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Optimal robust control of nonlinear time-delay systems: Maintaining low operating errors during feedback outages

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ABSTRACT
The problem of maintaining low operating errors during feedback outages is considered for a class of nonlinear systems with time-delays in the input channel. It is shown that there are optimal controllers that keep operating errors below a specified bound for the longest time possible. Furthermore, it is shown that optimal performance can be approximated as closely as desired by using bang-bang controllers – controllers that are relatively easy to calculate and implement.

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1. Introduction

Time delays are a common presence in control systems engineering. Indeed, time delays caused by transportation or communication lags (e.g. Bushnell, 2001; Sheridan & Ferrell, 1963), time delays caused by real-time computing delays (e.g. Ailon & Gil, 2000), or time delays caused by telemetry or tele-operation lags (e.g. Imaida, Yokokohji, Doi, Oda, & Yoshikawa, 2004) are just a few examples of unavoidable time delays encountered in control engineering practice. Time delays may, of course, appear in conjunction with other adverse events, such as feedback outages. Feedback outages may arise not only from system malfunctions or component failures, but also from deliberate design or implementation policies. For example, networked control systems receive feedback signals only intermittently, so as to reduce network load (see Montestruque & Antsaklis, 2004; Nair, Fagnani, Zampieri, & Evans, 2007;Zhivogyladov & Middleton, 2003, the references cited in these papers, and others). Feedback outages may also occur as a result of unavoidable deteriorations in operating conditions, such as the loss of line-of-sight to a missile target, or the loss of line-of-sight to a satellite as it encircles a celestial body.

The present paper addresses situations in which control engineers are faced with a combination of feedback outages, time delays, and nonlinearities. The paper concentrates on a certain class of nonlinear systems with time delays in their input channel. For such systems, the paper develops a methodology for the design of controllers that keep operating errors below a specified bound for the longest time possible during periods of feedback outages.

The configuration we consider is described in Figure 1, where the controlled system Σ consists a dynamical system Σ^- – the *core system*, coupled with a time delay of $\tau > 0$ in the input channel. The input signal of Σ at a time t is denoted by $u(t)$ and is generated by the controller C. As depicted in the figure, C experiences a feedback outage starting at the time $t = 0$. It goes without saying that a feedback outage may cause an increase in operating errors. Our objective is to design the controller C so that the input signal $u(t)$ it generates for Σ keeps the magnitude of operating errors below a specified bound $\ell > 0$ for the longest time possible. Such a controller provides maximal time for restoring feedback, before operating error bounds are violated.

The first question that comes to mind in this context is whether there exists an optimal input signal that keeps operating errors below the specified bound for the longest time possible. Along with this question, one must also address implementation issues. As the input signal $u(t)$ is generally a vector valued function of time, optimal input signals may be exceedingly difficult to calculate and implement. An important objective of this paper is to derive input signals that are relatively easy to calculate and implement while, concomitantly, inducing performance that approximates the performance achieved by optimal input signals. For future reference, our objectives can be summarised as follows.

Problem 1.1. Referring to Figure 1,

(i) Determine whether there exists a controller C that keeps the magnitude of operating errors below a specified bound $\ell > 0$ for the longest time possible.

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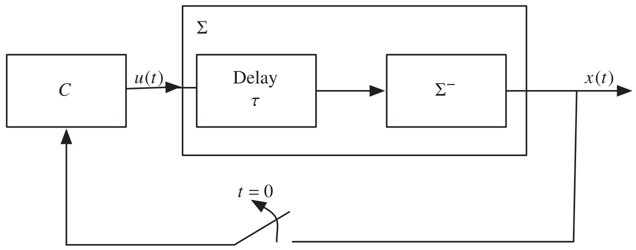

Figure 1. General configuration.

(ii) Devise a simple-to-implement controller that closely approximates optimal performance.

1.1 Preliminaries

As usual, let R be the set of real numbers, let R^+ be the set of non-negative real numbers, and, for integers n, $m > 0$, let R^n be the set of all column vectors with n real components, and let $R^{n\times m}$ be the set of all $n\times m$ matrices with real entries. Then, at every time t, the input signal $u(t)$ of the system Σ of Figure 1 is a member of R^m, while the output signal $x(t)$ of Σ is a member of R^n.

After possibly inducing a shift of the state-space coordinates, we assume that the design objective is to keep the state $x(t)$ of Σ in the vicinity of the zero state $x = 0$; deviations from the zero state are then considered operating errors. To express deviations from the zero state, it is convenient to use the inner product x^Tx, where ^{T} denotes the transpose. A maximal deviation of magnitude $\ell > 0$ from the zero state is permitted, where ℓ is a specified real number. Our objective is to design a controller C that creates an input signal $u(t)$ for Σ that keeps the inequality

$$x^T(t)x(t) \le \ell \tag{1.1}$$

valid for the longest time possible. We refer to ℓ as the *error bound.*

To display the dependence of the state $x(t)$ of Σ on the initial state $x(0) = x_0$ and on the input signal u, we use the notation

$$x(t) = \Sigma(x_0, u, t).$$

According to Figure 1, the system Σ was under feedback control up to the time $t = 0$. As a result, the initial state x_0 is known – it is the last data point provided to the controller C by the feedback channel. Naturally, we assume that the specified error bound ℓ has not been violated while feedback was available, namely, that

$$x_0^Tx_0 \le \ell. \tag{1.2}$$

In these terms, we can rewrite (1.1) in the form

$$\Sigma^T(x_0, u, t)\Sigma(x_0, u, t) \le \ell; \tag{1.3}$$

then, our objective is to design a controller C that generates an input signal u for Σ that maintains the inequality (1.3) valid for the longest time possible.

Most practical systems impose a bound on the largest input signal amplitude they can tolerate. The magnitude of such a bound is usually determined by physical limitations of the system's components. To discuss input signal bounds, we use the L^∞ —norm with the following notation. Denote by $|r|$ the absolute value of a real number $r \in R$; for a matrix $A \in R^{n\times m}$ with entries A_{ij}, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$, denote by

$$|A| := \max_{\substack{i=1,2,\dots,n\\ j=1,2.,,,,m}} |A_{ij}|$$

the L^∞ —norm of the matrix A. For a function v: $R^+ \to R^{n\times m}$: $t \mapsto v(t)$, the L^∞ —norm is denoted by

$$|v|_\infty := \sup_{t\ge 0} |v(t)|.$$

Note the difference between $|v(t)|$ – the largest absolute value of an entry of $v(t)$ at a time t, and $|v|_\infty$ – the L^∞ —norm of the function v. We often refer to $|v|_\infty$ as the *amplitude* of v.

In this notation, the controlled system Σ of Figure 1 imposes an input amplitude bound of $K > 0$, namely, only input signals u: $R^+ \to R^m$: $t \mapsto u(t)$ satisfying

$$|u|_\infty \le K$$

may be used for Σ.

Another important consideration from a practical perspective is the presence of inaccuracies, uncertainties, and errors in the information available about the controlled system Σ. For our discussion to be of practical significance, we must account for inaccuracies, uncertainties, and errors inherent in the model of Σ. To this end, let Σ_0 be the nominal model of the system Σ, and denote by $\Phi(\Sigma_0)$ the family of all systems whose parameters differ from those of Σ_0 by no more than a specified uncertainty of, say, $\gamma > 0$. All members of $\Phi(\Sigma_0)$ have m dimensional input signals and n dimensional states; all are subject to the same input time-delay of $\tau > 0$; all start from the same specified initial state x_0; and all impose the same input amplitude bound K.

To state our design objective formally, let $x(t) = \Sigma(x_0, u, t)$ be the state at the time t of a member $\Sigma \in \Phi(\Sigma_0)$ that started from the initial state x_0 and is being driven by an input signal $u(t)$, assuming that (1.2) is valid. Then, the longest time $t^*(x_0, \Sigma, u)$ during which Σ satisfies the requirement (1.1) is

$$t^*(x_0, \Sigma, u) = \inf\left\{t \geq 0 : x^T(t)x(t) > \ell\right\},$$

where $t^*(x_0, \Sigma, u) = \infty$ if the infimum does not exist.

Next, the longest time $t^*(x_0, u)$ during which *every* member of $\Phi(\Sigma_0)$ complies with the requirement (1.1), when started from the initial state x_0 and driven by the input signal u, is

$$t^*(x_0, u) = \inf_{\Sigma\in\Phi(\Sigma_0)} t^*(x_0, \Sigma, u), \tag{1.4}$$

where, again, $t^*(x_0, u) = \infty$ if the infimum does not exist.

Finally, the longest time $t^*(x_0)$ during which every member of $\Phi(\Sigma_0)$ can comply with the requirement (1.1), when started from the initial state x_0 and driven by an input signal of amplitude not exceeding K, is

$$t^*(x_0) = \sup_{|u|_\infty\leq K} t^*(x_0, u), \tag{1.5}$$

where, once more, $t^*(x_0) = \infty$ if the supremum does not exist. In these terms, Problem 1.1 can be restated more formally, as follows.

Problem 1.2. In the notation of (1.4) and (1.5), explore the following:

(i) Under what conditions is there an optimal input signal $u^*(x_0)$ of amplitude not exceeding K that achieves the maximal time $t^*(x_0) = t^*(x_0, u^*(x_0))$.
(ii) If such an optimal input signal $u^*(x_0)$ exists, find a simple-to-calculate-and-implement input signal $u^{\pm}(t)$ that closely approximates the performance induced by $u^*(x_0)$.

The design of automatic controllers which, during feedback outage, keep operating errors below a specified bound for the longest time possible was initiated in Chakraborty and Hammer (2007, 2008a, 2008b, 2008c, 2009a, 2009b, 2010) for the case where the controlled system Σ of Figure 1 is a linear system with no time delay. In the present paper, we extend these results to cases where the controlled system Σ is a nonlinear system with time delay in its input channel. For such a system, we show in Section 4 that an optimal input signal $u^*(x_0)$ does exist; and, in Section 5, we show that optimal performance can be approximated as closely as desired by using a bang-bang input signal – a piecewise constant signal whose components switch between the extremal values of $+K$ and $-K$. (We should mention that, in some cases, a bang-bang signal can itself be an optimal solution, as discussed in Chakraborty and Hammer (2009b, 2010); still, even in such cases, the approximation process may yield a simpler bang-bang signal). The fact that optimal performance can be approximated by controllers that generate bang-bang input signals is an important conclusion, since bang-bang input signals are relatively easy to calculate and implement; they are basically determined by a string of scalars – their string of switching times.

Our discussion in this paper depends on classical optimisation theory (Chakraborty & Hammer, 2009b;Chakraborty & Shaikshavali, 2009; Gamkrelidze, 1965; Kelendzheridze, 1961; Luenberger, 1969; Neustadt, 1966, 1967; Pontryagin, Boltyansky, Gamkrelidze, & Mishchenko, 1962;Warga, 1972; Young, 1969, the references cited in these works, and many others). A discussion of recent advances in the theory of systems with delays can be found in Niculescu and Gu (2012) and the references cited therein. Yet, to the best of our knowledge, there are no earlier reports in the literature that specifically address the existence, implementation, or approximation of solutions of Problem 1.2 for nonlinear systems with time delay in their input channel.

The paper is organised as follows. Section 2 introduces the mathematical background and notation employed in the paper. Section 3 develops auxiliary results that are used later in Section 4 to prove the existence of optimal solutions of Problem 1.2. Further, in Section 5 we show that optimal performance can be approximated as closely as desired by using easy to calculate and implement bang-bang input signals. The paper concludes in Section 6 with two detailed examples that demonstrate the capability of bang-bang input signals to achieve close to optimal performance.

2. The formal framework

2.1 *The class of input signals*

We adopt the basic mathematical framework of Chakraborty and Hammer (2009b, 2010).

Definition 2.1. For an integer $m > 0$, denote by L^m the linear space of all Lebesgue measurable functions $f: R \to R^m$ that are zero for negative arguments. Then, given a real number $\sigma > 0$, let $L_2^{\sigma,m}$ be the inner product space formed by members $f, g \in L^m$ with the inner product

$$\langle f, g\rangle := \int_0^\infty e^{-\sigma t} f^T(t)g(t)dt. \quad \square \tag{2.1}$$

Note that the inner product (2.1) is well defined and bounded for all bounded members of L^m. The space $L_2^{\sigma,m}$ forms the realm from which our input signals are taken. Recalling that $K > 0$ is the largest input signal amplitude permitted by the controlled system Σ of Figure 1, we denote by $U(K)$ the set of all input signals of Σ, where

$$U(K) := \left\{u \in L_2^{\sigma,m} : |u|_\infty \le K\right\}. \tag{2.2}$$

It will be convenient to use the following shorthand notation for 'inner products' with matrices. Let $D(t)$ be an $n\times m$ matrix function of time with the rows $D_1(t)$, $D_2(t), \dots, D_n(t)$, and assume that $D_j^T(t) \in L_2^{\alpha,m}$, $j = 1, 2, \dots, n$. For a function $g \in L_2^{\alpha,m}$, set

$$\left\langle D(t), g\right\rangle := \sum_{j=1}^{n} \left\langle D_j^T(t), g\right\rangle. \tag{2.3}$$

The controlled system Σ of Figure 1 is a nonlinear, time-varying, and input-affine system with a time delay of $\tau > 0$ in its input channel. Thus, an input signal $u(t)$ that commences at the time $t = 0$ starts affecting Σ at the time $t = \tau$. We refer to such an input signal as a *control input signal*, since it is a signal created by the controller to achieve a control objective.

Preceding the time $t = \tau$, namely, during the time interval $[0, \tau]$, the input of the system Σ originates from an input signal $v(t)$, $t \in [-\tau, 0]$, over which no control is provided; we refer to this signal as the *residual input signal* of Σ. For notational convenience, we set the residual input signal to zero outside the domain $[-\tau, 0]$, so that

$$v(t) := 0 \text{ for all } t \notin [-\tau, 0].$$

Just like the control input signal u, the residual input signal v is a Lebesgue measurable function bounded by K

$$|v(t)| \le K, -\tau \le t \le 0. \tag{2.4}$$

To simplify our notation, we use the symbol K^m to denote the set of all m −dimensional real vectors with components of absolute value not exceeding K. Then, the residual input signal is a Lebesgue measurable function

$$v : [-\tau, 0] \to K^m.$$

In these terms, the scenario of Figure 1 can be described as follows: a feedback outage starts at the time $t = 0$, so the last data provided by the feedback channel forms the initial state $x(0) = x_0$ of the controlled system Σ. Due to the time-delay that affects the input channel, action taken to address the feedback outage does not start to have an effect until the time $t = \tau$; during the time interval $[0, \tau)$, the response of Σ is determined by the residual input signal v – an input signal that remains from times before the feedback outage occurred. After the time τ, the response of Σ is controlled by the control input signal u – an input signal specifically designed to steer Σ during feedback outage.

In formal terms, this means that, at a time $t \ge 0$, the state $x(t)$ of Σ is determined by the initial state x_0, by the residual input signal v, and, for $t \ge \tau$, by the control input signal u. To incorporate these facts into our notation, we write

$$\Sigma(x_0, v, u, t) := \begin{cases} \Sigma(x_0, v, t) & t \in [0, \tau), \\ \Sigma(x(\tau), u, t) & t \ge \tau, \end{cases} \tag{2.5}$$
$$x(t) = \Sigma(x_0, v, u, t).$$

Our goal in this paper is to design a control input signal $u \in U(K)$ that achieves the performance objectives of Problem 1.2. We concentrate on the case where the controlled system Σ of Figure 1 is an input affine nonlinear system with a time-delay of $\tau > 0$ in its input channel, described by an equation of the form

$$\Sigma : \begin{array}{l} \dot{x}(t) = \begin{cases} a(t, x(t)) + b(t, x(t))v(t-\tau), & t \in [0, \tau), \\ a(t, x(t)) + b(t, x(t))u(t-\tau), & t \ge \tau, \end{cases} \\ x(0) = x_0, \end{array} \tag{2.6}$$

where a: $R^+ \times R^n \to R^n : (t, x) \mapsto a(t, x)$ and b: $R^+ \times R^n \to R^{n\times m} : (t, x) \mapsto b(t, x)$ are continuous functions. In brief, Σ is a nonlinear, time-varying, input-affine system with a time delay of τ in its input channel. Our goal in this paper is to design a control input signal $u(t)$ in line with the objectives of Problem 1.2.

In this context, it is convenient to define the *combined input signal*

$$w(t) = \begin{cases} v(t) & t \in [-\tau, 0), \\ u(t) & t \ge 0, \end{cases} \tag{2.7}$$

which allows us to rewrite the system equation (2.6) in the shorter form

$$\begin{array}{l} \Sigma : \dot{x}(t) = a(t, x(t)) + b(t, x(t))w(t - \tau), t \ge 0, \\ \quad x(0) = x_0. \end{array} \tag{2.8}$$

A slight reflection shows that the shifted signal $w_\tau := w(t - \tau)$ is defined at all times $t \ge 0$ and satisfies

$$w_\tau \in U(K). \tag{2.9}$$

The initial state $x(0) = x_0$ of the system Σ is assumed to be known, since it is the last data point transmitted by the feedback channel, before feedback outage occurred at $t = 0$. Beyond the time $t = 0$, the system Σ operates in open loop, thus potentially suffering from increased operating errors. Referring to Figure 1, our objective is to design a controller C that maintains these operating errors below a specified bound for the longest time possible. This provides the best opportunity to regain feedback, before operating errors reach un-acceptable levels.

2.2 Modeling uncertainties

Systems encountered in engineering practice are always subject to inaccuracies, errors, and uncertainties in the values of their models' parameters. To take such inaccuracies, errors, and uncertainties into consideration, we decompose the functions $a(t, x)$ and $b(t, x)$ that appear in the differential equation (2.8) into a sum of two terms: a term that describes the nominal model of Σ and a term that describes uncertainties, errors, and inaccuracies that may affect the model. Specifically, we write

$$\begin{aligned} a(t,x) &= a_0(t,x) + a_\gamma(t,x), \\ b(t,x) &= b_0(t,x) + b_\gamma(t,x), \end{aligned} \tag{2.10}$$

where a_0: $R^+ \times R^n \to R^n : (t, x) \mapsto a_0(t, x)$ and b_0: $R^+ \times R^n \to R^{n\times m} : (t, x) \mapsto b_0(t, x)$ describe the nominal model of Σ, while a_γ: $R^+ \times R^n \to R^n$: $(t, x) \mapsto a_\gamma(t, x)$ and b_γ: $R^+ \times R^n \to R^{n\times m}$: $(t, x) \mapsto b_\gamma(t, x)$ are *unspecified* continuous functions representing inaccuracies, errors, and uncertainties. Thus, the nominal model Σ_0 of Σ is

$$\begin{aligned} \Sigma_0 : \dot{x}(t) &= a_0(t, x(t)) + b_0(t, x(t))w(t-\tau), t \geq 0, \\ x(0) &= x_0. \end{aligned} \tag{2.11}$$

We assume that both constituents of $a(t, x)$ are continuous functions satisfying the Lipschitz conditions

$$\begin{aligned} \left|a_0(t,x') - a_0(t,x)\right| &\leq \alpha \left|x' - x\right|, a_0(t,0) = 0, \\ \left|a_\gamma(t,x') - a_\gamma(t,x)\right| &\leq \gamma \left|x' - x\right|, a_\gamma(t,0) = 0, \end{aligned} \tag{2.12}$$

for all $x', x \in R^n$ and all $t \geq 0$, where $\alpha \geq 0$ and $\gamma \geq 0$ are specified real numbers; the coefficient γ, which relates to inaccuracies, errors, and uncertainties, is interpreted as a 'small' number. Similarly, the constituents of $b(t, x)$ are continuous functions satisfying the Lipschitz conditions

$$\begin{aligned} \left|b_0(t,x') - b_0(t,x)\right| &\leq \beta \left|x' - x\right|, |b_0(t,0)| \leq \beta, \\ \left|b_\gamma(t,x') - b_\gamma(t,x)\right| &\leq \gamma \left|x' - x\right|, |b_\gamma(t,0)| \leq \gamma, \end{aligned} \tag{2.13}$$

for all $x', x \in R^n$ and all $t \geq 0$, where $\beta \geq 0$ is a specified real number. Note that the numbers γ and β are used in multiple roles in (2.12) and (2.13); this is just in order to simplify notation and formulas in the forthcoming discussion. Conceptually, the facts derived in the paper remain valid when distinct bounds are used for the different roles in (2.12) and (2.13). The specifics of the error terms a_γ and b_γ are not specified.

In addition to the modelling inaccuracies included in the model of the system Σ, we also consider the information available about the residual input signal v of (2.4) as incomplete. Specifically, it is known only that v is a member of a family $V(v_0, \gamma)$ of Lebesgue measurable functions v: $[-\tau, 0] \to K^m$ characterised by a nominal residual input signal v_0: $[-\tau, 0] \to K^m$ and an uncertainty bound of γ in the form

$$V(v_0, \gamma) := \{v : |v(t) - v_0(t)| \leq \gamma \text{ for all } t \in [-\tau, 0]\}. \tag{2.14}$$

Note that the uncertainty parameter γ of (2.12) and (2.13) is used here as well to simplify notation. It is not known which particular member of $V(v_0, \gamma)$ was active as residual input signal.

We can introduce now the family of systems that underlies our discussion in this paper.

Definition 2.2. Let $\alpha, \beta, \gamma, K, \tau > 0$ be real numbers. The family of systems $\mathcal{F}_\gamma(x_0, \tau)$ consists of all systems with a time delay of τ in their input channel that are described by a differential equation of the form (2.8), where a and b are continuous functions satisfying (2.10), (2.12), and (2.13). All members of $\mathcal{F}_\gamma(x_0, \tau)$ start from the same initial state x_0 and all have received an unspecified residual input signal $v \in V(v_0, \gamma)$, where $V(v_0, \gamma)$ is given by (2.14). All input signals are bounded by $K > 0$.

Note that all members of the family of systems $\mathcal{F}_\gamma(x_0, \tau)$ of Definition 2.2 have responses that are continuous functions of time, since these are responses of differential equations with continuous coefficients and bounded Lebesgue measurable input signals.

Considering that the response of the controlled system Σ during the time interval $[0, \tau]$ is determined by quantities over which we have no control – the initial condition x_0 and the residual input signal v, it is clear that the requirement (1.1) can be met at all times $t \geq 0$ only if

$$\Sigma^T(x_0, v, u, t)\Sigma(x_0, v, u, t) \leq \ell \text{ for all } t \in [0, \tau]. \tag{2.15}$$

Therefore, we assume throughout our discussion that (2.15) is valid.

2.3 Problem formulation

Reformulating our objectives in the current notation, denote by $t(x_0, \Sigma, v, u, \ell)$ the longest time during which the response of a member $\Sigma \in \mathcal{F}_\gamma(x_0, \tau)$ stays below the bound ℓ, assuming that Σ started from the initial state x_0 and received a residual input signal $v \in V(v_0, \gamma)$ and a control input signal $u \in U(K)$. In formal notation,

$$t(x_0, \Sigma, \upsilon, u, \ell) := \inf\left\{t \geq 0 : \Sigma^T(x_0, \upsilon, u, t)\Sigma(x_0, \upsilon, u, t) > \ell\right\}, \quad (2.16)$$

where $t(x_0, \Sigma, v, u, \ell) := \infty$ if the infimum does not exist. Note that by (2.15), we have $t(x_0, \Sigma, v, u, \ell) \geq \tau$.

Further, still applying the control input signal u, the longest time $t(x_0, \gamma, u, \ell)$ during which the response of *every* member of $\mathcal{F}_\gamma(x_0, \tau)$ stays below the bound ℓ for every residual input signal $v \in V(v_0, \gamma)$ is

$$t(x_0, \gamma, u, \ell) = \inf_{\substack{\Sigma \in \mathcal{F}_\gamma(x_0, \tau) \\ \upsilon \in V(\upsilon_0, \gamma)}} t(x_0, \Sigma, \upsilon, u, \ell), \quad (2.17)$$

where, again, $t(x_0, \gamma, u, \ell) := \infty$ if the infimum does not exist.

Finally, recalling that the control input signal u can come only from the class $U(K)$ of (2.2), it follows that the longest time $t(x_0, \gamma, \ell)$ during which the response of every member of $\mathcal{F}_\gamma(x_0, \tau)$ can be kept below the bound ℓ, irrespective of which residual input signal $v \in V(v_0, \gamma)$ it has received, is

$$t(x_0, \gamma, \ell) = \sup_{u \in U(K)} t(x_0, \gamma, u, \ell), \quad (2.18)$$

where, as before, $t(x_0, \gamma, \ell) := \infty$ if the supremum does not exist.

The supremal time $t(x_0, \gamma, \ell)$ is the object of our investigation in this paper. Our goal is to find out whether there is an optimal control input signal that achieves $t(x_0, \gamma, \ell)$; and, if such an optimal signal exists, can it be approximated by input signals that are easy to calculate and implement. In these terms, we can rephrase Problem 1.2 as follows.

Problem 2.1. Let $K, \gamma, \tau, \ell > 0$ be specified real numbers, and consider the family of systems $\mathcal{F}_\gamma(x_0, \tau)$ of Definition 2.2, all of whose members start from the initial state x_0, are subject to an input channel time-delay of τ, receive an unspecified residual input signal $v \in V(v_0, \gamma)$, receive a control input signal $u \in U(K)$, and satisfy (2.15). The state $x(t)$ at the time t of a member $\Sigma \in \mathcal{F}_\gamma(x_0, \tau)$ that received a residual input signal $v \in V(v_0, \gamma)$ and a control input signal $u \in U(K)$ is denoted by $x(t) = \Sigma(x_0, v, u, t)$. Let $t(x_0, \gamma, u, \ell)$ and $t(x_0, \gamma, \ell)$ be given by (2.17) and (2.18), respectively.

(i) Find conditions under which there is an optimal control input signal $u(x_0, \gamma, \ell) \in U(K)$ satisfying $t(x_0, \gamma, \ell) = t(x_0, \gamma, u(x_0, \gamma, \ell), \ell)$.

(ii) Find a simple-to-calculate-and-implement control input signal $u^{\pm}(x_0, \gamma, \ell) \in U(K)$ for which

$$t(x_0, \gamma, \ell) \approx t(x_0, \gamma, u^{\pm}(x_0, \gamma, \ell), \ell), \quad (2.19)$$

namely, a simple-to-calculate-and-implement control input signal that approximates optimal performance.

We show in Section 4 that an optimal control input signal $u(x_0, \gamma, \ell)$ that fulfils requirement *(i)* of Problem 2.1 does exist under rather broad conditions. In Section 5, we show that the performance achieved by an optimal control input signal can be approximated as closely as desired by a bang-bang control input signal – a piecewise constant signal whose components switch between the extremal values of K and $-K$. This is a significant fact, since bang-bang signals, being determined by a finite number of switching times, are relatively easy to calculate and implement.

3. Basic properties

3.1 Magnitude bounds

In this section, we examine several basic features of the family of systems $\mathcal{F}_\gamma(x_0, \tau)$ of Definition 2.2, starting with a statement showing that members of $\mathcal{F}_\gamma(x_0, \tau)$ have no finite escape times.

Proposition 3.1. *In the notation of Problem 2.1, the following is true. For every time $T \geq 0$, there is a real number $M(T) \geq 0$ such that*

$$|\Sigma(x_0, \upsilon, u, t)| \leq M(T)$$

at all times $t \in [0, T]$, for all members $\Sigma \in \mathcal{F}_\gamma(x_0, \tau)$, for all residual input signals $v \in V(v_0, \gamma)$, and for all control input signals $u \in U(K)$.

Proof. Following the notation of (2.5), let $x(t) := \Sigma(x_0, v, u, t)$ be the response at the time t of a member $\Sigma \in \mathcal{F}_\gamma(x_0, \tau)$ that started from the initial state x_0 and received a residual input signal $v \in V(v_0, \gamma)$ and a control input signal $u \in U(K)$. Using the combined input signal $w(t)$ of (2.7), invoking the system equation (2.8) with (2.10), (2.12), and (2.13), and considering a time

$t_1 \in [0, t)$, we can write

$$\begin{aligned} x(t) &= x(t_1) + \int_{t_1}^{t} [a(s, x(s)) + b(s, x(s))w(s-\tau)]ds \\ &= x(t_1) + \int_{t_1}^{t} [a(s, x(s)) - a(s, 0)]ds \\ &\quad + \int_{t_1}^{t} [b(s, x(s)) - b(s, 0)]w(s-\tau)ds \\ &\quad + \int_{t_1}^{t} b(s, 0)w(s-\tau)ds. \end{aligned}$$

From (2.12) and (2.13) combined with the fact that $|w(\theta)| \leq K$ for all θ, we get

$$\begin{aligned} \sup_{t_1 \leq \theta \leq t} |x(\theta)| \leq |x(t_1)| &+ (t - t_1)(\alpha + \gamma) \sup_{t_1 \leq \theta \leq t} |x(\theta)| \\ &+ (t - t_1)(\beta + \gamma)K \sup_{t_1 \leq \theta \leq t} |x(\theta)| \\ &+ (t - t_1)(\beta + \gamma)K, \end{aligned}$$

or

$$\begin{aligned} &(1 - (t - t_1)(\alpha + \gamma + (\beta + \gamma)K)) \sup_{t_1 \leq \theta \leq t} |x(\theta)| \\ &\quad \leq |x(t_1)| + (t - t_1)(\beta + \gamma)K. \end{aligned}$$

Now, choose a number μ satisfying the inequality

$$\mu(\alpha + \gamma + (\beta + \gamma)K) < 1,$$

and set the time $t = t_1 + \mu$. Then, we get

$$\begin{aligned} \sup_{t_1 \leq \theta \leq t_1 + \mu} |x(\theta)| &\leq (|x(t_1)| + \mu(\beta + \gamma)K) / \\ &\quad (1 - \mu(\alpha + \gamma + (\beta + \gamma)K)) < \infty, \end{aligned} \tag{3.1}$$

which shows that $x(t)$ is bounded over the time interval $[t_1, t_1 + \mu]$, if $x(t_1)$ is bounded.

Now, define the quantities

$$\begin{aligned} \eta_1 &:= 1/\left(1 - \mu(\alpha + \gamma + (\beta + \gamma)K)\right), \\ \eta_2 &:= \mu(\beta + \gamma)K/\left(1 - \mu(\alpha + \gamma + (\beta + \gamma)K)\right), \end{aligned}$$

which depend only on specified characteristics of the family $\mathcal{F}_\gamma(x_0, \tau)$ and its input signal amplitude bound. Then, (3.1) becomes

$$\sup_{t_1 \leq \theta \leq t_1 + \mu} |x(\theta)| \leq \eta_1 |x(t_1)| + \eta_2. \tag{3.2}$$

Next, recalling the time T of the proposition's statement, let p be the smallest integer satisfying $p \geq T/\mu$, and consider the interval $[0, p\mu]$. Partitioning this interval into segments of length μ yields the partition

$$[0, T] \subseteq [0, p\mu] = \left\{[0, \mu], [\mu, 2\mu], \ldots, [(p-1)\mu, p\mu]\right\}.$$

Applying (3.2) to interval i of this partition, we get

$$\sup_{(i-1)\mu \leq \theta \leq i\mu} |x(\theta)| \leq \eta_1 |x((i-1)\mu)| + \eta_2, \tag{3.3}$$

$i = 1, 2, \ldots, p$. Specifically, for $i = 1$, we get $\sup_{0 \leq \theta \leq \mu} |x(\theta)| \leq \eta_1 |x_0| + \eta_2$. Substituting this into the case $i = 2$, yields the inequality $\sup_{\mu \leq \theta \leq 2\mu} |x(\theta)| \leq \eta_1 (\eta_1 |x_0| + \eta_2) + \eta_2$, showing that $\sup_{0 \leq \theta \leq 2\mu} |x(\theta)| < \infty$. Continuing in this manner with the recursive inequality (3.3), we conclude that $|x(t)|$ is bounded over the entire interval $[0, p\mu]$. As $[0, T] \subseteq [0, p\mu]$, this proves the proposition. ■

Proposition 3.1 implies, in particular, that the functions $a(t, x(t))$ and $b(t, x(t))$ of (2.8) are bounded at finite times, as follows.

Corollary 3.1. *In the notation of Proposition 3.1, the following is true. For every time $T \geq 0$, there is a real number $M_{ab}(T) \geq 0$ such that*

$$\begin{aligned} &|a(t, \Sigma(x_0, v, u, t))| \leq M_{ab}(T) \text{ and} \\ &|b(t, \Sigma(x_0, v, u, t))| \leq M_{ab}(T) \end{aligned}$$

for all residual input signals $v \in V(v_0, \gamma)$, for all control input signals $u \in U(K)$, and for all members $\Sigma \in \mathcal{F}_\gamma(x_0, \tau)$.

Proof. Considering that continuous functions are bounded over a compact domains, the corollary is a consequence of the continuity of the functions a: $R^+ \times R^n \to R^n : (t, x) \mapsto a(t, x)$ and b: $R^+ \times R^n \to R^{n \times m} : (t, x) \mapsto b(t, x)$ over the compact domain $[0, T] \times [0, M(T)]$, where $M(T)$ is given by Proposition 3.1. ■

3.2 The impact of uncertainties

The next statement shows that, for the systems being considered, small variations in the residual input signal have a small impact on the system's response.

Proposition 3.2. *In the notation of Problem 2.1, the following is true at all times $t \in [0, \tau]$: for every real number $\delta > 0$, there is a real number $\gamma > 0$ such that*

$$|\Sigma(x_0, v, u, t) - \Sigma(x_0, v_0, u, t)| < \delta$$

for all residual input signals $v \in V(v_0, \gamma)$, for all control input signals $u \in U(K)$, and for all members $\Sigma \in \mathcal{F}_\gamma(x_0, \tau)$.

Proof. Let $x(t) := \Sigma(x_0, v_0, u, t)$ be the response to the nominal residual input signal v_0, and let $x'(t) := \Sigma(x_0, v, u, t)$ be the response to an arbitrary member $v \in V(v_0, \gamma)$. Note that, due to the input time delay of τ, the states $x(t)$ and $x'(t)$ are independent of the control input signal u, as long as $t \in [0, \tau]$. Invoking the system equation (2.6) at times $t_1, t \in [0, \tau]$, $t_1 < t$, we can write

$$x(t) = x(t_1) + \int_{t_1}^{t} [a(s, x(s)) + b(s, x(s))v_0(s-\tau)]ds,$$

$$x'(t) = x'(t_1) + \int_{t_1}^{t} [a(s, x'(s)) + b(s, x'(s))v(s-\tau)]ds.$$

Subtracting the first expression from the second and recalling that $t_1 < t \leq \tau$, we obtain

$$\begin{aligned}
&\sup_{t_1 \leq \theta \leq t} \left|x'(\theta) - x(\theta)\right| \\
&= \left|x'(t_1) - x(t_1)\right| \\
&+ \sup_{t_1 \leq \theta \leq t} \left|\int_{t_1}^{\theta} \left(a(s, x'(s)) - a(s, x(s))\right) ds\right| \\
&+ \sup_{t_1 \leq \theta \leq t} \left|\int_{t_1}^{\theta} \left(b(s, x'(s))v(s-\tau) - b(s, x(s))v_0(s-\tau)\right) ds\right| \\
&\leq \left|x'(t_1) - x(t_1)\right| + \int_{t_1}^{t} \sup_{t_1 \leq s \leq t} \left|a(s, x'(s)) - a(s, x(s))\right| ds \\
&+ \int_{t_1}^{t} \sup_{t_1 \leq s \leq t} \left|(b(s, x'(s)) - b(s, x(s)))v(s-\tau)\right| ds \\
&+ \int_{t_1}^{t} \sup_{t_1 \leq s \leq t} [|b(s, x(s))|\,|v(s-\tau) - v_0(s-\tau)|]\, ds
\end{aligned}$$

Now, denote $\mu := t - t_1$, and let $T \geq \tau$ be a real number. Employing the bound $M_{ab}(T)$ of Corollary 3.1 together with the bounds of (2.12) and (2.13), the inequality becomes

$$\begin{aligned}
\sup_{t_1 \leq \theta \leq t_1+\mu} &\left|x'(\theta) - x(\theta)\right| \leq \left|x'(t_1) - x(t_1)\right| \\
&+ [(\alpha + \gamma + (\beta + \gamma)K)]\,\mu \\
&\sup_{t_1 \leq \theta \leq t_1+\mu} \left|x'(\theta) - x(\theta)\right| + M_{ab}(T)\mu\gamma.
\end{aligned}$$

Rearranging terms, we get

$$\begin{aligned}
&[1 - \mu\,(\alpha + \gamma + (\beta + \gamma)K)] \sup_{t_1 \leq \theta \leq t_1+\mu} \left|x'(\theta) - x(\theta)\right| \\
&\leq \left|x'(t_1) - x(t_1)\right| + M_{ab}(T)\mu\gamma. \qquad (3.4)
\end{aligned}$$

Now, choose $\gamma \leq \min\{\alpha, \beta\}$, $\gamma > 0$. Then, (3.4) implies that

$$\begin{aligned}
&[1 - 2\mu\,(\alpha + \beta K)] \sup_{t_1 \leq \theta \leq t_1+\mu} \left|x'(\theta) - x(\theta)\right| \\
&\leq \left|x'(t_1) - x(t_1)\right| + M_{ab}(T)\mu\gamma. \qquad (3.5)
\end{aligned}$$

Next, choose the real number μ to satisfy $2\mu(\alpha + \beta K) < 1$, and denote

$$\eta := 1/\left[1 - 2\mu\,(\alpha + \beta K)\right].$$

Then, (3.5) can be rewritten in the form

$$\begin{aligned}
&\sup_{t_1 \leq \theta \leq t_1+\mu} \left|x'(\theta) - x(\theta)\right| \\
&\leq \eta\left|x'(t_1) - x(t_1)\right| + \eta\mu M_{ab}(T)\gamma. \qquad (3.6)
\end{aligned}$$

To continue, let p be the smallest integer satisfying $p \geq \tau/\mu$, and consider the time interval $[0, p\mu]$ partitioned into segment of length μ:

$$[0, p\mu] = \left\{[0, \mu], [\mu, 2\mu], \ldots, [(p-1)\mu, p\mu]\right\};$$

set $T := p\mu$. Then, a slight reflection shows that (3.6) implies that

$$\begin{aligned}
&\left|x'((i+1)\mu) - x((i+1)\mu)\right| \\
&\leq \eta\left|x'(i\mu) - x(i\mu)\right| + \eta\mu M_{ab}(p\mu)\gamma,
\end{aligned}$$

$i = 0, 1, \ldots, p-1$. This forms a recursive relation for the quantity $|x'(i\mu) - x(i\mu)|$ with the initial value $|x'(0) - x(0)| = |x_0 - x_0| = 0$. As a result, we obtain

$$\left|x'(i\mu) - x(i\mu)\right| \leq \left(\Sigma_{j=0}^{i-1}\eta^{j+1}\right)\mu M_{ab}(p\mu)\gamma,$$

which implies that

$$\begin{aligned}
&\left|x'(\theta) - x(\theta)\right| \\
&\leq \left(\Sigma_{j=0}^{p-1}\eta^{j+1}\right)\mu M_{ab}(p\mu)\gamma \text{ for all } \theta \in [0, \tau].
\end{aligned}$$

Consequently, referring to the real number $\delta > 0$ of the proposition's statement, it follows that any real number γ satisfying

$$0 < \gamma < \min\left\{\delta/\left[\left(\Sigma_{j=0}^{p-1}\eta^{j+1}\right)\mu M_{ab}(p\mu)\right], \alpha, \beta\right\}$$

validates the proposition. This concludes our proof. ■

As discussed earlier, $x(\tau)$ – the state at the time τ of a member Σ of the family $\mathcal{F}_\gamma(x_0, \tau)$ – is not under our control due to the time delay τ in the input channel of Σ (see (2.5)). Due to this delay, the control input signal $u \in U(K)$, being a bounded input signal that starts at the time $t = 0$, does not affect the response of Σ until after the time τ. In other words, the state $x(\tau)$ is independent of the control input signal u; it is determined by the initial state x_0 and by the residual input signal v, both of which are pre-determined and not under our control. In fact, this holds true for all states $x(t)$, $t \in [0, \tau]$. We concentrate next on the implications of this fact.

3.3 Uncertainties and operating errors

The uncertainties about Σ described in (2.12) and (2.13), together with the uncertainty about the residual input signal $v \in V(v_0, \gamma)$, induce an uncertainty about the values of the state of Σ at all times $t > 0$. In this subsection, we concentrate on this uncertainty during period $t \in [0, \tau]$ – the period during which we have no control over the state of Σ. The potential magnitude of $x(t)$ during this period of time is consequential due to the inequality (2.15) that must be satisfied.

Referring to the notation of Problem 2.1, recall that all that is known about the controlled system Σ is (i) Σ is a member of the family of systems $\mathcal{F}_\gamma(x_0, \tau)$; (ii) Σ experiences an input time delay of τ; (iii) the initial state of Σ is x_0; (iv) the nominal residual input signal of Σ is v_0; and (v) the actual residual input signal of Σ is a member of $V(v_0, \gamma)$. The specific manifestation of the functions a and b of the differential equation (2.8) of Σ as well as the specific residual input function v are unknown. The set $\rho(x_0, v_0, \gamma, \tau)$ of all possible states $x(t)$ through which Σ may pass during the time interval $0 \le t \le \tau$ is

$$\rho(x_0, v_0, \gamma, \tau) := \bigcup_{\substack{v \in V(v_0, \gamma), \\ \Sigma \in \mathcal{F}_\gamma(x_0, \tau), \\ t \in [0, \tau]}} \Sigma(x_0, v, u, t).$$

The norm of the set $\rho(x_0, v_0, \gamma, \tau)$ is given by

$$|\rho(x_0, v_0, \gamma, \tau)| := \sup_{x \in \rho(x_0, v_0, \gamma, \tau)} |x|, \tag{3.7}$$

and, as discussed earlier, we have no control over $|\rho(x_0, v_0, \gamma, \tau)|$. As a result, in order for our control objective (1.1) to be achievable, it must be guaranteed a-priori that this inequality is valid during the time interval $t \in [0, \tau]$; otherwise, the performance requirement was violated before the control input signal u could start to have an effect. This, of course, was expressed in the inequality (2.15).

To translate our performance requirement into a condition on the norm $|\rho(x_0, v_0, \gamma, \tau)|$, recall that the state x of Σ is of dimension n. Taking into account the distinction between the norms used in (2.15) and in (3.7), a sufficient condition for satisfying the operating error bound (2.15) at all times $t \in [0, \tau]$ is

$$|\rho(x_0, v_0, \gamma, \tau)|^2 \le \ell/n, \tag{3.8}$$

where ℓ is the specified error operating bound.

To demonstrate conditions under which (3.8) is valid, recall that the performance error is the deviation from the zero state. One might expect therefore that if Σ starts from a 'small' initial state x_0 and is driven by a 'small amplitude' residual input signal v, then (3.8) will be met. The next statement verifies this expectation.

Proposition 3.3. *In the notation of Problem 2.1 and (3.7), assume that the initial state satisfies $|x_0| \le \gamma$ and that the nominal residual input signal satisfies $|v_0(t)| \le \gamma$ at all $t \in [-\tau, 0]$. Then, for every real number $d > 0$, there is a real number $\gamma > 0$ such that $|\rho(x_0, v_0, \gamma, \tau)| < d$.*

Proof. We examine first the nominal response of Σ from the initial state $x(0) = x_0$, which we denote by $x_\phi(t)$. Let t_1, t be two times satisfying $0 \le t_1 < t \le \tau$. Recalling the nominal model (2.11) of Σ, and using the nominal residual input function $v_0(t)$, we obtain that the state $x_\phi(t)$ of Σ satisfies

$$x_\phi(t) = x_\phi(t_1) + \int_{t_1}^{t} [a_0(s, x_\phi(s)) + b_0(s, x_\phi(s))v_0(s - \tau)]ds.$$

Using (2.12) (recall that $a_0(s, 0) = 0$) together with (2.13) and the fact that $|v_0(s)| \le \gamma$ by the proposition's assumption, we obtain

$$\begin{aligned}
&\sup_{t_1 \le \theta \le t} \left|x_\phi(\theta)\right| \\
&\le \left|x_\phi(t_1)\right| + \sup_{t_1 \le \theta \le t} \int_{t_1}^{\theta} [a_0(s, x_\phi(s)) \\
&\quad + b_0(s, x_\phi(s))v_0(s - \tau)]ds \\
&\le \left|x_\phi(t_1)\right| + \sup_{t_1 \le \theta \le t} \left|\int_{t_1}^{\theta} [a_0(s, x_\phi(s)) - a_0(s, 0)]ds\right| \\
&\quad + \sup_{t_1 \le \theta \le t} \left|\int_{t_1}^{\theta} [b_0(s, x_\phi(s)) - b_0(s, 0)]v_0(s - \tau)]ds\right| \\
&\quad + \sup_{t_1 \le \theta \le t} \left|\int_{t_1}^{\theta} [b_0(s, 0)]v_0(s - \tau)]ds\right| \\
&\le \left|x_\phi(t_1)\right| + \int_{t_1}^{t} \alpha \sup_{t_1 \le \theta \le t} \left|x_\phi(\theta)\right| ds \\
&\quad + \int_{t_1}^{t} \beta \sup_{t_1 \le \theta \le t} \left|x_\phi(\theta)\right| \sup_{t_1 \le \theta \le t} |v_0(\theta - \tau)| \, ds \\
&\quad + \int_{t_1}^{t} \beta \sup_{t_1 \le \theta \le t} |v_0(\theta - \tau)| \, ds \\
&\le \left|x_\phi(t_1)\right| + (\alpha + \beta\gamma)(t - t_1) \\
&\quad \sup_{t_1 \le \theta \le t} \left|x_\phi(\theta)\right| + \beta\gamma(t - t_1).
\end{aligned}$$

Rearranging terms, we get

$$(1-(\alpha+\beta\gamma)(t-t_1)) \sup_{t_1\le\theta\le t} \left|x_\phi(\theta)\right| \le \left|x_\phi(t_1)\right| + \beta\gamma(t-t_1). \tag{3.9}$$

Now, fix a real number $\mu > 0$ for which there is a real number $\gamma' > 0$ such that $(\alpha + \beta\gamma')\mu < 1$. Then, $(\alpha + \beta\gamma)\mu < 1$ for all $0 < \gamma \le \gamma'$. Set $t = t_1 + \mu$, define the function

$$\eta(\gamma) := 1/\left(1-(\alpha+\beta\gamma)\mu\right), 0 < \gamma \le \gamma', \tag{3.10}$$

and note that

$$\frac{d\eta(\gamma)}{d\gamma} = \frac{\beta\mu}{(1-(\alpha+\beta\gamma)\mu)^2} > 0 \text{ for all } \gamma \in (0, \gamma'].$$

The latter implies that $\eta(\gamma)$ is an increasing function of γ over the domain $(0, \gamma']$.

Further, substituting (3.10) into (3.9) with $\gamma \in (0, \gamma']$ yields

$$\sup_{t_1\le\theta\le t_1+\mu} \left|x_\phi(\theta)\right| \le \eta(\gamma)\left|x_\phi(t_1)\right| + \eta(\gamma)\beta\gamma\mu. \tag{3.11}$$

To proceed, let p be the smallest integer satisfying $p \ge \tau/\mu$, and consider the partition

$$[0, \tau] \subseteq \left\{[0, \mu], [\mu, 2\mu], \ldots, [(p-1)\mu, p\mu]\right\}.$$

Over this partition, (3.11) yields the recursive relation

$$\sup_{(i-1)\mu\le\theta\le i\mu} \left|x_\phi(\theta)\right| \le \eta(\gamma)\left|x_\phi\left((i-1)\mu\right)\right| + \eta(\gamma)\beta\gamma\mu,$$

$\gamma \in (0, \gamma']$, $i = 1, 2, \ldots, p$. From this relation, we obtain

$$\sup_{0\le\theta\le\tau} \left|x_\phi(\theta)\right| \le \sup_{0\le\theta\le p\mu} \left|x_\phi(\theta)\right| \le \eta^{p-1}(\gamma)\,|x_0| + \beta\gamma\mu\sum_{j=1}^{p}\eta^j(\gamma).$$

Invoking the proposition's assumption that $|x_0| \le \gamma$, this yields

$$\sup_{0\le\theta\le\tau} \left|x_\phi(\theta)\right| \le \left(\eta^{p-1}(\gamma) + \beta\mu\sum_{j=1}^{p}\eta^j(\gamma)\right)\gamma \tag{3.12}$$

for all $\gamma \in (0, \gamma']$. Now, recalling the number d of the proposition statement, let $d' := d/2$, and choose a number $\gamma'' > 0$ that satisfies the inequality

$$\gamma'' < \min\left\{\gamma', d'/\left(\eta^{p-1}(\gamma') + \beta\mu\sum_{j=1}^{p}\eta^j(\gamma')\right)\right\}. \tag{3.13}$$

Then, using the facts that $\eta(\gamma)$ is an increasing function of γ over the interval $(0, \gamma']$ and that $0 < \gamma'' < \gamma'$, it follows from (3.12) and (3.13) that

$$\sup_{0\le\theta\le\tau} \left|x_\phi(\theta)\right| < d' = d/2 \text{ for all } \gamma \in (0, \gamma''].$$

Next, by Proposition 3.2, there is, for every real number $d'' > 0$, a real number $\gamma_1 > 0$ such that $|\Sigma(x_0, v, u, \tau) - \Sigma(x_0, v_0, u, \tau)| < d''$ for all residual input signals $v \in V(v_0, \gamma_1)$ and for all members $\Sigma \in \mathcal{F}_{\gamma_1}(x_0, \tau)$. Thus, referring to the number d of the proposition's statement, setting $d'' = d/2$, and using $\gamma := \min\{\gamma'', \gamma_1\}$, we obtain

$$\begin{aligned}|\rho(x_0, v_0, \gamma, \tau)| &= \sup_{x\in\rho(x_0,v_0,\gamma,\tau)} |x| \\ &\le \sup_{x\in\rho(x_0,v_0,\gamma,\tau)} \left|x - x_\phi\right| + \sup_{0\le\theta\le\tau}\left|x_\phi(\theta)\right| \\ &\le d/2 + d/2 = d\end{aligned}$$

whenever $|x_0| < \gamma$, $v \in V(v_0, \gamma)$, and $\Sigma \in \mathcal{F}_\gamma(x_0, \tau)$. This completes our proof. ■

Proposition 3.3 allows us to conclude that there are general conditions under which the inequality $x^T(t)x(t) \le \ell$ is valid throughout the initial time period $[0, \tau]$ during which we have no control over the system Σ. Indeed, referring to (3.8) and using a value

$$d \le \sqrt{\frac{\ell}{n}}$$

in Proposition 3.3 demonstrates such conditions. Other circumstances under which this inequality is met are, of course, possible as well.

4. Existence of optimal solutions

In this section, we prove the existence of optimal solutions of Problem 2.1. Specifically, we show that there is an optimal control input signal that keeps the controlled system Σ operating below the specified error bound ℓ for the longest time possible. In the next section, we show that the performance achieved by such an optimal control input signal can be approximated as closely as desired by a bang-bang control input signal. The main result of the current section can be stated in the following form.

Theorem 4.1. *In the notation of Problem 2.1, the following are true:*

(*i*) *If $t(x_0, \gamma, \ell) = \infty$, then, for every time $t' \geq 0$, there is a control input signal $u' \in U(K)$ for which $t(x_0, \gamma, u', \ell) \geq t'$.*
(*ii*) *If $t(x_0, \gamma, \ell) < \infty$, then there is an optimal control input signal $u(x_0, \gamma, \ell) \in U(K)$ satisfying $t(x_0, \gamma, \ell) = t(x_0, \gamma, u(x_0, \gamma, \ell), \ell)$.*

Before stating the proof of Theorem 4.1, we need a few preliminary results, starting with an examination of the time functional $t(x_0, \gamma, \ell)$ of (2.18). We can distinguish between two obvious cases:

$$\begin{aligned} &\textbf{Case 1: } t(x_0, \gamma, \ell) = \infty; \\ &\textbf{Case 2: } t(x_0, \gamma, \ell) < \infty. \end{aligned} \tag{4.2}$$

In Case 1, the response of the controlled system Σ can be kept below the specified error bound ℓ for as long as desired, by using appropriate control input signals. There is no optimal solution in this case, and no further discussion of this case is required in the context of optimality. We return to Case 1 later in Section 5, where we show that, in this case, one can use easy-to-implement bang-bang control input signals to keep the response of Σ below the specified error bound ℓ for any desired period of time. We devote the remaining part of the present section to an examination of Case 2, which requires detailed analysis.

Before starting our analysis of Case 2, we list for future reference the following fact, which is a direct consequence of the definition of supremum.

Lemma 4.1. *In the notation of Problem 2.1 and (2.17), assume that Case 2 of (4.2) is valid. Then, there is a time $T > 0$ such that $t(x_0, \gamma, u, \ell) \leq T$ for all control input signals $u \in U(K)$.*

Our discussion of the existence of an optimal solution of Problem 2.1 in Case 2 of (4.2) is based on the Generalized Weierstrass Theorem, which, in simplified terms, states that a continuous function attains extremal values in a compact domain. In our discussion here, the function of interest is the time functional $t(x_0, \gamma, u, \ell)$ of (2.16) as a function of the control input signal u; the domain of interest here is, of course, the domain $U(K)$ of (2.2), which describes the class of all permissible control input signals.

We show in this section that $t(x_0, \gamma, u, \ell)$ is continuous (in an appropriate sense) over $U(K)$, and that $U(K)$ is compact (in an appropriate sense). Once these facts have been established, the Generalized Weierstrass Theorem implies the existence of an optimal control input signal $u(x_0, \gamma, \ell)$ of Problem 2.1 in Case 2 of (4.2), namely, in the case when the maximal time is finite. Needless to say, the optimal control input signal $u(x_0, \gamma, \ell)$ – a vector valued function of time – may be hard to compute and implement. To overcome this difficulty, we show in Section 5 that the optimal performance achieved by $u(x_0, \gamma, \ell)$ can be approximated as closely as desired by an easy-to-calculate and easy-to-implement bang-bang control input signal $u^{\pm}(x_0, \gamma, \ell) \in U(K)$.

4.1 Some mathematical facts

The proof of Theorem 4.1(ii) depends on a number of notions and auxiliary results; first, we review two notions from functional analysis (e.g. Lusternik & Sobolev, 1961; Willard, 2004).

Definition 4.1. Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$.

(i) A sequence $\{x_i\}_{i=1}^{\infty} \subseteq H$ *converges weakly* to an element $x \in H$ if $\lim_{i \to \infty} \langle x_i, y \rangle = \langle x, y \rangle$ for every element $y \in H$.
(ii) A subset $W \subseteq H$ is *weakly compact* if every sequence of elements of W has a subsequence that converges weakly to an element of W.

We can quote now the following statement from Chakraborty and Hammer (2009b, Lemma 3.2), which states that the set of control input signals $U(K)$ is weakly compact.

Lemma 4.2. *The set $U(K)$ of (2.2) is weakly compact in the topology of the Hilbert space $L_2^{\sigma, m}$.*

The following notions of continuity (e.g. Willard, 2004) are critical to our discussion.

Definition 4.2. Let H be a Hilbert space, let S be a subset of H, let z be a point of S, and let R denote the real numbers.

(i) A functional $F: S \to R$ is *weakly upper semi-continuous* at a point $z \in S$ if the following is true whenever $F(z)$ is bounded: for every sequence $\{z_i\}_{i=1}^{\infty} \subseteq S$ that converges weakly to z, and for every real number $\epsilon > 0$, there is an integer $N > 0$ such that $F(z_i) - F(z) < \epsilon$ for all integers $i \geq N$.
(ii) If the functional F is weakly upper semi-continuous at every point of S, then F is *weakly upper semi-continuous on S.*
(iii) A function $G: S \to R^n$ is *weakly continuous* at a point $z \in S$ if the following is true for every sequence $\{z_i\}_{i=1}^{\infty} \subseteq S$ that converges weakly to z: for every real number $\epsilon > 0$, there is an integer $N > 0$ such that $|G(z_i) - G(z)| < \epsilon$ for all $i \geq N$.

(iv) If the function G is weakly continuous at every point of S, then G is *weakly continuous on S*.

The following are a few mathematical facts related to the notions of Definition 4.2 (see, e.g.Willard, 2004).

Theorem 4.3.

(*i*) *A continuous function of a weakly continuous function is weakly continuous.*
(*ii*) *A weakly continuous functional is weakly upper semi-continuous.*
(*iii*) *A weakly upper semi-continuous functional of a weakly continuous function forms a weakly upper semi-continuous functional.*
(*iv*) *Let S and A be topological spaces and assume that, for every member $a \in A$, there is a weakly upper semi-continuous functional f_a: $S \to R$. If $\inf_{a\in A} f_a(s)$ exists at each point $s \in S$, then the functional $f(s) := \inf_{a\in A} f_a(s)$ is weakly upper semi-continuous on S.*

4.2 Continuity and compactness

Our first objective is to show that the functional $t(x_0, \Sigma, v, u, \ell)$: $U(K) \to R$ of (2.17) is weakly upper semi-continuous in u. To that end, we need some convergence features. First, we show that delay does not affect weak convergence.

Lemma 4.3. *Let $\{u_i\}_{i=1}^{\infty} \subseteq U(K)$ be a sequence of signals that converges weakly to a signal $u \in U(K)$. Let $u_i^\tau(s) := u_i(s-\tau)$ and $u^\tau(s) := u(s-\tau)$ be the corresponding delayed signals. Then, the sequence $\{u_i^\tau\}_{i=1}^{\infty}$ converges weakly to u^τ.*

Proof. Let $y \in L_2^{\sigma,m}$ be a function, and consider the inner product

$$\begin{aligned}\langle u_i^\tau, y\rangle &= \int_0^\infty e^{-\sigma s}\left(u_i^\tau(s)\right)^T y(s)ds\\ &= \int_0^\infty e^{-\sigma s} u_i^T(s-\tau)y(s)ds\\ &= \int_{-\tau}^\infty e^{-\sigma(\theta+\tau)} u_i^T(\theta)y(\theta+\tau)d\theta.\end{aligned}$$

Considering that $u_i(\theta) = 0$ for all $\theta < 0$ by Definition 2.1, it follows that

$$\langle u_i^\tau, y\rangle = \int_0^\infty e^{-\sigma(\theta+\tau)}\left(u_i\right)^T(\theta)y(\theta+\tau)d\theta.$$

Now, define the function

$$y'(\theta) := \begin{cases} e^{-\sigma\tau}y(\theta+\tau), & \theta \geq 0,\\ 0 & \theta < 0.\end{cases}$$

Then, a slight reflection shows that $y' \in L_2^{\sigma,m}$ and that $\langle u_i^\tau, y\rangle = \langle u_i, y'\rangle$; similarly, $\langle u^\tau, y\rangle = \langle u, y'\rangle$. Using the fact that the sequence $\{u_i\}_{i=1}^{\infty}$ converges weakly to u, we obtain that $\lim_{i=\infty}\langle u_i^\tau, y\rangle = \lim_{i=\infty}\langle u_i, y'\rangle = \langle u, y'\rangle = \langle u^\tau, y\rangle$. As this is true for every member $y \in L_2^{\sigma,m}$, it follows that u_i^τ converges weakly to u^τ, and our proof concludes. ■

The next statement shows that the response to a weakly convergent sequence of control input signals is convergent.

Lemma 4.4. *In the notation of Problem 2.1, let $\{u_i\}_{i=1}^{\infty} \subseteq U(K)$ be a sequence of control input signals that converges weakly to a control input signal $u \in U(K)$. Then, $\lim_{i\to\infty}\Sigma(x_0, v, u_i, t) = \Sigma(x_0, v, u, t)$ at all times $t \geq 0$, for every system $\Sigma \in \mathcal{F}_\gamma(x_0, \tau)$, and for every residual input signal $v \in V(v_0, \gamma)$.*

Proof. Fix a member $\Sigma \in \mathcal{F}_\gamma(x_0, \tau)$ and a residual input signal $v \in V(v_0, \gamma)$, and recall that all members of $\mathcal{F}_\gamma(x_0, \tau)$ start from the same initial state x_0 at the time $t = 0$. Referring to the sequence $\{u_i\}_{i=1}^{\infty}$ of the lemma's statement, denote by $x_i(t) := \Sigma(x_0, v, u_i, t)$ the response of Σ to the control input signal u_i, $i = 1, 2, \ldots$, and denote by $x(t) := \Sigma(x_0, v, u, t)$ the response of Σ to u. Set

$$\xi_i(t) := x_i(t) - x(t),\ i = 1, 2, \ldots$$

Our proof will conclude upon showing that $\lim_{i\to\infty}\xi_i(t) = 0$ at all times $t \geq 0$.

Considering that in all cases, Σ starts from the initial state x_0 and has the same residual input signal $v(t)$, $t \in [-\tau, 0]$, it follows from the system equation (2.6) that

$$\xi_i(t) = 0 \text{ for all } t \in [0, \tau],\ i = 1, 2, \ldots$$

To examine other times, choose a time $t > \tau$ and let $t_1, t_2 \in [\tau, t]$ be two times, where $t_1 < t_2$. Then, according to the differential equation (2.6) of Σ, we have

$$\begin{aligned}\xi_i(t_2) = \xi_i(t_1) &+ \int_{t_1}^{t_2}[a(s, x_i(s)) - a(s, x(s))]ds\\ &+ \int_{t_1}^{t_2}[b(s, x_i(s))u_i(s-\tau) - b(s, x(s))u(s-\tau)]ds\\ &= \xi_i(t_1) + \int_{t_1}^{t_2}[a(s, x_i(s)) - a(s, x(s))]ds\\ &+ \int_{t_1}^{t_2}[b(s, x_i(s)) - b(s, x(s))]u_i(s-\tau)ds\\ &+ \int_{t_1}^{t_2} b(s, x(s))[u_i(s-\tau) - u(s-\tau)]ds.\end{aligned}$$

Using (2.12) and (2.13), we can write

$$\sup_{t_1\le\theta\le t_2}|\xi_i(\theta)| \le |\xi_i(t_1)| + (\alpha+\gamma)(t_2-t_1)\sup_{t_1\le\theta\le t_2}|\xi_i(\theta)| + (\beta+\gamma)K(t_2-t_1)\sup_{t_1\le\theta\le t_2}|\xi_i(\theta)| + \sup_{t_1\le\theta\le t_2}\left|\int_{t_1}^{\theta} b(s,x(s))[u_i(s-\tau)-u(s-\tau)]ds\right| \quad (4.4)$$

To estimate the last term, refer to the inner product (2.1) and define the function

$$y_\theta(s) := \begin{cases} e^{\sigma s}b(s,x(s)), & 0\le s\le\theta,\\ 0, & \text{else}.\end{cases}$$

Then, using the notation of Lemma 4.3, the last term of (4.4) becomes

$$\sup_{t_1\le\theta\le t_2}\left|\int_{t_1}^{\theta} b(s,x(s))[u_i(s-\tau)-u(s-\tau)]ds\right| = \sup_{t_1\le\theta\le t_2}\left|\left\langle\left(u_i^\tau-u^\tau\right),y_\theta\right\rangle\right|. \quad (4.5)$$

Recalling that the sequence $\{u_i\}_{i=1}^{\infty}$ converges weakly to u and applying Lemma 4.3, it follows that, for every real number $\epsilon > 0$, there is an integer $N_\theta \ge 0$ such that $\left|\langle u_i^\tau - u^\tau, y_\theta\rangle\right| < \varepsilon$ for all $i \ge N_\theta$.

Next, we show that N_θ can be selected to be independent of θ, namely, that there is an integer $N \ge 0$ such that $\sup_{t_1\le\theta\le t_2}\left|\langle u_i^\tau - u^\tau, y_\theta\rangle\right| < \varepsilon$ for all $i \ge N$. To this end, assume, by contradiction, that there is no such integer, i.e. that there is no integer $N \ge 0$ for which $\left|\langle u_i^\tau - u^\tau, y_\theta\rangle\right| < \varepsilon$ for all $i \ge N$ and all $\theta \in [t_1, t_2]$. Then, there is a sequence of times $\{\theta_j\}_{j=1}^{\infty} \subseteq [t_1, t_2]$ and a divergent sequence of integers $\left\{i_j\right\}_{j=1}^{\infty} \to \infty$ such that

$$\left|\left\langle u_{i_j}^\tau - u^\tau, y_{\theta_j}\right\rangle\right| > \varepsilon \quad (4.6)$$

for all $j = 1, 2, \ldots$. As the interval $[t_1, t_2]$ is compact, the sequence $\{\theta_j\}_{j=1}^{\infty}$ contains a convergent subsequence, say, the subsequence $\{\theta_{j_k}\}_{k=1}^{\infty}$. Denote the limit of this subsequence by $\theta' := \lim_{k\to\infty}\theta_{j_k}$. Invoking the weak convergence of the sequence $\{u_i^\tau\}_{i=1}^{\infty}$ to u^τ, the subsequence $\{u_{j_k}^\tau\}_{k=1}^{\infty}$ also converges weakly to u^τ. Hence, there is an integer $N' \ge 0$ such that

$$\left|\left\langle u_{i_{j_k}}^\tau - u^\tau, y_{\theta'}\right\rangle\right| < \varepsilon/2 \quad (4.7)$$

for all $k \ge N'$.

Next, using (2.13) and taking into account the facts that $u_{i_{j_k}}, u \in U(K)$ and that $\theta_{j_k}, \theta' \in [t_1, t_2]$ for all $k = 1, 2, \ldots$, we obtain

$$\begin{aligned}&\left|\left\langle u_{i_{j_k}}^\tau - u^\tau, y_{\theta_{j_k}}\right\rangle - \left\langle u_{i_{j_k}}^\tau - u^\tau, y_{\theta'}\right\rangle\right|\\ &= \left|\int_{\theta_{j_k}}^{\theta'} b(s,x(s))[u_{i_{j_k}}(s-\tau)-u(s-\tau)]ds\right|\\ &\le (\beta+\gamma)(2K)\left|\theta'-\theta_{j_k}\right|.\end{aligned} \quad (4.8)$$

Further, considering that $\lim_{k\to\infty}\theta_{j_k} = \theta'$, there is an integer $N'' \ge N'$ such that

$$\left|\theta'-\theta_{j_k}\right| < \frac{\varepsilon}{4(\beta+\gamma)K} \quad (4.9)$$

for all $k \ge N''$. Then, using (4.8), (4.9), and (4.7), we obtain

$$\begin{aligned}&\left|\left\langle u_{i_{j_k}}^\tau - u^\tau, y_{\theta_{j_k}}\right\rangle\right|\\ &= \left|\left\langle u_{i_{j_k}}^\tau - u^\tau, y_{\theta_{j_k}}\right\rangle - \left\langle u_{i_{j_k}}^\tau - u^\tau, y_{\theta'}\right\rangle + \left\langle u_{i_{j_k}}^\tau - u^\tau, y_{\theta'}\right\rangle\right|\\ &\le \left|\left\langle u_{i_{j_k}}^\tau - u^\tau, y_{\theta_{j_k}}\right\rangle - \left\langle u_{i_{j_k}}^\tau - u^\tau, y_{\theta'}\right\rangle\right| + \left|\left\langle u_{i_{j_k}}^\tau - u^\tau, y_{\theta'}\right\rangle\right|\\ &< \varepsilon/2+\varepsilon/2 = \varepsilon\end{aligned}$$

for all $k \ge N''$, in contradiction to (4.6). Consequently, for every real number $\epsilon > 0$, there is an integer $N \ge 0$ such that

$$\sup_{t_1\le\theta\le t_2}\left|\left\langle u_i^\tau - u^\tau, y_\theta\right\rangle\right| < \varepsilon \quad (4.10)$$

for all integers $i \ge N$. Substituting this into (4.5), it follows that, for all $i \ge N$, we have

$$\sup_{t_1\le\theta\le t_2}\left|\int_{t_1}^{\theta} b(s,x(s))[u_i(s-\tau)-u(s-\tau)]ds\right| < \varepsilon. \quad (4.11)$$

Next, substitute (4.11) into (4.4) and rearrange terms to obtain

$$\{1-(t_2-t_1)[(\alpha+\gamma)+(\beta+\gamma)K]\}\sup_{t_1\le\theta\le t_2}|\xi_i(\theta)| \le |\xi_i(t_1)| + \varepsilon \quad (4.12)$$

for all $i \ge N$. Now, select a real number $\mu > 0$ such that

$$\mu[(\alpha+\gamma)+(\beta+\gamma)K] < 1,$$

denote

$$\eta := 1/\{1-\mu[(\alpha+\gamma)+(\beta+\gamma)K]\},$$

and set

$$t_2 := t_1 + \mu.$$

Then, (4.12) together with (4.10) yield $\sup_{t_1\le\theta\le t_1+\mu}|\xi_i(\theta)| < \eta\,|\xi_i(t_1)| + \varepsilon/\eta$ for all $i \ge N$. Now, since the latter is valid for every real number $\epsilon > 0$, it follows that, for every real number $\delta > 0$, we can set $\epsilon < \eta\delta$. With this choice, it follows that, for every real number $\delta > 0$, there is an integer $N > 0$ such that

$$\sup_{t_1\le\theta\le t_1+\mu}|\xi_i(\theta)| \le \eta\,|\xi_i(t_1)| + \delta \tag{4.13}$$

for all $i \ge N$.

We can use (4.13) to derive a bound for the function $\xi_i(\theta)$ over the entire interval $[\tau, t]$. To this end, let p be the smallest integers satisfying $p \ge (t-\tau)/\mu$, and consider the partition

$$[\tau, \mu p] = \big\{[\tau, \tau+\mu], [\tau+\mu, \tau+2\mu], \ldots, [\tau+(p-1)\mu, \tau+\mu p]\big\}.$$

Then, we can rewrite (4.13) in the form

$$\sup_{\tau+(j-1)\mu\le\theta\le\tau+j\mu}|\xi_i(\theta)| \le \eta\,\big|\xi_i(\tau+(j-1)\mu)\big| + \delta,\ j = 1, 2, \ldots, p \tag{4.14}$$

for all $i \ge N$. Denoting

$$\zeta_i^j := \sup_{\tau+(j-1)\mu\le\theta\le\tau+j\mu}|\xi_i(\theta)|, \tag{4.15}$$

$i \ge N$, we obtain from (4.14) and (4.15) the recursive relation

$$\zeta_i^j \le \eta\zeta_i^{j-1} + \delta,\ j = 1, 2, \ldots, p,$$
$$\zeta_i^0 = 0,$$

for all $i \ge N$. This recursion yields

$$\zeta_i^p \le \left(\sum_{k=0}^{p-1}\eta^k\right)\delta \tag{4.16}$$

for all $i \ge N$. Combining (4.16) with (4.14), we obtain

$$\sup_{0\le\theta\le t}|\xi_i(\theta)| = \sup_{\tau\le\theta\le\tau+p\mu}|\xi_i(\theta)| \le \left(\sum_{k=0}^{p-1}\eta^k\right)\delta \tag{4.17}$$

for all $i \ge N$.

Now, let $\varpi > 0$ be a real number, and take $\delta < \varpi/\left(\sum_{k=0}^{p-1}\eta^k\right)$, $\delta > 0$. Then (4.17) implies that, for every real number $\varpi > 0$, there is a integer $N > 0$ such that

$$\sup_{0\le\theta\le t}|\xi_i(\theta)| < \varpi$$

for all integers $i \ge N$. Consequently, $\lim_{i\to\infty}\sup_{0\le\theta\le t}|\xi_i(\theta)| = 0$, or

$$\lim_{i\to\infty}\sup_{0\le\theta\le t}|x_i(\theta) - x(\theta)| = 0, \tag{4.18}$$

and out proof concludes. ■

In view of Definition 4.2, Lemma 4.4 implies the following.

Corollary 4.1. *In the notation of Problem 2.1, let $\Sigma \in \mathcal{F}_\gamma(x_0, \tau)$ be a system with a residual input signals $v \in V(v_0, \gamma)$, and let $t \ge 0$ be a time. Then, $\Sigma(x_0, v, u, t)$ is weakly continuous as a function of u over $U(K)$.*

The next statement forms a critical step-stone along the path to the proof of Theorem 4.1 by showing that the time functional $t(x_0, \Sigma, v, u, \ell)$ of (2.16) has desirable continuity features.

Lemma 4.5. *In the notation of Problem 2.1, let $\Sigma \in \mathcal{F}_\gamma(x_0, \tau)$ be a system with residual input signal $v \in V(v_0, \gamma)$. Then, the functional $t(x_0, \Sigma, v, u, \ell)$: $U(K) \to R : u \mapsto t(x_0, \Sigma, v, u, \ell)$ of (2.16) is weakly upper semi-continuous as a function of the control input signal u.*

Proof. Let $\Sigma \in \mathcal{F}_\gamma(x_0, \tau)$ and $v \in V(v_0, \gamma)$ be arbitrary members, and fix a time $t \ge 0$. Let $\{u_i\}_{i=1}^\infty \subseteq U(K)$ be a sequence of control input signals that converges weakly to a member $u \in U(K)$. Following the notation of the proof of Lemma 4.4 and recalling that all members of $\mathcal{F}_\gamma(x_0, \tau)$ start from the same initial state x_0 at the time $t = 0$, let $x_i(t) := \Sigma(x_0, v, u_i, t)$ be the response of Σ to u_i, $i = 1, 2, \ldots$, and let $x(t) := \Sigma(x_0, v, u, t)$ be the response of Σ to u.

Define the class of functions

$$S := \big\{z : R^+ \to R^n : z(t) = \Sigma(x_0, v, g, t) \text{ for some } g \in U(K)\big\},$$

and, recalling the error bound ℓ of (1.1), define the functional Θ: $S \to R$ given, for a function $z \in S$, by

$$\Theta(z) = \inf\{t \ge 0 : z^T(t)z(t) > \ell\}. \tag{4.19}$$

We intend to show that the functional $\Theta(z)$ is upper semi-continuous on S. In view of Lemma 4.4 (see, in particular, (4.18)), we know that the sequence $x_1(t), x_2(t), \ldots$ converges to $x(t)$ at each time $t \ge 0$. To examine the sequence $\Theta(x_i)$ and its relation to $\Theta(x)$, note that, by Definition 4.2, we have to consider only the case where $\Theta(x)$ is bounded. Then, to show that $\Theta(\cdot)$ is upper semi-continuous on S, we have to show that, for every real number $\epsilon > 0$, there is an integer $N > 0$ such that $\Theta(x_i) - \Theta(x) < \epsilon$ for all integers $i \ge N$. The proof of this point can be divided into two cases:

Case 1. There is an integer $N' > 0$ for which $\Theta(x_i) \le \Theta(x)$ for all integers $i \ge N'$.
Case 2. Case 1 is not valid.

In Case 1, we clearly have $\Theta(x_i) - \Theta(x) \le 0$ for all $i \ge N'$, so that $\Theta(x_i) - \Theta(x) < \epsilon$ for every real number $\epsilon > 0$. Hence, the definition of upper semi-continuity holds in this case for $N = N'$. This completes the proof in Case 1.

In Case 2, there is a subsequence $\{i_k\}_{k=1}^{\infty}$ and an integer $N'' > 0$ such that $\Theta(x_{i_k}) > \Theta(x)$ for all $k \ge N''$. Now, fix a real number $\epsilon > 0$. Then, it follows by the infimum of (4.19) that there is a time $t' \in [\Theta(x), \Theta(x) + \epsilon)$ such that

$$x^T(t')x(t') > \ell. \tag{4.20}$$

By Lemma 4.4 (see, in particular, (4.18)), we have $\lim_{i\to\infty}|x_i(t') - x(t')| = 0$; consequently, also $\lim_{i\to\infty} |x_i^T(t')x_i(t') - x^T(t')x(t')|=0$. Thus, for every real number $\epsilon_1 > 0$, there is an integer $N_1 > 0$ such that $\left|x_{i_k}^T(t')x_{i_k}(t') - x^T(t')x(t')\right| < \varepsilon_1$ for all $k \ge N_1$. In view of (4.20), we can choose $\epsilon_1 = [x^T(t')x(t') - \ell]/2$, so that

$$\left|x_{i_k}^T(t')x_{i_k}(t') - x^T(t')x(t')\right| < [x^T(t')x(t') - \ell]/2$$

for all $k \ge N_1$. Consequently,

$$\begin{aligned} x_{i_k}^T(t')x_{i_k}(t') &= x^T(t')x(t') + \left[x_{i_k}^T(t')x_{i_k}(t') - x^T(t')x(t')\right] \\ &\ge x^T(t')x(t') - \left|x_{i_k}^T(t')x_{i_k}(t') - x^T(t')x(t')\right| \\ &> x^T(t')x(t') - \left[x^T(t')x(t') - \ell\right]/2 \\ &= x^T(t')x(t')/2 + \ell/2 \\ &> \ell, \end{aligned}$$

so that $x_{i_k}^T(t')x_{i_k}(t') > \ell$ for all $k \ge N_1$. In view of the infimum in (4.19), this implies that $\Theta(x_{i_k}) \le t'$ for all $k \ge N_1$. But then, since $t' \in [\Theta(x), \Theta(x) + \epsilon]$, it follows that $\Theta(x_{i_k}) < \Theta(x) + \varepsilon$ for all $k \ge N_1$. As this is true for any such subsequence $\{i_k\}_{k=1}^{\infty}$, we conclude that $\Theta(\cdot)$ is upper semi-continuous on S in Case 2 as well.

Finally, by Corollary 4.1, the function $\Sigma(x_0, v, u, t)$: $U(K) \to R^n$ is a weakly continuous function over $U(K)$ at every time t, for every residual input signal $v \in V(v_0, \gamma)$, and for every member $\Sigma \in \mathcal{F}_\gamma(x_0, \tau)$. As z^Tz: $R^n \to R$ is a continuous functional of z, it follows by Theorem 4.3*(i)* that $\Sigma^T(x_0, v, u, t)\Sigma(x_0, v, u, t)$: $U(K) \to R$ also is a weakly continuous functional over $U(K)$ at every time t, for every residual input signal $v \in V(v_0, \gamma)$, and for every member $\Sigma \in \mathcal{F}_\gamma(x_0, \tau)$. Combining this with the conclusion of the previous paragraph, it follows by Theorem 4.3 (iii) that $\Theta(\Sigma(x_0, v, u, t))$: $U(K) \to R : u \mapsto \Theta(\Sigma(x_0, v, u, t))$ is an upper semi-continuous functional on $U(K)$ at every time $t \ge 0$, for every residual input signal $v \in V(v_0, \gamma)$, and for every member $\Sigma \in \mathcal{F}_\gamma(x_0, \tau)$. The lemma then follows from the fact that $t(x_0, \Sigma, v, u, \ell) = \Theta(\Sigma(x_0, v, u, t))$, and our proof concludes. ∎

We have reached now the last step-stone along our path to the proof of Theorem 4.1.

Lemma 4.6. *In the notation of Problem 2.1, the functional $t(x_0, \gamma, u, \ell) : U(K) \to R : u \mapsto t(x_0, \gamma, u, \ell)$ of (2.17) is weakly upper semi-continuous on $U(K)$.*

Proof. According to Lemma 4.5, the functional $t(x_0, \Sigma, v, u, \ell)$: $U(K) \to R$ is weakly upper semi-continuous for every $\Sigma \in \mathcal{F}_\gamma(x_0, \tau)$ and for every $v \in V(v_0, \gamma)$. Consequently, the lemma follows by Theorem 4.3*(iv)*, since

$$t(x_0, \gamma, u, \ell) = \inf_{(\Sigma, v) \in \mathcal{F}_\gamma(x_0, \tau) \times V(v_0, \gamma)} t(x_0, \Sigma, v, u, \ell).$$

This concludes our proof. ∎

We can prove now Theorem 4.1 – the main result of this section.

Proof of Theorem 4.1. Part (i) of the theorem is proved in the paragraph following the theorem's statement. We concentrate here on the proof of part (ii).

Part (ii) of the theorem refers to Case 2 of (4.2). This part of the theorem is a consequence of the Generalized Weierstrass Theorem, which states that a weakly upper semi-continuous functional attains a maximum in a weakly compact set (e.g. Zeidler, 1985). Indeed, Lemma 4.6 states that the functional $t(x_0, \gamma, u, \ell)$: $U(K) \to R$ is weakly upper semi-continuous over $U(K)$, while Lemma 4.2 states that $U(K)$ is weakly compact. Thus, the functional $t(x_0, \gamma, u, \ell)$ attains a maximum $t(x_0, \gamma, \ell)$ in $U(K)$, and there is a member $u(x_0, \gamma, \ell) \in U(K)$ satisfying $t(x_0, \gamma, \ell) = t(x_0, \gamma, u(x_0, \gamma, \ell), \ell)$. This concludes our proof. ∎

In summary, we have seen in this section that there is an optimal control input signal $u(x_0, \gamma, \ell)$ that keeps operating errors below a specified bound for the longest time possible. Although this result has profound theoretical implications, it does not resolve the issue of implementation, since the optimal control input signal $u(x_0, \gamma, \ell)$ of Theorem 4.1, being a vector-valued function of time, may be exceedingly hard to calculate and implement. In the next section, we address the implementation issue by showing that the performance achieved by an optimal control input signal $u(x_0, \gamma, \ell)$ can be approximated as closely as desired by a bang-bang control input signal $u^{\pm}(x_0, \gamma, \ell)$ – a piecewise constant input signal that switches between the extremal input values K and $-K$. Considering that bang-bang signals are determined by a finite string of scalars (their switching times), they are relatively easy to calculate and implement. As a result, the possibility of using bang-bang input signals alleviates concerns about potential difficulties in the implementation of optimal controllers. Needless to say, both the optimal control input signal $u(x_0, \gamma, \ell)$, as well as the bang-bang control input signal $u^{\pm}(x_0, \gamma, \ell)$, depend on the state $x_0 = x(0)$ provided by the feedback channel before feedback was disrupted.

5. Bang-bang input signals and optimal performance

In this section, we show that the performance achieved by an optimal control input signal of Theorem 4.1 can be approximated as closely as desired by a bang-bang control input signal. This fact simplifies considerably the calculation and the implementation of the controller C of Figure 1, since bang-bang signals are relatively easy to calculate and implement. Recall that the task of the controller C is to keep operating errors below the specified error bound ℓ for the longest time possible during periods of feedback outage. In accomplishing its task, the controller C faces the added difficulty of having to cope with an input delay of τ that afflicts the controlled system Σ. The main result of the current section is the following: by increasing the operating error bound ℓ ever so slightly to ℓ', a bang-bang control input signal can keep the controlled system Σ below the operating error of ℓ' for at least as long as the maximal time $t(x_0, \gamma, \ell)$ achieved by an optimal input signal for the error bound ℓ.

Theorem 5.1. *In the notation of Problem 2.1, (2.17), and (2.18), let $\ell > 0$ be a specified error bound. Then, the following are true for every error bound $\ell' > \ell$:*

(*i*) *If $t(x_0, \gamma, \ell) = \infty$, then, for every time $t' > \tau$, there is a bang-bang control input signal $u'^{\pm} \in U(K)$ (with a finite number of switchings) for which $t(x_0, \gamma, u'^{\pm}, \ell') \geq t'$.*

(*ii*) *If $t(x_0, \gamma, \ell) < \infty$, then there is a bang-bang control input signal $u^{\pm}(x_0, \gamma, \ell') \in U(K)$ (with a finite number of switchings) for which $t(x_0, \gamma, u^{\pm}(x_0, \gamma, \ell'), \ell') \geq t(x_0, \gamma, \ell)$.*

The proof of Theorem 5.1 depends on a number of auxiliary results, the main one of which is listed next.

Theorem 5.2. *In the notation of Problem 2.1, let $\Sigma \in \mathcal{F}_\gamma(x_0, \tau)$ be a system, let $u \in U(K)$ be a control input signal, and let $t' > \tau$ be a time. Then, for every real number $\epsilon > 0$, there is a bang-bang control input signal $u^{\pm} \in U(K)$ (with a finite number of switchings) for which the following is true. The difference between the response $x(t) := \Sigma(x_0, v, u, t)$ of Σ to u and the response $x^{\pm}(t) := \Sigma(x_0, v, u^{\pm}, t)$ of Σ to $u^{\pm}$ satisfies the inequality $|x(t) - x^{\pm}(t)| < \epsilon$ at all times $0 \leq t \leq t'$, irrespective of the member $\Sigma \in \mathcal{F}_\gamma(x_0, \tau)$ and the residual input signal $v \in V(v_0, \gamma)$.*

The proof of Theorem 5.2 requires the following auxiliary result.

Lemma 5.1. *In the notation of Problem 2.1, let $\Sigma \in \mathcal{F}_\gamma(x_0, \tau)$ be a system with initial state x_0, residual input signal $v \in V(v_0, \gamma)$, control input signal $u \in U(K)$, and response $x(t) := \Sigma(x_0, v, u, t)$, $t \geq 0$, where Σ is described by the differential equation (2.6) with the functions $a(t, x)$ and $b(t, x)$ subject to (2.10), (2.12), and (2.13). Then, for every time $t' \geq 0$, the following is valid. For every real number $\rho > 0$, there is a real number $\beta(x_0, t', \rho) > 0$ such that*

$$|b(t_1, x(t_1)) - b(t_2, x(t_2))| < \rho$$

for all times $t_1, t_2 \in [0, t']$ satisfying $|t_1 - t_2| < \beta(x_0, t', \rho)$, irrespective of the residual input signal $v \in V(v_0, \gamma)$ and the control input signal $u \in U(K)$.

Proof. We use the fact that the function $b(t, x(t))$ is a continuous function of time over the compact domain $[0, t']$, and hence is uniformly continuous there. Denote $w(t)$ as in (2.7) and set $\Sigma(x_0, w, t) := \Sigma(x_0, v, u, t) = x(t)$. Then, uniform continuity implies the following: for every real number $\rho > 0$, there is a real number $\beta(x_0, \rho, w) > 0$, such that $|b(t_1, x(t_1)) - b(t_2, x(t_2))| < \rho$ at all times $t_1, t_2 \in [0, t']$ satisfying $|t_1 - t_2| < \beta(x_0, \rho, w)$. We show next that $\beta(x_0, \rho, w)$ can be chosen independently of the combined input signal w.

To this end, choose a real number $\rho' < \rho$, $\rho' > 0$, and consider the quantity

$$\beta(x_0, \rho', w) := \sup\left\{|t_1 - t_2| : t_1, t_2 \in [0, t'] \text{ and } |b(t_1, \Sigma(x_0, w, t_1)) - b(t_2, \Sigma(x_0, w, t_2))| \leq \rho'\right\}.$$

Set

$$\beta^*(x_0, \rho') := \inf_w \beta(x_0, \rho', w).$$

Then, there are two possibilities: *(a)* $\beta^*(x_0, \rho') > 0$ or *(b)* $\beta^*(x_0, \rho') = 0$. In the first case, part (ii) of the lemma holds for any real number $\beta(x_0, \rho) \in (0, \beta^*(x_0, \rho')]$; we show next that option *(b)* is invalid.

Indeed, if $\beta^*(x_0, \rho') = 0$, there is a sequence of signals $\{w_i\}_{i=1}^{\infty}$, where $|w_i(t)| \leq K$ for all t, for which $\lim_{i\to\infty} \beta(x_0, \rho', w_i)=0$. Now, by (2.9), we have that the signal $z_i(t) := w_i(t - \tau)$ satisfies $z_i \in U(K)$ for all integers $i \geq 1$. Using the fact that $U(K)$ is weakly compact by Lemma 4.2, it follows that the sequence $\{z_i\}_{i=1}^{\infty}$ has a weakly convergent subsequence $\{z_{i_k}\}_{k=1}^{\infty}$ that weakly converges to a combined input signal $z \in U(K)$. Set

$w(t) := z(t + \tau)$. Then, invoking Lemma 4.4, it follows that $\lim_{k\to\infty} \Sigma(x_0, w_{i_k}, t) = \Sigma(x_0, w, t)$. Furthermore, by (4.18), the function $\Sigma(x_0, w, t)$ is a weakly uniformly continuous function of u and t over $U(K) \times [0, t']$. Consequently, for every real number $\epsilon > 0$, there is an integer $N(\epsilon) > 0$ such that $\sup_{t\in[0,t']} |\Sigma(x_0, w_{i_k}, t) - \Sigma(x_0, w, t)| < \varepsilon$ for all $k \geq N(\epsilon)$. As $b(t, x(t))$ is uniformly continuous, there is a value $\epsilon = \epsilon'' > 0$ for which $|b(t, \Sigma(x_0, w_{i_k}, t)) - b(t, \Sigma(x_0, w, t))| < \rho/3$ for all $k \geq N'$, where $N' := N(\epsilon'')$.

Next, the uniform continuity of $b(t, x(t))$ over the compact interval $[0, t']$ also implies that there is a real number $\beta > 0$ such that $|b(t_1, \Sigma(x_0, w, t_1)) - b(t_2, \Sigma(x_0, w, t_2))| < \rho/3$ for all $t_1, t_2 \in [0, t']$ satisfying $|t_1 - t_2| < \beta$. Combining all these facts, we conclude that, for every integer $k \geq N'$, we have

$$\begin{aligned} &\left|b(t_1, \Sigma(x_0, w_{i_k}, t_1)) - b(t_2, \Sigma(x_0, w_{i_k}, t_2))\right| \\ &\quad\leq \left|b(t_1, \Sigma(x_0, w_{i_k}, t_1)) - b(t_1, \Sigma(x_0, w, t_1))\right| \\ &\qquad + |b(t_1, \Sigma(x_0, w, t_1)) - b(t_2, \Sigma(x_0, w, t_2))| \\ &\qquad + \left|b(t_2, \Sigma(x_0, w, t_2)) - b(t_2, \Sigma(x_0, w_{i_k}, t_2))\right| \\ &\quad\leq \rho/3 + \rho/3 + \rho/3 = \rho \end{aligned}$$

for all times $t_1, t_2 \in [0, t']$ satisfying $|t_1 - t_2| < \beta$, contradicting the possibility of $\beta^*(x_0, \rho) = 0$. Therefore, part (ii) of the lemma holds for any real number $\beta(x_0, \rho) \in (0, \beta^*(x_0, \rho')]$, and our proof concludes. ■

We turn now to the proof of Theorem 5.2.

Proof of Theorem 5.2. The control input signals u and $u^{\pm}$ are applied to Σ starting at the time $t = 0$, and, consequently, their effects on the response of Σ are not felt until after $t = \tau$; prior to that, Σ started from the initial state x_0 at $t = 0$ and received the residual input signal $v(t)$, $t \in [-\tau, 0]$, in both cases. Consequently,

$$x(t) = x^{\pm}(t) \text{ for all } t \in [0, \tau]. \tag{5.3}$$

Thus, recalling that $t' > \tau$ in the theorem's statement, it only remains to consider times $t \in (\tau, t']$. To this end, let $t_1, t_2 \in [0, t' - \tau]$, $t_1 < t_2$ be times to be selected later. Let $\lambda > 0$ be a real number to be selected later as well, for which the ratio $p := (t_2 - t_1)/\lambda$ is an integer. Partition the interval $[t_1, t_2]$ into the p segments

$$\begin{aligned} [t_1, t_2] = \big\{&[t_1, t_1 + \lambda], [t_1 + \lambda, t_1 + 2\lambda], \\ &\ldots, [t_1 + (p-1)\lambda, t_2]\big\}, \end{aligned} \tag{5.4}$$

noting that $t_2 = t_1 + p\lambda$.

In terms of components, we have $u = (u_1, u_2, \ldots, u_m)^T \in U(K)$ and every component satisfies $|u_i(t)| \leq K$, $i = 1, 2, \ldots, m$ (see (2.2)). In each sub-interval $[t_1 + q\lambda, t_1 + (q + 1)\lambda]$, $q = 1, 2, \ldots, p - 1$, we will select in a moment m points $\theta_1^q, \theta_2^q, \ldots, \theta_m^q \in [t_1 + q\lambda, t_1 + (q + 1)\lambda]$ to serve as switching times for a bang-bang control input signal $u^{\pm} = \left(u_1^{\pm}, \ldots, u_m^{\pm}\right)^T \in U(K)$ given, in terms of its components, by

$$u_i^{\pm}(t) := \begin{cases} +K & \text{for } t \in [t_1 + q\lambda, \theta_i^q), \\ -K & \text{for } t \in [\theta_i^q, t_1 + (q+1)\lambda), \\ & \text{if } \theta_i^q < t_1 + (q+1)\lambda, \end{cases} \tag{5.5}$$

$q = 0, 1, \ldots, p - 1$, $i = 1, 2, \ldots, m$.

The construction of the points $\left\{\theta_i^q\right\}$ is based on the following facts:

(i) The control input signal $u(t)$, being a member of $U(K)$, is bounded by K; therefore, $-K\lambda \leq \int_{t_1+q\lambda}^{t_1+(q+1)\lambda} u_i(s)ds \leq K\lambda$ for all components $i = 1, 2, \ldots, m$ and all $q = 0, 1, 2, \ldots, p - 1$.

(ii) A slight reflection shows that, for any real number $d \in [-\lambda, \lambda]$, there is a real number $\theta \in [t_1 + q\lambda, t_1 + (q + 1)\lambda]$ satisfying

$$\begin{aligned} d &= \big(\theta - (t_1 + q\lambda)\big) - \big((t_1 + (q+1)\lambda) - \theta\big) \\ &= 2(\theta - (t_1 + q\lambda)) - \lambda. \end{aligned}$$

Combining these facts, we conclude that, for every component $i \in \{1, 2, \ldots, m\}$ and for every integer $q \in \{0, 1, \ldots, p - 1\}$, there is a point $\theta_i^q \in [t_1 + q\lambda, t_1 + (q + 1)\lambda]$ at which

$$K[2(\theta_i^q - (t_1 + q\lambda)) - \lambda] = \int_{t_1+q\lambda}^{t_1+(q+1)\lambda} u_i(s)ds.$$

The points θ_i^q, $q = 0, 1, \ldots, p - 1$, $i = 1, 2, \ldots, m$, are inserted into (5.5) to construct the bang-bang control input signal $u^{\pm}(t)$. This construction directly leads to the equality $\int_{t_1+q\lambda}^{t_1+(q+1)\lambda} u_i(s)ds = \int_{t_1+q\lambda}^{t_1+(q+1)\lambda} u_i^{\pm}(s)ds$, or

$$\int_{t_1+q\lambda}^{t_1+(q+1)\lambda} \left(u_i(s) - u_i^{\pm}(s)\right) ds = 0 \tag{5.6}$$

for all $i \in \{1, 2, \ldots, m\}$ and all $q \in \{0, 1, 2, \ldots, p - 1\}$.

Next, we estimate the quantity

$$\xi(t) := x(t) - x^{\pm}(t),\ t \in [0, t'].$$

According to (5.3), we have

$$\xi(t) = 0 \text{ for all } t \in [0, \tau]. \tag{5.7}$$

Referring to the two times $t_1, t_2 \in [0, t' - \tau]$, $t_1 < t_2$, mentioned earlier, and invoking the system equation (2.6) together with (2.10), (2.12), and (2.13), yields

$$\begin{aligned}\xi(\tau+t_2) &= \xi(\tau+t_1)+\int_{\tau+t_1}^{\tau+t_2}\Big[a(s,x(s))-a(s,x^{\pm}(s))\\ &\quad+b(s,x(s))u(s-\tau)-b(s,x^{\pm}(s))u^{\pm}(s-\tau)\Big]ds.\end{aligned}$$

Using magnitudes, we obtain

$$\begin{aligned}\sup_{t\in[\tau+t_1,\tau+t_2]}|\xi(t)| &\le |\xi(\tau+t_1)|\\ &\quad+\sup_{t\in[\tau+t_1,\tau+t_2]}\left|\int_{\tau+t_1}^{t}\left[a(s,x(s))-a(s,x^{\pm}(s))\right]ds\right|\\ &\quad+\sup_{t\in[\tau+t_1,\tau+t_2]}\left|\int_{\tau+t_1}^{t}\left[b(s,x(s))u(s-\tau)\right.\right.\\ &\quad\left.\left.-b(s,x^{\pm}(s))u^{\pm}(s-\tau)\right]ds\right|.\end{aligned}$$

This leads to

$$\begin{aligned}\sup_{t\in[\tau+t_1,\tau+t_2]}|\xi(t)| &\le |\xi(\tau+t_1)|\\ &\quad+\sup_{t\in[\tau+t_1,\tau+t_2]}\int_{\tau+t_1}^{t}\left|a(s,x(s))-a(s,x^{\pm}(s))\right|ds\\ &\quad+\sup_{t\in[\tau+t_1,\tau+t_2]}\left|\int_{\tau+t_1}^{t}\left[b(s,x(s))u(s-\tau)\right.\right.\\ &\quad\left.\left.-b(s,x(s))u^{\pm}(s-\tau)\right]ds\right|\\ &\quad+\sup_{t\in[\tau+t_1,\tau+t_2]}\left|\int_{\tau+t_1}^{t}\left[b(s,x(s))u^{\pm}(s-\tau)\right.\right.\\ &\quad\left.\left.-b(s,x^{\pm}(s))u^{\pm}(s-\tau)\right]ds\right|.\end{aligned}$$

Using (2.10), (2.12), and (2.13), we obtain

$$\begin{aligned}\sup_{t\in[\tau+t_1,\tau+t_2]}|\xi(t)| &\le |\xi(\tau+t_1)|\\ &\quad+(\alpha+\gamma)\int_{\tau+t_1}^{\tau+t_2}\sup_{s\in[\tau+t_1,\tau+t_2]}\left|x(s)-x^{\pm}(s)\right|ds\\ &\quad+\sup_{t\in[\tau+t_1,\tau+t_2]}\left|\int_{\tau+t_1}^{t}b(s,x(s))\left(u(s-\tau)-u^{\pm}(s-\tau)\right)ds\right|\\ &\quad+\int_{\tau+t_1}^{\tau+t_2}\sup_{s\in[\tau+t_1,\tau+t_2]}\left|b(s,x(s))-b(s,x^{\pm}(s))\right|\\ &\quad\times\sup_{s\in[\tau+t_1,\tau+t_2]}\left|u^{\pm}(s-\tau)\right|ds.\end{aligned}$$

Continuing to use (2.10), (2.12), and (2.13) combined with the inequalities $|u|_\infty \le K$ and $|u^{\pm}|_\infty \le K$, yields

$$\begin{aligned}\sup_{t\in[\tau+t_1,\tau+t_2]}|\xi(t)| &\le |\xi(\tau+t_1)|\\ &\quad+(\alpha+\gamma)\left(\sup_{s\in[\tau+t_1,\tau+t_2]}|\xi(s)|\right)(t_2-t_1)\\ &\quad+\sup_{t\in[\tau+t_1,\tau+t_2]}\left|\int_{\tau+t_1}^{t}b(s,x(s))\left(u(s-\tau)-u^{\pm}(s-\tau)\right)ds\right|\\ &\quad+(\beta+\gamma)(t_2-t_1)\sup_{s\in[\tau+t_1,\tau+t_2]}|\xi(s)|\,K.\end{aligned}$$

Collecting terms, we obtain

$$\begin{aligned}\sup_{t\in[\tau+t_1,\tau+t_2]}|\xi(t)| &\le |\xi(\tau+t_1)|\\ &\quad+(\alpha+\gamma+(\beta+\gamma)K)(t_2-t_1)\left(\sup_{t\in[\tau+t_1,\tau+t_2]}|\xi(t)|\right)\\ &\quad+\sup_{t\in[\tau+t_1,\tau+t_2]}\left|\int_{\tau+t_1}^{t}b(s,x(s))\left(u(s-\tau)-u^{\pm}(s-\tau)\right)ds\right|,\end{aligned}$$

or

$$\begin{aligned}&[1-(\alpha+\gamma+(\beta+\gamma)K)(t_2-t_1)]\sup_{t\in[\tau+t_1,\tau+t_2]}|\xi(t)|\\ &\le |\xi(\tau+t_1)|+\sup_{t\in[\tau+t_1,\tau+t_2]}\\ &\times\left|\int_{\tau+t_1}^{t}b(s,x(s))\left(u(s-\tau)-u^{\pm}(s-\tau)\right)ds\right|. \quad (5.8)\end{aligned}$$

Now, let $\eta\in(0,t'-(\tau+t_1)]$ be a real number satisfying $(\alpha+\gamma+(\beta+\gamma)K)\eta<1$, set

$$t_2 := t_1+\eta, \tag{5.9}$$

and define the number

$$\mu(\eta) := \frac{1}{1-(\alpha+\gamma+(\beta+\gamma)K)\eta}.$$

Then, shifting the integration variable, (5.8) yields

$$\begin{aligned}\sup_{t\in[\tau+t_1,\tau+t_2]}|\xi(t)| &\le \mu(\eta)\,|\xi(\tau+t_1)|\\ &\quad+\mu(\eta)\sup_{t\in[t_1,t_1+\eta]}\\ &\quad\times\left|\int_{t_1}^{t-\tau}b(s+\tau,x(s+\tau))\left(u(s)-u^{\pm}(s)\right)ds\right|.\end{aligned} \tag{5.10}$$

To estimate the last term, recall the partition (5.4) and the accompanying relation (5.6). Let $q(t)\in\{0,1,2,\dots,p-1\}$ be the integer for which $t-\tau\in[q(t)\lambda,(q(t)+1)\lambda]$.

Then,

$$\sup_{t\in[t_1,t_1+\eta]}\left|\int_{t_1}^{t-\tau} b(s+\tau,x(s+\tau))\left(u(s)-u^{\pm}(s)\right)ds\right|$$
$$=\sup_{t\in[t_1,t_1+\eta]}\left|\sum_{i=0}^{q(t)-1}\int_{t_1+i\lambda}^{t_1+(i+1)\lambda} b(s+\tau,x(s+\tau))\left(u(s)-u^{\pm}(s)\right)ds\right.$$
$$\left.+\int_{t_1+q(t)\lambda}^{t-\tau} b(s+\tau,x(s+\tau))\left(u(s)-u^{\pm}(s)\right)ds\right|$$
$$=\sup_{t\in[t_1,t_1+\eta]}\left|\sum_{i=0}^{q(t)-1}\int_{t_1+i\lambda}^{t_1+(i+1)\lambda}\{b(t_1+\tau+i\lambda,x(t_1+\tau+i\lambda))\right.$$
$$-b(t_1+\tau+i\lambda,x(t_1+\tau+i\lambda))$$
$$+b(s+\tau,x(s+\tau))\}\left(u(s)-u^{\pm}(s)\right)ds$$
$$\left.+\int_{t_1+q(t)\lambda}^{t-\tau} b(s+\tau,x(s+\tau))\left(u(s)-u^{\pm}(s)\right)ds\right|$$
$$\le\sup_{t\in[t_1,t_1+\eta]}\left|\sum_{i=0}^{q(t)-1} b(t_1+\tau+i\lambda,x(t_1+\tau+i\lambda))\right.$$
$$\left.\times\int_{t_1+i\lambda}^{t_1+(i+1)\lambda}\left(u(s)-u^{\pm}(s)\right)ds\right|$$
$$+\sup_{t\in[t_1,t_1+\eta]}\left|\sum_{i=0}^{q(t)-1}\int_{t_1+i\lambda}^{t_1+(i+1)\lambda}\left[b(s+\tau,x(s+\tau))\right.\right.$$
$$\left.\left.-b(t_1+\tau+i\lambda,x(t_1+\tau+i\lambda))\right]\left(u(s)-u^{\pm}(s)\right)ds\right|$$
$$+\sup_{t\in[t_1,t_1+\eta]}\left|\int_{t_1+q(t)\lambda}^{t-\tau} b(s+\tau,x(s+\tau))\left(u(s)-u^{\pm}(s)\right)ds\right|.$$

Invoking (5.6) leads to

$$1\sup_{t\in[t_1,t_2]}\left|\int_{t_1}^{t-\tau} b(s+\tau,x(s+\tau))\left(u(s)-u^{\pm}(s)\right)ds\right|$$
$$\le\sum_{i=0}^{q(t)-1}\int_{t_1+i\lambda}^{t_1+(i+1)\lambda}\sup_{s\in[t_1+i\lambda,t_1+(i+1)\lambda]}\left|b(s+\tau,x(s+\tau))\right.$$
$$\left.-b(t_1+\tau+i\lambda,x(t_1+\tau+i\lambda))\right|\left|u(s)-u^{\pm}(s)\right|ds$$
$$+\sup_{t\in[t_1,t_2]}\left|\int_{t_1+q(t)\lambda}^{t-\tau}[b(s+\tau,x(s+\tau))]\left(u(s)-u^{\pm}(s)\right)ds\right|. \tag{5.11}$$

Referring to Corollary 3.1 and Lemma 5.1, choosing a real number $\rho>0$, and selecting $\lambda\le\beta(\rho,x_0,t')$, we obtain from (5.11) that

$$\sup_{t\in[t_1,t_2]}\left|\int_{t_1}^{t-\tau} b(s+\tau,x(s+\tau))\left(u(s)-u^{\pm}(s)\right)ds\right|$$
$$\le q(t)\lambda\rho 2K+M_{ab}(t')2K\lambda.$$

By (5.9), we have $0\le q(t)\lambda\le t_2-t_1=\eta$, so that

$$\sup_{t\in[t_1,t_2]}\left|\int_{t_1}^{t-\tau} b(s+\tau,x(s+\tau))\left(u(s)-u^{\pm}(s)\right)ds\right|$$
$$\le 2K\rho\eta+2KM_{ab}(t')\lambda. \tag{5.12}$$

Returning to (5.10) and substituting (5.12) into it, we obtain

$$\sup_{t\in[\tau+t_1,\tau+t_2]}|\xi(t)|\le\mu(\eta)\,|\xi(\tau+t_1)|$$
$$+\mu(\eta)\left[2K\rho\eta+2KM_{ab}(t')\lambda\right]. \tag{5.13}$$

Now, choose a number $\delta>0$; then, choose $\rho>0$ so that $\mu(\eta)K\rho\eta<\delta/4$ and choose $\lambda>0$ so that $\mu(\eta)KM_{ab}(t')\lambda<\delta/4$. Substituting into (5.13), this yields

$$\sup_{t\in[\tau+t_1,\tau+t_2]}|\xi(t)|\le\mu(\eta)\,|\xi(\tau+t_1)|+\delta. \tag{5.14}$$

Next, we use sub-intervals of length η to cover the interval $[\tau,t']$: let r be the smallest integer satisfying $r\ge(t'-\tau)/\eta$, and create the partition

$$[\tau,t']\subseteq\big\{[\tau,\tau+\eta],[\tau+\eta,\tau+2\eta],\dots,$$
$$[\tau+(r-1)\eta,\tau+r\eta]\big\}.$$

Recalling from (5.7) that $\xi(\tau)=0$, we can rewrite (5.14) in the recursive form

$$\sup_{t\in[\tau+i\eta,\tau+(i+1)\eta]}|\xi(t)|\le\mu(\eta)\,|\xi(\tau+i\eta)|+\delta,$$
$$\xi(\tau)=0,$$

$i=0,\dots,r-1$. This leads to the relation

$$\sup_{t\in[\tau,t']}|\xi(t)|\le\delta\sum_{i=0}^{r}(\mu(\eta))^i.$$

Therefore, referring to the number ϵ of the theorem's statement, and using

$$\delta<\varepsilon/\left(\sum_{i=0}^{r}(\mu(\eta))^i\right),\ \delta>0,$$

it follows that the bang-bang control input signal $u^{\pm}$ of (5.5) satisfies the requirements of the theorem. Note that, by (5.5), the bang-bang control input signal $u^{\pm}$ has a finite number of switchings – no more than $(t'-\tau)/\lambda$. This concludes our proof. ■

At this point, we can state the proof of the main result of this section.

Proof of Theorem 5.1. Consider first statement (i) of the theorem. According to Theorem 4.1(i), there is, for every time $t' > \tau$, a control input signal $u' \in U(K)$ such that $t(x_0, \gamma, u', \ell) \geq t'$. Further, according to Theorem 5.2, there is, for every time $t' \geq \tau$ and for every real number $\epsilon > 0$, a bang-bang control input signal $u'^{\pm} \in U(K)$ with a finite number of switchings for which $|\Sigma(x_0, v, u', t) - \Sigma(x_0, v, u'^{\pm}, t)| < \epsilon$ for all $t \in [0, t']$, independently of the residual input signal v and the member $\Sigma \in \mathcal{F}_\gamma(x_0, \tau)$. Now, for any vectors $y, z \in R^n$, we can write $z^Tz = y^Ty - 2y^T(y - z) + (y - z)^T(y - z) \leq y^Ty + 2n|y||y - z| + n|y - z|^2$. This leads us to the inequality

$$\begin{aligned} &\Sigma^T(x_0, \upsilon, u'^{\pm}, t)\Sigma(x_0, \upsilon, u'^{\pm}, t) \\ &\leq \Sigma^T(x_0, \upsilon, u', t)\Sigma(x_0, \upsilon, u', t) \\ &+2n\left|\Sigma(x_0, \upsilon, u', t)\right|\left|\Sigma(x_0, \upsilon, u'^{\pm}, t) - \Sigma(x_0, \upsilon, u', t)\right| \\ &+n\left|\Sigma(x_0, \upsilon, u'^{\pm}, t) - \Sigma(x_0, \upsilon, u', t)\right|^2 \\ &\leq \ell + 2n\sqrt{\ell}\varepsilon + n\varepsilon^2 \end{aligned}$$

at all times $t \in [0, t']$. Considering that $\ell' > \ell$, we can choose $\epsilon > 0$ sufficiently small, so that $\ell + 2n\sqrt{\ell}\varepsilon + n\varepsilon^2 \leq \ell'$. This then implies that $t(x_0, \gamma, u'^{\pm}, \ell') \geq t'$, which proves statement (i) of the theorem.

Statement (ii) of the theorem follows by a similar argument from Theorems 4.1(ii) and 5.2, upon replacing u' by $u(x_0, \gamma, \ell)$, t' by $t(x_0, \gamma, \ell)$, and $u'^{\pm}$ by $u^{\pm}(x_0, \gamma, \ell')$ in the previous paragraph. This concludes our proof. ■

To summarise, we have seen in this section that optimal control input signals can be replaced by bang-bang control input signals without appreciably affecting performance. This is a consequential fact, since bang-bang input signals are easier to calculate and implement than optimal input signals, as discussed earlier.

6. Examples

In this section, we provide two examples about the effectiveness of the controllers developed in this paper.

Example 6.1. Consider the following system that has two states and a scalar input:

$$\Sigma: \dot{x}(t) = \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} -x_2(t)(0.3 + \sin x_1(t)) + (2 + \cos t)u(t - 0.5) \\ a\frac{x_1(t)}{1+x_1^2(t)}\sin t + (2 + \sin x_2(t))u(t - 0.5) \end{pmatrix}.$$

Here, a is a constant parameter with an uncertain value in the range $-0.5 \leq a \leq 0.5$, and the input time-delay is $\tau = 0.5$ seconds. The initial state of Σ is $x_0 = [1, -0.5]^T$, and the system's input signal bound is $K = 1$. The residual input signal $v(t)$ is an unspecified constant signal $v(t) = c$, $t \in [-0.5, 0]$, where $-0.1 \leq c \leq 0.1$. The permissible error bound in this case is specified as $\ell = 4$, so that the control objective is to control the system Σ so as to keep the state $x(t)$ within the range

$$x^T(t)x(t) \leq 4 \tag{6.1}$$

for the longest time possible.

Note that, in this example, we use different uncertainty bounds for the parameters of the system Σ and for the residual input signal v. In previous sections, the same uncertainty bound γ was used in all cases in order to simplify expressions. As this example demonstrates, different uncertainty bounds can be used for different quantities without much ado.

Denote by $t(x_0)$ the maximal time during which the inequality (6.1) can be kept valid. To find $t(x_0)$, we performed a numerical search process (see Procedure 6.2 below); it turns out that $t(x_0)$ is approximately 4.1 seconds. As shown in Figure 3, the bang-bang control input signal $u^{\pm}(t)$ of Figure 2, which has just two switching times, keeps the state $x(t)$ within the range (6.1) for almost the entire maximal time of 4.1 seconds. It seems that, very often, relatively simple bang-bang input signals can achieve close to optimal performance.

For demonstration purposes, the plots of Figure 3 describe three samples of parameter values:

Set 1: $a = -0.5,\ \nu(\theta) = -0.1;$
Set 2: $a = 0,\ \nu(\theta) = 0;$
Set 3: $a = 0.5,\ \nu(\theta) = 0.1.$

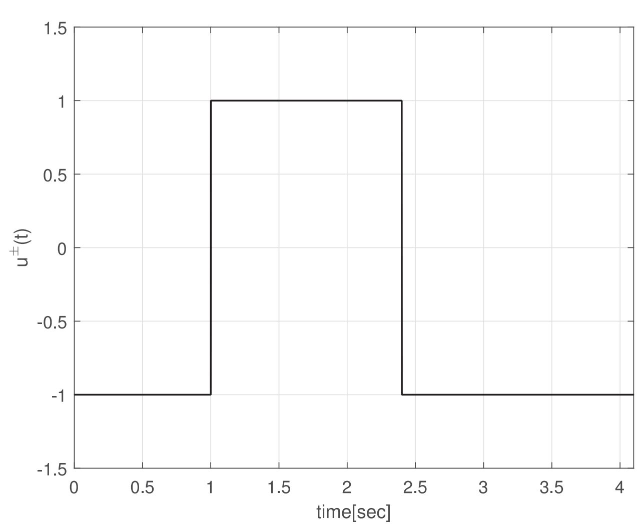

Figure 2. Bang-bang control input signal.

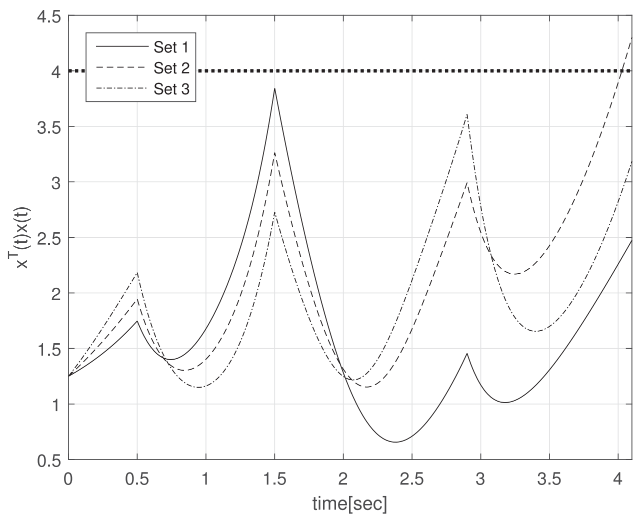

Figure 3. The response of Σ to the control input signal of Figure 2.

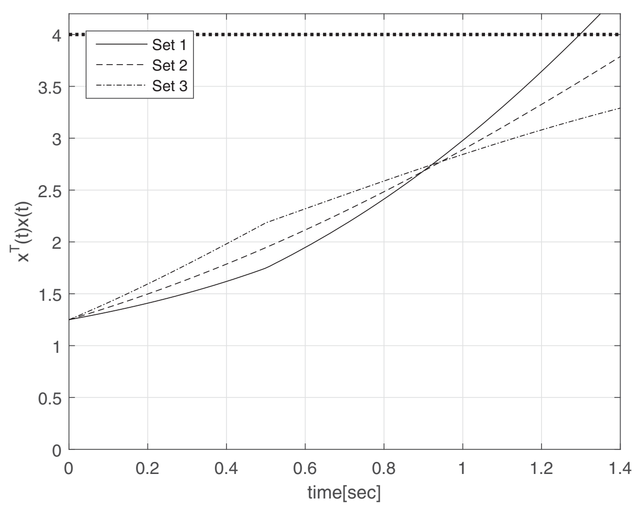

Figure 4. The response of Σ to the zero control input signal.

For comparison, Figure 4 show the response for the zero control input signal $u(t) = 0$, $t \geq 0$; as we can see form the figure, with this input signal, the error bound is breached at about $t = 1.3$ seconds. Clearly, the bang-bang control input signal of Figure 2, which achieves close to optimal performance, does indeed provide a substantially better outcome.

The numerical search process that was used to derive the bang-bang control input signal of Figure 2 can be described briefly as follows.

Procedure 6.2. Numerical Search Process. According to Theorem 5.1, bang-bang control input signals can keep the inequality (6.1) valid for almost $t(x_0)$ – the maximal time possible. Consequently, $t(x_0)$ can be estimated by searching over bang-bang control input signals, as described in the following steps.

Step 1: Perform a few preliminary tests with bang-bang control input signals having up to 10 switching times to find a bound on the maximal time $t(x_0)$. In this case, such tests show that $t(x_0)$ is not likely to exceed 5 seconds. To account for a possible under-estimate, the remaining search process is conducted over the time interval [0, 6].

Step 2: Divide the time interval [0,6] into 100 equal segments. Denote by $\mathcal{S}$ the set of endpoints of these segments. Then, the set $\mathcal{S}$ forms the set of potential switching times for bang-bang control input signals.

Step 3: Let $\mathcal{B}_1$ be the class of all bang-bang control input signals having one switching time within the set $\mathcal{S}$. For a control input signal $u \in \mathcal{B}_1$, let $t(u)$ be the time it takes the state of Σ to violate the operating error bound (6.1). Denote $t_1 := \max_{u\in\mathcal{B}_1} t(u)$; this is the longest time during which the inequality (6.1) can be kept valid with a control input signal from $\mathcal{B}_1$.

Step 4: For an integer $p \geq 1$, let $\mathcal{B}_p$ be the class of all bang-bang control input signals with p switching times within the set $\mathcal{S}$. For a control input signal $u \in \mathcal{B}_p$, let $t(u)$ be the time it takes the state of Σ to violate the operating error bound (6.1). Denote $t_p := \max_{u\in\mathcal{B}_p} t(u)$; this is the longest time during which the inequality (6.1) can be kept valid with a control input signal from $\mathcal{B}_p$.

Step 5: Select an error bound $\epsilon > 0$, and terminate the process when $|t_{p+1} - t_p| \leq \epsilon$.

Of course, if necessary, numerical search algorithms that are more sophisticated than the one described in Procedure 6.2 can be utilised.

Example 6.2. The linearised equation of a satellite on a circular equatorial orbit is represented by the following four dimensional system with two inputs (Jafarov, 2008):

$$\Sigma : \dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\Omega^2 & 0 & 0 & 2\Omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\Omega & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u(t-\tau). \tag{6.2}$$

Here, the state of the system is $x(t) = (x_1(t), x_2(t), x_3(t), x_4(t))^T = (r(t), \dot{r}(t), \theta(t), \dot{\theta}(t))^T$, where $r(t)$ and $\theta(t)$ represent the polar coordinates of the satellite in the equatorial plane; the constant parameter Ω represents the (constant) underlying angular velocity of the satellite. The input signal $u(t)$ has two components $u(t) = (u_1(t), u_2(t))^T$, where $u_1(t)$ is the radial thrust and $u_2(t)$ is the tangential thrust. The time delay τ represents the combination of two factors: the communication delay between the control station on earth and the satellite, and the reaction time of the satellite system.

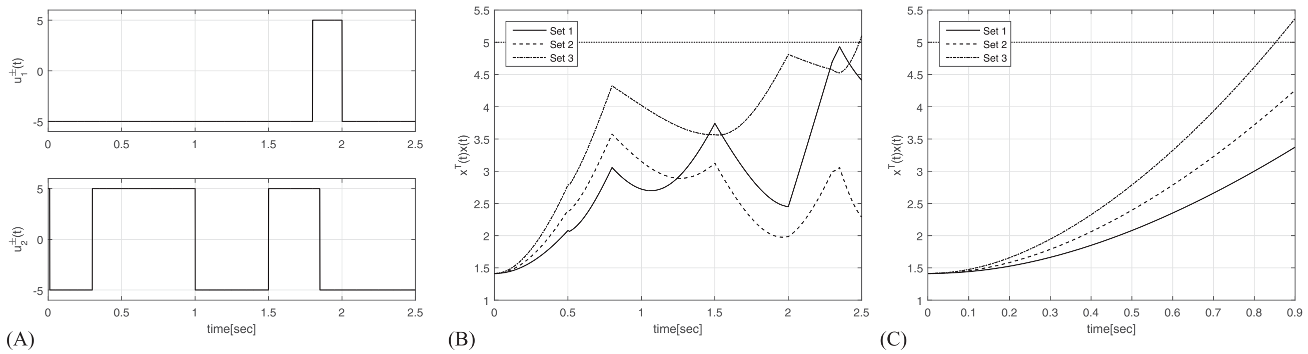

Figure 5. The response to optimal and non-optimal input : (A) Bang-bang control input signal; (B) The response of Σ to the control input signal of Figure 5 (A); (C) The response of Σ to the zero control input signal.

In the calculation below, we set the delay time at $\tau = 0.5$ seconds. We take into consideration uncertainties in the underlying constant angular velocity Ω by assuming $\Omega \in [0.9, 1.1]$. We further assume that the residual input signals are unspecified constants in the range $v_1(t), v_2(t) \in [-0.1, 0.1], t \in [-0.5, 0]$. The initial state of Σ, which was provided by the feedback right before it was interrupted, is given as $x_0 = x(0) = [1, 0, -1, 0]^T$. The input signal bound of the system, which represents the maximal thrust of the control rockets, is given as $K = 5$. The permissible error bound is specified as $\ell = 5$.

For demonstration purposes, the plots of Figure 5(B) describe three scenarios of parameter values, as follows:

$$\begin{aligned}
&\text{Set 1:}\quad \Omega = 0.9,\ v_1(\theta), v_2(\theta) = -0.1;\\
&\text{Set 2:}\quad \Omega = 1.0,\ v_1(\theta), v_2(\theta) = 0;\\
&\text{Set 3:}\quad \Omega = 1.1,\ v_1(\theta), v_2(\theta) = 0.1.
\end{aligned}$$

A qualitative analysis of Equation (6.2) indicates that the error can be kept below the specified bound for no more than 5 seconds. A numerical search over the time domain [0, 5] shows that the actual maximal time during which the state can be kept below the specified error bound of $\ell = 5$ is approximately $t^*(x_0) = 2.5$ seconds.

As we can see from Figure 5, the relatively simple bang-bang control input signal of Figure 5 (A) keeps the operating error below the specified value for almost the maximal time of 2.5 seconds (see the responses plotted in Figure 5(B) for the various parameter sets). Note that each component of the bang-bang control input signal $u = (u_1, u_2)^T$ of Figure 5 (A) has its own switching times.

For comparison, we provide in Figure 5 (C) the response of the system to the zero control input signal. As can be seen from the figure, with this signal the response violates the permissible error bound after 0.85 seconds. Thus, optimal input achieves an improvement by a factor of almost 3, compared to the zero input case.

7. Conclusion

In this paper, we investigated the existence and implementation of optimal controllers which, in the absence of feedback, keep operating errors below a specified error bound for the longest time possible. We have seen that such optimal controllers do exist for a broad family of nonlinear systems with input channel delays, and that optimal performance can be approximated as closely as desired by controllers that generate bang-bang input signals for the controlled system. Considering that bang-bang signals are relatively easy to calculate and implement, these results have significant implications on engineering practice.

The range of applications to which these results are relevant includes recovery from feedback failure; lessening of operating errors in networked control systems; improved performance of sampled-data control systems, where feedback is unavailable between samples; and many other applications.

Disclosure statement

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