Proceedings of the 1984 Automatic Control Conference (ACC), San Diego, California, June 1984

ON THE CONCEPTUAL IMPLICATIONS OF SOME RECENT RESULTS ON POLE ASSIGNMENT

Jacob Hammer

Department of Mathematics and Statistics
Case Western Reserve University
Cleveland, OH 44106, USA

ABSTRACT

Recent studies have shown that, for a given system Σ, the possibilities for pole assignment through internally stable output-feedback control configurations depend on certain integer invariants, related to the number of unstable poles, the number of unstable zeros, and the number of zeros at infinity of the system Σ. The present note provides a qualitative discussion of the conceptual and of the intuitive origins of these results.

1. INTRODUCTION

The purpose of the present note is to discuss the conceptual and the intuitive implications of some recent results in the theory of pole assignment for linear time-invariant systems, reported in HAMMER [1983]. In that report we discuss pole assignment for various control configurations, including dynamic output feedback, inside-loop precompensation, and unity output feedback. The main point of the discussion there is that the possibilities of pole assignment for actual engineering control configurations are not determined by classical system invariants, like the MacMillan degree, the reachability indices, or the observability indices, as is the case for state feedback. Rather, for output feedback, the possibilities of pole assignment are determined by certain other integer invariants which depend, roughly speaking, on the number of unstable poles, on the number of unstable zeros, and on the number of zeros at infinity of the given system. Our main objective in the present note is to show that these results are to be expected not only from the mathematical point of view (which is discussed in the above report), but from the intuitive practical-control point of view as well. Since our main objective here is to provide intuitive insight, we shall almost completely avoid mathematical details. We start with a <u>qualitative</u> review of some classical results on pole assignment.

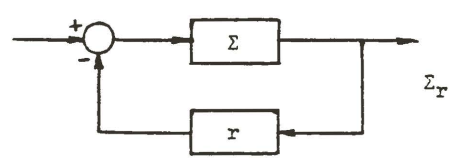
Probably, one of the most fascinating features of a linear time-invariant finite-dimensional control system is the fact that its fundamental control capabilities are determined by the (seeming) meager information contained in a finite set of integers. Specifically, we refer to pole assignment. Consider a linear time-invariant system Σ with a canonical state representation

$$\Sigma : \ \dot{x} = Ax + Bu, \quad y = Cx,$$

where the state vector x is of dimension n, and the input vector u is of dimension m. As is well known. the dynamic behaviour of the system Σ is determined by the roots of the characteristic polynomial $\phi(z) := \det(zI - A)$, which is of degree n. One of the fundamental interrogations into the control capabilities of Σ is the question of how can the dynamic behaviour of Σ be altered through the application of state feedback. Explicitly, one defines the state feedback $u = -Fx + v$, to obtain the new system (which is still controllable, but not always observable) $\dot{x} = A_F x + Bv$, $y = Cx$, where $A_F = A - BF$, and the new characteristic polynomial $\phi_F(z) = \det(zI - A_F)$, which again has degree n. The question then is, what are the new characteristic polynomials $\phi_F(z)$ that can be obtained by choosing appropriate state feedback matrices F. As is well known, the answer to this question is contained in the state feedback pole assignment theorem [6], one direction of which states the following. If the system Σ is controllable, then, <u>for any monic polynomial</u> $\phi(z)$ <u>of degree</u> n, <u>there exists a state feedback</u> F <u>for which</u> $\phi_F(z) = \phi(z)$. Thus, the solution to the problem of pole assignment through state-feedback is characterized by a single integer - the dimension of the state space n (which is commonly called the <u>MacMillan degree</u> of Σ). This is indeed a striking result - though the description of Σ consists of more than n^2 real numbers. and though the transformation $A \to A - BF$ looks rather complicated, nevertheless the set of all attainable characteristic polynomials is completely determined by a single integer - the MacMillan degree.

The theorem of pole assignment by state feedback sparkled a large number of investigations into the more applicative problem of pole assignment by dynamic output feedback, which refers to the following situation. Given a system Σ, one connects around it a dynamic output feedback compensator r

(1.1)



to obtain the new system Σ_r. Let ϕ_r denote the characteristic polynomial of Σ_r. The question now is - what are the characteristic polynomials ϕ_r that can be obtained for Σ_r, by choosing an appropriate dynamic output feedback compensator r.

We pause here for a moment to remark that the present situation is significantly more intricate than the one for the case of state feedback. The complication is mainly due to the fact that the output feedback compensator r is a dynamic system. Under such circumstances, the stability of the (input-output) transfer matrix of Σ_r no longer guaranties the complete stability of the composite system (1.1). Some cancellations of unstable poles by unstable zeros may have occured in the loop, so that the overall system may contain unstable hidden (unreachable or unobservable) modes, which do not appear in the input output relationship Σ_r. One has to guaranty the stability of all modes of the composite system (1.1), not just the stability of the modes appearing in the input-output relationship. A linear time-invariant system is called internally stable if all its modes, including the unobservable and the unreachable ones, are stable. Explicit conditions for the internal stability of feedback systems have been derived in the literature in several forms (e.g., [2], [4], [5]). We now state the problem of pole assignment by output feedback.

(1.2) Problem: Let $\phi(z)$ be a monic polynomial with stable roots. When does there exist a causal feedback compensator r such that $\phi(z) = \phi_r(z)$ (the characteristic polynomial of Σ_r), and Σ_r is internally stable.

A fundamental result in the context of Problem (1.2) was derived in [1]. This result consists of a sufficient condition on the polynomial ϕ, and it can be stated as follows.

(1.3) THEOREM. Let μ be the MacMillan degree of Σ, let μ_1 be the maximal observability index of Σ, and let $\phi(z)$ be any monic polynomial with stable roots. If $\deg \phi(z) \geq \mu + \mu_1 - 1$, then there exists a causal feedback compensator r such that $\phi(z)$ is a characteristic polynomial of Σ_r, and Σ_r is internally stable.

Thus, we again encounter a condition determined by an integer, and every polynomial with stable roots can be assigned as a characteristic polynomial of Σ_r, provided only that its degree satisfies the above integer inequality. When one starts to examine the conceptual implications of Theorem 1.3, one arrives at certain difficulties, which give rise to a new insight into the theory of linear systems. This is the topic of our next section. (We wish to emphasize that the intention of the following discussion is by no means to diminish the significance of the fundamental result (1.3), but rather to use it as a starting point for a new departure.)

2. SOME CONCEPTUAL CONSIDERATIONS

The main conceptual difficulty arising from (1.3) is caused by the fact that usually the MacMillan degree is not a characteristic invariant of the system, but rather a parameter of an approximation of the system. Indeed, the accurate description of most engineering systems (e.g., electrical systems, mechanical systems, flow systems) is provided by a partial differential equation. Very roughly speaking, a system described by a partial differential equation has an infinite number of modes, and therefore an infinite MacMillan degree. The common finite dimensional description of the system is actually only an approximation, obtained after the 'very high frequency' modes of the system are neglected, leaving only a finite number of significant modes. However, the term 'very high frequency' is a relative term, and its quantitative meaning depends on the application at hand. Thus, for one and the same system, the number of significant modes which have to be included in its finite dimensional description may vary from application to application. Different finite dimensional approximations of the same system may have different MacMillan degrees.

For instance, consider an electrical resistor. When this resistor is installed in an audio frequency circuit, it is usually described as a static system, having zero MacMillan degree. However, when the same resistor is installed in an HF (High Frequency) circuit, its description as a static system will no longer be adequate. One has to take into consideration the so called 'stray inductance and capacitance' of the resistor. In the HF circuit, the resistor will be represented by an 'equivalent circuit', which contains not only a resistor, but capacitors and inductors as well. The important point for our present discussion is that the resistor will no longer be approximated by a static system, but rather by a dynamic system having nonzero MacMillan degree. As we see, for one and the same component, different applications require different approximations with different MacMillan degrees. Thus, in many cases, the MacMillan degree is just a parameter of the approximation at hand, and not a physical characteristic of the system.

We return now to our discussion of the problem of pole assignment through the output feedback (1.1). Consider two approximations of the system Σ - a coarser approximation with MacMillan degree μ, and a finer approximation with MacMillan degree $\mu' > \mu$. Theorem 1.3 then provides us with two different statements regarding the possibilities of pole assignment for Σ : in the first case we may assign as characteristic polynomials all those with degree $\deg \phi \geq \mu + \mu_1 - 1$, whereas in the latter case only those polynomials with $\deg \phi \geq \mu' + \mu_1' - 1$, a subset of the first case. There is an imminent paradox in this situation, namely, the more accuracy we use in our description of the system, the more restricted the possibilities for pole assignment described by (1.3) become. Theorem 1.3 differs in its conclusion from one approximation of the same system to the other, and therefore does not provide a meaningful sufficient condition for pole assignment as related to the system Σ.

The main objective of [3] was to address the problem of pole assignment (1.2) for a system, rather than for a particular approximation of it. More explicitly, we seek an answer to the problem of pole assignment which, when applied to approximations of the system Σ, will yield the same result for any significant approximation of Σ. Such an answer has to depend on suitable invariants of the system Σ which are structural in the sense that they are shared by all significant approximations of Σ. Thus, the preliminary quest is to find invariants of the system Σ which are imprinted on all of its significant approximations. Even though this quest sounds somewhat abstract and vague, its resolution is intuitively clear and simple.

Consider again the system Σ. As we have already mentioned, it is a common practice in control engineering to neglect the effect of 'very high frequency' modes of Σ, which are outside the natural frequency range of the application at hand. However, one major precaution has to be taken when these high frequency modes are neglected. Namely, one may neglect only stable modes. An unstable mode will eventually destroy the system, no matter how high its frequency is. All unstable modes of the system Σ have to be represented in any significant approximation of it. Thus, all significant approximations of Σ will have the same number of unstable poles, equal to the number of unstable modes of Σ. The number of unstable poles is therefore a structural invariant of the system in the above sense.

An additional structural invariant of the system is given by the number of its unstable zeros, that is, the number of zeros located on the right hand side of the complex plane. The significance of the unstable zeros follows from internal stability considerations. As we have mentioned earlier, a linear system is said to be internally stable if all its modes, including the hidden ones, are stable. In order to guaranty the internal stability of (1.1) one has to make sure, in particular, that there are no cancellations of unstable poles of the feedback compensator r by unstable zeros of the system Σ in the open loop transmission $r\Sigma$. In case some unstable zeros of Σ are not represented in the approximation of it used for the computation of the stabilizing feedback r, then r may have unstable poles coinciding with these zeros, in which case some unstable cancellations may occur, and the final system may possess internal instabilities when the loop is closed. On the other hand, if the approximation containes more unstable zeros than the system itself, then, by a similar argument, the possibilities of choosing r will be over-restricted. Thus, the number of unstable zeros of Σ is a structural invariant of the system in our above sense.

The final structural invariant that we wish to mention in this context is the number of zeros at infinity, that is, roughly speaking, the difference between the total number of poles and the total number of zeros of the system. The easiest way to understand the intuitive significance of the number of zeros at infinity is to consider its implications for a discrete time system. For such system, the structure of zeros at infinity determines the internal delay of the system, i.e., the number of sampling periods that one has to wait between the occurence of the first nonzero sample of an input sequence, and the occurence of the first nonzero sample of the corresponding output sequence. For instance, the single variable system $z/(z+1)^3$ has two zeros at infinity, and this is also the number of delay-steps between the start of an input sequence and the start of the corresponding output sequence. Since the internal delay of any significant approximation of Σ evidently has to match the internal delay of Σ for any input sequence, any such approximation will, in particular, have the same number of zeros at infinity. We then conclude that the number of zeros at infinity is a structural invariant.

In summary, intuitive control-engineering considerations lead us to the conclusion that the number of unstable poles, the number of unstable zeros, and the number of zeros at infinity are structural invariants of a system. They have to be equally represented in any significant approximating model of the system. even though these models may widely vary in their MacMillan degree - the state space dimension.

3. POLE ASSIGNMENT

We now describe one of the pole assignment results derived in [3] for the control configuration (1.1). Motivated by our previous discussion, we would like to express the conditions for pole assignment in terms of structural invariants of the system Σ, like the number of its unstable poles, the number of its unstable zeros, and the number of its zeros at infinity. Such conditions will then rigidly depend on the system Σ, and not on the particular model at hand used to approximate it. Of course, our discussion here was very qualitative, and therefore it cannot lead us to the exact mathematical definition of the structural invariants of the system. In [3] we define several structural invariants of the system Σ which relate to various versions of the pole assignment problem. For our present case we need two of these invariants: the <u>stability index</u> θ, and the <u>maximal left pole index</u> ρ_1. To explain these invariants, consider first the case when Σ is a single-input single-output system. Let ζ be the number of unstable zeros of Σ, let η be the number of its zeros at infinity, and let ρ be the number of its unstable poles. Then, we have [3]

$$\theta = \zeta + \eta\ , \quad \rho_1 = \rho\ .$$

Thus, the building blocks of θ and ρ_1 are indeed the structural invariants that we have discussed before. In the general case of multi-input multi-output systems, the integers θ and ρ_1 are also related to the same structural invariants. An exact mathematical definition of these invariants in general is outside the scope of our present qualitative note, and it is given in [3]. Using these invariants, the following can be shown [3, section 5]. (For an integer q, we denote $[q]^+ := q$ if $q \geq 0$, and $[q]^+ := 0$ otherwise.)

(3.1) THEOREM. <u>Let $\phi(z)$ be a monic polynomial with stable roots. If $\deg \phi \geq \theta + [\rho_1 - 1]^+$, then there exists a causal feedback compensator r such that ϕ is a characteristic polynomial of Σ_r, and Σ_r is internally stable.</u>

EXAMPLE. In order to avoid unnecessary complications, we consider a single-input single-output case. Let Σ be represented by the transfer function $f = (z-1)(z+1)^4/[(z-2)^2(z+2)^6]$. Here we have $\rho = 2$, $\zeta = 1$, $\eta = 3$, so that $\theta = 4$ and $\rho_1 = 2$. Clearly, $\mu = 8$ and $\mu_1 = 8$. Now, let ϕ be a monic polynomial with stable roots. The different conditions for assignning ϕ as a characteristic polynomial of Σ_r, with Σ_r internally stable, are:

Condition (3.1) [3] :	$\deg \phi \geq 5$.
Condition (1.3) [1] :	$\deg \phi \geq 15$.

4. REFERENCES

[1] BRASCH & PEARSON [1970], Trans. IEEE, <u>AC-15</u>, 34-43.

[2] DESOER & CHAN [1975], J. Frank. Inst., <u>300</u>, 335-351.

[3] HAMMER [1983], Int. J. Contr., <u>37</u>, 63-88.

[4] HAMMER [1983a], Int. J. Contr., <u>37</u>, 37-61.

[5] HAMMER [1983b], Trans. IEEE, <u>AC-28</u>, 489-496.

[6] WONHAM [1967], Trans. IEEE, <u>AC-12</u>, 660-665.

For a more detailed list of references see [3].