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ON THE EFFECTS OF DISTURBANCES IN NONLINEAR CONTROL

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Abstract

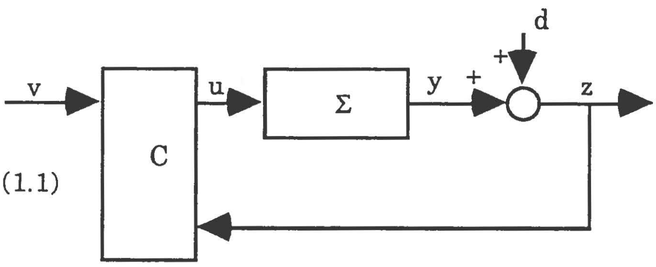
The paper deals with the control of a nonlinear system whose output is subject to an additive disturbance. The main result is a simple parametrization of the set of all system responses that can be obtained through internally stable control of the given system. An internally stable implementation scheme is provided for each possible response.

The class of achievable responses is determined by the 'numerator' of a right coprime fraction representation of the system being controlled.

1. Introduction

We consider the control of a nonlinear system Σ whose output signal is corrupted by an additive disturbance d. The control scheme can always be represented in the equivalent form

(1.1)



Here, C is an equivalent controller that incorporates all the control elements used. The external input signal is v, and the output signal is z. The disturbance d is regarded as an external signal over which only limited data is provided. The composite system (1.1) is required to be internally stable, and all implementations discussed below satisfy this requirement. Internal stability signifies that the configuration can tolerate small disturbances on its external and internal ports (including ports within the equivalent controller C), without loosing stability. Formally, we can write

(1.2) $$z = \Sigma_c(v,d),$$

where Σ_c is an equivalent system that depends on the controller C and on Σ.

The main result of the paper depends on basic concepts from the theory of fraction representations. A *right fraction representation* of a nonlinear system Σ is a factorization of Σ into a composition $\Sigma = PQ^{-1}$, where P and Q are stable systems and Q is invertible. The fraction representation $\Sigma = PQ^{-1}$ is *coprime* when P and Q are right coprime (see HAMMER [1985 and 1987] for coprimeness). Recall that a system is *bicausal* if it is causal and possesses a causal inverse.

The main result can be summarized as follows. Let Σ be a strictly causal and stabilizable system, and let $\Sigma = PQ^{-1}$ be a right coprime fraction representation with a bicausal 'denominator' Q (such fraction representations were derived in HAMMER [1989c]). Then,

(i) For every causal equivalent controller C for which (1.1) is internally stable, there is a stable and causal system $\phi(v,d)$ such that

(1.3) $$\Sigma_c(v,d) = d + P\phi(v,d) = [I + P\phi(v,\cdot)]d.$$

Here, P is the 'numerator' of the right coprime fraction representation of Σ, and I is the identity system.

(ii) Conversely, for every stable and causal system $\phi(v,d)$, there is an internally stable control configuration around the system Σ (with an equivalent controller denoted by C) such $\Sigma_c(v,d) = [I + P\phi(v,\cdot)]d$. The implementation of such a configuration is discussed in section 3.

Thus, (1.3) provides a complete parametrization of the class of all systems that can be obtained by internally stable control of Σ. The stable and causal system ϕ serves as the sole parameter. Every equivalent controller C that internally stabilizes Σ induces a certain ϕ, and, conversely, for every ϕ there is an equivalent controller C that internally stabilizes Σ and yields the response (1.3).

Using this parametrization, the design process can be dissolved in two steps:

(i) Specification of the desired response via the selection of ϕ, and

(ii) Implementation of a controller that internally stabilizes the system and yields the response $\Sigma_c(v,d) = [I + P\phi(v,\cdot)]d$.

Once ϕ has been selected, a procedure for obtaining a suitable controller is outlined in section 3. The selection of ϕ depends, of course, on the design at hand. In many cases, an important consideration for the selection of ϕ is the desire to achieve maximal attenuation of the effects of the disturbance d on the output z. In such case, ϕ is derived through an optimization procedure. The selection of ϕ is not discussed in the present note. Rather, we concentrate on the derivation of (1.3) and on the internally stable implementation of Σ_c once ϕ has been selected.

As (1.3) indicates, the basic limitation on the performance that can be achieved by internally stable control of Σ is imposed by the 'numerator' system P of a right coprime fraction representation of Σ. Indeed, apart from the identity, P the only fixed quantity in (1.3).

The present situation is closely analogous to the well known linear theory. In fact, the linear counterpart of (1.3) played an important role in the formulation of the linear theory of optimal disturbance attenuation (ZAMES [1981]).

This note is an abridged version of HAMMER [1992], where detailed proofs of all the statements mentioned below are provided. It depends on various results on fraction representations of nonlinear systems, presented in HAMMER [1984, 1985, 1987, 1988, 1989a, and 1989c], DESOER and KABULI [1988], TAY and MOORE [1988], VERMA [1988], SONTAG [1989], CHEN and de FIGUEIREDO [1990], PAICE and MOORE [1990], VERMA and HUNT [1991], the references cited in these works, and others.

The presentation here is for discrete-time systems.

2. Background and refinements

As usual, R^m is the set of all m-dimensional real vectors. Let $S(R^m)$ be the set of all sequences $u_0, u_1, u_2, \ldots$ of m-dimensional real vectors $u_j \in R^m$, $j = 0, 1, 2, \ldots$.

Adopting the input/output point of view, a system Σ is simply a map $\Sigma : S(R^m) \to S(R^p)$, transforming input sequences of m-dimensional vectors into output sequences of p-dimensional vectors. The image of a subset $S \subset S(R^m)$ through Σ is denoted by $\Sigma[S]$, and $\operatorname{Im} \Sigma := \Sigma[S(R^m)]$ is the entire image of the system Σ.

The i-the element of a sequence $u \in S(R^m)$ is denoted by u_i; the list of elements $u_i, u_{i+1}, \ldots, u_j$, where $j \geq i \geq 0$ are integers, is denoted by u_i^j. Letting $y := \Sigma u$, the notation $\Sigma u]_i^j$ refers to the elements y_i^j.

A system $\Sigma : S(R^m) \to S(R^p)$ is *causal* (respectively, *strictly causal*) if the following holds. For every pair of input sequences $u, v \in S(R^m)$ and for every integer $j \geq 0$ for which the equality $u_0^j = v_0^j$ holds, one has $\Sigma u]_0^j = \Sigma v]_0^j$ (respectively, $\Sigma u]_0^{j+1} = \Sigma v]_0^{j+1}$). Only causal systems can be implemented in a real-time environment. Many systems encountered in practice are in fact strictly causal. For instance, every system $\Sigma : S(R^m) \to S(R^p)$ that can be represented in the form

(2.1) $$x_{k+1} = f(x_k, u_k), \quad y_k = h(x_k), \quad k = 0, 1, 2, \ldots,$$

is strictly causal. When the functions $f : R^n \times R^m \to R^n$ and $h : R^n \to R^p$ are continuous, (2.1) constitutes a *continuous realization* of Σ. The system represented by the recursion $x_{k+1} = f(x_k, u_k)$ is called the *input/state part* of Σ, and is denoted by Σ_s.

A system $M : S(R^m) \to S(R^m)$ is *bicausal* if it is causal, and if it possesses a causal inverse M^{-1}. We shall need the following (see HAMMER [1992] for details).

(2.2) LEMMA. Let $M : S(R^p) \to S(R^p)$ be a bicausal system. Then, for every strictly causal system $\Gamma : S(R^p) \to S(R^p)$, the sum $(M - \Gamma) : S(R^p) \to S(R^p)$ is a bicausal system.♦

We continue now with review of notation. Let $\theta > 0$ be a real number, and denote by $[-\theta,\theta]^m$ the set of vectors in R^m having all their components in the interval $[-\theta,\theta]$. Let $S(\theta^m)$ be the set of all sequences $u \in S(R^m)$ satisfying $u_i \in [-\theta,\theta]^m$ for all integers $i \geq 0$. Then, $S(\theta^m)$ is the set of all sequences 'bounded' by θ. A system $\Sigma : S(R^m) \to S(R^p)$ is *BIBO* (Bounded-Input Bounded-Output)-*stable* whenever there is, for every real $\theta > 0$, a real $M > 0$ satisfying $\Sigma[S(\theta^m)] \subset S(M^p)$.

For our discussion of stability, two norms are used: the ℓ^∞-norm and a weighted ℓ^∞-norm. The ℓ^∞-norm is denoted by $|\cdot|$; for a vector $a = (a_1, \ldots, a_m) \in R^m$, it is simply $|a| := \max\{|a_1|, \ldots, |a_m|\}$. For a sequence $u \in S(R^m)$, it is given by $|u| := \sup_{i\geq 0} |u_i|$. The weighted ℓ^∞-norm is denoted by ρ, and is given by

(2.3) $$\rho(u) := \sup_{i\geq 0} 2^{-i} |u_i|$$

for all $u \in S(R^m)$.

(2.4) DEFINITION. A system $\Sigma : S(R^m) \to S(R^p)$ is *stable* (with respect to the norm ρ) if it is BIBO-stable, and if the restriction $\Sigma : S(\alpha^m) \to S(R^p)$ is continuous (with respect to ρ) for every real number $\alpha > 0$.♦

Another notion that is important to us is (HAMMER [1986]).

(2.5) DEFINITION. A stable system $\Sigma : S(R^m) \to S(R^p)$ is *differentially bounded* if there is a pair of real numbers $\varepsilon, \theta > 0$ such that, for every pair of sequences $u \in S(R^m)$ and $\upsilon \in S(\varepsilon^m)$, one has $|\Sigma(u+\upsilon) - \Sigma(u)| \leq \theta$.♦

The notion of differential boundedness is a weak form of uniform continuity with respect to the ℓ^∞-norm. It guaranties that a deviation of less than ε in the input sequence always causes a deviation of less than θ in the output sequence.

When several individual systems are combined into a composite system, one has to consider the notion of *internal stability*. A typical composite system $\Sigma^{(s)}$

consists of s individual systems, labeled $\Sigma^1, \ldots, \Sigma^s$, where $\Sigma^i : S(R^{m(i)}) \to S(R^{p(i)})$, $i = 1, \ldots, s$. Let $u \in S(R^m)$ be the external input sequence of the composite system, and let $y \in S(R^p)$ be its output sequence. Let $u^j \in S(R^{m(j)})$ be the input sequence of the component Σ^j, and let $y^j \in S(R^{p(j)})$ be its output sequence. We introduce s signals $\varepsilon^i \in S(R^{m(i)})$, $i = 1, \ldots, s$, with ε^i being added to the input u^i as a disturbance. The disturbances are all bounded by $\delta > 0$, so that in fact $\varepsilon^i \in S(\delta^{m(i)})$, $i = 1, \ldots, s$.

Let $\Sigma^{*s*} : S(R^m) \times S(\delta^{m(1)}) \times \cdots \times S(\delta^{m(s)}) \to S(R^p) \times S(R^{p(1)}) \times \cdots \times S(R^{p(s)}) : (u, \varepsilon^1, \ldots, \varepsilon^s) \mapsto \Sigma^{*s*}(u, \varepsilon^1, \ldots, \varepsilon^s)$ denote the system induced by the interconnected system $\Sigma^{(s)}$ and the disturbances, having the input signals $u, \varepsilon^1, \ldots, \varepsilon^s$ and the output signals $y, y^1, \ldots, y^s$, respectively.

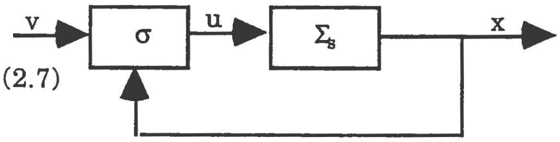
(2.6) DEFINITION. The composite system $\Sigma^{(s)}$ is *internally stable* if the system Σ^{*s*} is stable. The composite system $\Sigma^{(s)}$ is *strictly internally stable* if the system Σ^{*s*} is stable and differentially bounded.♦

Internal stability requires boundedness of all internal signals under noisy conditions, along with continuity of all internal and external outputs with respect to outside inputs and disturbances.

We turn now to fraction representations. A right fraction representation of a system $\Sigma : S(R^m) \to S(R^p)$ is determined by three quantities: a subset $S \subset S(R^q)$, $q > 0$, called the *factorization space;* and two stable systems $P : S \to S(R^p)$ and $Q : S \to S(R^m)$, with Q being a set isomorphism, such that $\Sigma = PQ^{-1}$. The fraction representation is *coprime* whenever the stable systems P and Q are right coprime (see HAMMER [1985, 1987] for coprimeness).

We shall need some results on right coprime fraction representations in which the 'denominator' system is bicausal. This depends on the theory of reversible state feedback developed in HAMMER [1989b and c]. We review now the basic facts, referring to (2.1) and the terminology introduced with it. Let $\Sigma : S(R^m) \to S(R^p)$ be a system having a continuous realization, with input/state part Σ_s and state dimension q. Close a static state feedback loop around Σ_s to obtain

(2.7)



Here, $\sigma : R^q \times R^m \to R^m : (x,v) \mapsto \sigma(x,v)$ is a continuous function, serving as state feedback. The state feedback function σ is *reversible* if it is injective in v for every state x. The system Σ_s is *stabilizable by state feedback* if there is a reversible state feedback function σ for which (2.7) is internally stable. A verifiable characterization of stabilizability by state feedback is provided in HAMMER [1989b]. It basically amounts to a certain nonlinear analog of the linear reachability requirement. Stabilizability by state feedback is related to the existence of right coprime fraction representations with bicausal denominators, as follows (HAMMER [1989c]).

(2.8) THEOREM. Let $\Sigma : S(\alpha^m) \to S(R^p)$ be a system with the bounded input space $S(\alpha^m)$, $\alpha > 0$. Assume Σ has a continuous realization with an input/state part that is stabilizable by state feedback. Then, Σ has a right coprime fraction representation $\Sigma = PQ^{-1}$, with Q being a bicausal system.♦

An important property of right coprime fraction representations is the fact that the denominator system contains all the information about instabilities of the system, as follows (HAMMER [1985], [1987], [1992]).

(2.9) PROPOSITION. Let $\Sigma : S(R^m) \to S(R^p)$ be a system having a right coprime fraction representation $\Sigma = PQ^{-1}$, where $P : S \to S(R^p)$, $Q : S \to S(R^m)$, and Q is bicausal. Let $D : S(R^r) \to S(R^m)$ be any stable and causal system for which the composition $\Sigma D : S(R^r) \to S(R^p)$ is stable. Then, there is a stable and causal system $\phi : S(R^r) \to S$ such that $D = Q\phi$.♦

Finally, a left fraction representation of a nonlinear system $\Sigma : S(R^m) \to S(R^p)$ is determined by three quantities: a subset $S_L \subset S(R^r)$, $r > 0$, called the *factorization space;* and a pair of two stable systems $T : S(R^m) \to S_L$ and $G : \mathrm{Im}\,\Sigma \to S_L$, where G is a set isomorphism, and $\Sigma = G^{-1}T$.

3. The parametrization

We refer to (1.1). Let $\Sigma : S(R^m) \to S(R^p)$ be a strictly causal system, and let $C : S(R^m) \times S(R^p) \to S(R^m) : (v,z) \mapsto C(v,z)$ represent a causal equivalent controller. (Implementations of C are discussed later in this section). The output sequence is $z = \Sigma_c(v,d)$.

The present section is divided into two main parts. In the first part, we show that when (1.1) is internally stable, Σ_c can be represented in the form (1.3). The second part deals with the converse direction; given a representation of the form (1.3), we provide a scheme for an internally stable configuration that implements it. The combination of both parts shows that (1.3) is a complete parametrization of all internally stable control configurations around the disturbed system Σ.

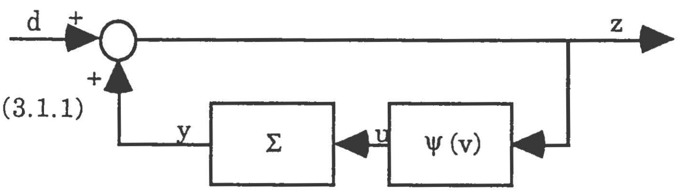
3.1 Implications of internal stability.

Assume that the (equivalent) controller C renders (1.1) internally stable. We show that then (1.3) is valid. It will be convenient to use the notation

$$\psi(v)z := C(v,z);$$

Here, $\psi(v) : S(R^p) \to S(R^m) : z \mapsto \psi(v)z = u$ is simply the partial function $C(v,\cdot)$. The notation $\psi(\cdot)$ is used for the function $S(R^m){\times}S(R^p) \to S(R^m) : (v,z) \mapsto \psi(v)z$, which is identical to C. With this notation, (1.1) can be re-drawn in the following alternative form, where v is formally regarded as an implicit variable, and d as an external input.

(3.1.1)



The equations that describe (3.1.1) (and whence also (1.1)) become

$$z = d + y,\qquad (3.1.2)$$
$$y = \Sigma\psi(v)z.$$

This yields $z = d + \Sigma\psi(v)z$, or

$$[I - \Sigma\psi(v)]z = d. \qquad (3.1.3)$$

Now, since Σ is strictly causal, so is $\Sigma\psi(v) : S(R^p) \to S(R^p) : z \mapsto \Sigma\psi(v)z$. Therefore, by Lemma (2.2), $[I - \Sigma\psi(v)]$ is bicausal, and has a causal inverse $[I - \Sigma\psi(v)]^{-1} : S(R^p) \to S(R^p)$. Thus, (3.1.3) yields

$$z = [I - \Sigma\psi(v)]^{-1}d, \qquad (3.1.4)$$

or $\Sigma_c = [I - \Sigma\psi(\cdot)]^{-1}$. Internal stability of (1.1) includes stability of Σ_c, and whence $[I - \Sigma\psi(\cdot)]^{-1}$ is stable.

From (1.1) $u = C(v,z) = \psi(v)z$, and, by (3.1.4),

$$u = \psi(v)[I - \Sigma\psi(v)]^{-1}d. \qquad (3.1.5)$$

Define now the system

$$\psi_d(v) := \psi(v)[I - \Sigma\psi(v)]^{-1}, \qquad (3.1.6)$$

so that $d \mapsto \psi_d(v)d = u$. Since $\psi(v)$ and $[I - \Sigma\psi(v)]^{-1}$ are causal, so is $\psi_d(v)$. Whence, $\Sigma\psi_d(v)$ is strictly causal by the strict causality of Σ. Lemma (2.2) then implies that $[I + \Sigma\psi_d(v)]$ is bicausal, and consequently has a causal inverse $[I + \Sigma\psi_d(v)]^{-1}$. We claim that

$$[I + \Sigma\psi_d(v)]^{-1} = [I - \Sigma\psi(v)]. \qquad (3.1.7)$$

Indeed, by direct calculation,

$$[I + \Sigma\psi_d(v)][I - \Sigma\psi(v)] = I - \Sigma\psi(v) + \Sigma\psi_d(v)[I - \Sigma\psi(v)]$$
$$= I - \Sigma\psi(v) + \Sigma\psi(v) = I,$$

where (3.1.6) was used.

When (3.1.7) is substituted into $\psi(v) = \psi_d(v)[I - \Sigma\psi(v)]$ (see (3.1.6)), one obtains

$$\psi(v) = \psi_d(v)[I + \Sigma\psi_d(v)]^{-1}. \qquad (3.1.8)$$

Since $[I - \Sigma\psi(\cdot)]^{-1}$ is stable, (3.1.7) implies that $[I + \Sigma\psi_d(\cdot)]$ is also stable. Then, $\Sigma\psi_d(v)d = [I + \Sigma\psi_d(v)]d - Id$ is the difference of two stable systems, and whence is itself stable. Using (3.1.4) and (3.1.7), one can write

$$z = [I + \Sigma\psi_d(v)]d, \qquad (3.1.9)$$

or $\Sigma_c(v,d) = [I + \Sigma\psi_d(v)]d$.

Combining (3.1.5) with (3.1.6) yields

$$u = \psi_d(v)d. \qquad (3.1.10)$$

By internal stability of the composite system, the transmission from (v,d) to u must be stable. Whence, (3.1.10) implies that $\psi_d(\cdot) : S(R^m){\times}S(R^p) \to S(R^m)$ is stable.

Assume next that the given system has a right coprime fraction representation $\Sigma = PQ^{-1}$ with a bicausal denominator $Q : S(R^m) \to S(R^m)$. In view of Theorem (2.8), this is basically a stabilizability assumption on Σ. Now, we have noted earlier that $\Sigma\psi_d(\cdot)$ and $\psi_d(\cdot)$ are stable. Invoking Proposition (2.9), we conclude that there is a stable and causal system $\phi(\cdot) : S(R^m){\times}S(R^p) \to S(R^m)$ satisfying

$$\psi_d(v)d = Q\phi(v)d \qquad (3.1.11)$$

for all $v \in S(R^m)$ and all $d \in S(R^p)$. Inserting this into (3.1.9), and using $\Sigma = PQ^{-1}$, we obtain $z = [I + P\phi(v)]d$. This validates the following statement (HAMMER [1992]).

(3.1.12) PROPOSITION. Let $\Sigma : S(R^m) \to S(R^p)$ be a strictly causal system having a right coprime fraction representation $\Sigma = PQ^{-1}$, where the denominator system $Q : S(R^m) \to S(R^m)$ is bicausal. Let $C : S(R^m){\times}S(R^p) \to S(R^m)$ be any causal equivalent controller for which (1.1) is internally stable. Then, there is a stable and causal system $\phi : S(R^m){\times}S(R^p) \to S(R^m)$ such that $z = [I + P\phi(v)]d$.♦

Furthermore, the equivalent controller C can be directly recovered from ϕ. Indeed, using (3.1.11) and (3.1.8), we have $C(v,z) = \psi(v)z = Q\phi(v)[I + \Sigma Q\phi(v)]^{-1}z$, or

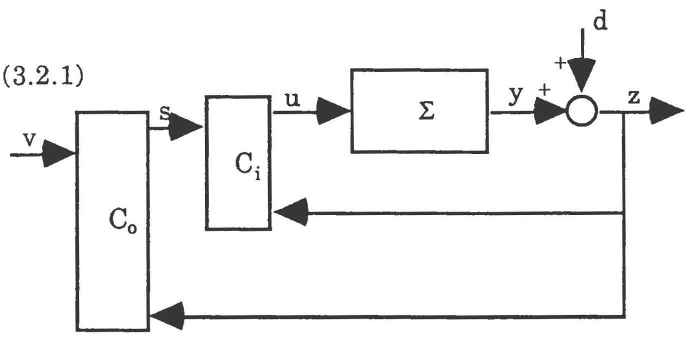
$$C(v,z) = Q\phi(v)[I + P\phi(v)]^{-1}z. \qquad (3.1.13)$$

Note that (3.1.13) cannot be implemented as is, since this would require an exact model of the denominator system Q of Σ, and will not be internally stable when Σ is unstable.

3.2 Internally stable implementations.

We discuss now an internally stable implementation of the equivalent controller C of (3.1.13). To this end, we decompose (1.1) into two nested loops, as below.

(3.2.1)



The inner loop controller C_i will internally stabilize Σ, and the outer loop controller C_o will complement the inner loop to yield an overall equivalent controller equal to C, while preserving internal stability. This can be viewed as a 'separation method', whereby the system Σ is first stabilized (by the inner loop), and then 'tweaked' (by the outer loop) to achieve desired performance.

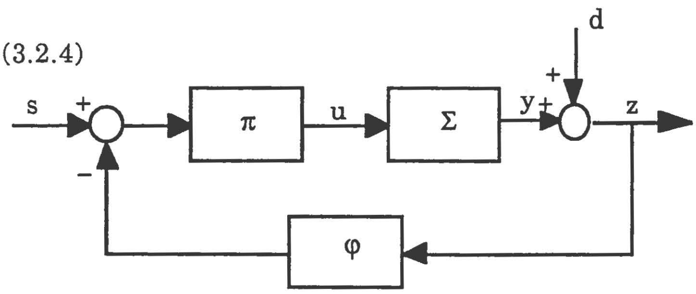
The inner loop controller C_i needs to satisfy two basic requirements: (i) Provide internal stabilization of the system Σ; and (ii) Facilitate the computation of an outer loop controller C_o that leaves the configuration internally stable, while complementing C_i to yield the equivalent controller C of (3.1.13). These objectives are particularly easy to achieve when the controller C_i has a left fraction representation of the form

(3.2.2) $$C_i(s,z) = G^{-1}(z)[s + Tz] = u,$$

where $T : S(R^p) \to S(R^m)$ and $G : S(R^m)\times S(R^p) \to S(R^m) : (u,z) \mapsto G(z)u$ are causal and stable systems, and the partial system $G(z) : S(R^m) \to S(R^m) : u \mapsto G(z)u$ is invertible for all $z \in S(R^p)$. As it turns out, many configurations that are used to stabilize nonlinear systems do possess controllers of the form (3.2.2). One such configuration is the example below, and others are discussed in HAMMER [1992].

(3.2.3) EXAMPLE. Consider the loop

(3.2.4)



Choose the compensators $\pi : S(R^m) \to S(R^m)$ and $\varphi : S(R^p) \to S(R^m)$ in the special form

(3.2.5) $$\varphi = A, \quad \pi = B^{-1},$$

where $A : S(R^p) \to S(R^m)$ and $B : S(R^m) \to S(R^m)$ are stable, A is causal, and B is bicausal. Letting $C_{(\pi,\varphi)} : S(R^m)\times S(R^p) \to S(R^m) : (s,z) \mapsto C_{(\pi,\varphi)}(s,z) = u$ denote the equivalent controller induced by π, φ, and the summer, we obtain

(3.2.6) $$C_{(\pi,\varphi)}(s,z) = \pi[s - \varphi z] = B^{-1}[s - Az],$$

which is of the form (3.2.2) with $G(z) := B$ and $T := -A$. Thus, when π and φ internally stabilize (3.2.4), we can take $C_i := C_{(\pi,\varphi)}$ in (3.2.1), and (3.2.2) is satisfied. (See HAMMER [1987] for details.)♦

Throughout the remaining part of this paper, we restrict ourselves to inner loop controllers C_i of the form (3.2.2).

As discussed earlier, a right coprime fraction representation $\Sigma = PQ^{-1}$ with a bicausal denominator $Q : S(R^m) \to S(R^m)$ is available. Given any stable and causal system $\phi(\cdot) : S(R^m)\times S(R^p) \to S(R^m)$, the construction of controllers C_i and C_o that internally stabilize the system and yield the response $z = [I + P\phi(v)]d$ proceeds as follows.

Step 1. Design any strictly internally stable closed loop around the system Σ (see Definition (2.6)), with a controller C_i of the form (3.2.2). Then, $C_i(s,z) = G^{-1}(z)[s + Tz]$.

Note that this step of the design process is concerned only with internal stabilization of Σ, and is independent of ϕ. The same C_i can be used for all ϕ.

Step 2. Set

(3.2.7) $$C_o(v,z) = G(z)C(v,z) - Tz,$$

where G and T are from Step 1, and

(3.2.8) $$C(v,z) := Q\phi(v)[I + P\phi(v)]^{-1}z.$$ ♦

It is quite easy to verify that these controllers yield the desired response. Reading from (3.2.1), we have

(3.2.9) $$s = C_o(v,z), \quad u = C_i(s,z),$$

so that $u = C_i(C_o(v,z),z)$.

Using (3.2.2) for C_i we obtain $u = G^{-1}(z)[C_o(v,z) + Tz] = G^{-1}(z)[G(z)C(v,z) - Tz + Tz] = C(v,z)$, and, by (3.2.8), $u = Q\phi(v)[I + P\phi(v)]^{-1}z$. But then,

$$\begin{aligned} z = y + d &= \Sigma u + d \\ &= PQ^{-1}u + d \\ &= PQ^{-1}\{Q\phi(v)[I + P\phi(v)]^{-1}z\} + d \\ &= P\phi(v)[I + P\phi(v)]^{-1}z + d \\ &= \{I + P\phi(v)\}[I + P\phi(v)]^{-1}z - [I + P\phi(v)]^{-1}z + Id \\ &= Iz - [I + P\phi(v)]^{-1}z + d = z - [I + P\phi(v)]^{-1}z + d; \end{aligned}$$

Canceling the z term on both sides and rearranging, we obtain $[I + P\phi(v)]^{-1}z = d$, or $z = [I + P\phi(v)]d$, as required. Thus, the proposed configuration achieves the desired response. It is in fact also internally stable, as follows (HAMMER [1992]).

(3.2.10) PROPOSITION. Let $\Sigma : S(R^m) \to S(R^p)$ be a strictly causal system having a right coprime fraction representation $\Sigma = PQ^{-1}$, with $Q : S(R^m) \to S(R^m)$

bicausal. Let $\phi : S(R^m) \times S(R^p) \to S(R^m) : (v,d) \mapsto \phi(v)d$ be any stable and causal system. Let C_i and C_o be controllers derived in Steps 1 and 2. Then, C_i and C_o render (3.2.1) internally stable, and assign to it the response $z = [I + P\phi(v)]d$.♦

Thus, any response of the form $z = [I + P\phi(v,\cdot)]d$ can be assigned to the system Σ by an internally stable control configuration. Combining this with Proposition (3.1.12), we obtain (HAMMER [1992])

(3.2.11)THEOREM. Let $\Sigma : S(R^m) \to S(R^p)$ be a strictly causal system having a right coprime fraction representation $\Sigma = PQ^{-1}$ with a bicausal denominator $Q : S(R^m) \to S(R^m)$. Assume that Σ can be strictly internally stabilized by a controller of the form (3.2.2). Then, referring to (1.1), the following is true. The class of all Σ_c that can be achieved through internally stable control of Σ is given by

$\{\Sigma_c(v,d) = [I + P\phi(v)]d,\ \phi(\cdot)$ is stable and causal$\}$.♦

Theorem (3.2.11) provides a parametrization of the class of all systems that can be obtained from a given system Σ by internally stable control. An interesting consequence of the design procedure of Steps 1 and 2 is a

Separation Principle. Stabilization can be separated from performance in the design process.

Indeed, Step 1 deals exclusively with stabilization, whereas Step 2 deals exclusively with performance assignment.

4. References

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