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Optimal Low Error Control of Disturbed Systems

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***Abstract*—A linear time invariant system with uncertain initial conditions, perturbed parameters, and active disturbance signals operates in open loop as a result of feedback failure or interruption. The objective is to find an optimal input signal that drives the system for the longest time without exceeding specified error bounds, to allow maximal time for feedback reactivation. It is shown that such a signal exists, and that it can be replaced by a bang-bang signal without significantly affecting performance. The use of bang-bang signals simplifies calculation and implementation.**

I. INTRODUCTION

Needless to say, feedback is an essential tool for reducing operating errors in control systems. However, disruptions in feedback service cannot be completely avoided, as temporary loss of feedback may originate from technical failure. Furthermore, suspension of feedback can be part of routine operating conditions in certain applications, such as guidance and control of space vehicles, where feedback communication links may be disrupted by the loss of line-of-sight; digital control of continuous time systems, where feedback is obtained only at sampling intervals; networked control systems, where feedback channels are closed intermittently to reduce network traffic (e.g., [6], [3]); and medical applications, such as glucose control in diabetics, where feedback requires irksome biological testing and is obtained relatively infrequently (e.g. [11], [13] and [14]). Although an increase of performance errors is often unavoidable while feedback is halted, it would be desirable to develop an operating policy that keeps open-loop performance errors below specified bounds for the longest possible time. This will provide the best opportunity to restore feedback before unacceptable degradation of performance occurs.

The present paper derives an open loop controller that maximizes the duration of time during which a system can operate without feedback and not exceed acceptable error bounds. Additionally, issues related to the calculation and the implementation of such a controller are also examined. In particular, we show that the optimal input signal generated by the controller can be replaced by a bang-bang signal without significantly degrading system performance. Bang-bang signals, i.e. signals that switch between their maximal values, are relatively easy to compute and implement, as they are completely determined by their switching times.

The control diagram is presented in Figure 1. Here, Σ is a linear time invariant system whose parameters and initial conditions are not precisely known, and whose operation is affected by an unspecified disturbance signal $v(t)$.

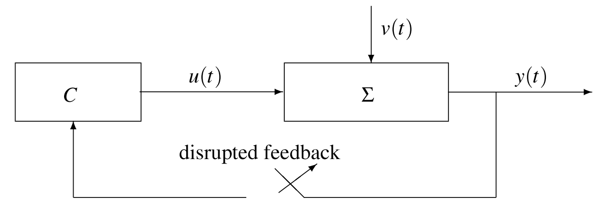

Figure 1. Basic Configuration

The controlled system is described by

$$\Sigma: \ \dot{x}(t) = A'x(t) + B'u(t) + G'v(t), \quad x(0) = x_0, \tag{1}$$

where $x(t) \in R^n$ is the state of the system, $u(t) \in R^m$ is the control input, $v(t) \in R^p$ is a disturbance signal, A' is an $n \times n$ matrix, B' is an $n \times m$ matrix, and G' is an $n \times p$ matrix. The initial condition $x_0 \in R^n$ of Σ, the entries of the matrices A', B', and G', and the disturbance signal $v(t)$ are not accurately specified. As the feedback signal is lost at the time $t = 0$, the system Σ operates in open loop for all times $t > 0$. After possibly having applied an appropriate shift transformation on the signals, we assume that the desired state trajectory of Σ is the zero signal $x(t) = 0$ for all $t \geq 0$. Our objective during the open loop operation is to ensure that $x(t)$ remains close to 0 for as long as possible, despite uncertainties and disturbances.

To describe the extent of uncertainty, we use the ℓ^∞-norm $\| \bullet \|$ given, for an n-dimensional vector $(c_1, \ldots, c_n)$, by $\|c\| := \max_{i=1,\ldots,n} |c_i|$, and for an $n \times m$ matrix C by $\|C\| := \max_{i=1,\ldots,n; j=1,\ldots,m} |c_{ij}|$; here c_{ij} is the (i,j) entry of C. The information available about the system Σ consists of the nominal initial condition x_0^0 and the nominal matrices A, B, and G of (1). The nominal disturbance signal is the zero signal.

To describe the uncertainty about the initial state x_0, we use a specified bound $\chi > 0$ to characterize the maximal deviation from the nominal initial state, so that the set of all possible initial states is

$$X_0 := \{x_0 \in R^n : \|x_0 - x_0^0\| \leq \chi\}. \tag{2}$$

The uncertainties about the entries of the matrices A', B', and G' of (1) are characterized similarly in terms of the nominal matrices A, B, G and a real number $d > 0$ by the inequalities

$$\|A' - A\| \leq d, \|B' - B\| \leq d, \text{ and } \|G' - G\| \leq d.$$

Denoting by Δ_A (respectively, Δ_B, Δ_G) the set of all $n \times n$ (respectively, all $n \times m$, all $n \times p$) matrices with entries in the interval $[-d,d]$, we can represent the perturbed matrices of (1) in the form

$$A' = A + D_A,\ B' = B + D_B,\ G' = G + D_G, \tag{3}$$

where $D_A \in \Delta_A, D_B \in \Delta_B$, and $D_G \in \Delta_G$. In shorthand, denote

$$D := (D_A, D_B, D_G) \text{ and } \Delta := \Delta_A \times \Delta_B \times \Delta_G, \tag{4}$$

so that $D \in \Delta$. For a particular selection of matrices given by (3), an initial condition $x_0 \in X_0$, and a disturbance signal $v(t)$, we denote the system of (1) by $\Sigma_{x_0,D,v}$. The response to an input signal $u(t)$ is then $x(t) = \Sigma_{x_0,D,v}u(t)$.

As the desired output signal of the system is the zero signal $x(t) = 0$ for all $t \geq 0$, we define the performance error as $e(t) = x^T(t)x(t)$. Our objective is to select the input signal $u(t)$ so as to keep the error $e(t)$ below a specified bound $M > 0$ for the longest possible time. If the error does not exceed the bound M during the time interval $[0,t_f]$, we can write

$$e(t) \leq M \text{ for all } 0 \leq t \leq t_f. \tag{5}$$

The optimal choice of $u(t)$ would maximize the value of t_f, taking into consideration the uncertainties and disturbances that affect Σ. In view of (5), we require $x_0^T x_0 \leq M$, as otherwise the initial error would exceed the allowed error.

A restricted version of this problem was introduced in [9],[7], and [8], where the noise signal $v(t)$ was not included and the initial condition x_0 was assumed to be accurately specified. The present paper extends these results to systems that are subject to disturbances and have unspecified initial conditions. We show in section II that the problem of calculating an optimal input signal $u(t)$ is a max-min optimization problem. In section III we prove that this problem has a solution, and in section IV we show that an optimal input signal $u(t)$ can be replaced by a bang-bang signal, with only a negligible effect on system performance. This fact simplifies the process of calculating and implementing an optimal solution, as bang-bang signals are completely determined by their switching times.

II. A Max-Min Formulation

To formalize our objective, we use the weighted inner product $\langle a,b\rangle = \int_0^\infty e^{-\alpha t} a(t)^T b(t) dt$, where $a(t)$ and $b(t)$ are m-dimensional vector valued Lebesgue measurable functions, α is a positive real number, and the integral is taken in the Lebesgue sense. The weight function $e^{-\alpha t}$ makes this inner product well-defined for all uniformly bounded functions. Denote by $L_2^{\alpha,m}$ the Hilbert space of all m-dimensional Lebesgue measurable functions with the inner product $\langle\cdot,\cdot\rangle$. In addition, we use the point-wise ℓ^∞-norm, which, for a function $f(t) = (f_1(t),...,f_m(t))$, is given by $\|f(t)\| := max_{i=1,...,m}|f_i(t)|$ at each time t.

The physical characteristics of systems often impose strict bounds on the allowable input amplitude. We denote by $K > 0$ the input amplitude bound of the system Σ of (1); then, set of all permissible input functions of Σ is

$$U := \{u \in L_2^{\alpha,m} : \|u(t)\| \leq K \text{ for all } t \geq 0\}. \tag{6}$$

Similarly, the disturbance signal $v(t)$ of (1) must also be bounded. Denoting by $L > 0$ the bound on the disturbance amplitude, the set of all permissible disturbance signals is

$$V := \{v \in L_2^{\alpha,p} : \|v(t)\| \leq L \text{ for all } t \geq 0\}. \tag{7}$$

While the arguments in this paper require the bounds L and K to be finite, no special relationship is assumed about their magnitudes. In practice, disturbance signals often originate from environmental noises and interferences, and have amplitudes that are much smaller than the amplitude of the control input signal $u(t)$, i.e., often $L \ll M$.

To highlight the dependence of the state trajectory $x(t)$ of (1) on the quantities x_0, D, v, and u, we usually write $x(t,x_0,D,v,u)$ instead of $x(t)$. Then (5) takes the form

$$e(t,x_0,D,v,u) := x^T(t,x_0,D,v,u)x(t,x_0,D,v,u) \leq M, \quad 0 \leq t \leq t_f \tag{8}$$

The time during which the error $e(t,x_0,D,v,u)$ does not exceed its bound M is given by

$$T(M,x_0,D,v,u) := \inf\{t \geq 0 : e(t,x_0,D,v,u) > M\}, \tag{9}$$

where $T(M,x_0,D,v,u) := \infty$ if $e(t,x_0,D,v,u) \leq M$ for all $t \geq 0$. As the initial state satisfies $x_0^T x_0 \leq M$, we have $T(M,x_0,D,v,u) \geq 0$. Recall that our objective is to find an input function $u(t) \in U$ that drives Σ so as to satisfy the error bound (8) for the longest possible time t_f, irrespective of uncertainties and disturbances. In our current notation, we need to select the input function u so as to obtain the largest possible duration $T(M,x_0,D,v,u)$, irrespective of the uncertainties about the initial conditions, about the matrices A', B', G', and about the disturbance signal v.

Consider now a fixed input signal u. Taking into account the perturbed values $x_0 \in X_0$, $D \in \Delta$, and $v \in V$, the longest time $T^*(M,u)$ during which the error does not exceed M for any perturbation or disturbance is the lowest value of $T(M,x_0,D,v,u)$ over all such perturbations and disturbances, i.e.,

$$T^*(M,u) = \inf_{(x_0,D,v)\in X_0\times\Delta\times V} T(M,x_0,D,v,u). \tag{10}$$

Thus, to maximize the duration t_f in (5), the best input signal $u(t)$ would be one that maximizes $T^*(M,u)$. If such an input signal exists in U, it yields the maximal time

$$t_f^* := \sup_{u\in U} T^*(M,u) \tag{11}$$

during which the error remains within desirable bounds, irrespective of which permissible combination of perturbations and disturbances is active. Denoting such an optimal function by $u^*(t)$, we obtain $t_f^* = T^*(M,u^*)$, and our objective can formally be phrased as follows.

Problem 1. Determine whether an optimal input signal $u^* \in U$ exists; if such a signal exists, describe a method for its computation. □

From (10) and (11), it is follows that the calculation of an optimal input signal u^* involves the solution of a max-min optimization problem. We proceed next to show that a solution to this problem does exist.

III. EXISTENCE OF AN OPTIMAL SOLUTION

In this section, we show that Problem 1 does have a solution. In broad terms, this is accomplished by showing that the set U of (6) has a certain compactness feature and that the function $T^*(M,u)$ of (10) has an appropriate continuity property. The existence of the supremal time t_f^* of (11) follows then by a generalized version of the Weierstrass Theorem. The compactness feature of the set U is described in the following statement, which generalizes a result of [7], [8]; a proof is provided in [10].

Lemma 2. *The set U of (6) is weakly compact in the topology of the Hilbert space $L_2^{\alpha,m}$.*

We say that the system Σ of (1) is *nominally unstable* if the nominal matrix A has at least one eigenvalue with strictly positive real part. Nominal instability of Σ implies that the state trajectory $x(t)$ must escape the bound M; this, in turn, implies that the optimal time t_f^* must be finite (see [10] for a proof):

Lemma 3. *Assume that the system Σ of (1) is nominally unstable and adopt the notation of (2), (4), and (9). Then, for each input function $u(t) \in U$, there is a triplet $(x_0,D,v) \in X_0 \times \Delta \times V$ for which $T(M,x_0,D,v,u) < \infty$.*

Regarding the continuity of $T^*(M,u)$, it is sufficient for our purpose to show weak upper semi-continuity, as follows (see [10] for a proof).

Lemma 4. *When the system Σ of (1) is nominally unstable. the function $T^*(M,u)$ of (10) is weakly upper semi-continuous in u.*

Finally, employing the generalized Weierstrass Theorem (e.g., [5]), we conclude from Lemmas 2 and 4 that Problem 1 has a solution (see [10] for details):

Theorem 5. *Assume that the system Σ of (1) is nominally unstable, and let U be given by (6). Then, using the notation of (11), the following are true.*

(i) There is a finite maximal time $t_f^* := sup_{u\in U} T^*(M,u)$, and

(ii) There is an input function $u^* \in U$ satisfying $t_f^* = T^*(M,u^*)$.

The longest duration of time t_f during which the system's response can be kept below the specified error bound M may vary depending on the values of the initial condition x_0, the perturbation matrix D, and the disturbance signal $v(t)$. However, this duration of time always satisfies $t_f \geq t_f^*$, and t_f^* is the maximal time that satisfies this inequality.

IV. BANG-BANG APPROXIMATION

We turn now to the consideration of issues related to the computation and the implementation of optimal input signals $u^*(t)$ that solve Problem 1; recall that such functions are guaranteed to exist by Theorem 5. Broadly speaking, the computation and the implementation of optimal signals is never an easy task. This is even more so in the present case, due to the complex nature of the conditions that characterize the optimal solution. The current section points to a simple way out of this complexity: we show that an optimal signal $u^*(t)$ can be replaced by a bang-bang signal without causing significant performance deterioration. A bang-bang input signal of Σ consists of component functions whose values switch between K and $-K$ as necessitated by control action, where K is the input bound of Σ. Bang-bang functions are completely determined by their switching times, and hence are relatively easy to calculate and implement.

Bang-bang input signals may not yield exactly the same performance as an optimal input signal. However, as the next statement indicates, optimal performance can be approximated as closely as desired by bang-bang input signals (compare to [7], where a related result is derived under more restrictive conditions).

Theorem 6. *Let Σ be a nominally unstable system given by (1), let U be the set of input signals (6), and let $x(t,x_0,D,v,u)$ be the state trajectory of Σ induced by an input function u. Let t_f^* be the optimal time and let u^* be an optimal input function of Theorem 5. Then, for every $\varepsilon > 0$, there is a bang-bang input function $u^\pm \in U$ for which the following are true.*

(i) $u^\pm$ has only a finite number of switches, and

(ii) The discrepancy between the state trajectories satisfies $\|x(t,x_0,D,v,u^*) - x(t,x_0,D,v,u^\pm)\| < \varepsilon$ for all $t \in [0,t_f^*]$ and for all $(x_0,D,v) \in X_0 \times \Delta \times V$.

Proof: We use the notation of (4), (5), and (6). As Σ is nominally unstable, it follows by Theorem 5 that the optimal time t_f^* is finite. Now, let $\varepsilon, \eta > 0$ be two real numbers. In view of the fact that the exponential function is uniformly continuous over any finite interval of time, there is a real number $\delta(\eta) > 0$ such that the function $\mu(t',t) := e^{-A't'} - e^{-A't}$ satisfies $\|\mu(t',t)\| \leq \eta$ whenever $|t'-t| < \delta(\eta)$ and $t',t \in [0,t_f^*]$. Denote $\beta := sup\{\|B+D_B\| : D_B \in \Delta_B\}$ and $N := \sup\left\{\left\|e^{A't}\right\| : D_A \in \Delta_A, t \in [0,t_f^*]\right\}$; here, β and N exist due the fact that all involved quantities are bounded.

Next, let $0 < \gamma \leq \delta(\eta)$ be any number for which the ratio t_f^*/γ is an integer. We build a partition of the interval $[0,t_f^*]$ into segments of length γ, namely, the partition determined by the intervals $[q\gamma,(q+1)\gamma]$, $q = 0,1,2,...,(t_f^*/\gamma)-1$. Recalling that input functions of Σ are m-dimensional column vectors bounded by $K > 0$, we build a bang-bang input function $u^\pm(t) = (u_1^\pm(t), u_2^\pm(t), ..., u_m^\pm(t))^T, 0 \leq t \leq t_f^*$, as follows: for the component $u_i^\pm(t)$, select in each interval $[q\gamma,(q+1)\gamma]$ a switching time θ_{qi} and set

$$u_i^\pm(t) := \begin{cases} K \text{ for } t \in [q\gamma, \theta_{qi}), \\ -K \text{ for } t \in [\theta_{qi},(q+1)\gamma), q = 0,1,2,...,(t_f^*/\gamma)-1, \end{cases} \tag{12}$$

$i = 1,2,...m$. For each such component function, we have $\int_{q\gamma}^{(q+1)\gamma} u_i^\pm(\tau)d\tau = K\int_{q\gamma}^{\theta_{qi}} d\tau - K\int_{\theta_{qi}}^{(q+1)\gamma} d\tau = K[2(\theta_{qi} - q\gamma) - \gamma]$. Now, select θ_{qi} to satisfy the equality $K[2(\theta_{qi} - q\gamma) - \gamma] =$

$\int_{q\gamma}^{(q+1)\gamma} u_i^*(\tau)d\tau$. Note that θ_{qi} exists due to the fact that $|u_i^*(t)| \leq K$ for all $t \geq 0$. For this value of θ_{qi}, we obtain the equality

$$\int_{q\gamma}^{(q+1)\gamma} [u_i^*(\tau) - u_i^{\pm}(\tau)]d\tau = 0 \tag{13}$$

for all $i = 1,2,...,m$ and all $q = 0,1,2,...,(t_f^*/\gamma)-1$.

Recall that the solution of (1) for particular values (A',B',G') of the system parameters, for an input signal $u(t)$ and for a disturbance function $v(t)$, is given by

$$x(t;u,v) = e^{A't}\left[x_0 + \int_0^t e^{-A'\tau}B'u(\tau)d\tau + \int_0^t e^{-A'\tau}G'v(\tau)d\tau\right] \tag{14}$$

Further, let $x^{\pm}(t)$ be the state trajectory generated by the system Σ when driven by the input function $u^{\pm}(t)$, and let $x^*(t)$ be the state trajectory induced by the optimal input function $u^*(t)$. Noting that the initial condition x_0, the perturbation matrix D, and the disturbance input $v(t)$ are all the same in both cases (we are considering the performance of the same system sample), we obtain from (14) and (13) that

$$\begin{aligned}
&\|x^*(t) - x^{\pm}(t)\| \\
=&\|e^{A't}\left[x_0 + \int_0^t e^{-A'\tau}B'u^*(\tau)d\tau\right] \\
&\qquad - e^{A't}\left[x_0 + \int_0^t e^{-A'\tau}B'u^{\pm}(\tau)d\tau\right] \\
=&\left\|e^{A't}\int_0^t e^{-A'\tau}B'[u^*(\tau) - u^{\pm}(\tau)]d\tau\right\| \\
\leq& N\left\|\int_0^t e^{-A'\tau}B'\left[u^*(\tau) - u^{\pm}(\tau)\right]d\tau\right\| \\
=&N\Bigg\|\left[\sum_{r=0}^{q-1}\int_{r\gamma}^{(r+1)\gamma} e^{-A'\tau}B'\left[u^*(\tau) - u^{\pm}(\tau)\right]d\tau\right] \\
&\qquad + \int_{q\gamma}^t e^{-A'\tau}B'\left[u^*(\tau) - u^{\pm}(\tau)\right]d\tau\Bigg\| \\
\leq&N\Bigg\|\sum_{r=0}^{q-1}\Bigg[e^{-A'r\gamma}B'\int_{r\gamma}^{(r+1)\gamma}\left[u^*(\tau) - u^{\pm}(\tau)\right]d\tau \\
&\qquad + \int_{r\gamma}^{(r+1)\gamma}\mu(\tau,r\gamma)B'\left[u^*(\tau) - u^{\pm}(\tau)\right]d\tau\Bigg]\Bigg\| \\
&\qquad + N\left\|\int_{q\gamma}^t e^{-A'\tau}B'\left[u^*(\tau) - u^{\pm}(\tau)\right]d\tau\right\| \\
\leq&N\sum_{r=0}^{q-1}\int_{r\gamma}^{(r+1)\gamma}\|\mu(\tau,r\gamma)\|\left\|B'\right\|\left[\|u^*(\tau)\| + \|u^{\pm}(\tau)\|\right]d\tau \\
&\qquad + N\int_{q\gamma}^t\left\|e^{-A'\tau}\right\|\|B'\|\left[\|u^*(\tau)\| + \|u^{\pm}(\tau)\|\right]d\tau \\
\leq&2KN\beta(\eta t_f^* + N\gamma)
\end{aligned}$$

for all $t \in [0,t_f^*]$. Finally, choose the value of η so that $2KN\beta\eta t_f^* < \varepsilon/2$. Then, choose γ so that

$$0 < \gamma \leq \min\{\delta(\eta), \varepsilon/(4KN^2\beta)\} \text{ and } t_f^*/\gamma \text{ is an integer.} \tag{15}$$

For these selections, we obtain $\|x^*(t) - x^{\pm}(t)\| < \varepsilon$ for all $t \in [0,t_f^*]$, and our proof concludes. ■

We emphasize that the bang-bang input signal $u^{\pm}(t)$ of Theorem 6 approximates optimal performance for all permissible perturbations and disturbance signals of the system Σ (see [10] for further details).

Remark 7. In Theorem 6, the cost of making the error ε smaller is an increase in the number of switches of the bang-bang function $u^{\pm}(t)$. This can be seen by examining inequality (15): to maintain the inequality, γ must be decreased as ε is decreased. According to (12), the number of switches is (in general) t_f^*/γ, so that a decrease of γ leads to an increase in the number of switches. □

A. Design considerations

In view of (10) and (11), the calculation of an optimal input function involves finding the 'worst' disturbance signal $v(t)$. In analogy to Theorem 6, the next statement shows that the worst disturbance signal can also be replaced by a bang-bang signal, without significantly affecting system output. Thus, both signals - optimal input and worst disturbance - can be replaced by bang-bang signals without significantly affecting performance. This is important, since the replacement of general signals by bang-bang signals turns the original infinite-dimensional optimization problem into a finite dimensional one.

Theorem 8. *Let Σ be a nominally unstable system given by (1), let U be the set of input signals (6), and let V be the set of disturbance signals (7). Let $x(t,x_0,D,v,u)$ be the state trajectory induced by the input signal u in the presence of the disturbance signal v. Finally, let t_f^* be the optimal time and let u^* be an optimal input signal of Theorem 5. Then, for every $\varepsilon > 0$ and for every disturbance signal $v \in V$, there are a bang-bang input signal $u^{\pm} \in U$ and a bang-bang disturbance signal $v^{\pm} \in V$ for which the following hold true.*

(i) $u^{\pm}$ and $v^{\pm}$ have a finite number of switches, and

(ii) The state trajectory $x(t,x_0,D,v^{\pm},u^{\pm})$ created by $u^{\pm}$ and $v^{\pm}$ satisfies $\|x(t,x_0,D,v,u^*) - x(t,x_0,D,v^{\pm},u^{\pm})\| < \varepsilon$ for all $t \in [0,t_f^*]$ and all $(x_0,D) \in X_0 \times \Delta$.

Proof: We use the notation of the proof of Theorem 6. As in that proof, the fact that Σ is nominally unstable implies, by Theorem 5, that the optimal time t_f^* is finite. The set of permissible disturbance signals is given by the set V of (7). A disturbance signal of Σ is a p-dimensional column vector with entry functions bounded by $L > 0$. Now, fix a disturbance signal $v(t) \in V$. We build a bang-bang disturbance signal $v^{\pm}(t) = (v_1^{\pm}(t), v_2^{\pm}(t), ..., v_p^{\pm}(t))^T, 0 \leq t \leq t_f^*$, that 'approximates' the effects of $v(t)$ as follows: for the component $v_i^{\pm}(t)$, select in each interval $[q\gamma,(q+1)\gamma]$ a switching time ψ_{qi} and set

$$v_i^{\pm}(t) := \begin{cases} L \text{ for } t \in [q\gamma, \psi_{qi}), \\ -L \text{ for } t \in [\psi_{qi}, (q+1)\gamma), q = 0,1,2,...,(t_f^*/\gamma)-1, \end{cases}$$

$i = 1,2,...,p$. Then, we have $\int_{q\gamma}^{(q+1)\gamma} v_i(\tau)d\tau = L\int_{q\gamma}^{\psi_{qi}} d\tau - L\int_{\psi_{qi}}^{(q+1)\gamma} d\tau = L[2(\psi_{qi} - q\gamma) - \gamma]$. Select ψ_{qi} to satisfy the equality

$$L[2(\psi_{qi} - q\gamma) - \gamma] = \int_{q\gamma}^{(q+1)\gamma} v_i(\tau)d\tau.$$

Note that ψ_{qi} exists due to the fact that $|v_i(t)| \leq L$ for all $t \geq 0$. For this value of ψ_{qi}, we obtain

$$\int_{q\gamma}^{(q+1)\gamma} [v_i(\tau) - v_i^{\pm}(\tau)]d\tau = 0 \tag{16}$$

for all $i = 1,2,...,p$ and all $q = 0,1,2,...,(t_f^*/\gamma) - 1$.

Further, let $x^{\pm}(t)$ be the state trajectory generated by the system Σ when driven by the bang-bang input function $u^{\pm}(t)$ of Theorem 6 in the presence of the bang-bang disturbance signal $v^{\pm}(t)$, and let $x^*(t)$ be the state trajectory induced by the optimal input function $u^*(t)$ in the presence of the actual disturbance signal $v(t)$. Noting that the initial condition x_0 and the perturbation matrix D are the same in both cases (we are considering the performance of the same system sample), we obtain from (14), (13), and (16) that

$$\begin{aligned}
&\|x^*(t) - x^{\pm}(t)\| \\
=&\left\| e^{A't}\left[x_0 + \int_0^t e^{-A'\tau}B'u^*(\tau)d\tau + \int_0^t e^{-A'\tau}G'v(\tau)d\tau\right] + \right.\\
&\left. - e^{A't}\left[x_0 + \int_0^t e^{-A'\tau}B'u^{\pm}(\tau)d\tau + \int_0^t e^{-A'\tau}G'v^{\pm}(\tau)d\tau\right]\right\| \\
=&\left\| e^{A't}\int_0^t e^{-A'\tau}B'[u^*(\tau) - u^{\pm}(\tau)]d\tau \right.\\
&\left. \quad + e^{A't}\int_0^t e^{-A'\tau}G'[v(\tau) - v^{\pm}(\tau)]d\tau\right\| \\
\leq& N\left\|\int_0^t e^{-A'\tau}B'\left[u^*(\tau) - u^{\pm}(\tau)\right]d\tau\right\| \\
&\quad + N\left\|\int_0^t e^{-A'\tau}G'\left[v(\tau) - v^{\pm}(\tau)\right]d\tau\right\|
\end{aligned} \tag{17}$$

Now, according to the proof of Theorem 6, we have

$$N\left\|\int_0^t e^{-A'\tau}B'\left[u^*(\tau) - u^{\pm}(\tau)\right]d\tau\right\| \leq 2KN\beta(\eta t_f^* + N\gamma). \tag{18}$$

Further, using the quantity $g := \sup\{\|G + D_G\| : D_G \in \Delta_G\}$, an argument similar to the one used in the proof of Theorem 6 yields the inequality

$$N\left\|\int_0^t e^{-A'\tau}G'\left[v(\tau) - v^{\pm}(\tau)\right]d\tau\right\| \leq 2LNg(\eta t_f^* + N\gamma). \tag{19}$$

Combining (18) and (19), we obtain from (17) that

$$\|x^*(t) - x^{\pm}(t)\| \leq 2N(K\beta + Lg)(\eta t_f^* + N\gamma).$$

Finally, choose the value of η so that $2N(K\beta + Lg)\eta t_f^* < \varepsilon/2$. Then, choose γ so that $0 < \gamma \leq \min\{\delta(\eta), \varepsilon/[4N^2(K\beta + Lg)]\}$ and t_f^*/γ is an integer. For these selections, we obtain $\|x^*(t) - x^{\pm}(t)\| < \varepsilon$ for all $t \in [0, t_f^*]$, and our proof concludes. ■

As in Remark 7, the accuracy of the approximation provided by the bang-bang functions $u^{\pm} \in U$ and $v^{\pm} \in V$ of Theorem 8 can be improved by increasing the number of switches.

The following algorithm uses Theorem 8 and a finite dimensional optimization process to obtain a bang-bang input signal for Σ that approximates the performance of an optimal solution of Problem 1.

B. Algorithm

Algorithm 9. Calculating a bang-bang approximant of an optimal input function:

Let $u^{\pm}(t) = [u_1^{\pm}(t), u_2^{\pm}(t), ..., u_m^{\pm}(t)]^T$ be a bang-bang approximant of an optimal input function $u^*(t)$, let $v^{\pm}(t) = [v_1^{\pm}(t), v_2^{\pm}(t), ..., v_p^{\pm}(t)]^T$ be a bang-bang approximant of the 'worst' disturbance function, and let $x^{\pm}(t)$ be the state trajectory induced by $u^{\pm}$ and $v^{\pm}$. Denote by $t_f^{\pm}$ the time at which $x^{\pm}$ exceeds the specified error bound, i.e., $t_f^{\pm} := \inf\{t \geq 0 : [x^{\pm}(t)]^T x^{\pm}(t) > M\}$. Let μ be the largest permissible deviation between $t_f^{\pm}$ and the optimal time t_f^*, so that $t_f^* - t_f^{\pm} \leq \mu$. Finally, assume that a bound t_f of t_f^* is provided, so that $t_f^* \leq t_f$. Let k denote the number of switches of each component of $u^{\pm}(t)$ and $v^{\pm}(t)$.

Step 1. Set $t_f^0 := 0$ and $k := 1$.

Step 2. Partition the interval $[0, t_f]$ into $Q \gg k$ equal segments. On this partition, create two families of bang-bang functions whose switching times are compatible with the partition: the family $U^{\pm}(k,Q) \subset U$ of all bang-bang functions $u(t) = [u_1(t), u_2(t), ..., u_m(t)]^T$ that have at most k switches in each component; and the family $V^{\pm}(k,Q) \subset V$ of all bang-bang functions $v(t) = [v_1(t), v_2(t), ..., v_p(t)]^T$ that have at most k switches in each component. Both of families are, of course, finite.

Step 3. For each $u(t)$ created in Step 2, calculate the quantity $T(u,k) := \inf_{(x_0,D,v) \in X_0 \times \Delta \times V^{\pm}(k,Q)} T(M, x_0, D, v, u)$. This is a finite dimensional minimization process.

Step 4. Let $t_f^k := \sup_{u \in U^{\pm}(k,Q)} T(u,k)$, and denote by $u^k \in U^{\pm}(k,Q)$ a function that achieves this maximum. Then, t_f^k is the best duration that can be achieved when using bang-bang approximants with at most k switches.

If $k = 1$, or if $k > 1$ and $t_f^k > t_f^{k-1} + \mu$, then replace k by $k+1$ and return to Step 2.

Otherwise, i.e., if $k > 1$ and $t_f^k < t_f^{k-1} + \mu$, then stop the algorithm. Use $t_f^* \approx t_f^{k-1}$ and $u^{\pm}(t) \approx u^{k-1}$. □

Algorithm 9 transforms our dynamic optimization problem into a finite dimensional optimization problem that can be solved numerically by a wide range of available optimization techniques (see, e.g., [16], [17], the references cited in these papers, and others).

Example 10. Consider a single state system described by the equation $\dot{x}(t) = ax(t) + u(t) + v(t)$ with the initial condition $x(0) = x_0$, the control input $u(t)$, and the disturbance signal $v(t)$. The uncertainties are described by $x_0 \in [0.9, 1.1]$, $a \in [1.2, 1.4]$, and $|v(t)| \leq 0.2$ for all $t > 0$; the input function amplitude bound is 2, i.e., $|u(t)| \in [-2, 2]$ for all $t \geq 0$. Taking

$M=25$, we need to calculate an optimal input function $u^*(t)$ that produces the maximal time t_f^*, irrespective of perturbations and disturbances. In the process, we also find worst instances of the parameters a and x_0, and of the disturbance signal $v(t)$. Specializing (11) to our present data, we seek an input function $u^*(t)$ that solves the max-min problem

$$t_f^* = \sup_{\{u(t):|u(t)|\le 2, t\ge 0\}} \left\{ \inf_{\substack{0.9\le x_0\le 1.1 \\ [1.2\le a\le 1.4] \\ \{v(t):|v(t)|\le 0.2, t\ge 0\}}} T(25, a, x_0, v(t), u(t)) \right\}.$$

To find a solution to this problem, we use Algorithm 9 to search over bang-bang approximants of optimal input signals $u^*(t)$ and worst disturbance signals $v(t)$.

In Step 3 of Algorithm 9, we find for each bang-bang input function $u^{\pm}(t)$, the values of a, of x_0, and the switching times of a disturbance signal $v^{\pm}(t)$ that yield the lowest value of $T(25, a, x_0, v(t), u(t))$. This search is implemented by using a global optimization algorithm based on multilevel coordinate search ([15]).

Using $\mu = 0.01$, the present calculation stops at $k = 2$ in Step 4 of Algorithm 9, yielding the approximation $t_f^* \approx t_f^{2-1} = t_f^1 = 2.18$ seconds; the approximate input solution $u^{\pm}(t)$ is given by

$$u^{\pm}(t) \quad = \quad \begin{cases} -2 & \text{for } t \le 1.248, \\ +2 & \text{for } t > 1.248. \end{cases} \tag{20}$$

With this input function, there are two sets of parameters and disturbance combinations that yield the worst terminal time $t_f^* \approx 2.18$, as follows:

$$\{a = 1.4, x_0 = 1.1, \text{ and } v(t) = 0.2 \text{ for all } t \ge 0.\} \tag{21}$$

$$\{a = 1.4, x_0 = 0.9, \text{ and } v(t) = -0.2 \text{ for all } t \ge 0.\} \tag{22}$$

In this approximation, the 'worst' disturbance signal $v^{\pm}(t)$ turns out to be constant in both cases. Figure 2 illustrates the result under the conditions of (21).

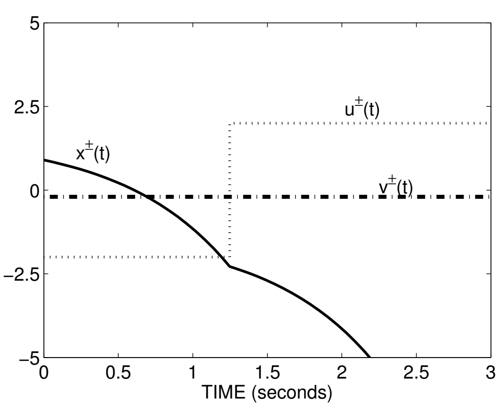

Figure 2. Disturbance set (21)

Similarly, Figure 3 displays the response under the conditions of (22).

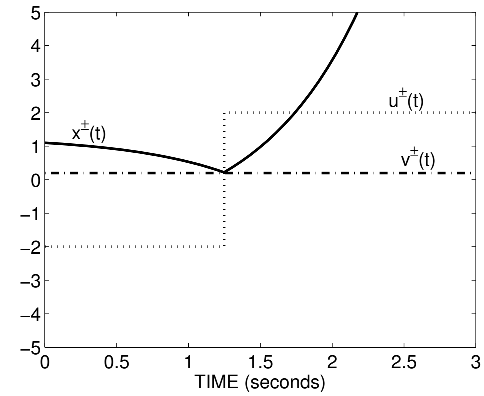

Figure 3. Disturbance set (22)

In conclusion, the paper presents a general theory for finding optimal input signals that keep performance errors below specified bounds for the longest possible time under a broad range of uncertainties and disturbances. The use of bang-bang signals to approximate optimal performance provides an effective approach to finding and implementing solutions of this optimization problem.

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