Proceedings of the IEEE Conference on Decision and Control, Brighton, England, December 1991

ON THE ALGEBRAIZATION OF NONLINEAR CONTROL

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Abstract

An algebraic theory on the stabilization of discrete and continuous time nonlinear systems by static state feedback is presented. The theory includes necessary and sufficient conditions for the existence of stabilizing feedback functions, as well as methods for their computation.

1. Introduction

The purpose of the present note is to indicate that several fundamental issues in the theory of stabilization of nonlinear systems are of an algebraic nature. We consider the stabilization of discrete-time and of continuous-time nonlinear systems by static state feedback. In both cases, we show that the existence of stabilizing feedback controllers can be completely characterized through certain algebraic properties of the functions determining the state representation of the system. Those properties permit the computation of all stabilizing feedback functions. Furthermore, an inherent resemblance between the discrete-time and the continuous-time cases is elucidated.

Stabilization here is understood in the strong sense of internal stability, meaning that stability is preserved under various small perturbations and noises. In this context, we also adopt the realistic assumption that the input signals of the closed loop systems must all be bounded and of an amplitude not exceeding a pre-specified, but otherwise arbitrary, fixed bound. Thus, we shall deal with internal stabilization over bounded domains.

In the discrete-time case, we consider systems possessing a 'state representation' of the form

(1.1) $$x_{k+1} = f(x_k,u_k),\ k = 0, 1, 2, \dots$$

where $\{u_k\}$ is the input sequence, and $\{x_k\}$ is the output sequence. The function f is assumed to be continuous.

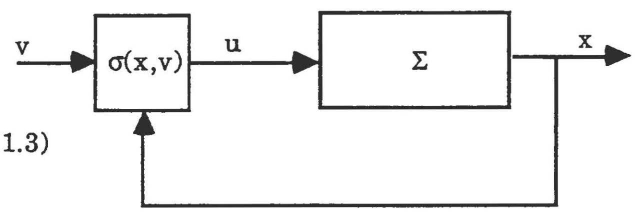
In complete analogy, the continuous-time systems we consider possess a state representation of the form

(1.2) $$\dot{x}(t) = f(x(t),u(t)),\ t \geq 0,$$

where, again, $u(t)$ is the input function, and $x(t)$ is the output function. Here, the function f is assumed to be continuously differentiable. In both cases, we shall refer to the function f as the *state representation function* of the system.

Throughout, p shall denote the dimension of x, and m the dimension of u. The initial condition x_0 or $x(0)$ is assumed specified, while perturbations of the initial condition are permitted.

We consider stabilization by static state-feedback, i.e., by a configuration of the form



(1.3)

where $\sigma : R^p \times R^m \to R^m$ is a continuous function (continuously differentiable in the continuous-time case). For the sake of simplicity of the presentation, we shall assume that the external input v is not present, i.e., that the control is achieved entirely through the feedback, with no operator present. Then, σ becomes a function $\sigma(x)$ of x only, and the closed loop is described by the equations

(1.4) $$x_{k+1} = f(x_k,\sigma(x_k)) \text{ (discrete-time)},$$

(1.5) $$\dot{x}(t) = f(x(t),\sigma(x(t))) \text{ (continuous-time)}.$$

In either case, the closed loop system induced by the configuration (1.3) is denoted by Σ_σ. We describe necessary and sufficient conditions for the existence of feedback functions σ that internally stabilize the closed loop system. The conditions are stated directly in terms of certain algebraic properties of the state representation function f. Despite the fundamental differences between discrete-time and continuous-time systems, the conditions for the existence of stabilizing feedback functions take on a rather similar form in both cases. The conditions we present also permit the computation of stabilizing feedback functions.

In addition to its own interest, the stabilization theory presented here also plays a fundamental role in the theory of fraction representation of nonlinear systems. It provides means for the construction of right coprime fraction representations for nonlinear systems, even in cases where the state of the system is not available as output. The resulting fraction representations can then be used to derive stabilizing dynamic compensators which achieve desirable design objectives, like dynamics assignment. For a more detailed discussion of this topic, see HAMMER [1989b and 1988].

The material presented in this note is a perspective on some recent work by the author (HAMMER [1989a] and [1991]). Alternative recent studies on the theory of stabilization of nonlinear systems can be found in HAMMER [1984] and [1986], DESOER and KABULI [1988], VERMA [1988], MOORE and TAY [1988], SONTAG [1989], PAICE and MOORE [1990], CHEN and de FIGUEIREDO [1990], the references cited in these papers, and others.

2. The discrete-time case

Let $S(R^m)$ be the set of all sequences $\{u_0, u_1, \dots\}$ of m-dimensional real vectors $u_i \in R^m$, $i = 0, 1, 2, \dots$. Adopting the input/output point of view, a system is simply a map $\Sigma : S(R^m) \to S(R^p)$, transforming m-dimensional input sequences into p-dimensional output sequences. The set $\Sigma[S]$ is the image of a subset $S \subset S(R^m)$ through Σ.

In preparation for a discussion of stability, we review some norms. First, given a vector $u \in R^m$, let $|u| := \max\{|u_i|, i = 1, \dots, m\}$ be the maximal absolute value of its coordinates. For a sequence $u \in S(R^m)$, denote $|u| := \sup_{i\geq 0} |u_i|$, so that $|\cdot|$ becomes the usual ℓ^∞-norm. When discussing the continuity of systems, we shall employ a weighted ℓ^∞-norm ρ, given by

$$\rho(u) := \sup_{i\geq 0} 2^{-i} |u_i| \qquad (2.1)$$

for all $u \in S(R^m)$. By $\bar{S}$ we denote the closure of a set $S \subset S(R^m)$, with respect to the topology induced by ρ.

To deal with bounded sequences, let $S(\theta^m)$ be the set of all $u \in S(R^m)$ satisfying $|u| \leq \theta$, where $\theta > 0$. Then, a system $\Sigma : S(R^m) \to S(R^p)$ is *BIBO* (Bounded-Input Bounded-Output)-*stable* if, for every real number $\theta > 0$, there is a real number $M > 0$ such that $\Sigma[S(\theta^m)] \subset S(M^p)$.

A system $\Sigma : S(R^m) \to S(R^p)$ is *stable* if (i) it is BIBO-stable, and (ii) for every real number $\theta > 0$, the restriction $\Sigma : S(\theta^m) \to S(R^p)$ is a continuous map. This notion of stability conforms with the qualitative notion of stability of the (input/output) Lyapunov theory.

In order to examine the stability of the closed loop system (1.3), we need to account for the effect of inaccuracies and noises within the configuration. Let $f : R^p \times R^m \to R^p$ be the state representation function of the system Σ, and let $\sigma : R^p \to R^m$ be the feedback function (recall that we discuss a pure feedback configuration with no external input). Inaccuracies within the state representation of Σ can be taken into account by introducing a noise sequence $n \in S(R^p)$ in the form

$$\begin{aligned} x_{k+1} &= f(x_k, u_k) + n_{k+1}, \ k = 0, 1, 2, \dots, \\ x_0 &= x_{00} + n_0, \end{aligned} \qquad (2.2)$$

where x_{00} is the nominal initial condition. In a similar fashion, the feedback is represented by

$$u_k = \sigma(x_k) + \upsilon_k, \ k = 0, 1, 2, \dots, \qquad (2.3)$$

where $\upsilon \in S(R^m)$ is a noise sequence. We denote by $\Sigma_{\sigma,n,\upsilon}$ the closed loop system with the noises n and υ present. The system $\Sigma_{\sigma,n,\upsilon}$ can be regarded as a system having the input sequences n and υ, so that $\Sigma_{\sigma,n,\upsilon} : S(R^p) \times S(R^m) \to S(R^p)$, where the first term of the cross product represents the noise n and the second term represents the noise υ. As is customary, the noises n and υ are assumed to have 'small' amplitudes, not exceeding a bound which is denoted by ε. We assume that the initial condition x_{00} is restricted to a subset $S \subset R^p$. The notion of internal stability is then defined as follows.

(2.4) DEFINITION. The configuration (1.3) is *internally stable* (for initial conditions within S) if there is a real number $\varepsilon > 0$ such that $\Sigma_{\sigma,n,\upsilon} : S(\varepsilon^p) \times S(\varepsilon^m) \to S(R^p)$ is a stable system for initial conditions $x_{00} \in S$.♦

In qualitative terms, internal stability means that the output of the closed loop system is bounded and depends continuously on the noise signals n and υ, where continuity is with respect to the topology induced by the metric ρ. According to our setup, this also implies continuous dependence on the initial condition, via the noise element n_0. All feedback configurations derived in the present note are internally stable.

Before continuing with our discussion of internal stability, we need some notation. Let $\varepsilon > 0$ be a real number. For a point $x \in R^n$, let $\mathcal{B}_\varepsilon(x)$ be the open ball of radius ε around the point x; namely, the set of all points $y \in R^n$ satisfying $|y - x| < \varepsilon$. Given a subset $S \subset R^n$, denote

$$\mathcal{B}_\varepsilon(S) := \cup_{x\in S} \mathcal{B}_\varepsilon(x). \qquad (2.5)$$

The set $B_\varepsilon(S)$ is clearly an open neighborhood of the set S, consisting of all points $y \in R^n$ for which there is a point $x \in S$ satisfying $|y - x| < \varepsilon$. Finally, let $\Pi_p : R^p \times R^m \to R^p$ be the standard projection onto the first p coordinates, so that $\Pi_p z := (z_1, \dots, z_p)$ for all $z = (z_1, \dots, z_{p+m}) \in R^p \times R^m$.

(2.6) DEFINITION. An *eigenset* E of a function $f : R^p \times R^m \to R^p$ is a subset $E \subset R^p \times R^m$ satisfying $f[E] \subset \Pi_p[E]$.

An *ε-eigenset* $\mathcal{E}$ of the function f is a subset $\mathcal{E} \subset R^p \times R^m$ satisfying the condition $f[B_\varepsilon(\mathcal{E})] \subset \Pi_p[\mathcal{E}]$, where $\varepsilon > 0$ is a real number.♦

In qualitative terms, an eigenset E consists of pairs (x,u) of states and inputs having the following property. For every state $x \in \Pi_p[E]$, and for every input u for which $(x,u) \in E$, the next state $f(x,u)$ of the system stays within $\Pi_p[E]$. In other words, an eigenset E associates with every state $x \in \Pi_p[E]$ a set of inputs u for which the next step also belongs to $\Pi_p[E]$. Thus, by using only pairs $(x,u) \in E$, we can generate an entire output sequence all of whose elements are within $\Pi_p[E]$.

The notion of an ε-eigenset is a somewhat stronger notion to this effect, guarantying that small inaccuracies (not exceeding ε) in x and in u will still leave the next step within $\Pi_p[E]$.

Eigensets and ε-eigensets of a function f can be computed by solving an appropriate set of inequalities involving the function.

Eigensets have simple properties; For instance, one can create new eigensets from available ones as described in the following statement. Given a subset $S \subset R^p \times R^m$, let $S(x)$ be the set of all elements $u \in R^m$ for which $(x,u) \in S$. In the case of an eigenset E and a point $x \in \Pi_p[E]$, the set $E(x)$ consists of input vectors u for which the next step of the system is still within $\Pi_p[E]$.

(2.7) PROPOSITION. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two ε-eigensets of the function $f : R^p \times R^m \to R^p$. Then, the following hold true.

(i) The union $\mathcal{E}_1 \cup \mathcal{E}_2$ is an ε-eigenset of the function f.

(ii) The intersection $\mathcal{E}_1 \cap \mathcal{E}_2$ is an ε-eigenset of the function f.

(iii) For any real number $\alpha > 0$, the intersection $\mathcal{E}_1 \cap R^p \times [-\alpha,\alpha]^m$ is an ε-eigenset of the function f.

(iv) Let $\mathcal{E} \subset R^p \times R^m$ be any subset satisfying the conditions $\Pi_p[\mathcal{E}] = \Pi_p[\mathcal{E}_1]$ and $\mathcal{E}(x) \subset \mathcal{E}_1(x)$ for all $x \in \Pi_p[\mathcal{E}]$. Then $\mathcal{E}$ is an ε-eigenset of the function f.♦

Recall that the graph of a function $g : R^p \to R^m$ is simply a subset of $R^p \times R^m$, consisting of all points of the form $(x,g(x))$, $x \in R^p$.

Based on the concept of a graph, we next define a notion critical to our theory. A subset $S \subset R^p \times R^m$ is a *uniform graph* if there is a continuous function $g : R^p \to R^m$ and a real number $\zeta > 0$ such that $\mathcal{B}_\zeta(g(x)) \subset S(x)$ for all $x \in \Pi_p[S]$. The function g is called a *graphing function* for the set S. The notion of a uniform graph is quite simple on an intuitive level. Clearly, a uniform graph S contains the graph of the continuous function g. Furthermore, it also contains the graph of any continuous function g' which differs from g by less than ζ, namely, any continuous function g' satisfying $|g'(x) - g(x)| < \zeta$ for all $x \in \Pi_p[S]$. The notion of a uniform graph is a natural tool for the description of functions whose values may be corrupted by noise.

We can now state our main result on the stabilization of discrete-time systems by state feedback (HAMMER [1989a]). (A set $S \subset R^n$ is *bounded* if there is a real number $\alpha > 0$ such that $S \subset [-\alpha,\alpha]^n$.)

(2.8) THEOREM. Let $\Sigma : S(R^m) \to S(R^p)$ be a system represented by the recursion $x_{k+1} = f(x_k,u_k)$ with the initial condition x_{00}, where $f : R^p \times R^m \to R^p$ is a continuous function. Then, the following two statements are equivalent.

(i) There exists a continuous state feedback function $\sigma : R^p \to R^m$ for which the closed loop system (1.3) is internally stable.

(ii) The state representation function f has a bounded ε-eigenset $\mathcal{E}$ for which the set $\mathcal{B}_\varepsilon(\mathcal{E})$ is a uniform graph, and $x_{00} \in \Pi_p[\mathcal{E}]$.♦

We comment that condition (ii) of the Theorem can be stated in terms of the existence of a solution to a simultaneous set of inequalities involving the given state representation function f of the system that needs to be controlled. The solution of this set of inequalities also provides the means for the computation of feedback functions σ that internally stabilize the system. For more details on this, as well as for a proof of the Theorem see HAMMER [1989a]. Thus, we have derived a complete characterization of internal stabilizability by static state feedback for discrete-time systems, expressed directly in terms of properties of the given state representation function f. From a practical point of view, this result yields a procedure for the computation of stabilizing feedback functions, as mentioned before. Examples on the computation of stabilizing state feedback functions are provided in the reference.

Finally, we comment that condition (ii) of the Theorem is in fact necessary and sufficient for the existence of a *reversible* state feedback function that internally stabilizes the system, as discussed in HAMMER [1989a]. We chose not to include this stronger result in the present note due to space limitations.

3. The continuous-time case

We consider now the stabilization of a nonlinear continuous-time system described by a differential equation of the form $\dot{x}(t) = f(x(t),u(t))$, using the state feedback configuration (1.3). In the present case we are able to solve a stronger version of the stabilization problem. We characterize the feedback functions σ that internally stabilize the system, while guarantying that the output trajectory remains confined within a prescribed box. The latter permits the designer to satisfy various amplitude constraints imposed by the physical characteristics of components.

Explicitly, denoting by p the dimension of x, we seek feedback functions σ which provide internal stabilization, and for which the output trajectory $x(t)$, $t \geq 0$, of the closed loop system satisfies $\alpha_i < x_i(t) < \beta_i$ for all $i = 1, ..., p$ and all $t \geq 0$, where $\alpha_1, ..., \alpha_p, \beta_1, ..., \beta_p$ are specified real numbers with $\alpha_i < \beta_i$ for all $i = 1, ..., p$. We provide necessary and sufficient conditions for the existence of such feedback functions σ. The conditions can be employed to compute the feedback functions.

As before, we consider pure feedback configurations with no external input, namely, of the form $u = \sigma(x)$. The differential equation describing the closed loop system is then $\dot{x}(t) = f(x(t),\sigma(x(t)))$, $x(0) = x_0$. We shall require both functions f and σ to be continuously differentiable. It will be convenient to introduce some notation.

Let R^+ be the set of all non-negative real numbers, which will serve as our time set. The response of a system Σ is then simply a function $x : R^+ \to R^p$. Denote by $C(R^p)$ the set of all continuous functions $h : R^+ \to R^p$. For a real number $\theta > 0$, let $C(\theta^p)$ be the set of all functions $u \in C(R^p)$ satisfying $|u_i(t)| \leq \theta$ for all $t \geq 0$ and all $i = 1, ..., p$, namely, the set of all continuous functions bounded by θ.

Next, denote by x^T the transpose of a vector $x \in R^p$. Given two vectors $\alpha, x \in R^p$, where $\alpha = (\alpha_1, ..., \alpha_p)^T$ and $x = (x_1, ..., x_p)^T$, let $x \geq \alpha$ (respectively, $x > \alpha$) indicate that $x_i \geq \alpha_i$ (respectively, $x_i > \alpha_i$) for all $i = 1, ..., p$. For two vectors $\alpha, \beta \in R^p$, where $\alpha < \beta$, denote by $[\alpha,\beta]$ (respectively, (α,β)) the set of all vectors $x \in R^p$ satisfying $\alpha_i \leq x_i \leq \beta_i$ (respectively, $\alpha_i < x_i < \beta_i$) for all $i = 1, ..., p$. Also, denote by $C((\alpha,\beta))$ the set of all functions $x \in C(R^p)$ satisfying $x(t) \in (\alpha,\beta)$ for all $t \geq 0$. For future reference, it is convenient to define the following.

<u>The (α,β)-confinement Problem by pure state feedback</u>. Let Σ be described by the differential equation (1.2), where the function $f : R^p \times R^m \to R^p$ is continuously differentiable. Let $\alpha, \beta \in R^p$, where $\alpha < \beta$, be two prescribed fixed vectors. Find a continuous feedback function $\sigma : [\alpha,\beta] \to R^m$, $x \mapsto \sigma(x)$, which is continuously differentiable over (α,β), and for which the following holds. The closed loop differential equation (1.5) has a unique solution $x(t)$, $t \geq 0$, for any initial condition $x_0 \in (\alpha,\beta)$, and this solution satisfies $\alpha < x(t) < \beta$ for all $t \geq 0$.♦

A critical ingredient of the problem of (α,β)-confinement is the fact that the initial condition x_0 of the system Σ is not known in advance, and may be any vector within the domain (α,β). For any such initial condition, a unique solution of the differential equation describing the closed loop system is required to exist, and this solution must be confined to the domain (α,β) at all times. We present below necessary and sufficient conditions for the existence of a solution σ to the problem of (α,β)-confinement by pure state feedback. These conditions are stated entirely in terms of certain algebraic properties of the given feedback function f, and they can be used to compute appropriate feedback functions σ, whenever they exist. Furthermore, the solution of the (α,β)-confinement problem also yields internal stabilization of the given system Σ.

In general, the existence of a solution to the (α,β)-confinement problem for a system Σ depends, among others, on the choice of the bounds α and β; a solution may exist only for some choices of these bounds. The necessary and sufficient conditions derived below can also be used to find the set of bounds α and β for which a solution exists, if there are such bounds.

In order to discuss the notion of stability, we introduce some norms, which are analogous to the norms we used in the discrete-time case. The usual L^∞-norm on R^p is denoted by $|\cdot|$, and is given by the maximal absolute value of the coordinates $|x| := \max\{|x_1|, ..., |x_p|\}$, where $x \in R^p$ is a vector with the components $x_1, ..., x_p$. The L^∞-norm on $C(R^p)$ is also denoted by $|\cdot|$, and is given by $|h| := \text{Sup}_{t\geq 0} |h(t)|$ for a function $h \in C(R^p)$.

On $C(R^p)$ we also induce the following weighted L^∞-norm

(3.1) $\rho'(h) := \text{Sup}_{t\geq 0}\, 2^{-t} |h(t)|$

for a function $h \in C(R^p)$. This norm will be used when dealing with the continuity of systems.

Consider a system Σ described by the differential equation (1.2). Assume the equation has a unique solution $x(t)$, $t \geq 0$, for any relevant initial condition x_0 and input function u. Formally, we regard Σ as a map $\Sigma : R^p \times C(R^m) \to C(R^p)$ which assigns to each pair $(x_0,u) \in R^p \times C(R^m)$ an output function $x \in C(R^p)$, where $x_0 \in R^p$ is the initial condition and $u \in C(R^m)$ is the input function. Then, given a subset $A \subset R^p \times C(R^m)$, let $\Sigma\{A\}$ be the image of the set A through Σ, namely, the set of all output functions generated by the system Σ from elements of A.

(3.2) DEFINITION. A system $\Sigma : R^p \times C(R^m) \to C(R^p)$ described by the differential equation (2.1) is *BIBO* (*B*ounded-*I*nput *B*ounded-*O*utput)-*stable* if the following conditions hold: (i) For every initial condition $x_0 \in R^p$ and every input function $u \in C(R^m)$, the equation (1.2) has a unique solution $x(t)$, $t \geq 0$; and (ii) For every pair of real numbers $\omega, \theta > 0$, there exists a real number $M > 0$ such that $\Sigma[[-\omega,\omega]^p \times C(\theta^m)] \subset C(M^n)$. ♦

Thus, our notion of BIBO-stability includes the existence of a unique solution of the differential equation describing the system. Next, define a norm ρ on product $R^p \times C(R^m)$ by setting

(3.3) $\rho(x,u) := |x| + \rho'(u)$

for all $x \in R^p$ and all $u \in C(R^m)$. We can now introduce our notion of stability.

(3.4) DEFINITION. A system $\Sigma : R^p \times C(R^m) \to C(R^p)$ is *stable* if it is BIBO-stable, and if, for every pair of real numbers $\omega, \theta > 0$, its restriction $\Sigma : [-\omega,\omega]^p \times C(\theta^m)$

$\to C(R^p)$ is a continuous function (with respect to the norm ρ).♦

When dealing with closed-loop systems, we need to take into account the effects of in-loop noises and inaccuracies on the performance. In order to represent noises related to the variable x and inaccuracies in the representation of the function f, we introduce a noise signal $\upsilon_1 \in C(R^p)$ into the differential equation in the form

(3.5) $$\dot{x}(t) = f(x(t),u(t)) + \upsilon_1(t).$$

We assume that υ_1 is a continuous function of t, bounded by $\varepsilon > 0$ in the L^∞ sense, i.e., that $\upsilon_1 \in C(\varepsilon^p)$.

We also permit the values of the feedback function σ to be corrupted by noise and inaccuracies, so that the input u seen by Σ is given by

(3.6) $$u(t) = \sigma(x(t)) + \upsilon_2(t),$$

where $\upsilon_2 \in C(\varepsilon^m)$ is a continuous noise function bounded again by $\varepsilon > 0$.

When the noises υ_1 and υ_2 are incorporated into the configuration (1.3) (with σ a function of x only), they may be regarded as external inputs (over which no control is provided). Then, the closed loop system becomes a map $\Sigma_\sigma : R^p \times C(\varepsilon^p) \times C(\varepsilon^m) \to C(R^p)$, where the terms in the cross product represent the initial condition x_0, the noise υ_1, and the noise υ_2, respectively. The differential equation describing the closed loop system Σ_σ with the noises υ_1 and υ_2 present is given by

(3.7) $$\dot{x}(t) = f(x(t),\sigma(x(t))+\upsilon_2(t)) + \upsilon_1(t), \quad x(0) = x_0.$$

(3.8) DEFINITION. Let $\omega > 0$ be a real number, and let $S \subset [-\omega,\omega]^p$ be a subset.The closed loop system (1.3) is *internally stable* (over the domain S of initial conditions) if there is a pair of real numbers $\varepsilon, N > 0$ such that the following hold.

(i) $\Sigma_\sigma \{S \times C(\varepsilon^p) \times C(\varepsilon^m)\} \subset C(N^p)$, and

(ii) The map $\Sigma_\sigma : S \times C(\varepsilon^p) \times C(\varepsilon^m) \to C(R^p)$ is continuous (with respect to ρ).

The number ε is referred to as the *noise level.* ♦

When the noises υ_1 and υ_2 are present, we shall refer to our (α,β)-confinement problem as the *disturbed (α,β)-confinement problem with pure feedback.* In precise terms, the problem consists of finding a continuous function $\sigma : [\alpha,\beta] \to R^m$ which is continuously differentiable over (α,β), and for which the following holds true: there is a real number $\varepsilon > 0$ such that $\Sigma_\sigma[(\alpha,\beta) \times C(\varepsilon^p) \times C(\varepsilon^m)] \subset C((\alpha,\beta))$. In addition to that, we shall require the closed loop system to be internally stable.

We describe now the solution of the disturbed (α,β)-confinement problem with internal stability. First, some notation. Let $\alpha, \beta \in R^p$ be two fixed vectors satisfying $\alpha < \beta$, and let $\Gamma(\alpha,\beta)$ denote the boundary of the rectangular box $[\alpha,\beta]$. In explicit terms, the boundary consists of $2p$ faces given by

(3.9) $$\Gamma_i^-(\alpha,\beta) := \{(x_1, \dots, x_p) \in [\alpha,\beta] : x_i = \alpha_i\}$$
$$\Gamma_i^+(\alpha,\beta) := \{(x_1, \dots, x_p) \in [\alpha,\beta] : x_i = \beta_i\},$$

$i = 1, \dots, p$, and

(3.10) $$\Gamma(\alpha,\beta) = \cup_{i=1}^{p} [\Gamma_i^-(\alpha,\beta) \cup \Gamma_i^+(\alpha,\beta)].$$

Ignoring for a moment the noises, the closed loop system Σ_σ is represented by the differential equation $\dot{x}(t) = f(x(t),\sigma(x(t)))$, $x(0) = x_0$. The state representation function f has p components $f_1, \dots, f_p$, each of which represents the derivative of the corresponding coordinate x_i, $i = 1, \dots, p$, along the system's trajectory. We shall reach the solution of our confinement problem by constructing a feedback function σ which meets the following requirements for some real number $\zeta > 0$: for each $i = 1, \dots, p$, the composed component function $f_i(x,\sigma(x))$ satisfies $f_i(x,\sigma(x)) \le -\zeta$ for all $x \in \Gamma_i^+(\alpha,\beta)$ and $f_i(x,\sigma(x)) \ge \zeta$ for all $x \in \Gamma_i^-(\alpha,\beta)$. A slight reflection shows that these properties guaranty that the trajectory of the closed loop system cannot exit the set $[\alpha,\beta]$. In fact, these conditions also provide for internal stability.

To continue with our discussion, recall that a subset $S \subset R^p \times R^m$ is the graph of a function $g : R^p \to R^m$ if $S = \{(x,u) \in R^p \times R^m : u = g(x)\}$; and $\Pi_p : R^p \times R^m \to R^p$ is the standard projection onto the first p coordinates. Given two subsets $S \subset R^p \times R^m$ and $X \subset R^p$, we say that S is a *uniform graph on X* if there is a continuous function $g : X \to R^m$ and a real number $\xi > 0$ such that $S = \{(x,u) \in X \times R^m : u \in \mathcal{B}_\xi(g(x))\}$. The function g is then called a *graphing function on X* of the set S, and the number ξ is called a *graphing radius.* We have already mentioned earlier that a uniform graph is simply a 'thickened' graph of a continuous function. Finally, for a function $h : R^n \to R$, a subset $A \subset R^n$, and a real number β, the notation $h(A) \ge \beta$ indicates that $h(x) \ge \beta$ for all $x \in A$.

Returning to our disturbed (α,β)-confinement problem, let $\zeta > 0$ be a real number. For each point x of the boundary $\Gamma(\alpha,\beta)$, construct the set $U_{f,\zeta}(\alpha,\beta,x)$ of input values

$$U_{f,\zeta}(\alpha,\beta,x) := \left\{ u \in R^m \;\middle|\; \begin{array}{l} f_i(x,u) \ge \zeta \\ \text{for all } i \in \{1, \dots, p\} \\ \text{for which } x \in \Gamma_i^-(\alpha,\beta), \\ \text{and} \\ f_i(x,u) \le -\zeta \\ \text{for all } i \in \{1, \dots, p\} \\ \text{for which } x \in \Gamma_i^+(\alpha,\beta). \end{array} \right\}$$

Note that $U_{f,\zeta}(\alpha,\beta,x)$ is obtained simply by solving a set of inequalities determined by the given state representation function f of the system Σ. We can somewhat condense the notation by listing each subset $U_{f,\zeta}(\alpha,\beta,x)$ side by side with the point x, which yields the following subset of $R^p \times R^m$ characterizing all sets $U_{f,\zeta}(\alpha,\beta,x)$

(3.11)

$S_f(\alpha,\beta,\zeta) := \{(x,u) \in R^p \times R^m : x \in \Gamma(\alpha,\beta),\ u \in U_{f,\zeta}(\alpha,\beta,x)\}.$

Of particular interest is the situation when there is a real number $\zeta > 0$ for which the set $S_f(\alpha,\beta,\zeta)$ contains a uniform graph on the boundary $\Gamma(\alpha,\beta)$. In intuitive terms, the significance of this case is quite simple. It means that a continuous function $g : \Gamma(\alpha,\beta) \to R^m$ defined on the boundary $\Gamma(\alpha,\beta)$ exists for which the following holds for all $i = 1, ..., p$: There is a real number $\xi > 0$ such that $f_i(x,\mathcal{B}_\xi(g(x))) \geq \zeta$ whenever $x \in \Gamma_i^-(\alpha,\beta)$, and $f_i(x,\mathcal{B}_\xi(g(x))) \leq -\zeta$ whenever $x \in \Gamma_i^+(\alpha,\beta)$. The function g, when used as feedback on the boundary $\Gamma(\alpha,\beta)$, will guaranty that the trajectory can not emerge from the box (α,β), even when appropriately small disturbances are present. When g is extended to the entire domain $[\alpha,\beta]$ (and such an extension is always possible here; see HAMMER [1991]), we obtain a feedback function that confines the system as desired. Furthermore, this function also provides for internal stabilization. A refinement of these arguments leads to the following result (HAMMER [1991]), which is a close analog of the situation in the discrete time case discussed earlier.

(3.12) THEOREM. Let Σ be a system described by the differential equation $\dot{x}(t) = f(x(t),u(t))$, $x(0) = x_0$, where $f : R^p \times R^m \to R^p$ is continuously differentiable, and let $\alpha < \beta$ be two fixed vectors in R^p. Then, the following two statements are equivalent.

(i) There exists a feedback function $\sigma : [\alpha,\beta] \to R^m$ solving the disturbed (α,β)-confinement problem for the system Σ, with the closed loop system Σ_σ being internally stable for all initial conditions $x_0 \in (\alpha,\beta)$.

(ii) The given state representation function f of Σ has the following property: There is a real number $\zeta > 0$ such that the set $S_f(\alpha,\beta,\zeta)$ contains a uniform graph on the boundary $\Gamma(\alpha,\beta)$. ♦

Condition (ii) of the Theorem can be reduced to the solution of a set of simultaneous inequalities, which are derived directly from the given state representation function f of the system Σ. This set of inequalities also permits the construction of stabilizing feedback functions (see HAMMER [1991] for details and examples).

Note that the solution depends on the amplitude bounds α and β; a solution may exist only for some choices of these bounds. The values of α and β for which a solution exists can be derived through the Theorem, by finding the bounds α, β for which condition (ii) can be satisfied.

To conclude, we have derived algebraic conditions that govern the existence of stabilizing state feedback functions for nonlinear systems. For both discrete and continuous time systems, the conditions reduce to the solution of a set of inequalities directly determined by the given state representation function f of the system that needs to be controlled. In both cases, the theory permits the computation of all stabilizing state feedback functions.

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This research was supported in part by the National Science Foundation, USA, Grant number 8913424.