On Simple Design of Nonlinear Observers for Robust Stabilization of Nonlinear Systems

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Robust internal stabilization is a strong notion of stabilization, whereby stability is maintained regardless of small disturbances, noises, and uncertainties. In this paper, simple tools are developed for achieving robust internal stabilization of a rather large family of nonlinear systems. The main notion is that of a strict observer function, a function characterized by the following feature: subtracting a strict observer function from the differential equation of the controlled system results in an asymptotically stable differential equation. Strict observer functions are relatively easy to derive, and they directly yield robust asymptotic observers; the latter can be combined with robust state feedback controllers to achieve robust internal stabilization. [DOI: 10.1115/1.4029886]

1 Introduction

The observer–controller configuration—a configuration in which an asymptotic observer is combined with a static state feedback controller as depicted in Fig. 1—has played an important role in the development of modern control theory. To restate the well known, the asymptotic observer in the configuration uses the input and output signals of the controlled system to generate an estimate of the controlled system's state. This state estimate is fed into a state feedback controller, replacing the unknown true state of the controlled system and resulting in an asymptotically stable closed-loop configuration. The present paper revisits the efforts to develop a simple methodology for building observer–controller configurations to stabilize nonlinear control systems.

The observer–controller configuration involves the notion of "state" as part of its conceptual makeup, since the observer estimates the state of the controlled system. As a result, the observer–controller configuration is relevant only to systems given in terms of a state representation. To describe such systems, let R denote the real numbers and, for an integer $q > 0$, let R^q be the set of all q-dimensional real vectors. We concentrate on time-invariant systems with a state representation

$$\Sigma: \begin{array}{l} \dot{x}(t) = f(x(t),\ u(t)),\ x(0) = x_0 \\ y(t) = h(x(t)) \end{array} \tag{1.1}$$

where $x(t) \in R^n$ is the state, $u(t) \in R^m$ is the input, and $y(t) \in R^p$ is the output of the system Σ at a time $t \geq 0$; here, n, m, and p are positive integers. The function $f : R^n \times R^m \to R^n$ is the *recursion function* and $h : R^n \to R^p$ is the *output function*. The initial state x_0 of Σ is not known. For the sake of convenience, we assume that Σ has a (not necessarily stable) stationary point at the origin, so that

$$f(0,0) = 0, \quad h(0) = 0 \tag{1.2}$$

As we will be dealing with stability, we need to adopt some norms on our spaces. In R^n, we use the $\ell^\infty -$ norm $|\bullet|$: for a scalar $a \in R$, it is the absolute value $|a|$; and, for a vector $a = (a_1, ..., a_n) \in R^n$, it is the largest coordinate magnitude $|a| = \max_{i=1,2,...,n} |a_i|$. Further, denoting by R^+ the set of nonnegative real numbers, let $F(R^n)$ be the set of all functions $u : R^+ \to R^n : t \mapsto u(t)$, namely, all functions of time with values in R^n. As usual, for $u \in F(R^n)$, the $\ell^\infty -$ norm is $|u| = \sup_{t\geq 0} |u(t)|$. The function u is *bounded* if $|u| < \infty$.

Considering that an asymptotic observer provides a close estimate of a system's state asymptotically, namely, potentially after a long time, asymptotic observers are relevant in situations in which the state of the observed system is well defined at all times. Thus, we restrict our attention to systems subject to the following (standard) requirements; these conditions are satisfied in most practical applications.

ASSUMPTION 1.1. *For a system Σ of the form (1.1), the functions f and h are continuous; the input signal u(t) is piecewise continuous and bounded; and the system is operated with input signals and initial conditions under which the differential equation (1.1) has a unique solution for all times $t \geq 0$.* □

Note that, if Σ does not freely satisfy the last requirement of Assumption 1.1 for all input signals and all initial conditions, it may be possible to satisfy the requirement by operating Σ under some constraints on the input signals and the initial conditions, or by operating it within a stabilizing closed-loop configuration.

The first line of (1.1) is the *input/state part* Σ_s of Σ

$$\Sigma_s: \quad \dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0 \tag{1.3}$$

A *static state feedback controller* for the input/state system Σ_s is formed by a function $\varphi : R^n \to R^m$ that generates the system's input $u(t)$ according to $u(t) = \varphi(x(t))$. This results in the closed-loop autonomous input/state system $\Sigma_{s\varphi}$ of Fig. 2, given by

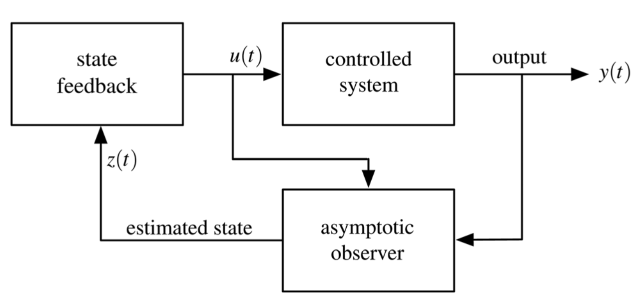

Fig. 1 The observer–controller configuration

Contributed by the Dynamic Systems Division of ASME for publication in the JOURNAL OF DYNAMIC SYSTEMS, MEASUREMENT, AND CONTROL. Manuscript received August 19, 2014; final manuscript received February 18, 2015; published online March 26, 2015. Assoc. Editor: Sergey Nersesov.

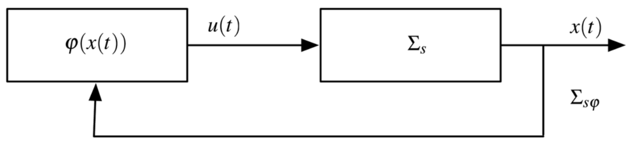

Fig. 2 Static state feedback

$$\Sigma_{s\varphi}: \quad \dot{x}(t) = f(x(t), \varphi(x(t))), \quad x(0) = x_0$$

Static state feedback controllers are the simplest state feedback controllers, but they are nonetheless powerful. In fact, a system that can be asymptotically stabilized by dynamic state feedback can also be so stabilized by static state feedback [1,2]. Thus, concentrating on static state feedback is not overly restrictive.

The process of stabilizing a system Σ of the form (1.1) through an observer–controller configuration consists of two steps:

Step 1: Use an observer to generate an estimate $z(t)$ of the state $x(t)$ of Σ.

Step 2: Feed the estimate $z(t)$ as input to a static state feedback controller that asymptotically stabilizes the input/state part Σ_s of Σ.

The first question is, of course, how accurate must the estimate $z(t)$ be. As there is no information available about the initial state x_0 of Σ, the estimate $z(t)$ is inevitably inaccurate near the initial time $t=0$. For an asymptotic observer, we must have that $\lim_{t\to\infty}[z(t) - x(t)] = 0$, irrespective of the discrepancy between the initial values $z(0)$ and $x(0)$; no restrictions are imposed on the initial state $z(0)$ of the observer. When attempting to implement this requirement, we are faced with two basic questions:

(i) How does one build an asymptotic observer for a system Σ of the form (1.1)?

(ii) If an asymptotic estimate of the state of Σ is fed into a nonlinear static state feedback controller φ that asymptotically stabilizes the input/state part Σ_s of Σ, would the resulting closed-loop system be asymptotically and internally stable?

We start our discussion by examining the possible structure of an asymptotic observer $\mathcal{O}$ for a system Σ of the form (1.1). To construct and operate an asymptotic observer for Σ, we must use all the information available about Σ. This includes the recursion function f, the output function h, the input signal $u(t)$, and the output signal $y(t)$. The last two will serve as the input signals of the observer $\mathcal{O}$. Denoting by $z(t)$ the asymptotic estimate of the state $x(t)$ of Σ generated by $\mathcal{O}$, we obtain the operational environment of Fig. 3. For $z(t)$ to be an asymptotic estimate of $x(t)$, we must have for all initial conditions

$$\lim_{t\to\infty}[z(t) - x(t)] = 0 \tag{1.4}$$

An asymptotic observer $\mathcal{O}$ for Σ is an input/state system with state $z(t)$ and input signals $u(t)$ and $y(t)$ represented by the differential equation

$$\mathcal{O}: \dot{z}(t) = s(z(t), u(t), y(t)), \quad t \geq 0, \quad z(0) = z_0 \tag{1.5}$$

Here, $z(t) \in R^n$ is the estimate of the state of Σ generated by $\mathcal{O}$. There is no relationship between the initial state z_0 of $\mathcal{O}$ and the initial state x_0 of Σ, since x_0 is not known; no restrictions are imposed on the initial state z_0 of the observer. The function $s: R^n \times R^m \times R^p \to R^n$ is the recursion function of the observer. Our main objective in this paper is to develop a simple methodology for deriving s, when s exists.

Given a system Σ with recursion function f and output function h, we show in Sec. 2 that the recursion function s of an asymptotic observer $\mathcal{O}$ for Σ is determined by a function $\omega: R^p \times R^m \to R^n$ that has the following feature:

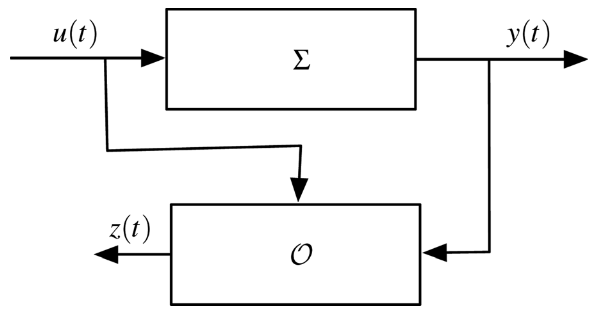

Fig. 3 An asymptotic observer $\mathcal{O}$ for the observed system Σ

The difference $\kappa(x,u) := f(x,u) - \omega(h(x),u)$ is the recursion function of an asymptotically stable differential equation.

Once ω is found, the recursion function s of an asymptotic observer for Σ can be assembled with no further ado. In this way, finding the recursion function of an asymptotic observer boils down to finding a function ω which, when subtracted from f, yields the recursion function of an asymptotically stable differential equation. We refer to such a function ω as a "strict observer function" (see Sec. 2 for an exact definition). Strict observer functions can be derived from the given functions f and h of Eq. (1.1) through Lyapunov's second method (see Sec. 8).

Once an asymptotic observer $\mathcal{O}$ has been derived for Σ, it can be combined with a static state feedback controller to obtain the observer–controller configuration depicted in Fig. 4. In the figure, the estimated state $z(t)$ generated by $\mathcal{O}$ is fed into a state feedback function $\varphi: R^n \to R^m$, instead of the (unknown) true state $x(t)$ of Σ. Here, φ forms a static state feedback controller that asymptotically stabilizes the input/state part Σ_s of Σ, when the true state $x(t)$ of Σ is provided as input to φ. The closed-loop system of Fig. 4 is denoted by $\Sigma_\varphi^{\mathcal{O}}$. Techniques for the derivation of state feedback functions φ that asymptotically stabilize a given nonlinear input/state system are discussed by Refs. [1,2] for general nonlinear systems, by [3] for affine systems, by the references cited in these publications and by many others.

An important issue arises in this context. Recall that $\mathcal{O}$, being an asymptotic observer, may initially provide a crude estimate $z(t)$ of the state $x(t)$ of Σ. True, the state estimate $z(t)$ does converge to the state $x(t)$ of Σ in time, but initially—at the initial time $t=0$ and for some time thereafter, there is no specific relationship between the estimate $z(t)$ and the state $x(t)$. It is important therefore to clarify under what conditions the estimate $z(t)$ is sufficient to induce asymptotic stabilization of Σ, when fed into the state feedback function φ. In addition, practical application of these results is feasible only when the configuration $\Sigma_\varphi^{\mathcal{O}}$ is immune to small errors, noises, and disturbances that may affect its constituents. These issues are addressed in Sec. 7, where we show that, for the observers we construct, $\Sigma_\varphi^{\mathcal{O}}$ is asymptotically and internally stable.

Another interesting aspect of the observer–controller configuration is the well-known classical *separation theorem*, which states that any combination of an asymptotic observer and a stabilizing static state feedback can be used to stabilize a system

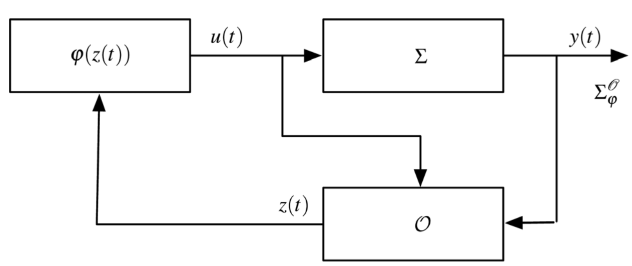

Fig. 4 The observer–controller configuration

under appropriate conditions. In Sec. 7, we revisit the separation theorem for the observers we construct. We show that, as long as the static state feedback controller asymptotically and robustly stabilizes the input/state part Σ_s of Σ, the separation theorem remains valid for our observers, and the resulting closed-loop system can tolerate small errors, noises, and disturbances.

Generally speaking, the class of controllers obtained from the observer–controller configuration is restricted; it includes only controllers formed by a combination of an asymptotic observer and a static state feedback. As a result, the behavior that can be assigned to a closed-loop system via the observer–controller configuration is restricted as well. Nevertheless, the observer–controller configuration does play an important role. First, it provides a rather general technique for stabilizing nonlinear systems. Second, a stabilizing controller obtained through the observer–controller configuration can be used to derive a fraction representation of the system Σ. This fraction representation can then be utilized to obtain more general stabilizing controllers for Σ (see Ref. [4]). In this way, the observer–controller configuration offers an opening for the derivation of stabilizing controllers of a more general nature.

Asymptotic observers have received considerable attention in the control theoretic literature of the past half a century or so, and this paper does not intend to provide a literature survey on the topic of asymptotic observers. Early developments of asymptotic observers can be found in the works of Refs. [5] and [6]. More recently, asymptotic observers for nonlinear systems were investigated by Refs. [7] and [8], by the papers cited by these authors, by their references, and by many others.

Alternative investigations into the stabilization of nonlinear systems can be found in Refs. [4,9–22] and many others.

The paper is organized as follows: Section 2 presents the basics of our formalism, while Sec. 3 introduces a strict notion of Lyapunov stability used in the derivation of strict asymptotic observers. Strict asymptotic observers and strict observer functions are introduced in Sec. 4, where it is also shown how to construct a strict asymptotic observer from a strict observer function. Section 5 examines the robustness of strict asymptotic observers. A simplified method for deriving strict observer functions is described in Sec. 6. Section 7 considers the separation theorem for observer–controller configurations that employ strict asymptotic observers, showing that such configurations are internally stable. Section 8 briefly describes the use of Lyapunov's second method to derive strict observer functions. The paper concludes in Sec. 9 with examples that demonstrate the construction of strict asymptotic observers.

2 The Structure of Asymptotic Observers

Consider the system Σ of (1.1), where f and h are continuous functions. The input signal $u(t)$ and the output signal $y(t)$ of Σ are both available for use as inputs to an observer, while the initial condition x_0 of Σ is unknown. Our objective is to devise an asymptotic observer $\mathcal{O}$ of the form (1.5) to generate an asymptotic estimate $z(t)$ of the state $x(t)$ of Σ, namely, an estimate satisfying (1.4). To simplify our discussion, we assume that Σ is a reachable system as follows:

Assumption 2.1. *The system Σ of (1.1) is reachable, namely, for every pair of states $x_0, x' \in R^n$, there is a time $t' \geq 0$ and an input signal $u(t)$, $t \in [0, t']$, such that $x(0) = x_0$ and $x(t') = x'$.* □

We also assume that the initial state x_0 of Σ can be any vector in R^n. Similarly, we impose no restrictions on the input signal $u(t)$, other than requiring it to be a piecewise continuous and bounded function of time. Thus, $u(t)$ can take any value in R^m at a time $t \geq 0$.

As the main feature of an asymptotic observer $\mathcal{O}$ is the asymptotic convergence requirement (1.4), our interest concentrates on the difference $z(t) - x(t)$. To obtain a differential equation of this difference, combine Eq. (1.5) with Eq. (1.1) to obtain

$$\dot{z}(t) - \dot{x}(t) = s(z(t), u(t), y(t)) - f(x(t), u(t)) \tag{2.1}$$

The requirement (1.4) implies that, for every $\varepsilon > 0$, there is a time $T(\varepsilon) > 0$ such that $|z(t) - x(t)| < \varepsilon$ for all $t \geq T(\varepsilon)$. Bearing in mind that all systems under consideration are time invariant, we can start Σ from the initial condition $x_0 := x(T(\varepsilon))$; start $\mathcal{O}$ from the initial condition $z_0 := z(T(\varepsilon))$; and apply the input signal $u'(t) := u(t + T(\varepsilon)), t \geq 0$. This will shift the original behavior over the time interval $[T(\varepsilon), \infty)$ to the time interval $[0, \infty)$. Let us denote by $x'(t)$ and $z'(t)$ the states at the time t of Σ and of $\mathcal{O}$, respectively, after this shift. Then, the paths of the two systems satisfy $|z'(t) - x'(t)| < \varepsilon$ for all $t \geq 0$. As this process can be accomplished for any input signal $u(t)$, the following is valid for any input signal: if Σ and $\mathcal{O}$ start from certain initial conditions that are close to each other, then their trajectories remain close at all times.

From this conclusion, it is just a small additional step to imposing the following general requirement: if the asymptotic observer $\mathcal{O}$ starts from the same initial condition as the system Σ, then the state trajectories of $\mathcal{O}$ and of Σ should remain identical at all times. This requirement is, actually, at the root of the asymptotic observer concept: we expect that, when a system and its asymptotic observer are operated under identical conditions, the two systems should exhibit identical behavior. From here, we reach the following formal definition of an asymptotic observer.

Definition 2.2. *Let Σ be a system of the form (1.1) with input signal $u(t)$, state $x(t)$, and initial condition $x(0) = x_0$, and let $\mathcal{O}$ be a system of the form (1.5) with the state $z(t)$ and the initial condition $z(0) = z_0$. Then, $\mathcal{O}$ is an asymptotic observer of Σ if it satisfies the following:*

(i) $\lim_{t\to\infty}[z(t) - x(t)] = 0$ for any input signal $u(t)$ and for any initial conditions $x_0, z_0 \in R^n$; and
(ii) $z(t) = x(t)$ for all $t \geq 0$ and all input signals $u(t)$, when $z_0 = x_0$.

The observer error is the difference

$$\xi(t) := z(t) - x(t) \qquad \square \tag{2.2}$$

Considering the observer error $\xi(t)$ of Eq. (2.2), it follows by Definition 2.2(i) that $\lim_{t\to\infty} \xi(t) = 0$ for all initial conditions $x_0, z_0 \in R^n$ and for all input signals $u(t)$. From Eq. (2.2), we have $z(t) = \xi(t) + x(t)$; substituting this into Eq. (2.1), and recalling from Eq. (1.1) that $y(t) = h(x(t))$, we obtain

$$\begin{aligned} \dot{\xi}(t) &= s(\xi(t) + x(t), u(t), h(x(t))) - f(x(t), u(t)), \\ \xi_0 &:= \xi(0) = z_0 - x_0 \end{aligned} \tag{2.3}$$

We can regard Eq. (2.3) as a differential equation for the observer error $\xi(t)$, where the signals $u(t)$ and $x(t)$ are formally interpreted as input signals of Eq. (2.3).

Consider now the special case when the asymptotic observer $\mathcal{O}$ and the system Σ start from the same initial condition $z_0 = x_0$; then, $\xi_0 = 0$. By Definition 2.2(ii), we must have then $z(t) = x(t)$ for all $t \geq 0$ and for any input signal $u(t)$. In other words, if $\xi(0) = 0$, then $\xi(t) = 0$ for all $t \geq 0$, and hence also $\dot{\xi}(t) = 0$ for all $t \geq 0$. Considering Eq. (2.3), this implies that, when $\xi(0) = 0$, we must have $s(x(t), u(t), h(x(t))) - f(x(t), u(t)) = 0$ for all $t \geq 0$. In view of Assumption 2.1, any state of Σ can be reached from every initial condition, so the last equality leads to

$$s(x, u, h(x)) = f(x, u) \text{ for all } x \in R^n \text{ and all } u \in R^m \tag{2.4}$$

Define now the function

$$\sigma(z, u, y) := s(z, u, y)) - f(z, u) \tag{2.5}$$

where z is the state of the observer $\mathcal{O}$ and $y = h(x)$ is the output of the observed system Σ. Then, when $z = x$, namely, when the state of $\mathcal{O}$ is identical to the state of Σ, it follows from Eq. (2.4) that

$\sigma(x, u, h(x)) = 0$ for all $x \in R^n$ and all $u \in R^m$. Changing variable names, we can also write

$$\sigma(z, u, h(z)) = 0 \text{ for all } z \in R^n \text{ and all } u \in R^m \tag{2.6}$$

Rewriting Eq. (2.5) in the form

$$s(z, u, y) = f(z, u) + \sigma(z, u, y)$$

the asymptotic observer equation (1.5) becomes

$$\mathcal{O}: \dot{z}(t) = f(z(t), u(t)) + \sigma(z(t),\ u(t),\ y(t)), \quad z(0) = z_0 \tag{2.7}$$

An important fact about Eq. (2.7) is that the dependence of σ on z factors over the output function h of the observed system Σ, as follows (Im h denotes the image of the function h).

LEMMA 2.3. *Let $h: R^n \to R^p$ be the output function of the system Σ of Eq. (1.1), let $y = h(x)$ be the output of Σ, let $\mathcal{O}$ be an asymptotic observer for Σ, and let z be the state of $\mathcal{O}$. Then, referring to Eq. (2.7), there is a function $\mu: R^p \times R^m \times R^p \to R^n$ such that $\sigma(z, u, y) = \mu(h(z), u, y)$ for all $z \in R^n$, $u \in R^m$, and $y \in \operatorname{Im} h$.*

Proof. First, if h is injective, then it has a left inverse function $h^{-1}: \operatorname{Im} h \to R^n$, so that $z = h^{-1}(h(z))$. Substituting into σ, we get $\sigma(z, u, y) = \sigma(h^{-1}(h(z)), u, y) =: \mu(h(z), u, y)$, and the Lemma is valid in this case. More generally, when h is not injective, let $z, z' \in R^n$ be states at which $h(z) = h(z') = y$; let $x \in R^n$ be the true state of Σ generating the output value y, namely, $y = h(x)$. Now, if $x = z$, then, by Eq. (2.6), it follows that $\sigma(z, u, y) = 0$. Further, as Σ is reachable by Assumption 2.1, we can also have the case where $x = z'$, namely, the true state of Σ is z' when the output is y. Then, the same argument entails that $\sigma(z', u, y) = 0$ as well. Thus, $\sigma(z', u, y) = \sigma(z, u, y)$ whenever $h(z) = h(z')$, and our proof concludes. □

Lemma 2.3 yields the following general form of an asymptotic observer.

COROLLARY 2.4. *Let Σ be a system of the form (1.1) with the recursion function f and the output function h, and let $\mathcal{O}$ be an asymptotic observer for Σ. Then, there is a function $\mu: R^p \times R^m \times R^p \to R^n$ such that $\mathcal{O}$ is given by*

$$\mathcal{O}: \dot{z}(t) = f(z(t), u(t)) + \mu(h(z(t)), u(t), y(t)), z(0) = z_0 \quad \square \tag{2.8}$$

Substituting $y = h(x)$ into Eq. (2.8), we get

$$\dot{z}(t) = f(z(t), u(t)) + \mu(h(z(t)), u(t), h(x(t))) \tag{2.9}$$

Using Eqs. (2.9) and (2.1), the differential equation of the observer error becomes

$$\begin{aligned}\dot{\xi}(t) = \dot{z}(t) - \dot{x}(t) = f(z(t), u(t)) - f(x(t), u(t)) \\ + \mu(h(z(t)), u(t), h(x(t)))\end{aligned} \tag{2.10}$$

Substituting $z(t) = \xi(t) + x(t)$ into Eq. (2.10) yields

$$\begin{aligned}\dot{\xi}(t) = f(\xi(t) + x(t), u(t)) - f(x(t), u(t)) + \\ \mu(h(\xi(t) + x(t)), u(t), h(x(t)))\end{aligned} \tag{2.11}$$

The system (2.11) is globally asymptotically stable since the observer error $\xi(t)$ of an asymptotic observer asymptotically converges to zero under all operating conditions, as discussed earlier.

3 Strict Lyapunov stability

In Eq. (2.11), the signals $x(t)$ and $u(t)$ both serve as input signals. To guarantee robustness, we must allow uncertainties in the recursion function f of Σ. As $x(t)$ and $u(t)$ are related through the differential equation (1.3), uncertainties in f will prevent any predictable exact relationship between the values of $x(t)$ and of $u(t)$, especially at times t not close to the initial time $t = 0$. To accommodate this situation, it is prudent to go a step further and require that $\lim_{t\to\infty} \xi(t) = 0$ be valid for all functions $x(t)$ and $u(t)$, not only for functions $x(t)$ and $u(t)$ related through the exact differential equation (1.3). This brings us to a stronger notion of an asymptotic observer, which is the main topic of this paper. We start by introducing the following terminology.

DEFINITION 3.1. *Let*

$$\begin{aligned}\dot{\theta}(t) &= g(\theta(t), w(t)), t \ge 0, \theta(0) = \theta_0 \\ \Lambda(t) &= \phi(\theta(t))\end{aligned} \tag{3.1}$$

be a system, where $\theta(t) \in R^n, w(t) \in R^m$, and $\Lambda(t) \in R^p$ for all $t \ge 0$, and where $g: R^n \times R^m \to R^n$ and $\phi: R^n \to R^p$ are continuous functions satisfying $g(0,0) = 0$ and $\phi(0) = 0$. Assume that Eq. (3.1) has a unique solution $\theta(t)$, $t \ge 0$, for every initial condition θ_0 and for every piecewise continuous and bounded input function $w(t)$. A strict Lyapunov function for Eq. (3.1) is a function $V: R^n \to R$ that satisfies the following:

(i) $V(\theta) > 0$ for all $\theta \ne 0$ and $V(0) = 0$;
(ii) $\partial V/\partial \theta$ exists and is a continuous function;
(iii) The set $\{\theta : V(\theta) \le A\}$ is a bounded subset of R^n for every real number $A \ge 0$;
(iv) $\dot{V}(\theta(t)) < 0$ for every solution $\theta(t)$ of Eq. (3.1), as long as $\theta(t) \ne 0$; and $\dot{V}(0) = 0$.

The system (3.1) is strictly Lyapunov stable if there is a strict Lyapunov function for it. □

In the special case when Eq. (3.1) is an autonomous system, namely, when g does not depend on w, strict Lyapunov stability reduces to the standard notion of Lyapunov stability. However, when an input signal $w(t)$ does appear in Eq. (3.1), strict Lyapunov stability is a rather strong notion of stability, as the next statement shows: the solution of a strictly Lyapunov stable differential equation always decays to zero, irrespective of the input signal.

PROPOSITION 3.2. *Let $\theta(t)$ be the state of a strictly Lyapunov stable differential equation of the form (3.1) with a piecewise continuous and bounded input function $w(t)$. Then, $\theta(t)$ is a bounded function and $\lim_{t\to\infty} \theta(t) = 0$, regardless of the input signal $w(t)$.*

Proof. We show first that $\theta(t)$ is a bounded function of time. To this end, define the function $\eta(t) := V(\theta(t))$. Then, according to Definition 3.1(i) and (iv), we have

$$\eta(t) \ge 0 \tag{3.2}$$

and

$$\dot{\eta}(t) < 0 \text{ as long as } \eta(t) \ne 0 \tag{3.3}$$

for all $t \ge 0$. Now, as $V(\theta)$ is defined for all $\theta \in R^n$, the initial value $\eta(0)$ is bounded; invoking Eqs. (3.2) and (3.3), we conclude that $\eta(0) \ge \eta(t) \ge 0$ for all $t \ge 0$. Thus, $\eta(t)$ is a bounded function. By Definition 3.1(iii), the latter implies that $\theta(t)$ is also a bounded function, proving the first part of the proposition.

Next, we show that $\lim_{t\to\infty} V(\theta(t)) = 0$. Indeed, by Eq. (3.3), the function $\eta(t)$ is bounded, strictly monotone decreasing, and continuous. Using these facts, it can be shown that $\lim_{t\to\infty} \eta(t) = 0$, or that $\lim_{t\to\infty} V(\theta(t)) = 0$. But then, using Definition 3.1(i), it can be further shown that $\lim_{t\to\infty} \theta(t) = 0$, and our proof concludes. □

In view of Proposition 3.2, strict Lyapunov stability of an equation of the form (3.1) is a rather strong notion of asymptotic stability, since it implies convergence to zero of the solution $\theta(t)$ for all input signals $w(t)$ (and for all initial conditions θ_0). This leads to the following property that we will utilize in our ensuing discussion.

PROPOSITION 3.3. *Let $\theta(t)$ be the state of a strictly Lyapunov stable differential equation of the form (3.1) with a piecewise continuous and bounded input signal $w(t)$. If $\theta(0) = 0$, then $\theta(t) = 0$ for all $t \ge 0$, regardless of the input signal $w(t)$.*

Proof. Using the notation of Definition 3.1, let V be a strict Lyapunov function for Eq. (3.1). Considering the function $\mu(t) := V(\theta(t))$ with $\theta(0) = 0$, it follows by Definition 3.1(i) and (iv) that $\mu(0) = 0$ and $\dot{\mu}(0) = 0$. We also have that $\mu(t) \geq 0$ by Definition 3.1(i) and that $\mu(t)$ is a monotone decreasing function by Definition 3.1(iv). These facts imply that $0 \geq \mu(t) \geq 0$ for all $t \geq 0$, namely, that $\mu(t) = 0$ for all $t \geq 0$. But then, by Definition 3.1(i), it follows that $\theta(t) = 0$ for all $t \geq 0$. □

The following notion is central to our discussion.

Definition 3.4. *A strict asymptotic observer for the system Σ of Eq. (1.1) is an asymptotic observer $\mathcal{O}$ of the form (2.8) whose observer error (2.11) is strictly Lyapunov stable, with $w(t) := (u(t), x(t))$ being regarded as the input signal.* □

It is easy to see that the traditional linear asymptotic observer [6] is a special case of a strict asymptotic observer. Indeed, consider a linear time-invariant system described by

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

where A, B, and C are constant matrices of appropriate dimensions. For this system, a traditional linear asymptotic observer exists if and only if there is a matrix L for which all eigenvalues of the matrix $(A - LC)$ have strictly negative real parts. Then, the traditional linear asymptotic observer takes the form

$$\dot{z}(t) = Az(t) + L[y(t) - Cz(t)] + Bu(t)$$

The observer error equation in this case is $\dot{\xi}(t) = (A - LC)\xi(t)$, and, since all eigenvalues of $(A - LC)$ are in the open left half of the complex plane, the observer error equation is strictly Lyapunov stable. Hence, the traditional linear observer is a special case of a strict asymptotic observer.

4 Building Strict Asymptotic Observers

Consider a strict asymptotic observer $\mathcal{O}$ for the system Σ of Eq. (1.1). Applying Definition 3.4 to the observer error equation (2.11) implies that there must be a strict Lyapunov function V such that $\dot{V}(\xi(t)) = (\partial V/\partial \xi)\dot{\xi}(t) < 0$ for all $\xi(t) \neq 0$. Substituting $\dot{\xi}(t)$ from Eq. (2.11), we obtain the inequality

$$\frac{\partial V}{\partial \xi}[f(\xi(t) + x(t), u(t)) - f(x(t), u(t)) + \mu(h(\xi(t) + x(t)), u(t), h(x(t)))] < 0 \text{ for all } \xi(t) \neq 0 \quad (4.1)$$

As indicated in Sec. 3, there is no strict correlation between the values of $x(t)$ and $u(t)$ at a time $t > 0$, due to uncertainties about the recursion function f of the observed system Σ. Furthermore, considering that $u(t)$ is a piecewise continuous function, we can change the value of $u(t)$ arbitrarily, irrespective of the value of $x(t)$. In view of these facts, inequality (4.1) must remain valid for any pair of values $(x(t), u(t))$. In particular, we can take $x(t) = 0$ and set $u(t)$ at an arbitrary value, without violating the inequality. Substituting this into Eq. (2.11) and using the fact that $h(0) = 0$ by Eq. (1.2) yields the strictly Lyapunov stable differential equation

$$\dot{\xi}(t) = f(\xi(t), u(t)) + [\mu(h(\xi(t)), u(t), 0) - f(0, u(t))] \quad (4.2)$$

Defining the function

$$\omega(h(\xi), u) := -[\mu(h(\xi), u, 0) - f(0, u)] : R^n \times R^p \to R^n$$

we can rewrite Eq. (4.2) in the form

$$\dot{\xi}(t) = f(\xi(t), u(t)) - \omega(h(\xi(t)), u(t)) \quad (4.3)$$

As this equation is strictly Lyapunov stable, it follows by Propositions 3.2 and 3.3 that $\lim_{t\to\infty} \xi(t) = 0$ for any input signal $u(t)$; and, when started from zero initial conditions, the solution becomes $\xi(t) = 0$ for all $t \geq 0$, again irrespective of the input signal $u(t)$.

Rewriting Eq. (4.3) twice with differently named variables, we obtain two strictly Lyapunov stable equations

$$\dot{\zeta}(t) = f(\zeta(t), u(t)) - \omega(h(\zeta(t)), u(t))$$
$$\dot{\chi}(t) = f(\chi(t), u(t)) - \omega(h(\chi(t)), u(t)) \quad (4.4)$$

so that $\lim_{t\to\infty} \zeta(t) = 0$ and $\lim_{t\to\infty} \chi(t) = 0$ for all input functions $u(t)$ and for all initial conditions. But then, the difference

$$\dot{\zeta}(t) - \dot{\chi}(t) = [f(\zeta(t), u(t)) - \omega(h(\zeta(t)), u(t))] - [f(\chi(t), u(t)) - \omega(h(\chi(t)), u(t))] \quad (4.5)$$

satisfies $\lim_{t\to\infty}[\zeta(t) - \chi(t)] = 0$ for all input functions $u(t)$ and for all initial conditions. Denoting $\vartheta(t) := \zeta(t) - \chi(t)$, we obtain that $\lim_{t\to\infty} \vartheta(t) = 0$ for any input function $u(t)$ and for any initial conditions.

Using these facts, and recalling that the output signal of the observed system Σ of Eq. (1.1) is given by $y(t) = h(x(t))$, we show now that one can assemble a strict asymptotic observer $\mathcal{O}$ for Σ in a rather simple way by using the equation

$$\mathcal{O}: \dot{z}(t) = f(z(t), u(t)) - [\omega(h(z(t)), u(t)) - \omega(y(t), u(t))], z(0) = z_0 \quad (4.6)$$

Indeed, for such an observer, the observer error $\xi(t) = z(t) - x(t)$ satisfies the equation

$$\dot{\xi}(t) = \dot{z}(t) - \dot{x}(t) = [f(z(t), u(t)) - \omega(h(z(t)), u(t))] - [f(x(t), u(t)) - \omega(h(x(t)), u(t))].$$

This equation is identical to Eq. (4.5) when setting $\zeta(0) := z(0)$ and $\chi(0) := x(0)$. Consequently, recalling our discussion of Eq. (4.5), we conclude that the observer error satisfies $\lim_{t\to\infty} \xi(t) = 0$ for all input functions $u(t)$ and for all initial conditions.

Furthermore, for equal initial conditions $z(0) = x(0)$, we have $\zeta(0) = \chi(0)$, and the two equations (4.4) clearly have the same solution $\zeta(t) = \chi(t)$, $t \geq 0$, which, by our earlier observations, implies that $\xi(t) = 0$ for all $t \geq 0$. These considerations lead to the following statement, which is one of the main results of this paper.

Theorem 4.1. *Let Σ be a system of the form (1.1) with recursion function f and output function h. The following two statements are equivalent*:

(i) *There is a strict asymptotic observer for Σ.*
(ii) *There is a continuous function $\omega : R^p \times R^m \to R^n$ for which the differential equation $\dot{\zeta}(t) = f(\zeta, u) - \omega(h(\zeta), u)$ is strictly Lyapunov stable.* □

Proof. The discussion preceding the theorem shows that (i) implies (ii). Conversely, assume that (ii) is valid. Then, let V be a strict Lyapunov function for the equation given in (ii), and setup an observer as described in Eq. (4.6), where $z(t)$ is the observer state. Referring to Eq. (4.4) and recalling that $x(t)$ is the state of the observed system Σ, it follows from the discussion preceding the theorem that

$$\xi(t) = z(t) - x(t) = \zeta(t) - \chi(t) \quad (4.7)$$

Consider now the sum $V(\zeta(t)) + V(\chi(t))$; as both of these functions are non-negative, so is their sum. In view of Eq. (4.7), we can rewrite this sum in the form

$$V_1(\xi(t), \chi(t)) := V(\xi(t) + \chi(t)) + V(\chi(t)) \quad (4.8)$$

As V is a strict Lyapunov function, it follows directly from Eq. (4.8) that $V_1(\xi(t),\chi(t))$ is a strict Lyapunov function for the pair $(\xi(t),\chi(t))$. This implies that Eq. (4.6) is a strict asymptotic observer for Σ, and our proof concludes. □

In view of Theorem 4.1, we can characterize the general form of a strict asymptotic observer as follows.

COROLLARY 4.2. *With the function ω of Theorem 4.1, a strict asymptotic observer for the system Σ of Eq. (1.1) is given by Eq. (4.6).* □

From an implementation perspective, Corollary 4.2 shows that finding a strict asymptotic observer boils down to finding a function ω for which the combination $f(x,u)-\omega(h(x),u)$ is the recursion function of a strictly Lyapunov stable differential equation. This fact is very convenient in practice, since it provides a recipe for finding strict asymptotic observers through a relatively standard calculation, as we discuss in Sec. 7. In the meanwhile, it is convenient to introduce the following terminology.

DEFINITION 4.3. *A continuous function ω that satisfies the requirements of Theorem 4.1 is a strict observer function for Σ.* □

5 Accounting for the Effects of Disturbances

When an observer is used in practice, one must take into account various disturbance and noise signals that may affect the observer's inputs. Specifically, recall that an observer $\mathcal{O}$ of a system Σ employs two input signals: the output signal $y(t)$ and the input signal $u(t)$ of the observed system Σ. Generally, these signals are corrupted by small additive disturbances and noises. Denoting by $\upsilon(t)$ the disturbance signal that affects $y(t)$ and by $\upsilon'(t)$ the disturbance signal that affects $u(t)$, the situation is depicted in Fig. 5. The following assumption, which is valid for disturbance and noise signals that appear in practice, simplifies a few mathematical arguments.

ASSUMPTION 5.1. *All disturbance and noise signals are piecewise continuous and bounded functions of time.* □

The next feature helps evaluate the quantitative impact of disturbances and noises on strict asymptotic observers.

LEMMA 5.2. *Let Σ be a system of the form (1.1) with recursion function f and input signal u(t), and assume that there is a strict observer function $\omega : R^p \times R^m \to R^n$ for Σ. Then, for every pair of real numbers $A,\alpha > 0$, there is a real number $\delta > 0$ such that the following is true for all input signals satisfying $|u| \le A$: for every pair of functions $\upsilon : R^+ \to R^p$ and $\upsilon' : R^+ \to R^m$ of magnitudes $|\upsilon|, |\upsilon'| < \delta$, the solution $\zeta(t)$ of the equation*

$$\dot{\zeta}(t) = f(\zeta(t), u(t)+\upsilon'(t)) - \omega(h(\zeta(t))+\upsilon(t), u(t)+\upsilon'(t))$$

is a bounded function of time, and it satisfies $\limsup_{t\to\infty} |\zeta(t)| < \alpha$.

Proof. Let V be a strict Lyapunov function for the equation $\dot{\theta}(t) = f(\theta(t),u(t)) - \omega(h(\theta(t)),u(t))$, and use the initial condition $\theta(0)=\zeta(0)\neq 0$. In view of Definition 3.1(iii) and (i) and the continuity of V, the domain $\Xi_1 := \{\theta : V(\theta) \le V(\theta(0))\}$ is compact in R^n, and therefore so is the intersection $\Xi' := \Xi_1 \cap \{|\theta| \ge \alpha\}$. Also, Ξ_1 is a connected set that includes the origin, since it is the inverse image through the continuous function V of the nonempty connected set $[0, V(\theta(0))] \subseteq R$. As a result, the intersection Ξ' is not empty for a sufficiently small positive number $\eta < \alpha$, namely,

$$\Xi' = \Xi_1 \cap \{|\theta| \ge \eta\} \neq \varnothing \tag{5.1}$$

Further, denote $c := \inf_{|\theta|}\{\theta : V(\theta) = V(\theta(0))\}$. Then, considering that $\theta(0)\neq 0$, Definition 3.1(iii) implies that $c > 0$. We can choose η so that $0 < \eta < c$ and use such value of η below.

Denote by $[-A, A]^m$ the set of all vectors $v \in R^m$ satisfying $|v| \le A$, where A is given in the statement of the Lemma. Then, $\Xi := \Xi' \times [-A,A]^m$ is a compact subset of $R^n \times R^m$. By Definition 3.1(ii) and the continuity of f, ω, and h, the derivative $\dot{V} = (\partial V/\partial\theta)[f(\theta,u) - \omega(h(\theta),u)]$ is a continuous function of (θ,u). Hence, $\dot{V}$ attains a minimal value β over Ξ, and $\beta < 0$ by Definition 3.1(iv). Denote $\gamma := -\beta$, so that $\gamma > 0$.

Next, by Definition 3.1(ii) and the compactness of Ξ', there is a maximum

$$b := \max\{|\partial V/\partial\theta| : \theta \in \Xi'\} \tag{5.2}$$

and $b \neq 0$ by Eqs. (5.1), (5.2), and Definition 3.1(iv). Thus, there is a real number $d > 0$ satisfying $bd < \gamma/3$. As the continuous function f is uniformly continuous over the compact domain Ξ, there is a real number $\delta' > 0$ such that $|f(\zeta, u+\upsilon') - f(\zeta,u)| < d$ for all $(\zeta,u) \in \Xi$ and all $\upsilon' \in R^m$ satisfying $|\upsilon'| < \delta'$. Similarly, the continuity of the functions ω and h implies that there is a real numbers $\delta'' > 0$ such that $|\omega(h(\zeta)+\upsilon, u+\upsilon') - \omega(h(\zeta),u)| < d$ for all $(\zeta,u) \in \Xi$ and all $\upsilon \in R^p, \upsilon' \in R^m$ satisfying $|\upsilon|,|\upsilon'| < \delta''$. Setting $\delta := \min\{\delta',\delta''\}$, the solution $\zeta(t)$ of Eq. (6.10) satisfies

$$\begin{aligned}\dot{V}(\zeta) &= \frac{\partial V}{\partial \zeta}[f(\zeta(t),u(t)+\upsilon'(t)) - \omega(h(\zeta(t))+\upsilon(t),u(t)+\upsilon'(t))] \\ &< \frac{\partial V}{\partial \zeta}[f(\zeta(t),u(t)) - \omega(h(\zeta(t)),u(t)) + 2d] \\ &< \frac{\partial V}{\partial \zeta}[f(\zeta(t),u(t)) - \omega(h(\zeta(t)),u(t))] + \frac{2}{3}\gamma < -\gamma + \frac{2}{3}\gamma < 0\end{aligned}$$

whenever $(\zeta(t),u(t)) \in \Xi$ and $|\upsilon|,|\upsilon'| < \delta$. Thus, $\dot{V} < 0$ whenever $|\zeta| \ge \eta$. Consequently, $V(\zeta(t)) < V(\theta(0))$ as long as $|\zeta(t)| \ge \eta$, so that, by Definition 3.1(iii), the function $\zeta(t)$ is bounded. This argument also implies that $\limsup_{t\to\infty}|\zeta(t)| < \eta$, and, as $\eta < \alpha$, the proof concludes. □

Lemma 5.2 provides tools we need to examine the impact of small disturbances and noises on a strict asymptotic observer. Consider the situation depicted in Fig. 5, where $\upsilon(t)$ and $\upsilon'(t)$ are disturbance and noise signals that may affect an observer. The following statement shows that a strict asymptotic observer can tolerate such disturbances and noises.

THEOREM 5.3. *Let Σ be a system of the form (1.1) with input signal u(t), state x(t), and output signal y(t), and assume that there is a strict asymptotic observer $\mathcal{O}$ for Σ. Let z(t) be the estimate of x(t) produced by $\mathcal{O}$ in the presence of the disturbance signals $\upsilon(t)$ and $\upsilon'(t)$ of Fig. 5. Then, for every pair of real numbers $A, \varepsilon > 0$, there is a real number $\delta > 0$ such that $\limsup_{t\to\infty}|z(t)-x(t)| < \varepsilon$ as long as $|\upsilon|,|\upsilon'| < \delta$ and $|u| \le A$. In particular, if x(t) is bounded, then so is z(t).*

Proof. Refer to Eq. (4.7), where $x(t)$ and $z(t)$ are given in the theorem, and $\zeta(t)$ and $\chi(t)$ are solutions of Eq. (4.4). Let $\varepsilon > 0$ be a real number. Apply Lemma 5.2 to $\zeta(t)$ and $\chi(t)$ using $\alpha = \varepsilon/2$ and the disturbance signals $\upsilon(t)$ and $\upsilon'(t)$ of the present theorem, and let $\delta > 0$ be as described in Lemma 5.2. Then, Lemma 5.2 implies that $\limsup_{t\to\infty}|\zeta(t)| < \varepsilon/2$ and $\limsup_{t\to\infty}|\chi(t)| < \varepsilon/2$ as long as $|\upsilon|,|\upsilon'| < \delta$ and $|u| \le A$. Thus, $\limsup_{t\to\infty}|z(t)-x(t)| = \limsup_{t\to\infty}|\zeta(t)-\chi(t)| \le \limsup_{t\to\infty}|\zeta(t)| + \limsup_{t\to\infty}|\chi(t)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$ proving the first part of the theorem.

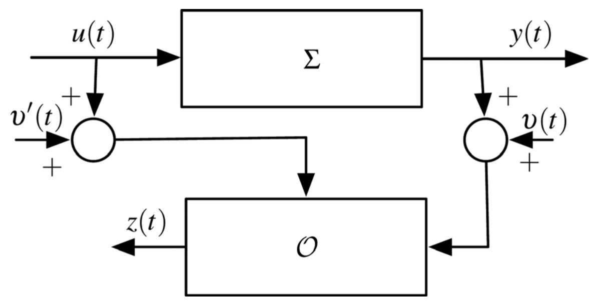

Fig. 5 Observer with disturbances

Further, Lemma 5.2 states that $\zeta(t)$ and $\chi(t)$ are both bounded functions; as $|\xi(t)| = |\zeta(t) - \chi(t)| \le |\zeta(t)| + |\chi(t)|$, it follows that $\xi(t)$ is bounded as well. Finally, by Eq. (4.7), we have $|z(t)| = |\xi(t)| + |x(t)|$; thus, if $x(t)$ is bounded, so is $z(t)$. □

We have seen in this section that a strict asymptotic observer fulfils two fundamental requirements: it provides an asymptotic estimate of the observed system's state, and this estimate is affected only slightly by small disturbances and noises. We have also seen that a strict asymptotic observer can be readily constructed from a strict observer function ω found from the recursion function f of the observed system Σ. Section 6 concentrates on simplifying the construction of strict observer functions.

6 Remainder Functions

The process of deriving strict observer functions can often be simplified by reducing the observed system's recursion function into a simpler form. To this end, let Σ of Eq. (1.1) be the observed system, where $f : R^n \times R^m \to R^n$ is its recursion function and $h : R^n \to R^p$ is its output function. Denote by Im h the image of h, namely, the set of all values of h. To simplify notation, we also use the symbol h for the surjective function $h : R^n \to \mathrm{Im}\, h$ induced by h. Then, there is a right inverse function $h^* : \mathrm{Im}\, h \to R^n$ of the surjective function $h : R^n \to \mathrm{Im}\, h$, so that $hh^* = I : \mathrm{Im}\, h \to \mathrm{Im}\, h$ is the identity function. Assume that h^* can be chosen as a continuous function. Then, the function

$$\pi := h^* h : R^n \to R^n \tag{6.1}$$

is continuous. Note that π is akin to a projection, since $\pi^2 = (h^*h)(h^*h) = h^*h = \pi$. Recalling that $hh^* = I$, we obtain

$$h(\pi(\zeta)) = hh^*h(\zeta) = h(\zeta) \text{ for all } \zeta \in R^n \tag{6.2}$$

Consequently, $\pi(\zeta)$ represents a part of the state ζ that determines the output value. Define a function $\lambda : R^n \to R^n$ by

$$\lambda(\zeta) := \zeta - \pi(\zeta), \quad \zeta \in R^n \tag{6.3}$$

so that

$$\zeta = \pi(\zeta) + \lambda(\zeta) \text{ for all } \zeta \in R^n \tag{6.4}$$

Applying the function h to Eq. (6.4) yields $h(\zeta) = h(\pi(\zeta) + \lambda(\zeta))$; together with Eq. (6.2), this yields

$$h(\pi(\zeta) + \lambda(\zeta)) = h(\pi(\zeta)) \text{ for all } \zeta \in R^n$$

Thus, $\lambda(\zeta)$ is a part of the state ζ that does not affect the output value.

Referring to Theorem 4.1, the function

$$\Delta(\zeta, u) := f(\zeta, u) - \omega(h(\zeta), u)$$

is the recursion function of a strictly Lyapunov stable differential equation; consider the difference

$$\begin{aligned}\rho(\zeta, u) :&= \Delta(\zeta, u) - \Delta(\pi(\zeta), u) \\ &= [f(\zeta, u) - \omega(h(\zeta), u)] - [f(\pi(\zeta), u) - \omega(h(\pi(\zeta)), u)]\end{aligned}$$

By Eq. (6.2), the last expression becomes $\rho(\zeta, u) = [f(\zeta, u) - \omega(h(\zeta), u)] - [f(\pi(\zeta), u) - \omega(h(\zeta), u)]$, so that

$$\rho(\zeta, u) = f(\zeta, u) - f(\pi(\zeta), u) \tag{6.5}$$

Using Eq. (6.1), we can write $f(\pi(\zeta), u) = f(h^*h(\zeta), u)$; consequently, there is a function $g : R^n \times R^m \to R^n$ satisfying

$$g(h(\zeta), u) = f(\pi(\zeta), u) \tag{6.6}$$

and g is continuous by the continuity of h^*. Substituting into Eq. (6.5) yields

$$\rho(\zeta, u) = f(\zeta, u) - g(h(\zeta), u) \tag{6.7}$$

Definition 6.1. *The function ρ of Eq. (6.7) is the remainder function of the system Σ.* □

To examine the significance of the remainder function ρ, assume that there is a function $\psi : R^p \times R^m \to R^n$ for which the differential equation

$$\dot{\zeta}(t) = \rho(\zeta(t), u(t)) - \psi(h(\zeta(t)), u(t)) \tag{6.8}$$

is strictly Lyapunov stable. Then, adding and subtracting $g(h(\zeta), u)$, we can write

$$\begin{aligned}\dot{\zeta}(t) = &[\rho(\zeta(t), u(t)) + g(h(\zeta(t)), u(t))] \\ &- [\psi(h(\zeta(t)), u(t)) + g(h(\zeta(t)), u(t))]\end{aligned}$$

Defining the function

$$\omega(h(\zeta), u) := \psi(h(\zeta), u) + g(h(\zeta), u) \tag{6.9}$$

and using Eq. (6.7), it follows that Eq. (6.8) takes the form

$$\dot{\zeta}(t) = f(\zeta(t), u(t)) - \omega(h(\zeta(t)), u(t))$$

This differential equation is strictly Lyapunov stable, since, having been derived from Eq. (6.8) by adding and subtracting the same term, is identical to Eq. (6.8). Consequently, when seeking strict observer functions for Σ, we can replace the recursion function f of Σ by the remainder function ρ of Eq. (6.7). Often, as in the example below, ρ is simpler than f, and this makes it simpler to find a strict observer function. It leads to the following alternative form of Theorem 4.1.

Theorem 6.2. *Let Σ be a system of the form (1.1) with the recursion function f and the output function h. Assume that h, when considered as a surjective function $h : R^n \to \mathrm{Im}\, h$, has a continuous right inverse function h^*; let ρ be the corresponding remainder function of Σ. Then, the following two statements are equivalent*:

(i) There is a strict asymptotic observer for Σ.
(ii) There is a continuous function $\psi : R^p \times R^m \to R^n$ such that $\rho(\zeta,u) - \psi(h(\zeta),u)$ is the recursion function of a strictly Lyapunov stable differential equation. □

Example 6.3. Consider the system Σ of the form (1.1), where

$$f(x_1, x_2, u) = \begin{pmatrix} x_2 + x_1^2 + u \\ x_1^3 \end{pmatrix} : R^2 \times R \to R^2$$
$$h(x_1, x_2) = x_1 : R^2 \to R$$

The function f can be construed as a third-order approximation of a more general nonlinear recursion function, thus pointing to a broader family of practical examples. Assume that this system is operated within a stabilizing closed-loop configuration, so that its state is well defined at all times, in compliance with Assumption 1.1. Referring to Eqs. (6.1) and (6.3), we can select here $h^*(x_1) := (x_1, 0)^{\mathrm{T}}$ (the transpose of $(x_1, 0)$); then, $\pi(x_1, x_2) := (x_1, 0)^{\mathrm{T}}$ and $\lambda(x_1, x_2) := (0, x_2)^{\mathrm{T}}$. Applying Eq. (6.6), we have

$$g(h(x_1, x_2), u) = f(\pi(x_1, x_2), u) = f(x_1, 0, u) = \begin{pmatrix} x_1^2 + u \\ x_1^3 \end{pmatrix}$$

By Eq. (6.7), we obtain

$$\rho(x_1,x_2,u) = f(x_1,x_2,u) - g(h(x_1,x_2),u) = \begin{pmatrix} x_2 \\ 0 \end{pmatrix} \quad (6.10)$$

which is simpler than f. In fact, ρ is a linear function here, and we can use

$$\psi(x_1) := \begin{pmatrix} x_1 \\ x_1 \end{pmatrix} \quad (6.11)$$

to get

$$\Delta(x_1,x_2,u) := \rho - \psi = \begin{pmatrix} x_2 - x_1 \\ -x_1 \end{pmatrix}$$

A direct calculation shows that the equation $\dot{z}(t) = \Delta(z(t),u)$ is strictly Lyapunov stable. Consequently, by Eq. (6.9), a strict observer function for Σ is given by

$$\omega = \psi + g = \begin{pmatrix} x_1 + x_1^2 + u \\ x_1 + x_1^3 \end{pmatrix}$$

and, by Corollary 4.2, a strict asymptotic observer for Σ is given by

$$\dot{z}(t) = \begin{pmatrix} z_2(t) - z_1(t) + y(t) + y^2(t) + u(t) \\ -z_1(t) + y(t) + y^3(t) \end{pmatrix}, \quad z(0) = z_0 \ \square$$

7 The Separation Theorem

We connect now a strict asymptotic observer $\mathcal{O}$ and a static state feedback function φ around a system Σ of the form (1.1) to obtain the observer–controller configuration of Fig. 4. Our objective is to determine whether the closed-loop system $\Sigma_\varphi^{\mathcal{O}}$ is stable, when $\mathcal{O}$ is a strict asymptotic observer and φ is a static state feedback function that robustly and asymptotically stabilizes the input/state part Σ_s of Σ. As $\Sigma_\varphi^{\mathcal{O}}$ is a composite system, we must turn our attention to internal stability (e.g., Ref. [11]).

7.1 Internal stability. The notion of internal stability comes to guarantee that small disturbances and noises that may appear in a composite system do not destroy stability. Specifically, signals that travel between subsystems in a composite system may pick up disturbances and noises represented in our case by the signals υ_1, υ_2, υ_3, and υ_4 of Fig. 6. Internal stability guarantees that, as long as these disturbances and noises are sufficiently small, their impact on $\Sigma_\varphi^{\mathcal{O}}$ is small as well.

Definition 7.1. *Let Σ be a system of the form (1.1), let φ be a state feedback function that asymptotically stabilizes the input/state part Σ_s of Σ, and let $\mathcal{O}$ be a strict asymptotic observer for Σ. Referring to Fig. 6, let $x(\upsilon_1,\upsilon_2,\upsilon_3,\upsilon_4,t)$ be the state of Σ and let $z(\upsilon_1,\upsilon_2,\upsilon_3,\upsilon_4,t)$ be the state of $\mathcal{O}$ at a time $t \geq 0$. Denote by x_0 and z_0 the initial conditions of Σ and of $\mathcal{O}$, respectively. Then, the observer–controller configuration $\Sigma_\varphi^{\mathcal{O}}$ of Fig. 6 is internally and asymptotically stable if, for every pair of real numbers $\varepsilon, A > 0$, there is a real number $\delta > 0$ such that the following are true whenever $|x_0|, |z_0| \leq A$ and $|\upsilon_i| < \delta$, $i = 1,2,3,4$:*

(i) $x(\upsilon_1,\upsilon_2,\upsilon_3,\upsilon_4,t)$ and $z(\upsilon_1,\upsilon_2,\upsilon_3,\upsilon_4,t)$ are both bounded functions; and
(ii) $\limsup_{t\to\infty} |x(\upsilon_1,\upsilon_2,\upsilon_3,\upsilon_4,t)| < \varepsilon$. □

Internal asymptotic stability guarantees that small disturbances would not spoil the two main features of an asymptotically stable configuration: boundedness of all internal signals and tendency of the controlled system's state to approach the vicinity of the origin.

7.2 State Feedback. To obtain internal stability of $\Sigma_\varphi^{\mathcal{O}}$, the state feedback function φ must possess certain robustness features. Consider the static state feedback configuration $\Sigma_{s\varphi}$ of Fig. 7, where the feedback function φ is connected around the input/state part Σ_s of the controlled system Σ in the presence of disturbance signals $\upsilon(t)$ and $\upsilon'(t)$. In order to be of practical use, $\Sigma_{s\varphi}$ must tolerate these disturbance signals in the following sense.

Definition 7.2. *Let Σ be a system of the form (1.1) with input/state part Σ_s and initial state x_0. Referring to Fig. 7, let $\varphi : R^n \to R^m$ be a state feedback function, let $\upsilon(t)$ and $\upsilon'(t)$ be disturbance signals, and let by $x(\upsilon,\upsilon',t)$ be the state of the closed-loop system. Then, φ internally and asymptotically stabilizes Σ_s if*

(i) φ is a piecewise continuous function; and
(ii) For every pair of real numbers $\varepsilon, A > 0$, there is a real number $\delta > 0$ such that (a) and (b) are valid whenever $|x_0| \leq A$ and $|\upsilon|, |\upsilon'| < \delta$:
  (a) $x(\upsilon,\upsilon',t)$ is a bounded function, and
  (b) $\limsup_{t\to\infty} |x(\upsilon,\upsilon',t)| < \varepsilon$. □

The following statement shows that the main implications of Definition 7.2 remain valid when the bound δ on the disturbance signals is valid only asymptotically.

Proposition 7.3. *In the notation of Definition 7.2, let $\varphi : R^n \to R^m$ be a state feedback function that internally and asymptotically stabilizes the input/state part Σ_s of Σ. Then, for every pair of real numbers $A, \varepsilon > 0$, there is a real number $\delta > 0$ such that the following are valid whenever $\limsup_{t\to\infty} |\upsilon(t)| < \delta$ and $\limsup_{t\to\infty} |\upsilon'(t)| < \delta$:*

(i) $x(\upsilon,\upsilon',t)$ is a bounded function, and
(ii) $\limsup_{t\to\infty} |x(\upsilon,\upsilon',t)| < \varepsilon$.

Proof. Since $\limsup_{t\to\infty} |\upsilon(t)| < \delta$ and $\limsup_{t\to\infty} |\upsilon'(t)| < \delta$, there is a time $\tau \geq 0$ such that $|\upsilon(t)| < \delta$ and $|\upsilon'(t)| < \delta$ for all $t \geq \tau$. As $x(\upsilon,\upsilon',t)$ is a continuous function, there is a maximum $B' := \max_{t\in[0,\tau]} |x(\upsilon,\upsilon',t)|$. Start Σ from the initial state $x_0 := x(\upsilon,\upsilon',\tau)$ and apply the disturbance signals $\upsilon_1(t) := \upsilon(t+\tau)$ and $\upsilon_2(t) := \upsilon'(t+\tau), t \geq 0$. Then, $|\upsilon_1(t)| < \delta$ and $|\upsilon_2(t)| < \delta$ for all $t \geq 0$. Also, by time invariance, the response of Σ_s is $x'(\upsilon_1,\upsilon_2,t) = x(\upsilon,\upsilon',t+\tau)$, $t \geq 0$. As φ internally and asymptotically stabilizes Σ_s, there is a real number $B'' \geq 0$ such that $|x'(\upsilon_1,\upsilon_2,t)| \leq B''$ for all $t \geq 0$ and $\limsup_{t\to\infty} |x'(\upsilon_1,\upsilon_2,t)| < \varepsilon$. Finally, setting $B := \max\{B', B''\}$ and observing that

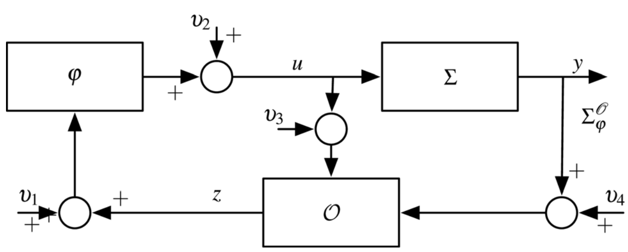

Fig. 6 Internal stability of the observer–controller configuration

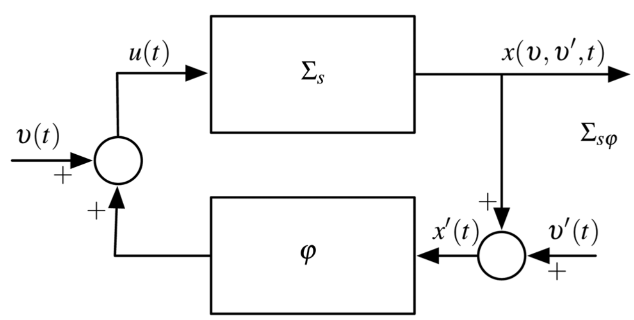

Fig. 7 Robust state feedback

$\limsup_{t\to\infty}|x(\upsilon,\upsilon',t)| = \limsup_{t\to\infty}|x'(\upsilon_1,\upsilon_2,t)| < \varepsilon$, the proof concludes. □

We are ready now to combine state feedback with observer.

7.3 The Separation Theorem. Like all composite systems, the observer–controller configuration may be affected by noises and disturbances as depicted in Fig. 6; here, υ_1, υ_2, υ_3, and υ_4 are noises or disturbances. Referring to Definition 7.1, we can state the following form of the separation theorem, which shows that a nonlinear system can be stabilized by any combination of a strict asymptotic observer and a stabilizing state feedback.

THEOREM 7.4. *Let* Σ *be a system of the form (1.1) with a strict asymptotic observer* $\mathcal{O}$*, and assume that there is a state feedback function* $\varphi : R^n \to R^m$ *that internally and asymptotically stabilizes the input/state part of* Σ*. Then, the observer–controller configuration* $\Sigma_\varphi^{\mathcal{O}}$ *is internally and asymptotically stable.*

Proof. Referring to Fig. 6, let $x(\upsilon_1,\upsilon_2,\upsilon_3,\upsilon_4,t)$ be the state of Σ at the time t, let x_0 be the initial condition of Σ, let $z(\upsilon_1,\upsilon_2,\upsilon_3,\upsilon_4,t)$ be the estimated state generated by $\mathcal{O}$, and let z_0 be the initial condition of $\mathcal{O}$. Let $A>0$ be a real number satisfying $|x_0|,|z_0|\le A$, and let $\mu>0$ be a real number. By Theorem 5.3, there is a real number $\delta_A>0$ such that $\limsup_{t\to\infty}|z(\upsilon_1,\upsilon_2,\upsilon_3,\upsilon_4,t)-x(\upsilon_1,\upsilon_2,\upsilon_3,\upsilon_4,t)|<\mu$ whenever $|\upsilon_3|,|\upsilon_4|<\delta_A$.

Next, set $\upsilon''(t) := z(\upsilon_1,\upsilon_2,\upsilon_3,\upsilon_4,t)-x(\upsilon_1,\upsilon_2,\upsilon_3,\upsilon_4,t)$ and, referring to Fig. 7, define the signals $\upsilon(t) := \upsilon_2(t)$ and $\upsilon'(t) := \upsilon''(t)+\upsilon_1(t), t\ge 0$. Then,

$$x(\upsilon,\upsilon',t) = x(\upsilon_1,\upsilon_2,\upsilon_3,\upsilon_4,t) \text{ for all } t\ge 0 \tag{7.1}$$

Choose a real number $\varepsilon>0$. According to Proposition 7.3, there is a real number $\delta'_A>0$ such that the following are valid whenever $\limsup_{t\to\infty}|\upsilon(t)|<\delta'_A$ and $\limsup_{t\to\infty}|\upsilon'(t)|<\delta'_A$:

(a) There is a real number $B'>0$ such that $|x(\upsilon,\upsilon',t)|\le B'$ for all $t\ge 0$, and
(b) $\limsup_{t\to\infty}|x(\upsilon,\upsilon',t)|<\varepsilon$.

Now, take $0<\mu<\delta'_A/2, |\upsilon_1|<\delta'_A/2$, and $|\upsilon_2|<\delta'_A$. This yields $\limsup_{t\to\infty}|\upsilon'(t)| = \limsup_{t\to\infty}|\upsilon''(t)+\upsilon_1(t)| \le \limsup_{t\to\infty}(|\upsilon''(t)|+|\upsilon_1(t)|) = \limsup_{t\to\infty}|\upsilon''(t)|+\limsup_{t\to\infty}|\upsilon_1(t)| < \mu + \delta'_A/2 < \delta'_A/2+\delta'_A/2 = \delta'_A$. Defining $\delta := \min\{\delta_A,\delta'_A/2\}$, we obtain that $\limsup_{t\to\infty}|x(\upsilon_1,\upsilon_2,\upsilon_3,\upsilon_4,t)|<\varepsilon$ whenever $|\upsilon_i|<\delta$ for all $i=1, 2, 3, 4$, verifying condition (ii) of Definition 7.1. Finally, (a), Eq. (7.1), and Theorem 5.3 imply that $z(\upsilon_1,\upsilon_2,\upsilon_3,\upsilon_4,t)$ is a bounded function, verifying condition (i) of Definition 7.1. □

Thus, strict asymptotic observers form a handy tool for achieving asymptotic stabilization of nonlinear systems, when combined with state feedback functions that internally and asymptotically stabilize the input/state part of the controlled system. Recalling that a strict asymptotic observer is determined by a strict observer function, we point out next a simple method of finding such functions.

8 Finding Strict Observer Functions

As we have seen, a strict asymptotic observer is easily obtained, once a function $\psi : R^p\times R^m\to R^n$ satisfying Theorem 6.2 is available. Such functions ψ can be derived by using the well known principles of Lyapunov's second method. In our case, we must search simultaneously for two functions: a strict Lyapunov function V: $R^n\to R$ and a function ψ. In detail, let ρ be the remainder function of Theorem 6.2. Then, ψ must turn Eq. (6.8) into a strictly Lyapunov stable differential equation. In particular, the requirement of Definition 3.1 (iv) becomes $dV(\zeta(t))/dt = (\partial V/\partial\zeta)\dot\zeta = (\partial V/\partial\zeta)[\rho(\zeta,u)-\psi(h(\zeta),u)]<0$, or

$$(\partial V/\partial\zeta)\rho(\zeta,u) < (\partial V/\partial\zeta)\psi(h(\zeta),u) \text{ for all } 0\neq\zeta\in R^n \text{ and all } u\in R^m \tag{8.1}$$

For future reference, we summarize this point as follows.

THEOREM 8.1. *Let* Σ *be a system of the form (1.1) with the recursion function f, output function h, and remainder function* ρ*. Assume that the surjective function* $h : R^n\to \mathrm{Im}\,h$ *has a continuous right inverse function* h^**. Then, there is a strict asymptotic observer for* Σ *if and only if there is a strict Lyapunov function* $V(\zeta)$ *and a continuous function* $\psi : R^p\times R^m\to R^n$ *satisfying Eq. (8.1).* □

9 Examples

Example 9.1. For the system of Example 6.3 with ρ of Eq. (6.10) and ψ of Eq. (6.11), Eq. (6.8) becomes

$$\begin{pmatrix}\dot\zeta_1(t)\\ \dot\zeta_2(t)\end{pmatrix} = \begin{pmatrix}\zeta_2(t)\\ 0\end{pmatrix} - \begin{pmatrix}\zeta_1(t)\\ \zeta_1(t)\end{pmatrix} \tag{9.1}$$

To examine Lyapunov's second method, take the strict Lyapunov function $V=(\zeta_1^2+\zeta_2^2)/2$. This yields $\partial V/\partial\zeta_1=\zeta_1$, $\partial V/\partial\zeta_2=\zeta_2$, and Eq. (8.1) takes the form $(\partial V/\partial\zeta_1)\zeta_2 < (\partial V/\partial\zeta_1)\zeta_1+(\partial V/\partial\zeta_2)\zeta_1$, or $0<\zeta_1^2$. Thus, $dV(\zeta(t))/dt<0$ whenever $\zeta_1(t)\neq 0$. Finally, if $\zeta_1(t)=0$ over an interval of time, say $t\in(t_1,t_2), t_2>t_1\ge 0$, then also $\dot\zeta_1(t)=0$ over (t_1,t_2); by Eq. (9.1), this implies that $\zeta_2(t)=0$ over (t_1,t_2), and the system is resting at the origin. As all other requirements of Definition 3.1 hold for Eq. (9.1), we conclude that ψ is indeed an appropriate choice in this case, as noted in Example 6.3. □

Example 9.2. Consider a system Σ of the form (1.1), where

$$f(x_1,x_2,u) = \begin{pmatrix}x_2^3+x_1^2+u\\ x_1^3+ux_1\end{pmatrix} : R^2\times R\to R^2$$
$$h(x_1,x_2)=x_1 : R^2\to R$$

The function f can be construed as a third-order approximation of a more general nonlinear recursion function, thus pointing to a broader family of practical examples. Assume that this system is operated within a stabilizing closed-loop configuration, so that its state is well defined at all times, in compliance with Assumption 1.1. Referring to Eq. (6.1), we can use here

$$h^*(\zeta_1) = \begin{pmatrix}\zeta_1\\ 0\end{pmatrix}, \quad \pi(\zeta_1,\zeta_2) = \begin{pmatrix}\zeta_1\\ 0\end{pmatrix}$$

Then, in Eq. (6.6), we have

$$g(h(\zeta_1,\zeta_2),u) = f(\zeta_1,0,u) = \begin{pmatrix}\zeta_1^2+u\\ \zeta_1^3+u\zeta_1\end{pmatrix}$$

Using Eq. (6.7), this yields

$$\rho(\zeta,u) = f(\zeta,u)-g(h(\zeta),u) = \begin{pmatrix}\zeta_2^3+\zeta_1^2+u\\ \zeta_1^3+u\zeta_1\end{pmatrix} - \begin{pmatrix}\zeta_1^2+u\\ \zeta_1^3+u\zeta_1\end{pmatrix} = \begin{pmatrix}\zeta_2^3\\ 0\end{pmatrix}$$

Let us choose the strict Lyapunov function

$$V=\zeta_1^2/2+\zeta_2^4/4$$

with the function

$$\psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix}\zeta_1\\ \zeta_1\end{pmatrix}$$

Substituting into Eq. (6.8), we get

$$\begin{pmatrix} \dot{\zeta}_1 \\ \dot{\zeta}_2 \end{pmatrix} = \begin{pmatrix} \zeta_2^3(t) - \zeta_1(t) \\ -\zeta_1(t) \end{pmatrix} \tag{9.2}$$

Further, substitution into Eq. (8.1) yields

$$(\partial V/\partial \zeta)\rho = \zeta_1 \zeta_2^3$$
$$(\partial V/\partial \zeta_1)\psi_1 + (\partial V/\partial \zeta_2)\psi_2 = \zeta_1^2 + \zeta_2^3 \zeta_1$$

so that $(\partial V/\partial \zeta)\rho < (\partial V/\partial \zeta)\psi$ for all $\zeta_1 \neq 0$. Finally, if $\zeta_1(t) = 0$ over an interval of time, say $t \in (t_1, t_2), t_2 > t_1 \geq 0$, then also $\dot{\zeta}_1(t) = 0$ over (t_1, t_2); by Eq. (9.2), this implies that $\zeta_2(t) = 0$ over (t_1, t_2), so that Eq. (9.2) is resting at the origin, and we have asymptotic stability of EQ. (9.2). Thus, ψ is appropriate for this case. Then, using Eq. (6.9), we get the strict observer function

$$\omega(h(\zeta), u) = \omega(\zeta_1, u) = \begin{pmatrix} \zeta_1 + \zeta_1^2 + u \\ \zeta_1 + \zeta_1^3 + u\zeta_1 \end{pmatrix}$$

By Eq. (4.6), a strict asymptotic observer for Σ is then given by

$$\dot{z}(t) = \begin{pmatrix} z_2^3(t) - z_1(t) + y_1(t) + y_1^2(t) + u(t) \\ -z_1(t) + y_1(t) + y_1^3(t) + u(t)y_1(t) \end{pmatrix}, \quad z(0) = z_0 \qquad \square$$

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