_{\Sigma \in \mathcal{F}_\gamma(\Sigma_0)} \Sigma^\top(x_0, u, t)\Sigma(x_0, u, t) \leq \sigma \text{ and } u \in U(x_0, K, \ell, \gamma, t) \right\}, \tag{19}$$

where $T(x_0, \ell, \gamma, \sigma, u) := \infty$ if the supremum does not exist. Equivalently, we can rewrite (19) in the form

$$T(x_0, \ell, \sigma, \gamma, u) = \inf_{\Sigma \in \mathcal{F}_\gamma(\Sigma_0)} T(x_0, \ell, \sigma, \Sigma, u). \tag{20}$$

Clearly, different input signals may yield different times $T(x_0, \ell, \gamma, \sigma, u)$. The longest time that can be achieved for the initial state x_0 by using an input signal $u \in U(K)$ is

$$T(x_0, \ell, \gamma, \sigma) = \sup_{u \in U(K)} T(x_0, \ell, \gamma, \sigma, u), \tag{21}$$

where $T(x_0, \ell, \gamma, \sigma) := \infty$ if the supremum does not exist. In view of the paragraph following (18), we have that $T(x_0, \ell, \gamma, \sigma)$ is well defined for every $x_0 \in \rho(\sigma)$.

If there is an input signal $u'_{x_0} \in U(K)$ for which $T(x_0, \ell, \gamma, \sigma, u'_{x_0}) = T(x_0, \ell, \gamma, \sigma)$, then u'_{x_0} is an optimal input signal that achieves the maximal time for the initial state x_0. We discuss conditions for the existence of such optimal signals in Section 3.

Now, according to Conclusion 2.2, the initial state x_0 can be any member of the ball $\rho(\sigma)$. The longest time that is compatible with all initial states in $\rho(\sigma)$ is

$$T(\ell,\gamma,\sigma) = \inf_{x_0\in\rho(\sigma)} T(x_0,\ell,\gamma,\sigma), \tag{22}$$

where $T(\ell,\gamma,\sigma) := \infty$ if the infimum does not exist. Note that, by the observation of the paragraph following (21), the time $T(\ell,\gamma,\sigma)$ is well defined for all $0 \le \sigma \le \ell$.

If, for every initial state $x_0 \in \rho(\sigma)$, there is an input signal $u^*_{x_0} \in U(K)$ satisfying $T(x_0,\ell,\gamma,\sigma,u^*_{x_0}) = T(\ell,\gamma,\sigma)$, then $u^*_{x_0}$ is an optimal input signal that achieves for x_0 the maximal time lapse that is compatible with all initial states $x_0 \in \rho(\sigma)$. Conditions for the existence of such optimal signals are discussed in Section 3.

Finally, the value of the sample radius $\sigma \ge 0$ is not specified; the only requirement on σ is $\sigma \le \ell$, where ℓ is the specified operating error bound. Utilizing this flexibility of σ, we obtain the maximal sampling period

$$T^*(\ell,\gamma) := \sup_{0\le\sigma\le\ell} T(\ell,\gamma,\sigma), \tag{23}$$

where $T^*(\ell,\gamma) := \infty$ if the supremum does not exist. Note also that $T^*(\ell,\gamma)$ is well defined and $T^*(\ell,\gamma) \ge 0$ for all ℓ and γ (see the paragraph following (22)). If there is a maximising value σ^* of σ satisfying $T(\ell,\gamma,\sigma^*) = T^*(\ell,\gamma)$, then σ^* is an optimal sampling radius. We discuss the existence of optimal sampling radii in Section 3.

In the framework of Conclusion 2.2, the time $T^*(\ell,\gamma)$ represents the supremal sampling period for periodic sampling, given specified operating error bound ℓ and system uncertainty parameter γ. We show in Section 3 that this supremal sampling period can be achieved under rather broad conditions. In Section 4 we show that sampling periods very close to the supremal sampling period can be achieved by bang-bang controllers – controllers that are relatively easy to design and implement. We can restate Problem 1.1 in the following form.

Problem 2.6: In the control configuration of Figure 1, the controlled system Σ is an unspecified member of the family $\mathcal{F}_\gamma(\Sigma_0)$.

(i) Find conditions under which there is an optimal sample radius σ^* satisfying $T(\ell,\gamma,\sigma^*) = T^*(\ell,\gamma)$.
(ii) Find conditions under which there is, for every initial state $x_0 \in \rho(\sigma^*)$, an optimal input signal $u^*(x_0,\ell,\gamma)$ that achieves the maximal sampling period $T^*(\ell,\gamma)$.
(iii) If an optimal input signal $u^*(x_0,\ell,\gamma)$ exists, find an easy-to-calculate-and-implement signal $u^\pm$ that approximates the optimal performance achieved by $u^*(x_0,\ell,\gamma)$.

Remark 2.7: If $T^*(\ell,\gamma) > 0$ and if the optimal sample radius σ^* of Problem 2.6 exists, then $\sigma^* > 0$. This is because the terminal state of every sampling period belongs to $\rho(\sigma^*)$, and the uncertainty about the controlled system Σ induces a dispersion of these states. Therefore, $\sigma^* = 0$ is not possible.

2.6 Constrained controllability

Let Σ be the controlled system of Figure 1, let σ be the sample radius, and let ℓ be the operating error bound. Then, according to Conclusion 2.2, we must find input signals that take Σ from any initial state $x_0 \in \rho(\sigma)$ back to $\rho(\sigma)$ at some time $t > 0$, without exceeding the operating error bound ℓ along the way. Whether this is possible or not forms the basis of the following controllability notion (see Choi & Hammer, 2018a, 2018c for related notions).

Definition 2.8: Let $K,\ell,\sigma > 0$ be real numbers, where $\sigma \le \ell$. A system $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ is $(K,\ell,\sigma)-$*controllable* if there are a finite time $t' > 0$ and a real number $\sigma' < \sigma$ for which the following is true: for every initial state $x_0 \in \rho(\sigma)$, there is an input signal $u_{x_0} \in U(K)$ satisfying $\Sigma(x_0,u_{x_0},t') \in \rho(\sigma')$ and $\Sigma(x_0,u_{x_0},t) \in \rho(\ell)$ for all $t \in [0,t']$. The family of systems $\mathcal{F}_\gamma(\Sigma_0)$ is (K,ℓ,σ)-controllable if every member $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ is (K,ℓ,σ)-controllable with the same σ'.

Note that (K,ℓ,σ)-controllability includes a contractive feature: the ball $\rho(\sigma)$ is taken into the smaller ball $\rho(\sigma')$. This contractive feature will help us handle the uncertainty about the model of the controlled system Σ of Figure 1.

When the contractive feature is removed from Definition 2.8 by letting $\sigma' = \sigma$, the definition reduces to the following: there is a time $t' > 0$ such that, for every initial state $x_0 \in \rho(\sigma)$, there is an input signal that guides Σ from x_0 to reach $\rho(\sigma)$ at the time t', without violating the operating error bound ℓ along the way. By Conclusion 2.2, this requirement is a necessary condition for periodic sampling. Thus, (K,ℓ,σ)-controllability is very close to being a necessary condition for periodic sampling in the framework of Conclusion 2.2.

A slight reflection shows that, for uncertainty parameters $\gamma' \le \gamma$, one has $\mathcal{F}_{\gamma'}(\Sigma_0) \subseteq \mathcal{F}_\gamma(\Sigma_0)$. This implies the following.

Proposition 2.9: *Let $\gamma',\gamma > 0$ be two uncertainty parameters, where $\gamma' \le \gamma$. If the family of systems $\mathcal{F}_\gamma(\Sigma_0)$ is (K,ℓ,σ)-controllable, then so is the family of systems $\mathcal{F}_{\gamma'}(\Sigma_0)$.*

To continue, we need to establish two facts about members of the family of systems $\mathcal{F}_\gamma(\Sigma_0)$: they have no finite escape times, and their responses are continuous, as follows.

Lemma 2.10: *Let Σ be a system, let $\sigma > 0$ be a real number, and let x_0 be an initial state.*

(i) For every time $T \ge 0$, there is a real number $M(T) \ge 0$ such that $|\Sigma(x_0,u,t)| \le M(T)$ at all times $t \in [0,T]$, for all members $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$, for all initial states $x_0 \in \rho(\sigma)$, and for all input signals $u \in U(K)$.
(ii) For every real number $\varepsilon > 0$, there is a time $t_\varepsilon > 0$ such that $|\Sigma(x_0,u,t) - x_0| < \varepsilon$ at all times $t \in [0,t_\varepsilon]$, for all members $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$, for all initial states $x_0 \in \rho(\sigma)$, and for all input signals $u \in U(K)$.

Proof: (i) The proof of a similar statement appears in Yu and Hammer (2016a).

(ii) Applying (3)–(5), we can write

$$\sup_{s\in[0,t]} |x(s) - x_0| = \left|\int_0^t [a(x(s)) + b(x(s))u(s)]\ ds\right|$$
$$\leq \alpha^+ \left(\sup_{s\in[0,t]} |x(s)|\right) t(1+K) + (\gamma + \alpha^+ K)t.$$

Now, select a time $T > 0$, and let $t \leq T$. Then, using the bound M of part (i) of the lemma, we get

$$\sup_{t\in[0,\mu]} |x(t) - x_0| \leq [\alpha^+ M(1+K) + (\gamma + \alpha^+ K)]t.$$

Thus, for $t < \varepsilon/[\alpha^+ M(1+K) + (\gamma + \alpha^+ K)]$ we get $\sup_{s\in[0,t]} |x(s) - x_0| < \varepsilon$, and our proof concludes. ■

Needless to say, having to verify (K, ℓ, σ)-controllability individually for every member of the family $\mathcal{F}_\gamma(\Sigma_0)$ would be a tedious task. The next statement shows that, when the uncertainty parameter γ is not excessively large, it is sufficient to verify (K, ℓ, σ)-controllability of the nominal system Σ_0; this would assure (K, ℓ, σ)-controllable of all members of $\mathcal{F}_\gamma(\Sigma_0)$, as follows.

Proposition 2.11: *Let $K, \ell_0, \sigma_0 > 0$ be real numbers, where $\sigma_0 \leq \ell_0$, and assume that the nominal system Σ_0 is (K, ℓ_0, σ_0)-controllable. Then, for every real number $\ell > \ell_0$, there is an uncertainty parameter $\gamma > 0$ such that the family of systems $\mathcal{F}_\gamma(\Sigma_0)$ is (K, ℓ, σ_0)-controllable.*

The proof of Proposition 2.11 depends on the following auxiliary fact.

Lemma 2.12: *Let Σ_0 be the nominal system, let $T > 0$ be a time, and let $\sigma > 0$ be a real number. Then, for every real number $\varepsilon > 0$, there is an uncertainty parameter $\gamma > 0$ such that $|\Sigma(x_0, u, t) - \Sigma_0(x_0, u, t)| < \varepsilon$ for all times $t \in [0, T]$, for all members $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$, for all input signals $u \in U(K)$, and for all initial states $x_0 \in \rho(\sigma)$.*

Proof: Let $\gamma > 0$ be an uncertainty parameter, let Σ be a member of $\mathcal{F}_\gamma(\Sigma_0)$, and let $x_0 \in \rho(\sigma)$ be an initial state. For a time $t \geq 0$ and an input signal $u \in U(K)$, denote $x(t) := \Sigma_0(x_0, u, t)$, $x'(t) := \Sigma(x_0, u, t)$, and $\xi(t) = x'(t) - x(t)$. As Σ and Σ_0 have the same initial state x_0, we get $\xi(0) = x_0 - x_0 = 0$. Now, fix two times $t_1, t_2 \in [0, T]$, $t_1 < t_2$, and consider a time $t \in [t_1, t_2]$. By (2)–(6), we can write

$$|\xi(t)| = \left|\xi(t_1) + \int_{t_1}^t [a(x'(s)) - a_0(x(s))]\, ds + \int_{t_1}^t [b(x'(s)) - b_0(x(s))]u(s)\, ds\right|$$
$$\leq |\xi(t_1)| + \int_{t_1}^t (\alpha|\xi(s)| + \gamma|x'(s)| + \gamma)\, ds + \int_{t_1}^t (\alpha|\xi(s)| + \gamma|x'(s)| + \gamma)|u(s)|\, ds.$$

In view of Lemma 2.10(i), there is a real number $M > 0$ such that $|x(t)| \leq M$ and $|x'(t)| \leq M$ for all $t \in [0, T]$, for all members $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$, for all input signals $u \in U(K)$, and for all initial states $x_0 \in \rho(\sigma)$. Inserting this bound and reordering terms, we obtain

$$\{1 - (\alpha(K+1))(t - t_1)\} \left(\sup_{t_1 \leq \theta \leq t} |\xi(\theta)|\right) \leq |\xi(t_1)| + \gamma(M+1)(K+1)(t - t_1).$$

Now, let $\mu > 0$ be a real number such that $(\alpha(K+1))\mu \leq 1/2$ and $p := T/\mu$ is an integer. Setting $t = t_1 + \mu$, we get

$$\sup_{t_1 \leq \theta \leq t_1 + \mu} |\xi(\theta)| \leq 2|\xi(t_1)| + 2\gamma\mu(M+1)(K+1). \tag{24}$$

Create the partition $[0, T] = \{[0, \mu], [\mu, 2\mu], \ldots, [(p-1)\mu, p\mu]\}$, and set $t_1 := i\mu$ for an integer $i \in \{0, 1, 2, \ldots, p-1\}$. Then, (24) implies that

$$\sup_{i\mu \leq \theta \leq (i+1)\mu} |\xi(\theta)| \leq 2|\xi(i\mu)| + 2\gamma\mu(M+1)(K+1),$$
$$i = 0, 1, \ldots, p-1, \quad \xi(0) = 0.$$

Invoking a linear iteration over $i = 0, 1, \ldots, p-1$ yields

$$\sup_{0 \leq \theta \leq T} |\xi(\theta)| \leq q_{p-1}\gamma\mu(M+1)(K+1),$$

where q_{p-1} is the integer resulting from the recursion $q_{k+1} = 2(q_k + 1)$, with $q_0 = 0$. Thus, the lemma is valid for any uncertainty parameter $\gamma > 0$ satisfying

$$\gamma < \varepsilon/[q_{p-1}\mu(M+1)(K+1)]. \tag{25}$$

This concludes our proof. ■

We can state now the proof of Proposition 2.11.

Proof of Proposition 2.11: (K, ℓ_0, σ_0)-controllability of the nominal system Σ_0 implies, by Definition 2.8, that there is a radius $\sigma < \sigma_0$ for which the following is true: there is a finite time $t' \geq 0$ such that, for every initial state $x_0 \in \rho(\sigma_0)$, there is an input signal $u_{x_0} \in U(K)$ for which $\Sigma_0(x_0, u_{x_0}, t') \in \rho(\sigma)$ and $\Sigma_0(x_0, u_{x_0}, t) \in \rho(\ell_0)$ for all $t \in [0, t']$.

Now, consider the positive real number

$$\varepsilon := \min\{(\sigma_0 - \sigma)/2, \ell - \ell_0\}. \tag{26}$$

Then, according to Lemma 2.12 (see (25)), there is an uncertainty parameter $\gamma > 0$ such that $|\Sigma(x_0, u_{x_0}, t) - \Sigma_0(x_0, u_{x_0}, t)| < \varepsilon$ for all members $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$, for all times $t \in [0, t']$, and for all initial states $x_0 \in \rho(\sigma_0)$. For this value of γ, set $\sigma' := (\sigma_0 + \sigma)/2$; note that $\sigma' < \sigma_0$ since $\sigma < \sigma_0$. Then, we obtain from (26) that $\Sigma(x_0, u_{x_0}, t) \in \rho(\ell)$ for all $t \in [0, t']$ and $\Sigma(x_0, u_{x_0}, t') \in \rho(\sigma')$. As $\sigma' < \sigma_0$, the proposition is valid for this γ, and our proof concludes. ■

Remark 2.13: Values of the uncertainty parameter γ that are compatible with Proposition 2.11 are described by (26).

When the nominal system Σ_0 is (K, ℓ_0, σ)-controllable and the uncertainty parameter γ is not excessively large, then, according to Proposition 2.11, all members of the family $\mathcal{F}_\gamma(\Sigma_0)$ are (K, ℓ, σ)-controllable for an operating error bound ℓ slightly bigger than ℓ_0. In such case, it follows from the discussion leading to (16) that there is a time $t' > 0$ at which the set $U(x_0, K, \ell, \sigma, \gamma, t')$ is not empty. This implies that the supremal time $T^*(\ell, \gamma)$ of (23) is not zero. We state this fact formally for future reference.

Proposition 2.14: *Let $K, \ell_0, \sigma, \ell > 0$ be real numbers, where $\ell > \ell_0$. Assume that the nominal system Σ_0 is (K, ℓ_0, σ)- controllable, and let $T^*(\ell, \gamma)$ be the supremal sampling period given by* (23). *Then, there is an uncertainty parameter $\gamma' > 0$ for which $T^*(\ell, \gamma) > 0$ for all $0 < \gamma \leq \gamma'$.*

Proposition 2.14 alludes to the fact that the notion of (K, ℓ, σ)-controllability plays a critical role in our discussion and is related to the existence of solutions of Problem 2.6. This point is discussed in the next section.

3. Existence of optimal solutions

In this section, we show that optimal solutions of Problem 2.6(i) exist under rather broad conditions.

3.1 Main statements

The existence of optimal solutions of Problem 2.6(i) is the focus of the following statements, whose proofs are built in this section. The first statement affirms that, for every initial state x_0, there is an optimal input signal that achieves the maximal sampling period possible for a particular sample radius σ.

Theorem 3.1: *Let $\sigma, \ell > 0$, $\sigma \leq \ell$, be real numbers, and let $\gamma > 0$ be an uncertainty parameter for which the family $\mathcal{F}_\gamma(\Sigma_0)$ is (K, ℓ, σ)-controllable. For an initial state x_0, let $T(x_0, \ell, \gamma, \sigma, u)$ and $T(x_0, \ell, \gamma, \sigma)$ be as given by* (20) *and* (21), *respectively. Then, for every $x_0 \in \rho(\sigma)$, there is an optimal input signal $u^*(x_0, \ell, \gamma, \sigma) \in U(K)$ satisfying $T(x_0, \ell, \gamma, \sigma, u^*(x_0, \ell, \gamma, \sigma)) = T(x_0, \ell, \gamma, \sigma)$.*

Theorem 3.1 states that (K, ℓ, σ)-controllability of the family $\mathcal{F}_\gamma(\Sigma_0)$ is a sufficient condition for the existence of an optimal input signal that achieves maximal sampling period. Note that this condition is very close to being a necessary condition for the same. Indeed, in order for such an optimal input signal to exist, it must be possible to take every member $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ from every initial state $x_0 \in \rho(\sigma)$ back to a state in $\rho(\sigma)$ at a time $t > 0$, without breaching the operating error bound ℓ along the way. The requirement of (K, ℓ, σ)-controllability is slightly stronger than that due to its contractive property: every state $x_0 \in \rho(\sigma)$ must be brought into a slightly smaller ball $\rho(\sigma')$ at a time $t > 0$, without breaching the operating error bound ℓ along the way. The contractive property comes to help in the handling of modelling uncertainties.

The next statement shows that there is an optimal sample radius at which the maximal sampling period is achieved.

Theorem 3.2: *Let $\ell > 0$ be the operating error bound, let $\sigma \in [0, \ell]$, and let $\gamma > 0$ be an uncertainty parameter for which the family of systems $\mathcal{F}_\gamma(\Sigma_0)$ is (K, ℓ, σ)-controllable. Then, in the notation of* (22) *and* (23), *there is an optimal sample radius $\sigma^* \in [0, \ell]$ that achieves the maximal sampling period $T^*(\ell, \gamma) = T(\ell, \gamma, \sigma^*)$.*

Remark 3.3: By Proposition 2.11, the family $\mathcal{F}_\gamma(\Sigma_0)$ is (K, ℓ, σ)-controllable if the nominal system Σ_0 is (K, ℓ', σ)-controllable for an operating error bound $\ell' < \ell$ (and the uncertainty parameter γ is not excessively large). Thus, (K, ℓ, σ)-controllability of the family $\mathcal{F}_\gamma(\Sigma_0)$ can be determined by checking just one system – the nominal system Σ_0. Therefore, the process of testing for the existence of a maximal sampling period is relatively simple; see Sections 4 and 5 for more details.

Values of the uncertainty parameter γ that satisfy the requirements of Proposition 2.11 are discussed in Remark 2.13. In view of Proposition 2.14, the supremal time is not zero and thus forms a viable sampling period, as follows.

Corollary 3.4: *If the family of systems $\mathcal{F}_\gamma(\Sigma_0)$ is (K, ℓ, σ)-controllable, then the optimal time $T^*(\ell, \gamma)$ of* (23) *is not zero.*

The proofs of Theorems 3.1 and 3.2 depend on a number of auxiliary results discussed in the next subsection.

3.2 Mathematical considerations

We start with a brief review of a few mathematical notions that are important to our discussion (e.g. Lusternik & Sobolev, 1961; Willard, 2004; Zeidler, 1985).

Definition 3.5: Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$.

(a) A sequence $\{x_i\}_{i=1}^{\infty}$ of members of H *converges weakly* to a member $x \in H$ if $\lim_{i \to \infty} \langle x_i, y \rangle = \langle x, y \rangle$ for every member $y \in H$.
(b) A subset W of H is *weakly compact* if every sequence of members of W has a subsequence that converges weakly to a member of W.

Let S be a subset of H, and let z be a member of S.

(c) A functional $F : S \to R$ is *weakly upper semi-continuous* at z if the following is true when $F(z)$ is bounded: for every sequence $\{z_i\}_{i=1}^{\infty} \subseteq S$ that converges weakly to z, and for every real number $\varepsilon > 0$, there is an integer $N > 0$ such that $F(z_i) - F(z) < \varepsilon$ for all integers $i \geq N$.
(d) A function $G : R^+ \times S \to R^n : (t, s) \mapsto G(t, s)$ is *weakly continuous* at z at a time $t \geq 0$ if, for every sequence $\{z_i\}_{i=1}^{\infty} \subseteq S$ that converges weakly to z and for every real number $\varepsilon > 0$, there is an integer $N > 0$ such that $|G(z_i, t) - G(z, t)| < \varepsilon$ for all integers $i \geq N$.

The function G is *uniformly weakly continuous* over a time interval $[t_1, t_2]$, $t_2 > t_1 \geq 0$, if, for every sequence $\{z_i\}_{i=1}^{\infty} \subseteq S$ that converges weakly to z and for every real number $\varepsilon > 0$, there is an integer $N > 0$ such that $|G(z_i, t) - G(z, t)| < \varepsilon$ for all integers $i \geq N$ and all $t \in [t_1, t_2]$.

The following statement is reproduced here from Chakraborty and Hammer (2009).

Lemma 3.6: *The set of input signals $U(K)$ of (7) is weakly compact in the topology of the Hilbert space $L_2^{\omega,m}$.*

The next statement indicates that the response $\Sigma(x_0, u, t)$ is a weakly continuous function of the input signal u (see Yu & Hammer, 2016a for proof).

Lemma 3.7: *For a system $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$, the function $\Sigma(x_0) : U(K) \times R^+ : (u, t) \mapsto \Sigma(x_0, u, t)$ is uniformly weakly continuous over every finite interval of time.*

To continue, we recount a few mathematical facts (e.g. Willard, 2004; Zeidler, 1985).

Theorem 3.8: *(i) A continuous function of a weakly continuous function is weakly continuous.*
(ii) A weakly continuous functional is weakly upper semi-continuous.
(iii) A weakly upper semi-continuous functional of a weakly continuous function forms a weakly upper semi-continuous functional.
(iv) Let S and A be topological spaces and assume that, for every member $a \in A$, there is a weakly upper semi-continuous functional $f_a : S \to R$. If $\inf_{a \in A} f_a(s)$ exists at each point $s \in S$, then the functional $f(s) := \inf_{a \in A} f_a(s)$ is weakly upper semi-continuous on S.

The following lists two forms of the generalised Weierstrass Theorem (e.g. Zeidler, 1985).

Theorem 3.9: *(i) An upper semi-continuous functional attains a maximum in a compact set.*
(ii) A weakly upper semi-continuous functional attains a maximum in a weakly compact set.

Next, we show that the supremal time is an upper semi-continuous functional of the input signal.

Lemma 3.10: *Let $\sigma, \ell > 0$ be real numbers, where $\sigma \leq \ell$, and let $x_0 \in \rho(\sigma)$ be an initial state of a system $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$. Then, the functional $T(x_0, \ell, \sigma, \Sigma, \cdot) : U(K) \to R : u \mapsto T(x_0, \ell, \sigma, \Sigma, u)$ of (18) is weakly upper semi-continuous over $U(K)$.*

Proof: Consider a member $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ with initial state $x_0 \in \rho(\sigma)$. Let $t \geq 0$ be a time and let $\{u_i\}_{i=1}^\infty \subseteq U(K)$ be a sequence of input signals that converges weakly to a member $u \in U(K)$. We have to show that, for every real number $\varepsilon > 0$, there is an integer $N > 0$ such that $T(x_0, \ell, \sigma, \Sigma, u_i) - T(x_0, \ell, \sigma, \Sigma, u) < \varepsilon$ for all integers $i \geq N$. To this end, denote by $x_i(t) := \Sigma(x_0, u_i, t)$ the response of Σ to u_i, $i = 1, 2, \ldots$, and by $x(t) := \Sigma(x_0, u, t)$ the response of Σ to u.

Define the class of functions

$$S := \{z : R^+ \to R^n : z(t) = \Sigma(x_0, v, t) \text{ for some } v \in U(K)\}.$$

Then, define the functional $\Theta : S \to R$ given, for a function $z \in S$, by

$$\Theta(z) = \sup\{t \geq 0 : z^\top(t)z(t) \leq \sigma \text{ and } z^\top(s)z(s) \leq \ell \text{ for all } s \in [0, t]\}. \tag{27}$$

We intend to show that the functional $\Theta(z)$ is upper semi-continuous on S. In view of Lemma 3.7, the sequence $x_1(t)$, $x_2(t)$, … converges to $x(t)$ at each time $t \geq 0$. To show that $\Theta(\cdot)$ is upper semi-continuous on S, we have to show that, for every real number $\varepsilon > 0$, there is an integer $N > 0$ such that $\Theta(x_i) - \Theta(x) < \varepsilon$ for all integers $i \geq N$. The proof can be divided into two cases:

Case 1: There is an integer $N' > 0$ for which $\Theta(x_i) \leq \Theta(x)$ for all integers $i \geq N'$.

Case 2: Case 1 is not valid.

In Case 1, we clearly have $\Theta(x_i) - \Theta(x) \leq 0$ for all $i \geq N'$, so that $\Theta(x_i) - \Theta(x) < \varepsilon$ for every real number $\varepsilon > 0$. Hence, upper semi-continuity holds in this case. This case includes the case where $\Theta(x) = \infty$.

In Case 2, we have $\Theta(x) < \infty$ and there is a subsequence $\{x_{i_k}\}_{k=1}^\infty$ and an integer $N'' > 0$ such that $\Theta(x_{i_k}) > \Theta(x)$ for all $k \geq N''$. Recalling the set of input signals $U_\ell(x_0, K, \ell, \gamma, T)$ of (11), this implies that $u_{i_k} \in U_\ell(x_0, K, \ell, \gamma, T(x_0, \ell, \sigma, \Sigma, u))$ for all $k \geq N''$. Considering that $\{u_{i_k}\}_{k=N''}^\infty$ converges weakly to u in $U(K)$ and that $u \in U_\ell(x_0, K, \ell, \gamma, T(x_0, \ell, \sigma, \Sigma, u))$, it follows that $\{u_{i_k}\}_{k=N''}^\infty$ converges weakly to u in $U_\ell(x_0, K, \ell, \gamma, T(x_0, \ell, \sigma, \Sigma, u))$.

Now, the supremum of (27) implies that one of the following two options must be valid: (a) $x^\top(t)x(t) > \sigma$ for all $t > \Theta(x)$; or (b) there is a time $t_1 > \Theta(x)$ such that $x^\top(t_1)x(t_1) > \ell$ and $x^\top(t)x(t) > \sigma$ for all $t \in (\Theta(x), t_1]$. In either case, the following must hold: for every real number $\varepsilon > 0$, there is a time $t' \in (\Theta(x), \Theta(x) + \varepsilon)$ such that

$$x^\top(t')x(t') > \sigma. \tag{28}$$

Further, according to Lemma 3.7, the sequence $x_{i_k}(t')$, $k = 1, 2, \ldots$, converges to $x(t')$; consequently, $x_{i_k}^\top(t')x_{i_k}(t')$ converges to $x^\top(t')x(t')$ as well. Thus, for every real number $\varepsilon_1 > 0$, there is an integer $N_1 > 0$ such that

$$|x_{i_k}^\top(t')x_{i_k}(t') - x^\top(t')x(t')| < \varepsilon_1 \tag{29}$$

for all $k \geq N_1$. In view of (28), we can choose $\varepsilon_1 := [x^\top(t')x(t') - \sigma]/2$. Substituting this into (29) yields

$$|x_{i_k}^\top(t')x_{i_k}(t') - x^\top(t')x(t')| < [x^\top(t')x(t') - \sigma]/2$$

for all $k \geq N_1$. The last inequality leads to the following sequence of inequalities, where (28) is used again in the last step:

$$\begin{aligned} x_{i_k}^\top(t')x_{i_k}(t') &= x^\top(t')x(t') + [x_{i_k}^\top(t')x_{i_k}(t') - x^\top(t')x(t')] \\ &\geq x^\top(t')x(t') - |x_{i_k}^\top(t')x_{i_k}(t') - x^\top(t')x(t')| \\ &> x^\top(t')x(t') - [x^\top(t')x(t') - \sigma]/2 > \sigma \end{aligned}$$

for all $k \geq N_1$; thus, $x_{i_k}^\top(t')x_{i_k}(t') > \sigma$ for all $k \geq N_1$. Combining this with observations (a) and (b) above, we obtain by (27) that

$\Theta(x_{i_k}) < t'$ for all $k \geq N_1$. But then, since $t' \in (\Theta(x), \Theta(x) + \varepsilon)$, we conclude that $\Theta(x_{i_k}) < \Theta(x) + \varepsilon$ for all $k \geq N_1$. Consequently, $\Theta(\cdot)$ is upper semi-continuous on S in Case 2. Together with our earlier discussion of Case 1, it follows that $\Theta(\cdot)$ is an upper semi-continuous functional on S.

Finally, recall that, by Lemma 3.7, the function $\Sigma(x_0, \cdot, t) : U(K) \to R^n$ is weakly continuous over $U(K)$ at all times $t \geq 0$ and for all members $\Sigma \in \mathcal{F}_\gamma(x_0)$. As $z^\top z : R^n \to R$ is a continuous functional of z, it follows by Theorem 3.8(i) that $\Sigma^\top(x_0, \cdot, t)\Sigma(x_0, \cdot, t) : U(K) \to R$ is a weakly continuous functional over $U(K)$ at all times $t \geq 0$ and for all members $\Sigma \in \mathcal{F}_\gamma(x_0)$. Combining this with the conclusion of the previous paragraph, it follows by Theorem 3.8(iii) that $\Theta(\Sigma(x_0, \cdot, t)) : U(K) \to R : u \mapsto \Theta(\Sigma(x_0, u, t)))$ is a weakly upper semi-continuous functional on $U(K)$ at all times $t \geq 0$ and for all members $\Sigma \in \mathcal{F}_\gamma(x_0)$. The lemma then follows from the fact that $T(x_0, \ell, \sigma, \Sigma, u) = \Theta(\Sigma(x_0, u, t))$. ■

Combining Lemma 3.10 with Theorem 3.8(iv) and (20) yields

Lemma 3.11: *Let $\sigma, \ell, \gamma > 0$ be real numbers, where $\sigma \leq \ell$, and let $x_0 \in \rho(\sigma)$ be an initial state of the family of systems $\mathcal{F}_\gamma(\Sigma_0)$. Then, the functional $T(x_0, \ell, \gamma, \sigma, \cdot) : U(K) \to R : u \mapsto T(x_0, \ell, \gamma, \sigma, u)$ of (19) is weakly upper semi-continuous on $U(K)$.*

We can prove now the existence of optimal input signals.

Proof of Theorem 3.1: By Lemma 3.11, the functional $T(x_0, \ell, \gamma, \sigma, \cdot) : U(K) \to R$ is weakly upper semi-continuous over $U(K)$ and, by Lemma 3.6, $U(K)$ is weakly compact. Thus, Theorem 3.9(ii) implies that $T(x_0, \ell, \gamma, \sigma, u)$ attains a maximum in $U(K)$. Consequently, there is an input signal $u^*(x_0, \ell, \gamma, \sigma) \in U(K)$ at which this maximum is attained. ■

3.3 The sample radius

Our next objective is to examine features of the sample radius σ, which characterises the domain within which the state of Σ resides at sampling times.

Proposition 3.12: *Let Σ be a member of the family $\mathcal{F}_\gamma(\Sigma_0)$ with an initial state x_0. Let $\ell > 0$ be a real number, and let $u \in U(K)$ be the input signal of Σ. Then, the functional $T(x_0, \ell, \cdot, \Sigma, u) : [0, \ell] \to R : \sigma \mapsto T(x_0, \ell, \sigma, \Sigma, u)$ of (18) has the following features:*

- *(i) $T(x_0, \ell, \sigma, \Sigma, u)$ is a monotone increasing function of the sample radius σ.*
- *(ii) $T(x_0, \ell, \sigma, \Sigma, u)$ is almost everywhere continuous.*
- *(iii) $T(x_0, \ell, \sigma, \Sigma, u)$ is an upper semi-continuous functional of the sample radius σ.*

Proof: Denote by $x(t) = \Sigma(x_0, u, t)$ the response of Σ. By Lemma 2.10(ii), the function $x(t)$ is a continuous function of time. Therefore, so is the function $x^\top(t)x(t)$.

Case 1: Consider first a case where there is an interval of time $[t_1, t_2]$, $0 \leq t_1 < t_2$, on which the function $x^\top(t)x(t)$ is strictly increasing and the following conditions are satisfied: (a) the operating error bound ℓ has not been reached by the time t_2, namely, $x^\top(t)x(t) < \ell$ for all $t \leq t_2$; (b) $x^\top(t)x(t) < x^\top(t_1)x(t_1)$ for all $t < t_1$; and (c) $x^\top(t)x(t) > x^\top(t_2)x(t_2)$ for all $t > t_2$. Set $\sigma_1 := x^\top(t_1)x(t_1)$ and $\sigma_2 := x^\top(t_2)x(t_2)$; then, $\sigma_1 < \sigma_2$. In view of the supremum in (18), the value of $T(x_0, \ell, \sigma, \Sigma, u)$ increases as σ increases. Therefore, the functional $T(x_0, \ell, \sigma, \Sigma, u)$ forms a continuous and monotone increasing functional of σ on the interval (σ_1, σ_2).

Case 2: Assume next that (c) of Case 1 is not valid, and that the function $x^\top(t)x(t)$ reaches a local maximum at the time t_2. Then, $x^\top(t)x(t)$ will decrease (or not increase) for some time after the time t_2, but assume it remains bigger than σ_1. Suppose that $x^\top(t)x(t)$ later resumes its increase and starts to exceed the value σ_2 at a time $t_3 > t_2$. By the supremum in (18), it follows that the value of $T(x_0, \ell, \sigma, \Sigma, u)$ will jump from t_2 to t_3 at $\sigma = \sigma_2$. Note that, due to the supremum in (18), we have $T(x_0, \ell, \sigma_2, \Sigma, u) = t_3$ in this case, so that $T(x_0, \ell, \sigma, \Sigma, u)$ takes the higher value at a jump. If there is no finite time $t > t_2$ at which $x^\top(t)x(t) > \sigma_2$, then $T(x_0, \ell, \sigma, \Sigma, u)$ will jump to ∞ at σ_2. Irrespective of the jump, $T(x_0, \ell, \sigma, \Sigma, u)$ remains a monotone increasing function of σ.

A slight reflection shows that the discontinuity described in the last paragraph is the only type of discontinuity experienced by $T(x_0, \ell, \sigma, \Sigma, u)$ as a function of σ. Thus, $T(x_0, \ell, \sigma, \Sigma, u)$ is a piecewise-continuous monotone increasing function of σ with simple jump discontinuities. As seen before, $T(x_0, \ell, \sigma, \Sigma, u)$ takes the higher value at a jump. This proves Parts (i) and (ii) of the lemma.

Next, since a continuous function is also upper semi-continuous, it follows that $T(x_0, \ell, \sigma, \Sigma, u)$ is upper semi-continuous as a function of σ on intervals over which it is continuous. Furthermore, since $T(x_0, \ell, \sigma, \Sigma, u)$ always takes the higher value at a jump point, it is upper semi-continuous at jump points as well. Thus, $T(x_0, \ell, \sigma, \Sigma, u)$ is an upper semi-continuous functional of the sample radius σ, thus verifying Part (iii) of the lemma. This concludes our proof. ■

In view of Proposition 3.12(iii) and (20), it follows by Theorem 3.8(iv) that the following is true.

Corollary 3.13: *Let $\ell, \gamma > 0$ be real numbers, let x_0 be the initial state of the family of systems $\mathcal{F}_\gamma(\Sigma_0)$, and let $u \in U(K)$ be the input signal. Then, the functional $T(x_0, \ell, \gamma, \cdot, u) : [0, \ell] \to R : \sigma \mapsto T(x_0, \ell, \gamma, \sigma, u)$ of (20) is an upper semi-continuous functional of the sample radius σ.*

Next, we show that the maximal time for a given initial state also is an upper semi-continuous functional of the sample radius.

Proposition 3.14: *Let $\ell, \gamma > 0$ be real numbers, and let x_0 be the initial state of the family of systems $\mathcal{F}_\gamma(\Sigma_0)$. Then, the functional $T(x_0, \ell, \gamma, \cdot) : [0, \ell] \to R : \sigma \mapsto T(x_0, \ell, \gamma, \sigma)$ of (21) is an upper semi-continuous function of the sample radius σ.*

Proof: For given initial state x_0 and sample radius σ, let $u^*(x_0, \sigma)$ be an input signal that yields the maximal time $T(x_0, \ell, \gamma, \sigma) = T(x_0, \ell, \gamma, \sigma_i', u^*(x_0, \sigma))$ described by Theorem

3.1 (proved earlier). Let $\{\sigma_i\}_{i=1}^{\infty}$ be a sequence of sample radii converging to the sample radius σ. Then, consider the difference

$$\begin{aligned}&T(x_0,\ell,\gamma,\sigma_i,u^*(x_0,\sigma_i)) - T(x_0,\ell,\gamma,\sigma,u^*(x_0,\sigma))\\ &\quad= [T(x_0,\ell,\gamma,\sigma_i,u^*(x_0,\sigma_i)) - T(x_0,\ell,\gamma,\sigma,u^*(x_0,\sigma_i))]\\ &\qquad+ [T(x_0,\ell,\gamma,\sigma,u^*(x_0,\sigma_i)) - T(x_0,\ell,\gamma,\sigma,u^*(x_0,\sigma))].\end{aligned} \tag{30}$$

Now, since the function $T(x_0,\ell,\gamma,\sigma,u^*(x_0,\sigma_i))$ is upper semi-continuous by Corollary 3.13, there is, for every $\varepsilon > 0$, an integer $N > 0$ such that $T(x_0,\ell,\gamma,\sigma_i,u^*(x_0,\sigma_i)) - T(x_0,\ell,\gamma,\sigma,u^*(x_0,\sigma_i)) < \varepsilon$ for all $i \geq N$. Also, since $u^*(x_0,\sigma)$ is a maximising input signal for the sample radius σ, it follows that $T(x_0,\ell,\gamma,\sigma,u^*(x_0,\sigma_i)) - T(x_0,\ell,\gamma,\sigma,u^*(x_0,\sigma)) \leq 0$. Substituting the last two inequalities into the last two rows of (30), we obtain that $T(x_0,\ell,\gamma,\sigma_i,u^*(x_0,\sigma_i)) - T(x_0,\ell,\gamma,\sigma,u^*(x_0,\sigma')) < \varepsilon$ for all integers $i \geq N$. Hence, $T(x_0,\ell,\gamma,\sigma,u^*(x_0,\sigma))$ is upper semi-continuous, and our proof concludes. ■

Further, from the combination of Theorem 3.8(iv) and Proposition 3.14, it follows that the functional $T(\ell,\gamma,\sigma)$ of (22) is an upper semi-continuous functional of the sample radius σ, as follows.

Corollary 3.15: *Let $\ell,\gamma > 0$ be real numbers. The functional $T(\ell,\gamma,\cdot) : [0,\ell] \to R : \sigma \mapsto T(\ell,\gamma,\sigma)$ of (22) is an upper semi-continuous functional of the sample radius σ.*

Now, we can state the proof of Theorem 3.2.

Proof of Theorem 3.2: Referring to (23), it follows by Corollary 3.15 and the generalised Weierstrass theorem cited as Theorem 3.9(i), that there is a real number $\sigma^* \leq \ell$ satisfying $T^*(\ell,\gamma) = T(\ell,\gamma,\sigma^*)$. This completes our proof. ■

In fact, an optimal sample radius σ^* of Theorem 3.2 is not zero, when the family $\mathcal{F}_\gamma(\Sigma_0)$ is (K,ℓ,σ)-controllable. Indeed, by Proposition 2.14, the optimal sampling period $T^*(\ell,\gamma)$ of (23) is then strictly greater than zero. But then, the uncertainties (3) present in the model of the controlled system Σ cause a dispersion among the states reached by members of $\mathcal{F}_\gamma(\Sigma_0)$ at the time $T^*(\ell,\gamma) > 0$. In formal terms, we have the following.

Proposition 3.16: *Under the conditions of Proposition 2.14, an optimal sample radius σ^* satisfies $\sigma^* > 0$.*

Proof: Let $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$ be a system with initial state x_0 and input signal $u \in U(K)$. By (2), the response $x(t) = \Sigma(x_0,u,t)$ can be expressed in the form

$$\begin{aligned}x(t) = x_0 + \int_0^t &\{[a_0(x(s)) + a_\gamma(x(s))]\\ &+ [b_0(x(s)) + b_\gamma(x(s))]u(s)\}\,\mathrm{d}s.\end{aligned} \tag{31}$$

Recall that $\gamma > 0$, and choose a constant vector $\zeta \in R^n$ satisfying $|\zeta| \leq \gamma$. Among all possible uncertainties a_γ and b_γ allowed by (5), one option is the case where $a_\gamma(x) = \zeta$ and $b_\gamma(x) = 0$. Substituting these values into (31), and setting the time $t = T^*(\ell,\gamma)$, we get

$$x(T^*(\ell,\gamma)) = x_0 + \int_0^{T^*(\ell,\gamma)} \{a_0(x(s)) + \zeta + b_0(x(s))u(s)\}\,\mathrm{d}s,$$

which is a function of the entries of ζ. Differentiating with respect to the entries of ζ, we get

$$\frac{\partial x_i(T^*(\ell,\gamma))}{\partial \zeta_j} = T^*(\ell,\gamma) > 0, \quad j = 1,2,\ldots,n,$$

where the last inequality is taken from Proposition 2.14. Consequently, $x(T^*(\ell,\gamma)) \neq 0$ at least for some permissible model uncertainties. As all potential values of $x(T^*(\ell,\gamma))$ must be included in the ball $\rho(\sigma^*)$, we conclude that $\sigma^* > 0$. ■

Our discussion so far indicates the existence of robust optimal controllers that facilitate the use of the maximal sampling period in a periodic sampling application. Larger sampling periods carry many benefits; they offer time for more sophisticated control algorithms, thus improving performance and reducing costs. Yet, the implementation of optimal controllers is, more often than not, and unwieldy task. In the next section, we show that optimal performance can be approximated by controllers that are relatively easy to derive and implement – controllers that generate bang-bang input signals for the controlled system Σ of Figure 1.

4. Approximating an optimal response

An optimal input signal $u^*(x_0,\ell,\gamma,\sigma)$ of Theorem 3.1, being a general Lebesgue measurable vector-valued function of time, may be difficult to calculate and implement. In this section, we show that the optimal response elicited by an optimal input signal $u^*(x_0,\ell,\gamma,\sigma)$ can be approximated as closely as desired by a bang-bang input signal. Bang-bang signals are relatively easy to calculate and implement, since they are piecewise-constant signals, whose values switch between the input signal bounds K and $-K$. Bang-bang signals are determined by a string of scalars – their switching times. To clarify what we mean by a bang-bang signal, we provide a formal definition.

Definition 4.1: A bang-bang signal $u^\pm \in U(K)$ is a piecewise-constant signal, whose components switch between the values of $+K$ and $-K$ a finite number of times in every finite interval of time.

The next statement shows that the sampling period achieved by a bang-bang input signal is not shorter than the maximal sampling period, provided that the operating error bound is slightly increased from ℓ to ℓ'; the uncertainty parameter γ may have to be decreased somewhat, namely, modelling accuracy of the controlled system may have to be improved. In precise terms, the following is true.

Theorem 4.2: *Let $\ell > 0$ be an operating error bound, and let σ^* be an optimal sample radius of Theorem 3.2. Assume that the nominal system Σ_0 is (K,ℓ,σ^*)-controllable and that the* *maximal sampling period $T^*(\ell,\gamma)$ is finite. Then, for every operating error bound $\ell' > \ell$, there is an uncertainty parameter $\gamma > 0$ for which the following is true. For each initial state $x_0 \in \rho(\sigma^*)$, there is a bang-bang input signal $u^{\pm}_{x_0} \in U(K)$ satisfying $T(x_0,\ell',\gamma,\sigma^*,u^{\pm}_{x_0}) \geq T^*(\ell,\gamma)$.*

The proof of Theorem 4.2 relies on the following statement, reproduced here from Choi and Hammer (2018a) (see also Chakraborty & Hammer, 2009, 2010; Choi & Hammer, 2018b; Yu & Hammer, 2016a, 2016b).

Theorem 4.3: *Let Σ be a member of the family of system $\mathcal{F}_\gamma(\Sigma_0)$ with the initial state x_0, let $u \in U(K)$ be an input signal of Σ, and let $t' > 0$ be a finite time. Then, for every real number $\varepsilon > 0$, there is a bang-bang input signal $u^{\pm} \in U(K)$ and an uncertainty parameter $\gamma > 0$ for which the following is true. The difference between the response $x(t) := \Sigma(x_0,u,t)$ of Σ to u and the response $x^{\pm}(t) := \Sigma(x_0,u^{\pm},t)$ of Σ to $u^{\pm}$ satisfies $|x(t) - x^{\pm}(t)| < \varepsilon$ at all times $0 \leq t \leq t'$ and for all members $\Sigma \in \mathcal{F}_\gamma(\Sigma_0)$.*

Remark 4.4: There is a slight difference between the model of the controlled system Σ used in the present paper and the model used in Choi and Hammer (2018a): the reference requires $a_\gamma(0) = 0$, while (5) does not impose this requirement. The proof of the reference can be modified to accommodate the current model.

Remark 4.5: One of the implications of Theorem 4.3 is that (K,ℓ,σ)-controllability of the nominal controlled system Σ_0 can be verified via a numerical search process performed over a family of bang-bang input signals. We discuss this point in more detail in Section 5.

We turn now to the proof of the main result of this section.

Proof of Theorem 4.2: Referring to Theorem 3.2, let σ^* be an optimal sample radius satisfying $T^*(\ell,\gamma) = T(\ell,\gamma,\sigma^*)$, where $\sigma^* > 0$ by Proposition 3.16. According to Theorem 3.1, there is an optimal input signal u^* satisfying $T(x_0,\ell,\gamma,\sigma^*) = T(x_0,\ell,\gamma,\sigma^*,u^*)$. Define $\ell'' := \ell + (\ell' - \ell)/2$; According to Proposition 2.11, there is an uncertainty parameter $\gamma' > 0$ for which the family of systems $\mathcal{F}_{\gamma'}(\Sigma_0)$ is (K,ℓ'',σ^*)-controllable.

Further, let $\varepsilon > 0$ be a real number satisfying $\varepsilon < (\ell' - \ell'')$. For a member $\Sigma \in \mathcal{F}_{\gamma'}(\Sigma_0)$, denote $x^*(t) = \Sigma(x_0,u^*,t)$. According to Theorem 4.3, there is an uncertainty parameter $\gamma_\varepsilon > 0$ and a bang-bang input signal $u^{\pm} \in U(K)$ for which the response $x^{\pm}(t) := \Sigma(x_0,u^{\pm},t)$ satisfies $|x^*(t) - x^{\pm}(t)| < \varepsilon$ at all times $t \in [0, T^*(\ell,\gamma_\varepsilon)]$ and for all members $\Sigma \in \mathcal{F}_{\gamma_\varepsilon}(\Sigma_0)$.

Now, set $\gamma'' := \min\{\gamma',\gamma_\varepsilon\}$. Then, since $\gamma'' \leq \gamma'$ and the family $\mathcal{F}_{\gamma'}(\Sigma_0)$ is $(K.\ell,\sigma^*)$-controllable, it follows by Proposition 2.9 that the family of systems $\mathcal{F}_{\gamma''}(\Sigma_0)$ is also $(K.\ell,\sigma^*)$-controllable. Therefore, by Definition 2.8, there is a real number $\sigma \in (0,\sigma^*)$ satisfying $\Sigma(x_0,u^*,T^*(\ell,\gamma)) \in \rho(\sigma)$ for all $x_0 \in \rho(\sigma^*)$ and for all $\Sigma \in \mathcal{F}_{\gamma''}(\Sigma_0)$. Finally, choose $\varepsilon > 0$ to satisfy $\varepsilon < \min\{\ell' - \ell'', \sigma^* - \sigma\}$; then, it follows that all systems $\Sigma \in \mathcal{F}_{\gamma''}(\Sigma_0)$ satisfy $\Sigma(x_0,u^{\pm},T^*(\ell_0,\gamma_0)) < \sigma^*$ and $\Sigma(x_0,u^{\pm},t) \in \rho(\ell')$ for all $t \in [0, T^*(\ell_0,\gamma_0)]$. This implies that $T(x_0,\ell',\gamma'',\sigma^*,u^{\pm}) \geq T^*(\ell,\gamma'')$.

We claim that the value of $\gamma'' > 0$ can be selected independently of the initial state $x_0 \in \rho(\sigma^*)$. Indeed, denote by $\gamma(x_0) := \gamma''$ the value of the uncertainty parameter obtained in the previous paragraph. In view of Proposition 2.9, it is enough to show that there is a value $\gamma > 0$ satisfying $\gamma \leq \gamma(x_0)$ for all $x_0 \in \rho(\sigma^*)$. By contradiction, assume there is no such γ. Then, there is a sequence of initial states $x_{0i} \in \rho(\sigma^*)$, $i = 1,2,\ldots$, for which $\lim_{i\to\infty} \gamma(x_{0i}) = 0$. But then, since $\rho(\sigma^*)$ is a compact domain in R^n, there is convergent subsequence $\{x_{0i_k}\}_{k=1}^{\infty}$ and a state $x'_0 \in \rho(\sigma^*)$ satisfying $x'_0 = \lim_{k\to\infty} x_{0i_k}$. At the state x'_0, we must then have $\gamma(x'_0) = 0$, contradicting the results of the previous paragraph. Thus, the uncertainty parameter γ can be selected independently of $x_0 \in \rho(\sigma^*)$, and our proof concludes. ∎

5. Example

Consider a single-link manipulator described by the equation (Kim, Kuc, Kim, & Lee, 2017)

$$\Sigma : \begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \frac{0.5m_0 + M_0}{J} gl \sin x_1 + \frac{1}{J} u. \end{aligned} \tag{32}$$

Nominal values are: tip load $m_0 = 2\,\text{kg}$; length of the link $l = 0.5\,\text{m}$; mass of the link $M_0 = 4\,\text{kg}$; and gravitational acceleration $g = 9.8\,\text{m/s}^2$. The input bound is $K = 20$, and the operating error bound is $\ell = 3$. There is an uncertainty of 5% about the values of m_0 and M_0, i.e.

$$1.9 \leq m_0 \leq 2.1 \quad \text{and} \quad 3.8 \leq M_0 \leq 4.2 \tag{33}$$

5.1 Estimating the optimal sample radius

The optimal sample radius can be estimated through a numerical search process. In this example, we employed a relatively simple numerical search algorithm to find a bang-bang input signal that approximates optimal response. To this end, select a set of potential sample radii $\{\sigma_1,\sigma_2,\ldots,\sigma_p\}$ in the interval $[0,\ell] = [0,3]$. For instance, one could use

$$\{\sigma_1 = 0.5, \sigma_2 = 1.0, \sigma_3 = 1.5, \sigma_4 = 2.0, \sigma_5 = 2.5, \sigma_6 = 3.0\},$$

where $p = 6$.

Within each one of the domains $\rho(\sigma_i)$, $i = 1,2,\ldots,p$, select a family of q states $\{x_{0_{ik}}\} \subseteq \rho(\sigma_i)$, $k = 1,2,\ldots,q$, $i = 1,2,\ldots,p$, to serve as initial states for testing the response. Finally, select r representative samples $\Sigma_1,\Sigma_2,\ldots,\Sigma_r$ of the family of systems induced by (33). For instance, we selected $r = 3$ members with parameter values

$$\Sigma_1 : m_0 = 1.9, M_0 = 3.8;$$
$$\Sigma_2 : m_0 = 2, M_0 = 4;$$
$$\Sigma_3 : m_0 = 2.1, M_0 = 4.2.$$

The above selections lead to a search process over pqr cases: p sample radii, q initial states, and r system models. From Theorem 4.2, we know that optimal performance can be approximated as closely as desired by bang-bang input signals. In line with this statement, we perform, for each combination of initial state and sample radius, a search over bang-bang input signals to achieve the largest sampling period possible for all system models. The process of searching for such an input signal is described in more detail in Choi and Hammer (2018b).

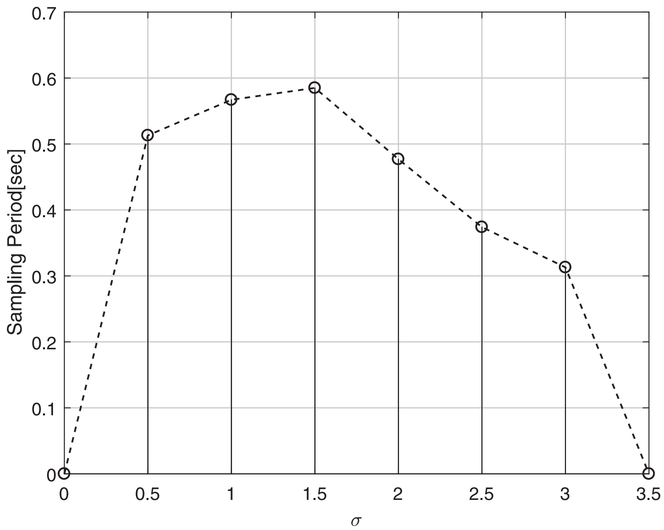


Figure 2. Estimated period for each sample radius σ.

The results of this numerical search process are summarised in Figure 2, which shows the maximal sampling period as a function of the sample radius σ. As can be seen from the figure, the longest sampling period is obtained at the sampling radius

$$\sigma^* = 1.5; \tag{34}$$

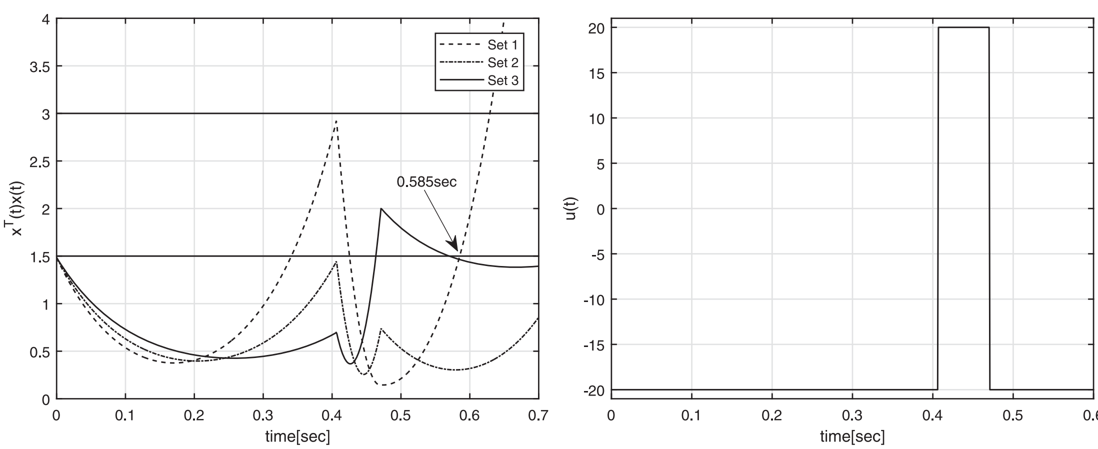
the corresponding estimated maximal sampling period T^* is approximately

$$T^* \approx 0.585 \text{ seconds}, \tag{35}$$

as see in Figure 3(a). The same numerical search process also shows that the family of systems described by (32) and (33) is $(20, 3, 1.5)$-controllable (see Remark 4.5).

5.2 Comparison to the sample-and-hold approach

The most common methodology used in the control of sampled-data systems is the sample-and-hold technique. In this technique, a constant input signal is applied to the controlled system during each sampling period; this constant value may vary from one sampling period to another. Here, we compare the longest sampling period achievable by the sample-and-hold technique to the sampling period achievable by the optimal approach presented in this paper. The results are shown in Figures 3 and 4; we use the sample radius $\sigma^* = 1.5$ of (34). (Due to



(A) (B)

Figure 3. Optimal control approach. (a) State trajectories: optimisation approach and (b) Control input signal: optimisation approach.

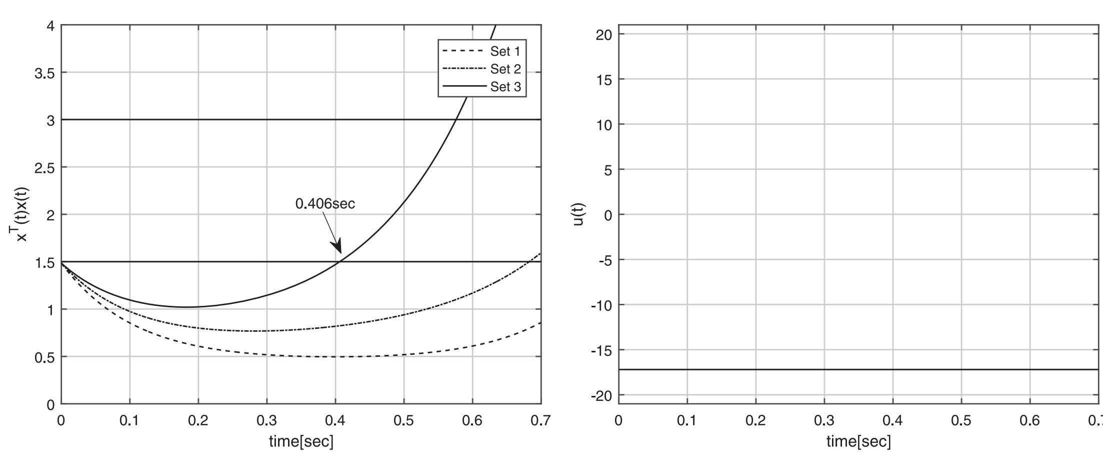


Figure 4. Sample-and-hold approach. (a) State trajectories for sample-and-hold and (b) Control input for sample-and-hold.

space limitations, the results are shown only for the initial state $x_0 = [\pi/6, 1.1]^T$.)

For the sample-and-hold technique, the maximal sampling period is 0.406 seconds under the current input amplitude constraints and model uncertainty (see Figure 4(a)). On the other hand, the sampling period achieved by the optimal approach presented in the current paper is 0.585 seconds, as shown in (35). Thus, the current approach offers an improvement of 44% over the best possible outcome of the sample-and-hold technique. Note that this improvement is obtained with the relatively simple input signal of Figure 3(b). Of course, other examples may yield larger or smaller improvements of the sampling period. In any case, by virtue of its optimality, the approach of this paper always yields the longest possible sampling period.

6. Conclusion

In this paper, we presented a methodology for achieving the longest possible sampling period for sampled-data control systems. We have shown that a maximal sampling period is achievable for a broad family of nonlinear systems. We have also shown that a sampling period that is as close as desired to the maximal one can be achieved by controllers that are relatively easy to design and implement – controllers that generate bang-bang signals as input for the controlled system. Achieving a longer sampling period is a desirable objective in many applications, since it allows more time for the processing of control data between samples. This facilitates more sophisticated control algorithms and reduces data load in feedback communication channels.

Future research efforts may focus on extending the results of the current paper to families of nonlinear systems that are broader than the family of nonlinear input-affine systems considered here. The theory of fraction representations of nonlinear systems may be instrumental in these efforts.

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

The work of Ho-Lim Choi was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIP) [No. NRF-2016R1A2B4008465].

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