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On the control of sequential machines with disturbances

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The problem of controlling a sequential machine under the influence of disturbances is considered. A methodology is developed for the design of controllers that guarantees that the effect of a 'small' disturbance on the performance of the controlled machine remains 'small'. The methodology is based on a theory of fraction representations of sequential machines reminiscent of the general theory of fraction representations of nonlinear systems.

1. Introduction

Quite frequently one encounters the need to deal with sequential machines that are influenced by disturbances. These disturbances may originate from physical noise sources or from modelling uncertainties, or they may have a numerical origin. As an example of the former, consider a digital control system with remote telemetry. Here, noises in the telemetry communication channel create a disturbance that affects the system. As another example, consider a biochemical signalling chain in molecular biology (Hammer 1995 a, b). Here, several distinct phenomena may be regarded as disturbances that act on a fixed nominal model: (i) the natural random nature of biochemical processes can be viewed as a disturbance acting on a deterministic model that represents the 'average' response of the system; (ii) measurement inaccuracies incurred in the determination of the state of a biochemical reaction chain can be regarded as disturbances; and (iii) small differences among copies of similar reaction chains functioning in different locations can be regarded as disturbances that act on a fixed nominal model. Digital control systems and digital filtering systems furnish examples of sequential machines where disturbances of numerical origin may become important. Here, inaccuracies caused by finite word length can be regarded as disturbances that act on an idealized model. Other examples of application areas abound.

Consider then a sequential machine Σ that operates within an environment where disturbances have undesirable effects on the performance of the machine. In order to correct the undesirable effects and improve the overall performance of the machine, connect Σ to another sequential machine C that serves as a controller, as depicted in the Figure. Here, the composite system is influenced by three disturbances: an external input disturbance v_3, an in-loop input disturbance v_1, and an output disturbance v_2. The only *a priori* information available about these disturbances is an amplitude bound, i.e. it is known that the amplitude of the disturbances v_1, v_2 and v_3 cannot exceed a specified value. Other than that, no assumption is made as to the nature or the origin of the disturbances. The purpose of the controller C is to drive

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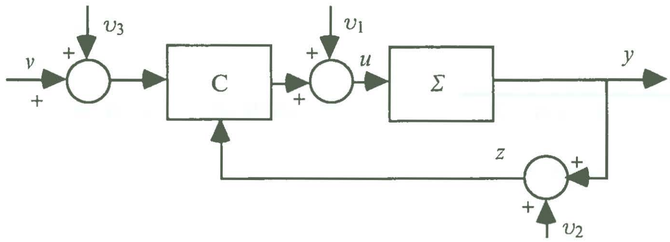

Figure 1. Composite system.

the system Σ so as to elicit from it desirable behaviour, while accommodating the disturbances. As the figure indicates, the signal y is regarded as the output signal of the configuration. Alternatively, one could regard the signal z as the output signal. The external input signal is denoted by v. The symbol Σ_c will be used to indicate the input/output map induced by the closed loop system, so that, when the disturbances are absent, $y = \Sigma_c v$.

In the present paper we concentrate on the study of controllers C for which the effect of a disturbance on the output signal y does not exceed the original amplitude of the disturbance. We shall refer to such controllers as *disturbance attenuating controllers*. A disturbance attenuating controller guarantees that the disturbance is not amplified, so that 'small' disturbances have only 'small' effects on performance. The main result of the paper is the derivation of necessary and sufficient conditions for the existence of disturbance attenuating controllers, as well as the construction of such controllers, when they exist (§ 3).

Following a long-standing tradition in digital circuit theory and practice, we conduct our discussion within an input/output framework, where a sequential machine is considered as a system that maps input sequences of discrete values into output sequences of discrete values. Input/output representations are usually the most convenient form of specifying the desired characteristics of a system, and hence are the most common starting point for design considerations. The process of implementing a system involves the translation of the input/output description into a state representation, or realization, of the system. Techniques for deriving realizations from input/output representations are well established (e.g. Kohavi 1978).

The effect of disturbances on closed loop systems is, of course, a central and widely studied subject in the literature on linear and nonlinear control theory. An important difference between the situation considered here and the standard literature is the fact that, presently, the systems operate over discrete spaces. Consequently, the standard notions of continuity and differentiability, which are commonly used to analyse the effects of small disturbances in classical control theory, do not apply here. To study the effects of small perturbations in discrete spaces, we introduce in § 2 the notion of 'continuity radius'. In intuitive terms, the continuity radius gauges the largest deviation that can occur in the response of a sequential machine as a result of a small perturbation in the input sequence. The notion of continuity radius is used to analyse quantitatively the effects of small disturbances on sequential machines, and to derive techniques for the design of controllers that keep these effects below specified bounds.

The tools and notions developed in this paper form a conceptual framework for the analysis of error propagation in systems over discrete spaces, and can be used, among other applications, to study error propagation in numerical algorithms. Presently however, we concentrate exclusively on control theoretic applications.

An important aspect of the framework presented here is the development of a theory of fraction representations of sequential machines (§ 3), in line with the general theory of fraction representations of nonlinear systems (Hammer 1984 a, b, 1985, 1994 a, Desoer and Kabuli 1988, Verma 1988, Verma and Hunt 1993, Sontag 1989, Chen and de Figueiredo 1990, Paice and Moore 1990, Paice and van der Schaft 1994, Baramov and Kimura 1995, and others). In brief terms, a right fraction representation of a sequential machine Σ is a representation of the form $\Sigma = PQ^{-1}$, where P and Q are sequential machines with desirable continuity radii. Fraction representations provide convenient tools for the investigation of disturbance attenuation for sequential machines, fulfilling here a role analogous to their role in the theory of control for systems over continuous spaces.

The sequential machines considered in this paper are recursive machines over the integers. To be specific, let Z be the set of integers and, for an integer $n > 0$, let Z^n be the set of all n-dimensional vectors of integers. We then consider sequential machines Σ that permit a representation of the form

$$\left.\begin{aligned} x_{k+1} &= f(x_k, u_k) \\ y_k &= h(x_k), \quad k = 0, 1, 2, \ldots \end{aligned}\right\} \tag{1.1}$$

where $x_k \in Z^n$ is an n-dimensional vector of integers called the *state* of Σ at the step k; $u_k \in Z^m$ is an m-dimensional vector of integers called the *input value* at the step k; and $y_k \in Z^p$ is a p-dimensional vector of integers called the *output value* of Σ at the step k. The function f is the *recursion function* of Σ, and h is the *output function*. We assume that an initial condition x_0 is provided, so that the response to an input sequence is uniquely determined. Following accepted practice in digital circuit theory, coding theory, and other areas, we adopt an input/output view and regard the machine Σ as a system that generates a sequence of output values in response to each sequence of input values.

Models of the form (1.1) are used to represent systems in many application areas. Some examples are models of biological signalling chains (Hammer 1995 a, b), a class of critical biochemical reactions that in many ways determine the well being of all live organisms; models of digital curcuits used in computer hardware and software design; models of digital control systems; and models of digital filters.

In order to provide a quantitative characterization of disturbances, we shall use (in § 2) the l^∞-norm as a notion of distance within the spaces of input and output sequences, in analogy to the notion of distance commonly used in coding theory (e.g. Berlekamp 1968).

Background related to the theory of automata and sequential machines can be found in Ginsburg (1962, 1966), Eilenberg (1974), Hoare (1976), Milner (1980), Arnold and Nivat (1980), and many other excellent sources.

In general terms, the material discussed in this paper is within the context of the theory of discrete-event systems, although the basic approach relies more heavily on concepts and techniques used in nonlinear control theory. The theory of discrete event systems offers a number of alternative treatments of problems related to the control of discrete systems; these include Ramadge and Wonham (1987), Vaz and Wonham (1986), Lin and Wonham (1988), Thistle and Wonham (1988), Cieslak *et al.* (1988), Ozveren and Willsky (1990), the references cited in these papers, and many others. Issues related to the stabilization of discrete-event systems were investigated by Ozveren *et al.* (1991). Finally, the discussion of the present paper is a continuation of the work presented by Hammer (1994 b, 1995 a, b, 1996 a).

2. Basic concepts

2.1. *Preliminaries*

As mentioned earlier, the paper deals with sequential machines that operate on vectors of integers. We denote by Z the set of integers, and by Z^m the set of all m-dimensional vectors with integer entries. Let $S(Z^m)$ be the set of all sequences $(u_0, u_1, u_2, \ldots)$ of m-dimensional integer vectors $u_k \in Z^m$, $k = 0, 1, 2, \ldots$. Given a sequence $u \in S(Z^m)$, we denote by u_k the kth element of the sequence, where $k \geq 0$ is an integer, and we call u_k the kth step of the sequence u. For an integer $k \geq 0$, we denote by u_0^k the list $u_0, u_1, \ldots, u_k$.

A sequential machine that accepts sequences of m-dimensional integer vectors as input and generates sequences of p-dimensional integer vectors as output is simply a map $\Sigma : S(Z^m) \rightarrow S(Z^p)$. In many cases, the set of input sequences accepted by Σ is restricted. For instance, Σ may only permit input sequences whose amplitudes do not exceed a given bound. We denote by D_Σ the subset of $S(Z^m)$ that consists of all sequences that are allowed to serve as input sequences of the system Σ, and call D_Σ the *input domain* of Σ. In these terms, a sequential machine is represented by a map $\Sigma : D_\Sigma \rightarrow S(Z^p)$, where $D_\Sigma \subset S(Z^m)$ is the input domain of Σ. Our main interest is in sequential machines Σ that permit a recursive representation of the form (1.1).

Let u be an input sequence of a system Σ, and let $y := \Sigma u$ be the corresponding output sequence. We denote by $\Sigma u]_0^k$ the list of output values $y_0, y_1, \ldots, y_k$.

A system $\Sigma : D_\Sigma \rightarrow S(Z^p)$ is *causal* (respectively, *strictly causal*) if the following holds for all input sequences $u, v \in D_\Sigma$: whenever $u_0^k = v_0^k$ for some integer $k \geq 0$, then $\Sigma u]_0^k = \Sigma v]_0^k$ (respectively, $\Sigma u]_0^{k+1} = \Sigma v]_0^{k+1}$). This is the standard definition of causality, and it indicates that the system cannot react to changes in its input sequence before they have occurred.

It can readily be shown that a system with a recursive representation of the form (1.1) is strictly causal. If the output function h in (1.1) also depends on the input value u_k (rather than depending only on the state x_k), then the system is causal, but not necessarily strictly causal. A system $M : D_1 \rightarrow D_2$, where $D_1 \subset S(Z^m)$ and $D_2 \subset S(Z^p)$, is *bicausal* if it is a set isomorphism with both M and M^{-1} being causal.

The input domain of a sequential machine usually has a simple structure, which we now describe. A subset $\Delta \subset Z^m$ is called an *interval* if it is of the form $[a_1, b_1] \times \cdots \times [a_m, b_m]$, where $b_i \geq a_i$, $i = 1, \ldots, m$, are integers (and therefore finite). The interval $\Delta \subset Z^m$ induces an interval $S(\Delta)$ in $S(Z^m)$, which consists of all sequences in $S(Z^m)$ whose elements belong to Δ. Normally, the input domain of a sequential machine Σ is an interval in $S(Z^m)$.

A subset $D \subset S(Z^m)$ is *bounded* if it is contained in an interval. Similarly, a system $\Sigma : D_\Sigma \rightarrow S(Z^p)$ is bounded if its image $\operatorname{Im} \Sigma$ is contained within an interval in $S(Z^p)$.

We conclude this subsection with a qualitative review of the definition of a fraction representation. Consider a system $\Sigma : D_\Sigma \rightarrow S(Z^p)$, where $D_\Sigma \subset S(Z^m)$. A *right fraction representation* of Σ is determined by a subset $D \subset S(Z^q)$, where $q > 0$ is some integer, and by a pair of causal systems $P : D \rightarrow S(Z^p)$ and $Q : D \rightarrow D_\Sigma$, where Q is a set isomorphism and the equality $\Sigma = PQ^{-1}$ is satisfied. The set D is called the *factorization set* of the fraction representation. The numerator and denominator systems P and Q are required to satisfy additional conditions (usually relating to their continuity properties), and we shall discuss these further requirements later.

Finally, in this paper the term *functional norm* refers to an assignment $|\cdot|$ that assigns a number $|f|$ to each function f of a given class, with the following properties: (*a*) for every function f and variables u and v, one has $|f(u) - f(v)| \leq |f||u - v|$; and (*b*) for every pair of functions f, g having appropriate domain and codomain, the composition gf satisfies $|gf| \leq |g||f|$. These properties make functional norms useful for the derivation of bounds on disturbance effects.

2.2. *The continuity radius and the gain functional norm*

The main subject of the present paper involves the investigation of sequential machines that are influenced by perturbations and disturbances. The investigation of the effects of perturbations and disturbances is, of course, a long standing subject of control theory. In the classical theory of control systems over continuous spaces, the analysis of the effects of small perturbations and disturbances depends, to a large degree, on the notions of continuity and differentiability. Needless to say, these notions cannot be directly used for systems that operate over discrete spaces. In the present subsection we define a notion for functions over discrete spaces which, in its qualitative properties, resembles the standard topological notion of continuity. This notion will allow us to develop a formalism for handling the effects of perturbations and disturbances for systems over discrete spaces. We base our discussion in this context on the use of the standard l^∞-norm, but other norms can be used instead.

Given a vector $v = (v^1, \ldots, v^m) \in Z^m$ we denote by $|v| := \max(|v^1|, \ldots, |v^m|)$ the largest absolute value of a component. For a sequence $u = (u_0, u_1, \ldots) \in S(Z^m)$, let

$$|u| := \sup_{i \geq 0} |u_i|$$

denote the usual l^∞-norm. Clearly, the norm $|u|$ of a sequence $u \in S(Z^m)$ is either a non-negative integer, or infinity.

Consider for a moment the classical notion of continuity over a topological space. In qualitative terms, a function is deemed continuous if a 'small' change in its variable creates (only) a 'small' change in its value. In order to gauge the effect of a 'small' change in the variable of a function over a discrete space, we define the following notion.

Definition 2.2.1: Let $g : \Delta \to Z^p$ be a function with the non-empty input domain $\Delta \subset Z^m$. The *continuity radius* δ of g is given by

$$\delta := \sup\{|gu' - gu| : u', u \in \Delta \text{ and } |u' - u| \leq 1\}. \qquad \square$$

The continuity radius is simply a gauge of the effect on the output of the smallest possible deviation in the input. Clearly, when $g : \Delta \to Z^p$ is a constant function, its continuity radius is zero. Note however that a zero continuity radius does not necessarily imply that the function is constant. In the case where the domain Δ has the property that any pair of distinct points u and u' satisfies $|u - u'| > 1$, then g has continuity radius zero over Δ. This anomaly is eliminated through the notion of gain functional norm defined later in this subsection.

Before defining the continuity radius for a sequential machine, we need to analyse some implications of the notion of causality. First, we define two projections for every integer $k \geq 0$. One is the projection

$$p_k : S(Z^n) \to Z^n : p_k(u_0, u_1, \ldots) := u_k$$

that projects each sequence onto its kth step; the second one is the projection

$$P_k : S(Z^n) \to (Z^n)^{k+1} : P_k(u_0, u_1, \ldots) := (u_0, \ldots, u_k) \tag{2.2.2}$$

that projects each sequence onto its initial $(k+1)$ elements. Now, consider a causal system $\Sigma : D_\Sigma \to S(Z^p)$ having the input domain $D_\Sigma \subset S(Z^m)$. Since Σ is a causal system, it follows that the output value $p_k\Sigma u$ is determined by the input values $P_k u$ for any input sequence $u \in S(Z^m)$. We can then define, for every integer $k \geq 0$, a function

$$\Sigma_{|k} : P_k D_\Sigma \to Z^p : \Sigma_{|k} P_k u := p_k \Sigma u \tag{2.2.3}$$

defined for all points $(u_0, \ldots, u_k) \in P_k D_\Sigma$ by the relation $\Sigma_{|k}(u_0, \ldots, u_k) := p_k\Sigma u$, where $u \in D_\Sigma$ is a sequence for which $(u_0, \ldots, u_k) = P_k u$.

Conversely, given a family of functions $F_k : P_k D_\Sigma \to Z^p, k = 0, 1, 2, \ldots$, we can induce a causal system $\Phi : D_\Sigma \to S(Z^p)$ by setting $\Phi u := (F_0 P_0 u, F_1 P_1 u, F_2 P_2 u, \ldots)$ for all $u \in D_\Sigma$. Thus we conclude that a causal system Σ is equivalent to the family $\{\Sigma_{|k}\}_{k=0}^{\infty}$ of functions. The family $\{\Sigma_{|k}\}$ characterizes the causal structure of the system Σ, and we use it now to define the continuity radius of Σ.

Definition 2.2.4: Let $\Sigma : D_\Sigma \to S(Z^p)$ be a causal system with the non-empty input domain $D_\Sigma \subset S(Z^m)$, and let δ_k be the continuity radius of the function $\Sigma_{|k}, k = 0, 1, 2, \ldots$. Then, the *continuity radius* δ of the system Σ is $\delta := \sup_{k \geq 0} \delta_k$. □

The continuity radius of a system is defined so that it coincides with the impression obtained by following the system output as time progresses. At each step in time, the continuity radius gauges the effect on the present output value of perturbations whose present and past values are of amplitude not exceeding one. The continuity radius of a constant system is zero.

We list now a few simple properties of the continuity radius. First, consider two systems Σ_1 and Σ_2 with a common input domain D_Σ. In order to obtain an estimate of the continuity radius of the sum $\Sigma_1 + \Sigma_2$, note that $(\Sigma_1 + \Sigma_2)_{|k} = \Sigma_{1|k} + \Sigma_{2|k}$ for all integers $k \geq 0$. Also, for any pair of input sequences $u', u \in D_\Sigma$, we have $|(\Sigma_{1|k} + \Sigma_{2|k})P_k u' - (\Sigma_{1|k} + \Sigma_{2|k})P_k u| = |(\Sigma_{1|k}P_k u' - \Sigma_{1|k}P_k u) + (\Sigma_{2|k}P_k u' - \Sigma_{2|k}P_k u)| \leq |(\Sigma_{1|k}P_k u' - \Sigma_{1|k}P_k u)| + |(\Sigma_{2|k}P_k u' - \Sigma_{2|k}P_k u)|$, which directly implies the following.

Proposition 2.2.5: *Let $\Sigma_1, \Sigma_2 : D_\Sigma \to S(Z^p)$ be two causal systems having continuity radii δ_1 and δ_2, respectively. Then, the sum $\Sigma_1 + \Sigma_2$ has continuity radius not exceeding $(\delta_1 + \delta_2)$.*

Our next objective is to discuss the continuity radius of a composition of two systems. (For a positive real number $\alpha > 0$, denote by $[\alpha]^+$ the smallest integer that is not less than α; for a negative number $\alpha < 0$, set $[\alpha]^+ := -[-\alpha]^+$.)

Proposition 2.2.6: *Let $D_1 \subset S(Z^m)$ and $D_2 \subset S(Z^p)$ be bounded subsets, where D_2 is an interval, and let $\Sigma : D_1 \to D_2$ and $\Sigma' : D_2 \to S(Z^q)$ be two causal systems. Assume that Σ has a finite continuity radius of δ, and that Σ' has a finite continuity radius of δ'. Then, the composition $\Sigma'\Sigma : D_1 \to S(Z^q)$ has a continuity radius not exceeding the product $\delta'\delta$.*

Proof: Fix an integer $k \geq 0$. Let $u, u' \in D_1$ be two input sequences satisfying $|P_k u - P_k u'| \leq 1$, let $v := \Sigma u, w := \Sigma u'$, and note that $v, w \in D_2$. From the definition of the continuity radius it follows that

$$|P_k v - P_k w| \leq \delta \tag{2.2.7}$$

In the next paragraph we build a list $z_0, z_1, \ldots, z_\delta$ of $\delta + 1$ points in D_2 that are 'between' v and w, and have the property that $|P_k z_r - P_k z_{r+1}| \leq 1$ for all $r = 0, \ldots, \delta - 1$.

Let $v_j \in Z^p$ be the jth step of the sequence v, where $0 \leq j \leq k$, and let $v_j^1, \ldots, v_j^p$ be the components of v_j. Similarly, let w_j be the jth step of the sequence w, and let $w_j^1, \ldots, w_j^p$ be the components w_j. By (2.2.7), we have $|v_j^i - w_j^i| \leq \delta$ for all $i = 1, \ldots, p$ and $j = 0, \ldots, k$. For fixed i and j, consider the scalar interval in Z having v_j^i and w_j^i as its endpoints. Since the length of this interval cannot exceed δ, we can select in it $\delta + 1$ integers $z_{j,0}^i, \ldots, z_{j,\delta}^i$ that satisfy the conditions $z_{j,0}^i := v_j^i, z_{j,\delta}^i = w_j^i$; and $|z_{j,r}^i - z_{j,r+1}^i| \leq 1$ for all $r = 0, \ldots, \delta - 1$. For fixed r and j, build now the vector $z_{j,r} := (z_{j,r}^1, z_{j,r}^2, \ldots z_{j,r}^p)$. Finally, for each $r = 0, \ldots, \delta$, create the sequence $z_r := (z_{0,r}, z_{1,r}, z_{2,r}, \ldots, z_{k,r}, z_{k+1,r}, \ldots) \in S(Z^p)$, setting $z_{t,r} := z_{k,r}$ for all integers $t \geq k$. Then, since D_2 is an interval, it follows that $z_0, \ldots, z_r \in D_2$, and, by construction, $|P_k z_r - P_k z_{r+1}| \leq 1$ for all $r = 0, \ldots, \delta - 1$. Furthermore, recalling the projection $p_k : S(Z^p) \to Z^p : y \mapsto y_k$, we obtain

$$\begin{aligned}
p_k|\Sigma'\Sigma u - \Sigma'\Sigma u'| &= |\Sigma'_{|k} P_k \Sigma u - \Sigma'_{|k} P_k \Sigma u'| \\
&= |\Sigma'_{|k} P_k v - \Sigma'_{|k} P_k w| \\
&= |\Sigma'_{|k} P_k z_0 - \Sigma'_{|k} P_k z_\delta| \\
&= |(\Sigma'_{|k} P_k z_0 - \Sigma'_{|k} P_k z_1) + (\Sigma'_{|k} P_k z_1 - \Sigma'_{|k} P_k z_2) + \cdots \\
&\quad + (\Sigma'_{|k} P_k z_{\delta-1} - \Sigma'_{|k} P_k z_\delta)| \\
&\leq |(\Sigma'_{|k} P_k z_0 - \Sigma'_{|k} P_k z_1)| + |(\Sigma'_{|k} P_k z_1 - \Sigma'_{|k} P_k z_2)| + \cdots \\
&\quad + |(\Sigma'_{|k} P_k z_{\delta-1} - \Sigma'_{|k} P_k z_\delta)| \\
&\leq \delta\delta'
\end{aligned}$$

where we used the facts that Σ and Σ' are causal systems; that $|P_k z_r - P_k z_{r+1}| \leq 1$ for all $r = 0, \ldots, \delta - 1$; and that Σ' has a continuity radius not exceeding δ'. Since the above is valid for all integers $k \geq 0$, this concludes our proof. □

In the following, we shall need to consider the combined effect of several independent perturbations, each one acting on its own variable. We encounter such an example in Fig. 1, which contains the three independent disturbances υ_1, υ_2, and υ_3. The next statement shows that the effects of several simultaneous pertubations may combine, but the overall effect cannot exceed the sum of the effects of the perturbations acting individually.

Proposition 2.2.8: *Let $f : Z^m \times Z^n \to Z^p(u, v) \mapsto f(u, v)$ be a function of the two variables $u \in Z^m$ and $v \in Z^n$. For each value of v, let $\delta_1(v)$ be the continuity radius of the partial function $f(., v) : Z^m \to Z^p$; and, for each value of u, let $\delta_2(u)$ be the continuity radius of the partial function $f(u, .) : Z^n \to Z^p$. Denote $\delta_1 := \sup_v \delta_1(v)$ and $\delta_2 := \sup_u \delta_2(u)$. Then, the continuity radius δ of f satisfies $\delta \leq \delta_1 + \delta_2$.*

Proof: Let (u, v) and (u', v') be two points in $Z^m \times Z^n$ satisfying $|(u', v') - (u, v)| \leq 1$. Using the triangle inequality, we have $|f(u', v') - f(u, v)| = |[f(u', v') - f(u, v')] + [f(u, v') - f(u, v)]| \leq |f(u', v') - f(u, v')| + |f(u, v') - f(u, v)| \leq \delta_1 + \delta_2$. □

Let $g : \Delta_{in} \to Z^p$ be a function defined over the domain $\Delta_{in} \subset Z^m$. It is convenient to defined the *gain functional norm* $|g|$ of g by the integer

$$|g| := \sup \left\{ \left[\frac{|gu - gu'|}{|u - u'|} \right]^+ : u, u' \in D_1, u \neq u' \right\} \tag{2.2.9}$$

In intuitive terms, the gain functional norm provides a measure of the 'largest application' that g generates for perturbations in its argument. It is closely related to a standard mathematical norm, and is an important tool for our discussion. Note that the gain functional norm of any constant function is zero, so that the zero function is not the only function that has a zero gain functional norm. Consequently, the gain functional norm is not a norm in the rigid sense of the word. Nevertheless, as we show below, the gain functional norm shares norm properties that are important for the derivation of bounds on the effects of disturbances. We next show that the gain functional norm has the multiplicative property, i.e. that the gain functional norm of a composition cannot exceed the product of the gain functional norms of the functions being composed.

Proposition 2.2.10: *Let $\Delta_1 \subset Z^m$ and $\Delta_2 \subset Z^p$ be non-empty subsets, and let $g_1 : \Delta_1 \to \Delta_2$ and $g_2 : \Delta_2 \to S(Z^q)$ be two functions with bounded gain functional norms. Then, $|g_2 g_1| \leq |g_2||g_1|$.*

Proof: Let $u, u' \in \Delta_1$ be two points satisfying $u \neq u'$. Clearly, when $g_1 u = g_1 u'$, we have $g_2 g_1 u - g_2 g_1 u' = 0$. When $g_1 u \neq g_1 u'$, we can write

$$\frac{|g_2 g_1 u - g_2 g_1 u'|}{|u - u'|} = \frac{|g_2 g_1 u - g_2 g_1 u'|}{|g_1 u - g_1 u'|} \frac{|g_1 u - g_1 u'|}{|u - u'|}$$

Consequently

$$\begin{aligned} \frac{|g_2 g_1 u - g_2 g_1 u'|}{|u - u'|} &\leq \sup \left\{ \frac{|g_2 g_1 u - g_2 g_1 u'|}{|g_1 u - g_1 u'|} \frac{|g_1 u - g_1 u'|}{|u - u'|} : u, u' \in \Delta_1, u \neq u', g_1 u \neq g_1 u' \right\} \\ &\leq \sup \left\{ \left[\frac{|g_2 g_1 u - g_2 g_1 u'|}{|g_1 u - g_1 u'|} \right]^+ \left[\frac{|g_1 u - g_1 u'|}{|u - u'|} \right]^+ : u, u' \in \Delta_1, u \neq u', \right. \\ &\qquad \left. g_1 u \neq g_1 u' \right\} \\ &\leq |g_2||g_1| \end{aligned}$$

which implies $|g_2 g_1| \leq |g_2||g_1|$. □

Before discussing further properties of the gain functional norm, we extend the definition to the case of causal systems.

Let $\Sigma : D_\Sigma \to S(Z^p)$ be a causal system with the non-empty input domain $D_\Sigma \subset S(Z^m)$. The *gain functional norm* $|\Sigma|$ of Σ is defined by

$$|\Sigma| := \sup_{k \geq 0} |\Sigma_{|k}|$$

where $\Sigma_{|k}$ is given by (2.2.3). It is closely related to the standard Lipschitz norm. If a causal system Σ has a finite gain functional norm $|\Sigma|$, we obtain $|\Sigma u - \Sigma u'| = \sup_{k\geq 0} |\Sigma_{|k}P_k u - \Sigma_{|k}P_k u'| \leq \sup_{k\geq 0} |\Sigma_{|k}||P_k u - P_k u'| \leq (\sup_{k\geq 0} |\Sigma_{|k}|)(\sup_{k\geq 0} |P_k u - P_k u'|)$ which directly yields

$$|\Sigma u - \Sigma u'| \leq |\Sigma|(|u - u'|) \tag{2.2.11}$$

The relationship between the gain functional norm of a system and its continuity radius is indicated by the following.

Proposition 2.2.12: *Let $\Sigma : D_\Sigma \to S(Z^p)$ be a causal system with the input domain $D_\Sigma \subset S(Z^m)$. If Σ has a bounded gain functional norm $|\Sigma|$, then the continuity radius of Σ over D does not exceed $|\Sigma|$.*

Proof: For any points $u, u' \in D$ satisfying $|u' - u| \leq 1$, it follows by (2.2.11) that $|\Sigma u - \Sigma u'| \leq |\Sigma|(|u - u'|) \leq |\Sigma|$. □

Consider now the composition $\Sigma := \Sigma_2\Sigma_1$ of two causal systems Σ_1 and Σ_2. For each integer $k \geq 0$, define the vector function $\Sigma^k_{1|0} := (\Sigma_{1|0}, \Sigma_{1|1}, \ldots, \Sigma_{1|k})$. It follows then by causality that, for every integer $k \geq 0$, one has

$$\Sigma_{|k} = \Sigma_{2|k}\Sigma^k_{1|0} \tag{2.2.13}$$

Using Proposition 2.2.10, we obtain $|\Sigma_{|k}| \leq |\Sigma_{2|k}||\Sigma^k_{1|0}|$, so that $|\Sigma_{|k}| \leqslant |\Sigma_{2|k}|(\sup_{0\leq j\leq k} |\Sigma_{1|j}|)$, and

$$\sup_{k\geq 0} |\Sigma_{|k}| \leq \left(\sup_{k\geq 0} |\Sigma_{2|k}|\right)\left(\sup_{j\geq 0} |\Sigma_{1|j}|\right)$$

In view of the definition of the gain functional norm of a causal system, this yields the following statement.

Proposition 2.2.14: *Let $D_1 \subset S(Z^m)$ and $D_2 \subset S(Z^p)$ be non-empty subsets, and let $\Sigma_1 : D_1 \to D_2$ and $\Sigma_2 : D_2 \to S(Z^q)$ be two causal systems with bounded gain functional norms. Then, $|\Sigma_2\Sigma_1| \leq |\Sigma_2||\Sigma_1|$.*

We turn now to the problem of extending the domain of a function without affecting its gain functional norm. This will lead us to an adaptation to our present discrete set-up of an analogue of the Tietze Lemma on the extension of continuous functions, and is critical to our discussion of disturbance attenuating controllers in the next section. To be specific, let $\Sigma : D_\Sigma \to D_0$ be a system having the bounded domain $D_\Sigma \subset S(Z^m)$, and whose image is contained within an interval $D_0 \subset S(Z^p)$. We show that, for any bounded domain $D \subset S(Z^m)$ containing D_Σ, there is an extension $\Sigma_e : D \to D_0$ of Σ whose gain functional norm does not exceed the original gain functional norm of Σ, i.e. $|\Sigma_e| = |\Sigma|$, and whose image is contained within the same interval D_0. To simplify the notation, we shall adopt the convention of using the same symbol for a function and for its extension, so that we shall use the symbol Σ for Σ_e in the following. We start by considering scalar valued functions.

Lemma 2.2.15: *Let $\Delta_{\text{in}} \subset \Delta \subset Z^m$ be non-empty bounded domains, let $\Delta_0 \subset Z$ be an interval, and let $g : \Delta_{\text{in}} \to \Delta_0$ be a scalar valued function with gain functional norm $\gamma := |g|$. There is then an extension $g : \Delta \to \Delta_0$ of g whose gain functional norm is still γ.*

Proof: We construct an extension for g in a stepwise manner, where at each step the function g is extended to one new point at which it has not been previously defined. Assume then that g has been extended so far to the set $\sigma \subset \Delta$ while preserving the gain functional norm γ and the codomain Δ_0. At the initial step, we set $\sigma := \Delta_{\text{in}}$. Consider now a point p in the difference set $\Delta - \sigma$, i.e. a point p in Δ at which the extension has not yet been defined. For each point $q \in \sigma$, define the interval

$$S_q(p) := [g(q) - \gamma|p-q|, g(q) + \gamma|p-q|] \cap \Delta_0 \tag{2.2.16}$$

A sight reflection shows that the set $S_q(p)$ consists of all possible values within Δ_0 that an extension of g could take at the point p without violating the condition

$$|g(p) - g(q)|/|p-q| \leq \gamma \tag{2.2.17}$$

Letting Δ_0 be the interval $[\alpha, \beta] \subset Z$, we obtain

$$S_q(p) = [\max\{g(q) - \gamma|p-q|, \alpha\}, \min\{g(q) + \gamma|p-q|, \beta\}]$$

Since $g(q) \in \Delta_0$, it follows that $S_q(p) \neq \emptyset$. Note that $S_q(p)$ is an interval in Z. Assume now for a moment that the intersection

$$S(p) := \bigcap_{q \in \sigma} S_q(p)$$

is not empty, and let $a \in S(p)$ be an integer. Define the extension of g at the point p by setting $g(p) := a$. It follows then directly from (2.2.16) and (2.2.17) that $g(p) \in \Delta_0$ and $|g(p) - g(q)|/|p-q| \leq \gamma$ for all points $q \in \sigma$, so that we obtain an extension to the larger set $\sigma \cup p$. Further extension of g to the entire domain Δ is then achieved by repeating this process for each point in the difference set $\Delta - \Delta_{\text{in}}$. Thus, our proof will conclude upon showing that the set $S(p)$ is not empty, which we do next.

The set $S(p)$ is clearly not empty when the set σ consists of only a single point. For larger sets σ, the fact that $S_q(p)$ is always an interval in Z implies that the set $S(p)$ is not empty if and only if the following holds. For any pair of points $q, r \in \sigma$, the intersection $S_q(p) \cap S_r(p) \neq \emptyset$. We prove that this intersection is not empty.

Let $q, r \in \sigma$ be two points. Without loss of generality, we can take $g(q) \geq g(r)$ (if this inequality does not hold initially, exchange the roles of q and r). By contradiction, assume that $S_q(p) \cap S_r(p) = \emptyset$. But then one must have

$$g(q) - \gamma|p-q| > g(r) + \gamma|p-r|$$

which implies $g(q) - g(r) > \gamma(|p-q| + |p-r|)$. Using the triangle inequality $|p-q| + |p-r| \geq |q-r|$, this yields $g(q) - g(r) > \gamma|q-r|$, in contradiction to the fact that the gain functional norm of g over σ does not exceed γ. Thus, the equality $S_q(p) \cap S_r(p) = \emptyset$ leads to a contradiction, and it follows that $S_q(p) \cap S_r(p) \neq \emptyset$ for all pairs of points $q, r \in \sigma$. This concludes our proof. □

Note that, in view of Proposition 2.2.12, the continuity radius of the extension $g : \Delta \to \Delta_0$ of Lemma 2.2.15 does not exceed γ.

Consider next the case of a vector valued function $g : \Delta_{\text{in}} \to Z^p$. The function g is then simply a vector of p scalar valued functions $g = (g_1, \ldots, g_p)$, where $g_i : \Delta_{\text{in}} \to Z$, $i = 1, \ldots, p$. By applying Lemma 2.2.15 to each one of the component functions $g_1, \ldots, g_p$, we obtain the following proposition.

Proposition 2.2.18: *Let $\Delta_{\text{in}} \subset \Delta \subset Z^m$ be non-empty bounded domains, let $\Delta_0 \subset Z^p$ be an interval, and let $g : \Delta_{\text{in}} \to \Delta_0$ be a function with gain functional norm $\gamma := |g|$. There is then an extension $g : \Delta \to \Delta_0$ of g whose gain functional norm is still γ.*

As before, Proposition 2.2.12 implies that the continuity radius of the extension $g : \Delta \to Z^p$ of Proposition 2.2.18 cannot exceed γ.

Finally, we consider the extension of causal systems.

Proposition 2.2.19: *Let $D_\Sigma \subset D \subset S(Z^m)$ be non-empty bounded domains, let $D_0 \subset S(Z^p)$ be an interval, and let $\Sigma : D_\Sigma \to D_0$ be a causal system having the gain functional norm $\gamma := |\Sigma|$. Then, there is a causal extension $\Sigma : D \to D_0$ of Σ having gain functional norm γ.*

Proof: Let $\Delta_0 \subset Z^p$ be the interval generating D_0, i.e. $D_0 = S(\Delta_0)$. Recall that $\gamma = \sup_{k\geq 0} \gamma_k$, where γ_k is the gain functional norm of the function $\Sigma_{|k}$, $k = 0, 1, 2, \ldots$. By Proposition 2.2.18 there is, for every integer $k \geq 0$, an extension $\Sigma_{|k,e} : P_k D \to \Delta_0$ of the function $\Sigma_{|k} : P_k D_\Sigma \to \Delta_0$, with gain functional norm $|\Sigma_{|k,e}| = \gamma_k$. The family of extended functions $\{\Sigma_{|k,e}\}_{k=0}^{\infty}$ induces then a causal extension $\Sigma_e : D \to D_0$ of Σ, whose gain functional norm δ is, by definition, given by $\delta = \sup_{k\geq 0} |\Sigma_{|k,e}| = \sup_{k\geq 0} \gamma_k = \gamma$, as required. □

2.3. *Disturbance attenuating systems*

In this subsection we introduce an analogue of the gain functional norm, which leads to a somewhat less restrictive treatment of disturbance attenuation. Consider for a moment a system whose gain functional norm does not exceed 1. For such a system, the effect of an input disturbance on the system output cannot exceed the amplitude of the disturbance. As appealing as this is, it may in some cases be overly restrictive. In the present subsection we 'atomize' the disturbance amplitude into multiples of an integer $\delta > 0$. We regard a system as 'disturbance attenuating' if any input disturbance of amplitude not exceeding δ causes an output deviation of amplitude not exceeding δ. In this sense, the system 'attenuates' the class of disturbances of amplitude not exceeding δ. However, the output deviation caused by an input disturbance of amplitude less than δ may exceed the disturbance amplitude (but may not exceed δ). This broader sense of disturbance attenuation is satisfactory in many applications, and it broadens the class of systems for which disturbance attenuation can be achieved. We start our discussion of this broader notion with the following definition.

Definition 2.3.1: Let $g : \Delta_{\text{in}} \to Z^p$ be a function defined over the non-empty domain $\Delta_{\text{in}} \subset Z^m$, and let $\delta > 0$ be an integer. The *δ-gain functional norm* $|g|_\delta$ of g is defined by the integer

$$|g|_\delta := \sup\left\{\left[\frac{|gu - gu'|}{d\delta}\right]^+ : \mathrm{d} := \left[\frac{|u-u'|}{\delta}\right]^+ ; u, u' \in \Delta_{\text{in}}, u \neq u'\right\}$$

The function g is *δ-attenuating* if $|g|_\delta \leq 1$. □

Note that when $\delta = 1$, the δ-gain functional norm reduces to the gain functional norm defined earlier in (2.2.9). When the δ-gain functional norm is finite, it is, of course, a non-negative integer.

A δ-attenuating function has the property that a perturbation of amplitude not exceeding δ in its input induces a deviation of amplitude not exceeding δ in its output. Strictly speaking, however, the term 'attenuating' applies only to the overall effect of the entire class of disturbances of amplitude not exceeding δ. A particular input disturbance of amplitude less than δ may induce an output deviation of amplitude larger than its own, but not exceeding δ. When $\delta = 1$, the amplitude of an output deviation never exceeds the amplitude of the input perturbation causing it.

In the context of controlling a system (discussed later), the main advantage of the use of the δ-gain functional norm with $\delta > 1$ is that it entails less restrictive requirements. It allows more latitude in controller selection and design, and increases the class of systems amenable to control. With a larger value of δ, the designer can also take advantage of the further freedom to improve other aspects of system performance.

For a vector $u \in Z^m$, we denote $|u|_\delta := [|u|/\delta]^+$. It follows then from the definition that

$$|gu - gu'|_\delta \leq |g|_\delta |u - u'|_\delta \quad \text{for all } u, u' \in \Delta_{\text{in}}$$

We start our investigation of the δ-gain functional norm by showing that it is multiplicative under composition.

Proposition 2.3.2: *Let $\delta \geq 1$ be an integer, let $\Delta_1 \subset Z^m$ and $\Delta_2 \subset Z^p$ be two non-empty domains, and let $g_1 : \Delta_1 \to \Delta_2$ and $g_2 : \Delta_2 \to Z^q$ be two functions with bounded δ-gain functional norms $|g_1|_\delta$ and $|g_2|_\delta$, respectively. Then, the δ-gain functional norm of the composition $g := g_2 g_1 : \Delta_1 \to Z^q$ satisfies $|g|_\delta \leq |g_1|_\delta |g_2|_\delta$.*

Proof: Consider two elements $u, u' \in \Delta_1$, where $u \neq u'$, and denote

$$d := \left[\frac{|u - u'|}{\delta}\right]^+ \quad \text{and} \quad d' := \left[\frac{|g_1 u - g_1 u'|}{\delta}\right]^+$$

In view of the fact that the round-up operation $[\cdot]^+$ increases a number by less than 1, we have

$$\left[\frac{d'}{d}\right]^+ = \left[\frac{|g_1 u - g_1 u'|}{d\delta}\right]^+ \tag{2.3.3}$$

Assume further that u and u' are such that $g_1 u \neq g_1 u'$. We can write

$$\begin{aligned}
\frac{|g_2 g_1 u - g_2 g_1 u'|}{d\delta} &= \frac{|g_2 g_1 u - g_2 g_1 u'|}{d'\delta} \frac{d'\delta}{d\delta} \\
&\leq \left[\frac{|g_2 g_1 u - g_2 g_1 u'|}{d'\delta}\right]^+ \left[\frac{d'\delta}{d\delta}\right]^+ \\
&= \left[\frac{|g_2 g_1 u - g_2 g_1 u'|}{d'\delta}\right]^+ \left[\frac{|g_1 u - g_1 u'|}{d\delta}\right]^+
\end{aligned}$$

where (2.3.3) was used in the last step. Consequently

$$\begin{aligned}
\frac{|g_2 g_1 u - g_2 g_1 u'|}{d\delta} &\leq \sup \left\{ \left[\frac{|g_2 g_1 u - g_2 g_1 u'|}{|g_1 u - g_1 u'|}\right]^+ \left[\frac{|g_1 u - g_1 u'|}{|u - u'|}\right]^+ : u, u' \in \Delta_1, u \neq u', \right. \\
&\qquad \left. g_1 u \neq g_1 u' \right\} \\
&\leq |g_2|_\delta |g_1|_\delta
\end{aligned}$$

which implies that $|g_1 g_2|_\delta \leq |g_2|_\delta |g_1|_\delta$. □

In our discussion of the control of systems subject to disturbances, we shall encounter the need to extend functions without altering their δ-gain functional norm. Such extensions were discussed in §2.2 for the gain functional norm, which corresponds to the case $\delta = 1$ of the δ-gain functional norm. Presently, we consider the extension problem for the case $\delta > 1$, starting with the following analogue of Lemma 2.2.15. As before, we slightly abuse the notation by using the same symbol for a function and its extension.

Lemma 2.3.4: *Let $\Delta_{\text{in}} \subset \Delta \subset Z^m$ be non-empty bounded domains, let $\Delta_0 \subset Z$ be an interval, and let $\delta > 0$ be an integer. Let $g : \Delta_{\text{in}} \to \Delta_0$ be a scalar valued function with δ-gain functional norm $\gamma := |g|_\delta$. There is then an extension $g : \Delta \to \Delta_0$ of g whose δ-gain functional norm is still γ.*

Proof: The proof is similar to the proof of Lemma 2.2.15. We construct the extension of g in a stepwise manner, where at each step the function g is extended to one new point at which it has not been previously defined. Assume then that g has been extended so far to the set $\sigma \subset \Delta$ while preserving the δ-gain functional norm γ and the codomain Δ_0. At the initial step, we set $\sigma := \Delta_{\text{in}}$. Consider now a point p in the difference set $\Delta - \sigma$, i.e. a point p in Δ at which the extension has not been defined yet. For each point $q \in \sigma$, define the quantity $d(p,q) := [|p-q|/\delta]^+$, and consider the interval

$$S_q(p) := [g(q) - \gamma d(p,q)\delta, g(q) + \gamma d(p,q)\delta] \cap \Delta_0 \tag{2.3.5}$$

Note that $S_q(p)$ consists of all possible values in Δ_0 that an extension of g could have at the point p and still satisfy the condition

$$\frac{|g(p) - g(q)|}{d(p,q)\delta} \leq \gamma \tag{2.3.6}$$

Assume for a moment that the intersection $S(p) := \bigcap_{q \in \sigma} S_q$ is not empty, let $a \in S(p)$ be an integer, and define the extension of g at the point p by setting $g(p) := a$. It follows then directly from (2.3.5) and (2.3.6) that $g(p) \in \Delta_0$ and $|g(p) - g(q)|/(d(p,q)\delta) \leq \gamma$ for all points $q \in \sigma$, so that we obtain an extension to the larger set $\sigma \cup p$. Thus, our proof will conclude upon showing that the set $S(p)$ is not empty, which we do next.

The set $S(p)$ is clearly not empty when σ contains only a single point. For larger sets σ, the fact that $S_q(p)$ is an interval in Z implies that $S(p)$ is not empty if and only if the following holds. For any pair of points $q, r \in \sigma$, the intersection $S_q(p) \cap S_r(p) \neq \varnothing$. We prove that this intersection is not empty. We can clearly take $g(q) \geq g(r)$ (otherwise, exchange the roles of q and r). Assume, by contradiction, that $S_q(p) \cap S_r(p) = \varnothing$ for some pair of points $q, r \in \sigma$. Then, a slight reflection shows that one must have $g(q) - \gamma d(p,q)\delta > g(r) + \gamma d(p,r)\delta$. This implies $g(q) - g(r) > \gamma(d(p,q)\delta + d(p,r)\delta)$. By Lemma 2.3.7 below, we have $d(p,q) + d(p,r) \geq d(q,r)$ so that we obtain $g(q) - g(r) > \gamma\delta d(q,r)$, in violation of the fact that the δ-gain functional norm of g over σ does not exceed γ. Consequently, $S_q(p) \cap S_r(p) \neq \varnothing$ for all pairs of points $q, r \in \sigma$, and our proof concludes. □

Lemma 2.3.7: *For two points $z, w \in Z^m$ and an integer $\delta > 0$, define the quantity $d(z,w) := [|z-w|/\delta]^+$. Then, for every three points $p, q, r \in Z^m$, one has $d(p,q) + d(p,r) \geq d(q,r)$.*

Proof: The triangle inequality directly yields that $|p-q|+|p-r| \geq |q-r|$, or $|p-q|/\delta + |p-r|/\delta \geq |q-r|/\delta$, which implies $[|p-q|/\delta + |p-r|/\delta]^+ \geq [|q-r|/\delta]^+$. Furthermore, a slight reflection shows that $[|p-q|/\delta]^+ + [|p-r|/\delta]^+ \geq [|p-q|/\delta + |p-r|/\delta]^+$. Combining this with the previous inequality, we obtain $[|p-q|/\delta]^+ + [|p-r|/\delta]^+ \geq [|q-r|/\delta]^+$, as required. □

Using techniques similar to the ones used in the proof of Proposition 2.2.18, we obtain the following consequence of Lemma 2.3.4.

Proposition 2.3.8: *Let $\Delta_{\text{in}} \subset \Delta \subset Z^m$ be non-empty bounded domains, let $\Delta_0 \subset Z^p$ be an interval, let $\delta > 0$ be an integer, and let $g : \Delta_{\text{in}} \to \Delta_0$ be a function with δ-gain functional norm $\gamma := |g|_\delta$. There is then an extension $g : \Delta \to \Delta_0$ of g whose δ-gain functional norm is still γ.*

We now adapt the notion of δ-gain functional norm to causal systems.

Definition 2.3.9: Let $\Sigma : D_\Sigma \to S(Z^p)$ be a causal system having the non-empty input domain $D_\Sigma \subset S(Z^m)$, and let $\delta > 0$ be an integer. For every integer $k \geq 0$, let $|\Sigma_{|k}|_\delta$ be the δ-gain functional norm of the function $\Sigma_{|k}$ of (2.2.3) induced by the system Σ. Then, the *δ-gain functional norm* $|\Sigma|_\delta$ of Σ is defined by

$$|\Sigma|_\delta := \sup_{k \geq 0} |\Sigma_{|k}|_\delta$$

The system Σ is *δ-attenuating* if $|\Sigma|_\delta \leq 1$. □

Adapting the proof of Proposition 2.2.19 to the present situation, we obtain from Proposition 2.3.8 the following result.

Proposition 2.3.10: *Let $D_\Sigma \subset D \subset S(Z^m)$ be non-empty bounded domains, let $D_0 \subset S(Z^p)$ be an interval, let $\delta > 0$ be an integer, and let $\Sigma : D_\Sigma \to D_0$ be a causal system having the δ-gain functional norm $\gamma := |\Sigma|_\delta$. Then, there is a causal extension $\Sigma : D \to D_0$ of Σ whose δ-gain functional norm is still γ.*

Consider now the δ-gain functional norm of a composition $\Sigma := \Sigma_2\Sigma_1$ of two causal systems Σ_1 and Σ_2. Using (2.2.13) and Proposition 2.3.2, we have $|\Sigma_{|k}|_\delta \leq |\Sigma_{2|k}|_\delta(\sup_{0 \leq j \leq k} |\Sigma_{1|j}|_\delta)$ for all integers $k \geq 0$. By the Definition of the δ-gain functional norm for a causal system, this implies the following.

Proposition 2.3.11: *Let $D_1 \subset S(Z^m)$ and $D_2 \subset S(Z^p)$ be two non-empty domains, and let $\Sigma_1 : D_1 \to D_2$ and $\Sigma_2 : D_2 \to S(Z^q)$ be two causal systems with bounded δ-gain functional norms $|\Sigma_1|_\delta$ and $|\Sigma_2|_\delta$, respectively. Then, the δ-gain functional norm of the composition $\Sigma := \Sigma_2\Sigma_1 : D_1 \to S(Z^q)$ satisfies $|\Sigma|_\delta \leq |\Sigma_2|_\delta|\Sigma_1|_\delta$.*

An important consequence of this Proposition is the fact that the class of δ-attenuating systems is closed under composition.

3. Disturbance attenuation

In the present section we consider the problem of disturbance attenuation for sequential machines. Specifically, we derive necessary and sufficient conditions for the existence of a disturbance attenuating controller for a given sequential machine Σ. When the conditions for the existence of such a controller are satisfied, we also describe the controller structure. The section starts with a derivation of conditions on the controlled machine Σ that are necessary for the existence of a disturbance attenuating controller. Later we show that these conditions are also sufficient. The discussion is based on a theory of fraction representations of sequential machines developed in this section. We start with some basic definitions and observations.

3.1. *Disturbance attenuation and fraction representations*

We employ the mathematical tools developed in earlier parts of the paper to investigate the propagation of disturbances through the control configuration (Fig. 1). Here, Σ is a given system whose performance in the presence of disturbances has to be controlled, and C is a controller. Our main interest is in controllers C that guarantee disturbance attenuation for the closed loop system, where the term 'disturbance attenuation' is discussed in detail below. In broad terms, disturbance attenuation means that the deviation caused by a disturbance of amplitude $\delta > 0$ does not exceed δ.

In Fig. 1, the system $\Sigma : D_\Sigma \to S(Z^p)$ has the non-empty input domain $D_\Sigma \subset S(Z^m)$. The input domain of the closed loop system is denoted by D_{in}, and it is required to be an interval in $S(Z^m)$. In this way, the closed loop system accepts any input sequence v whose element values stay within a prescribed range. The controller C is then a map $\mathrm{C} : D_{\text{in}} \times S(Z^p) \to S(Z^m)$. In the absence of disturbances (i.e. when $\upsilon_1 = 0$, $\upsilon_2 = 0$ and $\upsilon_3 = 0$), the controller generates the input signal u of Σ according to the equation $u = \mathrm{C}(v, y)$. The configuration contains three additive disturbances $\upsilon_1, \upsilon_3 \in S(Z^m)$ and $\upsilon_2 \in S(Z^p)$, which we assume are bounded in norm by a specified integer $\delta > 0$. Only additive disturbances are considered (but note that any disturbance can be represented as an additive disturbance, equal to the difference between the disturbed and undisturbed signals). Of course, the set-up must be such that the effective input signal $v + \upsilon_3$ is always contained within the input interval $D_{\text{in}} \subset S(Z^m)$ of the closed loop system. The disturbances υ_1, υ_2 and υ_3 may represent actual disturbance or noise signals that affect the configuration, or they may represent deviations between the actual and the nominal models of the system or the controller. The equations that describe Fig. 1 are

$$\left.\begin{aligned} u &= \mathrm{C}(v + \upsilon_3, y + \upsilon_2) + \upsilon_1 \\ y &= \Sigma u \end{aligned}\right\} \tag{3.1.1}$$

Following standard terminology, we say that the configuration in Fig. 1 is *well posed* if the signals u and y are uniquely determined by the input signal v and by the disturbance signals υ_1, υ_2 and υ_3. In most applications, only well posed control configurations are of practical interest. A simple condition that guarantees that Fig. 1 is well posed is the requirement that one of the systems Σ or C be strictly causal (more general conditions for well posedness can, of course, be stated). For the sake of simplicity, we shall assume throughout our discussion that the system Σ is strictly causal and the controller C is causal, so that Fig. 1 is well posed. We use the notation

$$y = \Sigma_c(v, \upsilon_1, \upsilon_2, \upsilon_3) \tag{3.1.2}$$

to denote the response of the closed loop system to the external signal v and the disturbances υ_1, υ_2 and υ_3. Since the configuration is well posed, the input signal u of Σ is uniquely determined by the external input signal v and the disturbances υ_1, υ_2 and υ_3, and we shall write

$$u = E(v, \upsilon_1, \upsilon_2, \upsilon_3) \tag{3.1.3}$$

where E is an appropriate system. It follows then directly that

$$\Sigma_c = \Sigma E \tag{3.1.4}$$

In view of this relation, it is common to refer to E as an *equivalent precompensator*. We shall also use the notation $E_0(v) := E(v,0,0,0)$ and $\Sigma_{c0} := \Sigma_c(v,0,0,0)$ to indicate the noise-free response of the corresponding systems.

A frequent restriction on the controller C is that, for any output sequence y, the map $C(v,y)$ be an injective (one to one) function of the external input sequence v. A controller that satisfies this requirement is called a *reversible controller* (see Hammer 1989 a for a more detailed discussion). For example, an additive feedback controller is always reversible. In fact, the notion of a reversible controller can be regarded, in many ways, as a generalization of the additive feedback controller to the nonlinear case. In intuitive terms, a reversible controller passes on to the controlled system Σ all the degrees of freedom available in the external input space D_{in}, so as to allow maximal utilization of the external input sequence to fine-tune the response of the closed loop system.

For a reversible controller C, the equivalent precompensator E_0 is an invertible system, as indicated by the following statement (see Hammer 1989 a for a proof).

Lemma 3.1.5: *Let $\Sigma : D_\Sigma \to S(Z^p)$ be a strictly causal system with the input domain $D_\Sigma \subset S(Z^m)$, and let $C : D_{in} \times S(Z^p) \to S(Z^m)$ be a causal controller. If C is a reversible controller, then the equivalent precompensator $E_0 : D_{in} \to \operatorname{Im} E_0$ is a set isomorphism.*

When discussing the quantitative effects of disturbances on the performance of a system, one has to specify the number of disturbance sources that are simultaneously active. As a simple example, consider the two-variable function $f(x,u) = x + 2u$. When x is the only variable subject to disturbance, the effect of the disturbance on the value of f can be obtained by calculating the continuity radius of f as a partial function of x only; in this case, this continuity radius is 1. When u is the only variable subject to disturbance, the effect of the disturbance is represented by the continuity radius of f as a partial function of u only; in this case, this continuity radius is 2. When both x and u are simultaneously subject to disturbances, f is considered a function of the vector (x,u), and its continuity radius is 3. In general, Proposition 2.2.8 indicates that the effect of several simultaneous disturbances (as described by the continuity radius) is bounded by the sum of the continuity radii of the individual disturbances acting alone. Thus, it is relatively simple to estimate the combined effect of several disturbances, when the individual effect of each disturbance is known. The evaluation of the effect of each disturbance acting alone has the advantage of providing information that indicates which disturbance sources have the most significant effects on the performance of the system. For these reasons, as well as for the sake of simplicity, we shall consider below the effects of each one of the disturbance sources v_1, v_2 and v_3 of the configuration in Fig. 1 individually.

In this spirit, we turn now to our investigation of the effects of the disturbances v_1, v_2 and v_3 on the control configuration (Fig. 1). We shall require for each of these disturbances that whenever their amplitude is bounded by $\delta > 0$, the deviation they cause in any of the internal or external signals of the configuration also be bounded by δ. In this way we guarantee that the entire control configuration is not disturbed beyond a permissible bound. This requirement leads to the following notion of disturbance attenuation, which is in the spirit of the definition of internal stability used in nonlinear control theory.

Definition 3.1.6: Let $\delta > 0$ be an integer. The configuration of Fig. 1 has a *disturbance attenuation radius* of δ if, for any disturbances $\upsilon_1, \upsilon_3 \in S(Z^m)$ and $\upsilon_2 \in S(Z^p)$ satisfying $|\upsilon_1| \leq \delta$, $|\upsilon_2| \leq \delta$, and $|\upsilon_3| \leq \delta$, and for any external input sequence v satisfying $v + \upsilon_3 \in D_{\text{in}}$, the following hold: (here E is given by (3.1.3) and Σ_c is given by (3.1.2))

(i) $|E(v + \upsilon_3, 0, 0) - E(v, 0, 0)| \leq \delta$

(ii) $|E(v, \upsilon_1, 0) - E(v, 0, 0)| \leq \delta$

(iii) $|E(v, 0, \upsilon_2) - E(v, 0, 0)| \leq \delta$

(iv) $|\Sigma_c(v + \upsilon_3, 0, 0) - \Sigma_c(v, 0, 0)| \leq \delta$

(v) $|\Sigma_c(v, \upsilon_1, 0) - \Sigma_c(v, 0, 0)| \leq \delta$

(vi) $|\Sigma_c(v, 0, \upsilon_2) - \Sigma_c(v, 0, 0)| \leq \delta$ □

A disturbance attenuating control configuration gives rise to a particular fraction representation of the system Σ being controlled. Indeed, consider a control configuration of the form in Fig. 1, where C is a reversible causal controller and Σ is a strictly causal system, and assume the configuration has a disturbance attenuation radius of δ. Combining formula (3.1.4) with Lemma 3.1.5 while setting all disturbances to zero, we obtain the fraction representation

$$\Sigma = \Sigma_{c0} E_0^{-1}$$

where $E_0 : D_{\text{in}} \to S(Z^m)$ and $\Sigma_{c0} : D_{\text{in}} \to S(Z^p)$. As we can see, the input interval D_{in} of the closed loop system serves here as the factorization set. We claim that requirements (i) and (iv) of Definition 3.1.6 imply that E_0 and Σ_{c0} are δ-attenuating systems.

Indeed, consider first E_0. Let $v_1, v_2 \in D_{\text{in}}$ be two distinct points, and let $d := [|v_1 - v_2|/\delta]^+$. Taking into account the fact that D_{in} is an interval, and employing a technique used in the proof of Proposition 2.2.6, we can construct a list $z_0, \ldots, z_d \in D_{\text{in}}$ of points satisfying $z_0 = v_1$; $z_d = v_2$; and $|z_i - z_{i+1}| \leq \delta$, $i = 0, \ldots, d-1$. We then have

$$\left[\frac{|E_0(v_1) - E_0(v_2)|}{d\delta}\right]^+ = \left[\frac{|[E_0(v_1) - E_0(z_1)] + [E_0(z_1) - E_0(z_2)] + \cdots + [E_0(z_{d-1}) - E_0(v_2)]|}{d\delta}\right]^+$$
$$\leq \frac{d\delta}{d\delta} = 1$$

where the inequality follows by the triangle inequality and part (i) of Definition 3.1.6. This shows that E_0 is δ-attenuating. The case of Σ_{c0} is similar. Thus, the fraction representation $\Sigma = \Sigma_{c0} E_0^{-1}$ has numerator and denominator systems that are δ-attenuating. This yields the following proposition.

Proposition 3.1.7: *Let $\Sigma : D_\Sigma \to S(Z^p)$ be a strictly causal system, with the input domain $D_\Sigma \subset S(Z^m)$. Assume there is a causal reversible controller $\mathrm{C} : D_{\text{in}} \times S(Z^p) \to S(Z^m)$ for which the closed loop system of Fig. 1 has disturbance attenuation radius $\delta > 0$. Then, the system Σ has a right fraction representation $\Sigma = ST^{-1}$ with D_{in} serving as the factorization set, where $S : D_{\text{in}} \to S(Z^p)$ and $T : D_{\text{in}} \to \operatorname{Im} T \subset S(Z^m)$*

are causal δ-attenuating systems. The fraction representation is valid over the domain $T[D_{\text{in}}] \subset D_\Sigma$.

The full significance of Proposition 3.1.7 will come to light later in this section, when we consider the construction of disturbance attenuating controllers. In the meantime, we briefly interrupt our examination of disturbance attenuation in order to review the notion of a graph.

Let $\Sigma : D_\Sigma \to S(Z^p) : u \mapsto \Sigma u$ be a strictly causal system with the non-empty input domain $D_\Sigma \subset S(Z^m)$. As usual, the *graph* G_Σ of Σ is a subset of the cross-product space $S(Z^p) \times S(Z^m)$, consisting of all pairs $(\Sigma u, u)$, $u \in D_\Sigma$. The graph of the system Σ plays an important role in our discussion, compatible with its role in the general theory of nonlinear control systems over topological spaces (Hammer 1984 a, 1985). Proposition 3.1.7 has certain implications on the structure of the graph of Σ, as we discuss next.

Assume there is a causal reversible controller $\mathrm{C} : D_{\text{in}} \times S(Z^p) \to S(Z^m)$, where $D_{\text{in}} \subset S(Z^m)$ is an interval, for which the control configuration of Fig. 1 around the given system Σ has a disturbance attenuation radius $\delta > 0$. Then, as Proposition 3.1.7 indicates, there is a right fraction representation $\Sigma = ST^{-1}$, where $S : D_{\text{in}} \to S(Z^p)$ and $T : D_{\text{in}} \to \operatorname{Im} T$ are δ-attenuating causal systems. For every sequence $v \in D_{\text{in}}$, the sequences $u := Tv$ and $y := Sv$ satisfy $y = Sv = (ST^{-1})Tv = \Sigma Tv = \Sigma u$, so that the pair $(Sv, Tv) = (\Sigma u, u)$ is a point of the graph of Σ. Consequently, the set

$$\Gamma = \{(Sv, Tv), v \in D_{\text{in}}\}$$

is a subset of the graph G_Σ of Σ. Define the map

$$M := D_{\text{in}} \to \Gamma : Mv := (Sv, Tv)$$

We claim that M is a set isomorphism. Indeed, M is surjective (onto) by the definition of the set Γ, and it is injective since T is injective. Furthermore, the fact that S and T are both δ-attenuating causal systems implies that M is a δ-attenuating causal system as well, and the following holds.

Lemma 3.1.8: *Let* $\Sigma : D_\Sigma \to S(Z^p)$ *be a strictly causal system with the non-empty input domain* $D_\Sigma \subset S(Z^m)$*, and let* G_Σ *be the graph of* Σ*. Assume there is a reversible causal controller* $\mathrm{C} : D_{\text{in}} \times S(Z^p) \to S(Z^m)$ *for which the closed loop in Fig. 1 has a disturbance attenuation radius* $\delta > 0$*. Then, there is a causal* δ*-attenuating injection* $M : D_{\text{in}} \to G_\Sigma$.

We continue now with our qualitative discussion of the effect of the disturbances v_1, v_2 and v_3 on the configuration of Fig. 1. So far, we have imposed the requirement that small disturbances cause only small deviations of the signals u and y. It is also important to address the question of whether or not it is possible to correct for these deviations, as small as they may be, through small changes in the external input sequence v. To be more specific, assume that the configuration has a disturbance attenuation radius $\delta > 0$, and consider, for example, a persistent (constant) disturbance v_1 of amplitude not exceeding δ. Since the closed loop system has a disturbance attenuation radius of δ, the deviation of the signals u and y caused by this disturbance will not exceed δ. Nevertheless, a deviation has occurred. It would be natural to demand that it be possible to counteract this deviation (and return the signals u and y to their undisturbed values) by making a 'small' adjustment to the external input signal v of the closed loop system. In broader terms, we shall require that it be possible to cancel the effect of any (known) disturbance signal of amplitude not exceeding $d\delta$, by applying an adjustment of magnitude not exceeding $d\delta$ to the external input sequence v, where $d \geq 1$ is any integer. This will lead us to the concept of strict disturbance attenuation, which is defined shortly. In preparation for the definition, we need the following notation.

Let $S_c(D_{\text{in}}, \delta) \subset S(Z^m)$ be the set of all possible input sequences u that the system Σ may obtain within the closed loop configuration of Fig. 1, with disturbance signals of amplitude not exceeding δ. Explicitly, it follows from (3.1.1) that

$$S_c(D_{\text{in}}, \delta) := \{C(v + v_3, y + v_2) + v_1 : v \in D_{\text{in}}, v + v_3 \in D_{\text{in}}, |v_1| \leq \delta, |v_2| \leq \delta, |v_3| \leq \delta\}$$

Definition 3.1.9: Let $\delta > 0$ be an integer. For a pair of internal input sequences $u, u' \in S_c(D_{\text{in}}, \delta)$ of Σ in Fig. 1, denote by $y := \Sigma u$ and $y' := \Sigma u'$ the corresponding output sequences. Then, the configuration in Fig. 1 with a reversible controller C is *strictly disturbance attenuating* with radius $\delta > 0$ if the following hold.

(i) The configuration has a disturbance attenuation radius of δ; and

(ii) whenever $|u - u'| \leq d\delta$ and $|y - y'| \leq d\delta$ for some integer $d \geq 1$, there are external input sequences $v, v' \in D_{\text{in}}$ for which $u = E_0 v$, $u' = E_0 v'$, and $|v - v'| \leq d\delta$. □

In order to examine the implications of Definition 3.1.9, assume first that the configuration in Fig. 1 has a disturbance attenuation radius of δ, i.e. that part (i) of the definition holds. In the discussion that leads to Lemma 3.1.8, we have seen that in such a case the map $M : D_{\text{in}} \to \Gamma : Mv = (\Sigma_{c0} v, E_0 v)$ is a δ-attenuating set isomorphism. Assume now that part (ii) of Definition 3.1.9 also holds. Then, in the notation of the Definition, we have $Mv = (y, u)$ and $Mv' = (y', u')$, or $M^{-1}(y, u) = v$ and $M^{-1}(y', u') = v'$. This shows that when part (ii) of Definition 3.1.9 holds, the inverse map $M^{-1} : \Gamma \to D_{\text{in}}$ must also be δ-attenuating. In other words, when the configuration in Fig. 1 is strictly disturbance attenuating with radius δ, the systems M and M^{-1} are both δ-attenuating. A system M for which M and M^{-1} are both δ-attenuating is called a *δ-unimodular system*. We have obtained the next statement.

Proposition 3.1.10: *Let $\Sigma : D_\Sigma \to S(Z^p)$ be a strictly causal system, let G_Σ be the graph of Σ, and let $\delta > 0$ be an integer. Assume there is a causal reversible controller $C : D_{\text{in}} \times S(Z^p) \to S(Z^m)$ for which the closed loop (Fig. 1) is strictly disturbance attenuating with radius δ. Then there is a causal injection $M : D_{\text{in}} \to G_\Sigma$ for which the restriction $M : D_{\text{in}} \to \operatorname{Im} M$ is δ-unimodular.*

In general terms, Proposition 3.1.10 indicates that if there is a controller for Σ that provides strict disturbance attenuation, the system Σ must have the following property: the graph of Σ must contain a subset that is homeomorphic to an interval $D_{\text{in}} \subset S(Z^m)$, in the sense described in the Proposition. We show below that this property is, in fact, also a sufficient condition for the existence of a controller that provides strict disturbance attenuation for Σ. Thus, the existence of a subset of the graph of Σ that is homeomorphic to an interval is a necessary and sufficient condition for strict disturbance attenuation. This fact is closely analogous to the situation encountered in the theory of robust stabilization of nonlinear systems over topological spaces (Hammer 1989 b), and can be viewed as a general principle of control theory.

3.2. *The existence of a controller for strict disturbance attenuation*

In the previous subsection, we have discussed conditions that are necessary for the existence of a controller that provides strict disturbance attenuation for a given system Σ. In particular, we have seen that, whenever such a controller exists, there must be a subset of the graph of Σ that is homeomorphic to an interval of $S(Z^m)$. In the present subsection we consider the reverse direction, showing that whenever such a subset exists, one can construct a controller that provides strict disturbance attenuation for the system Σ. We start with the examination of a fraction representation of Σ that is directly induced by the graph of Σ (see also Hammer 1984 a, 1985 for the use of analogous techniques in the theory of nonlinear control systems over topological spaces).

First, some notation.

Denote by $\Pi_p : S(Z^p) \times S(Z^m) \to S(Z^p) : (y, u) \mapsto y$ the standard projection onto the first p coordinates. Similarly, let $\Pi_m : S(Z^p) \times S(Z^m) \to S(Z^m) : (y, u) \mapsto u$ be the standard projection onto the last m coordinates. It follows directly that the projections Π_p and Π_m are both δ-attenuating for any integer $\delta > 0$.

Now, let $\Sigma : D_\Sigma \to S(Z^p)$ be a strictly causal system with the non-empty input domain $D_\Sigma \subset S(Z^m)$, and let $G_\Sigma \subset S(Z^p) \times S(Z^m)$ be the graph of Σ. Let

$$P : G_\Sigma \to S(Z^p) : P(y, u) := y$$

be the restriction of the projection Π_p to the graph G_Σ of Σ, and let

$$Q : G_\Sigma \to D_\Sigma : Q(y, u) := u \tag{3.2.1}$$

be the restriction of the projection Π_m to the graph G_Σ. Note that on the graph G_Σ, each point (y, u) corresponds to exactly one value of u, since $y = \Sigma u$. Consequently, Q is a set isomorphism, with the inverse $Q^{-1}u = (\Sigma u, u)$ for all $u \in D_\Sigma$. It can also be readily seen that Q is a bicausal system. Furthermore, for every input sequence $u \in D_\Sigma$ we can write $y = \Sigma u = P(\Sigma u, u) = PQ^{-1}u$, which yields the fraction representation

$$\Sigma = PQ^{-1}$$

over the input domain D_Σ of Σ. This fraction representation has then the special property that its numerator and denominator systems are both δ-attenuating for every integer $\delta > 0$, and its factorization space is the graph of Σ. The denominator Q is bicausal, and the numerator $P = \Sigma Q$ is strictly causal by the strict causality of Σ.

In Proposition 3.1.10 we have seen that a controller that provides strict disturbance attenuation with radius $\delta > 0$ for the system Σ exists only if there is a causal δ-unimodular system $M : D_{\text{in}} \to \operatorname{Im} M \subset G_\Sigma$, where D_{in} is an interval in $S(Z^m)$. We now prepare to address the converse direction of this fact, starting with some terminology.

A subset $\Gamma \subset G_\Sigma$ is *δ-homeomorphic to* a subset $D \subset S(Z^m)$ if there is a bicausal and δ-unimodular set isomorphism $M : D \to \Gamma$. The system M is then called a *δ-homeomorphism.*

Next, let $D_{\text{in}} \subset S(Z^m)$ be an interval, and let $\delta > 0$ be an integer. A point $v \in D_{\text{in}}$ is a *δ-interior point* of D_{in} if D_{in} contains all points $v' \in S(Z^m)$ satisfying $|v' - v| \leq \delta$. In other words, a δ-interior point of D_{in} is an interior point of D_{in}, not closer than δ to the boundary of D_{in}. Note that when the configuration of Fig. 1 has a disturbance attenuation radius of δ, only external input sequences that are δ-interior points of D_{in} are allowed, so as to guarantee that $v + \upsilon_3 \in D_{\text{in}}$ for all $|\upsilon_3| \leq \delta$. Of course, we assume throughout this paper that the interval D_{in} is large enough, so that the set of δ-interior points of D_{in} is not empty.

We turn now to a technical property of subsets that are δ-homeomorphic to an interval. In qualitative terms, we show that a subset of the graph of Σ that is δ-homeomorphic to an interval, must contain all input/output pairs that correspond to a neighbourhood of size δ around each input sequence it includes. An accurate statement of this property is provided by the following auxiliary result.

Lemma 3.2.2: *Let $\Sigma : D_\Sigma \to S(Z^p)$ be a strictly causal system with the non-empty input domain $D_\Sigma \subset S(Z^m)$, let $D_{\text{in}} \subset S(Z^m)$ be an interval, and let $\delta > 0$ be an integer. Assume that the graph of Σ has a subset Γ that is δ-homeomorphic to D_{in}, and let $M : D_{\text{in}} \to \Gamma$ be a δ-homeomorphism. Then,* (i) *for every δ-interior point $v \in D_{\text{in}}$, the set Γ contains all points $(\Sigma u, u) \in S(Z^p) \times S(Z^m)$ for which $|u - \Pi_m M v| \le \delta$;* (ii) *the map $\Pi_m M : D_{\text{in}} \to \Pi_m \Gamma$ is a δ-homeomorphism.*

Proof: (i) Fix a δ-interior point $v \in D_{\text{in}}$, and let Θ_δ be the set of all points $v' \in D_{\text{in}}$ satisfying $|v' - v| \le \delta$. Denote $M_\delta := M[\Theta_\delta]$. Then, M_δ is a subset of the graph of Σ, and the set of all input sequences corresponding to elements of M_δ is given by the set $U_\delta := \Pi_m M_\delta \subset S(Z^m)$. Since M_δ is a subset of the graph of Σ, each point of U_δ corresponds to exactly one point of M_δ. Let I_δ be the set of all sequences $u' \in S(Z^m)$ that satisfy $|u' - \Pi_m M v| \le \delta$. Since M is δ-attenuating, we have

$$U_\delta \subset I_\delta \tag{3.2.3}$$

We show next that $U_\delta = I_\delta$. The fact that M is bicausal means that for every integer $n \ge 1$, the first n elements of any sequence of Θ_δ uniquely determine the first n elements of a sequence of M_δ; and, vice versa, the first n elements of any sequence of M_δ uniquely determine that first n elements of a sequence of Θ_δ. Since every sequence of M_δ corresponds to exactly one sequence of U_δ (i.e. its input sequence), we obtain that for every integer $k \ge 0$, there is a set isomorphism $P_k[\Theta_\delta] \cong P_k[U_\delta]$, where P_k is the projection onto the first $k+1$ elements of a sequence, as defined in (2.2.2). Since $P_k[\Theta_\delta]$ and $P_k[U_\delta]$ are finite sets, this implies that they must have the same cardinality. Now, the cardinality of $P_k[\Theta_\delta]$ is equal to that of $P_k[I_\delta]$, since the sets Θ_δ and I_δ are both obtained by adding all sequences of $S(Z^m)$ with amplitude not exceeding δ to a fixed sequence. Thus, $P_k[U_\delta]$ has the same finite cardinality as $P_k[I_\delta]$; since (3.2.3) implies that $P_k[U_\delta] \subset P_k[I_\delta]$, it follows by finiteness that $P_k[U_\delta] = P_k[I_\delta]$. Being valid for all integers $k \ge 0$, this yields $U_\delta = I_\delta$, which proves (i).

To prove (ii), note that the bicausality of M implies that the map $\Pi_m M : D_{\text{in}} \to \Pi_m \Gamma$ is a bicausal set isomorphism. The fact that M is δ-attenuating directly implies that $\Pi_m M$ is δ-attenuating as well. Finally, the earlier part of the proof indicates that $(\Pi_m M)^{-1}[I_\delta] = \Theta_\delta$, which proves that the inverse system $(\Pi_m M)^{-1}$ is also δ-attenuating, concluding our proof. □

We are now in a position to state one of the main results of the paper, which shows that whenever the graph of Σ contains a subset that is δ-homeomorphic to an interval, there is a reversible controller that provides strict disturbance attenuation for Σ. The structure of such a controller is described in the proof below in terms of the δ-homeomorphism. This yields a converse direction of Proposition 3.1.10.

Theorem 3.2.4: *Let $\Sigma : D_\Sigma \to S(Z^p)$ be a strictly causal system with the non-empty input domain $D_\Sigma \subset S(Z^m)$, and let $\delta > 0$ be an integer. Assume there is a bounded subset Γ of the graph of Σ that is δ-homeomorphic to an interval $D_{\text{in}} \subset S(Z^m)$. Then,* *there is a causal reversible controller* C *for which the configuration of Fig. 1 around* Σ *is strictly disturbance attenuating with radius* δ, *and has the external input domain* D_{in}.

Proof: We start by constructing the controller, and then we show that it satisfies the requirements of the theorem. Let $\delta > 0$ be an integer, let Γ be a bounded subset of the graph G_Σ of Σ that is δ-homeomorphic to the interval $D_{\text{in}} \subset S(Z^m)$, and let $M : D_{\text{in}} \to \Gamma$ be a δ-homeomorphism. Define the system

$$N := QM : D_{\text{in}} \to D_\Sigma : v \mapsto u \tag{3.2.5}$$

where Q is the projection defined in (3.2.1). Since Q and M are both δ-attenuating, it follows by Proposition 2.3.11 that N is δ-attenuating as well. Furthermore, the fact that Q and M are both bicausal implies that the restriction $N : D_{\text{in}} \to \operatorname{Im} N$ also is bicausal.

Consider now a sequence $v \in D_{\text{in}}$. Denote $Mv = (y, u) \in G_\Sigma$, so that $y = \Sigma u$. By (3.2.5), we can write $u = Nv$, so that $(y, Nv) = Mv$, and we have $v = M^{-1}(y, Nv)$. Applying now N to both sides of this equality, we obtain the formula

$$u = NM^{-1}(y, Nv) \tag{3.2.6}$$

which creates the input sequence u of the system Σ from the output sequence y of Σ and the sequence $v \in D_{\text{in}}$, in analogy to the way the controller C operates in the configuration of Fig. 1. We shall use (3.2.6) as the basis for the ensuing construction of our controller.

The construction of the controller C depends on an extension of the inverse system $M^{-1} : \Gamma \to D_{\text{in}}$. Since Γ is a bounded set by assumption, there are two intervals $D_Y \subset S(Z^p)$ and $D_U \subset S(Z^m)$ such that $\Gamma \subset D_Y \times D_U$. We construct a causal and δ-attenuating extension $F : D_Y \times D_U \to D_{\text{in}}$ of M^{-1} in two steps. First, for every point $(y, u) \in (\Pi_p \Gamma) \times (\Pi_m \Gamma)$ we set

$$F_1(y, u) := (\Pi_m M)^{-1} u \tag{3.2.7}$$

By part (ii) of Lemma 3.2.2, the system $F_1 : (\Pi_p \Gamma) \times (\Pi_m \Gamma) \to D_{\text{in}}$ is well defined, causal, and δ-attenuating. It is independent of y, but is clearly injective in the variable u for every (fixed) value of y. Combined with our forthcoming construction, the last fact implies that the controller C we build is a reversible controller.

Next, by Proposition 2.3.10, there is a causal δ-attenuating extension $F : D_Y \times D_U \to D_{\text{in}}$ of F_1. We now define the controller $\mathrm{C} : D_{\text{in}} \times D_Y \to \operatorname{Im} N$ by setting

$$\mathrm{C}(v, y) := NF(y, Nv) \quad \text{for all } (v, y) \in D_{\text{in}} \times D_Y \tag{3.2.8}$$

The controller C is causal since N and F are causal. We next show that when this controller is used to control the system Σ according to the configuration of Fig. 1, it provides strict disturbance attenuation with radius δ. To this end, we shall consider each one of the requirements of Definitions 3.1.6 and 3.1.9.

Consider first the effect of a disturbance $|\upsilon_3| \le \delta$ in the configuration of Fig. 1, taking $\upsilon_1 = 0$ and $\upsilon_2 = 0$. Let $v \in D_{\text{in}}$ be an external input sequence satisfying $v + \upsilon_3 \in D_{\text{in}}$. By (3.2.8), the output sequence y of the closed loop system Σ_c and the input sequence u of Σ are given in this case by $(y, u) = M(v + \upsilon_3)$. The fact that M is δ-attenuating indicates directly that requirements (i) and (iv) of Definition 3.1.6 hold.

Consider next the effect of a disturbance $|\upsilon_1| \le \delta$, taking $\upsilon_2 = 0$ and $\upsilon_3 = 0$. Let v be a δ-interior point of D_{in} serving as the external input sequence of the configuration in Fig. 1. Let u' denote the sequence that is induced at the input of Σ in the configuration of Fig. 1, when v serves as the external input sequence and the disturbance υ_1 is active. Let u denote the sequence generated by the controller C under these conditions, so that $u' - u = \upsilon_1$. The output sequence of the configuration in Fig. 1 is then given by $y = \Sigma u'$. We then have $u = \mathrm{C}(v, y)$, which by (3.2.8) implies $u \in \operatorname{Im} N$. Define the sequence

$$w := N^{-1}u = F(y, u) \in D_{\text{in}} \tag{3.2.9}$$

Then $\mathrm{C}(w, y) = NF(y, Nw) = NF(y, u) = NN^{-1}u = u$, so that when w is used as the external input sequence of the configuration of Fig. 1 together with the disturbance υ_1, it yields the same sequences u, u' and y as the external input sequence v. Furthermore, by (3.2.8), the equality $\mathrm{C}(w, y) = \mathrm{C}(v, y) = u$ implies

$$F(y, Nw) = F(y, Nv) \tag{3.2.10}$$

Now, note that $\Pi_m M$ and N are identical maps (except for the definition of their codomains), so that $\operatorname{Im} N = \Pi_m[\operatorname{Im} M]$. Also, the equality $u' - u = \upsilon_1$ clearly implies $|u' - u| \le \delta$. It follows then by Lemma 3.2.2(i) that $u' \in \operatorname{Im} N$. Therefore, $(y, u') = (\Sigma u', u') \in \Gamma$, so that $y \in \Pi_p \Gamma$. Combining this with (3.2.7), we obtain that $F(y, Nw) = F_1(y, Nw) = (\Pi_m M)^{-1} Nw = w$, and $F(y, Nv) = F_1(y, Nv) = (\Pi_m M)^{-1} Nv = v$. By (3.2.10) this implies that $v = w = F(y, u)$, where the last equality is by (3.2.9).

Next, consider the sequence $v' := F(y, u')$. In view of the fact that F is δ-attenuating, the relations $|u' - u| \le \delta$ and $v = F(y, u)$ imply that $|v' - v| \le \delta$. Apply now v' as the external input sequence of Fig. 1 with all disturbance signals set to zero. Using (3.2.8) combined with the definition of v' and the fact that (y, u') belongs to the graph of Σ, it follows that $\mathrm{C}(v', y) = u'$. Thus, the new input sequence v' (with no disturbances) creates the same input sequence u' of Σ (and hence the same output sequence y) as the combination (v, υ_1). Consequently, we can say that the effect of the disturbance υ_1 is equivalent to shifting the input sequence from v to v', where $|v' - v| \le \delta$. In this way, the effect of the disturbance υ_1 is equivalent to the effect of a disturbance υ_3 of amplitude not exceeding δ, given by $\upsilon_3 := v' - v$. Our analysis of the effect of the disturbance υ_3 earlier in this proof shows then that requirements (ii) and (v) of Definition 3.1.6 are satisfied. A slight modification of this argument would also show that part (ii) of Definition 3.1.9 is valid.

Finally, consider the effect of a disturbance υ_2 with $|\upsilon_2| \le \delta$, keeping $\upsilon_1 = 0$ and $\upsilon_3 = 0$. As before, let v be a δ-interior point of D_{in} that serves as the external input sequence of Fig. 1. Under these conditions, let u be the input sequence of Σ generated by the controller C in Fig. 1, let y be the output sequence of Σ, and let z be the sequence fed into the controller from the output side. Then, $z - y = \upsilon_2$, so that $|z - y| \le \delta$, and $u = \mathrm{C}(v, z)$. Define now the sequence $u' := \mathrm{C}(v, y)$. The fact that F and N are both δ-attenuating implies, by (3.2.8) and Proposition 2.3.11, that C is δ-attenuating as well; since $|z - y| \le \delta$, it follows then that $|u' - u| = |\mathrm{C}(v, y) - \mathrm{C}(v, z)| \le \delta$. Define now the disturbance $\upsilon_1 := u - u'$, note that $|\upsilon_1| \le \delta$, and consider the configuration of Fig. 1 with the disturbance υ_1 and the external input sequence v, while setting $\upsilon_2 = 0$ and $\upsilon_3 = 0$. Since $u' + \upsilon_1 = u$, it follows that the input sequence of Σ in Fig. 1 (and hence its output sequence y) is the same under these conditions as it was before. Thus, the effect of the disturbance υ_2 is equivalent to the effect of a disturbance υ_1 that satisfies $|\upsilon_1| \le \delta$. In view of our earlier discussion of the effects of the disturbance υ_1, we obtain that requirements (iii) and (vi) of Definition 3.1.6 are satisfied. Thus, all requirements of Definition 3.1.6

hold. We have seen earlier that part (ii) of Definition 3.1.9 also holds. This concludes our proof. □

To summarize in somewhat crude terms, we have seen that a necessary and sufficient condition for strict disturbance attenuation for a system Σ is the requirement that the graph of Σ contain a subset Γ that is homeomorphic to an interval. Furthermore, given a homeomorphism $M : D_{\text{in}} \to \Gamma$ from an interval D_{in} onto a subset Γ of the graph of Σ, we can construct a disturbance attenuating controller using (3.2.8). Methods for the derivation of such homeomorphisms M have been developed for the case of nonlinear systems over topological spaces (Hammer 1989 a, c and 1991). These methods can be adapted to the present discrete framework as well.

We conclude our discussion with some comments relating to the extension of our present results to the case of systems over continuous spaces. First we note that the disturbance attenuating controller C constructed in the proof of Theorem 3.2.4 can be an open loop controller, namely, it may be independent of the output signal y. Such an open loop controller is feasible here due to the fact that our systems operate over discrete spaces. It is facilitated by Lemma 3.2.2, which is not valid for systems over continuous spaces in its present form. For systems over continuous spaces, one would usually need a feedback controller to achieve the disturbance attenuation properties described in Theorem 3.2.4. When extended to the case of systems over continuous spaces, the techniques and results of the present paper will indeed yield feedback controllers when necessary, and we shall discuss this topic in a separate report. We remark that feedback controllers are also indispensable when dealing with large deviations of systems over discrete spaces, as discussed by Hammer (1996 a).

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