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On non-linear systems, additive feedback, and rationality

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The problem of stabilizing a non-linear system by the application of additive output feedback is considered. It is shown that if an injective recursive system Σ can be so stabilized, then it must be rational, namely, it must be possible to express Σ as a ratio $\Sigma = PQ^{-1}$ of two stable and recursive systems P and Q.

1. Introduction

Frequently, when speaking about feedback, one actually refers to the notion of *additive* feedback, whereby a portion of the output signal is returned, subtracted from a reference signal to obtain an 'error' signal, and then the error signal is used to steer the overall system toward zero error. For linear systems, additive feedback is, of course, the only possible form of feedback, but, in general, other forms of feedback are certainly conceivable. Nevertheless, even in non-linear situations, additive feedback is heavily employed. Indeed, it seems that the notion of additive feedback is particularly close to the origins of the feedback concept.

Our present paper is devoted to a study of non-linear systems with additive feedback. Specifically, we consider the classical control configuration shown in Fig. 1 where Σ is a given non-linear dynamic system, π is a non-linear precompensator, and φ is a non-linear dynamic output-feedback compensator. We shall usually assume that the precompensator π is invertible, so as to ensure that the final system $\Sigma_{(\pi,\varphi)}$ has the same control capabilities as the original system Σ.

In order to make our discussion concrete and as close as possible to engineering applications, we shall not discuss non-linear systems at the highest level of generality. Rather, we shall confine our attention to non-linear systems which are discrete time and time invariant, admit input values from

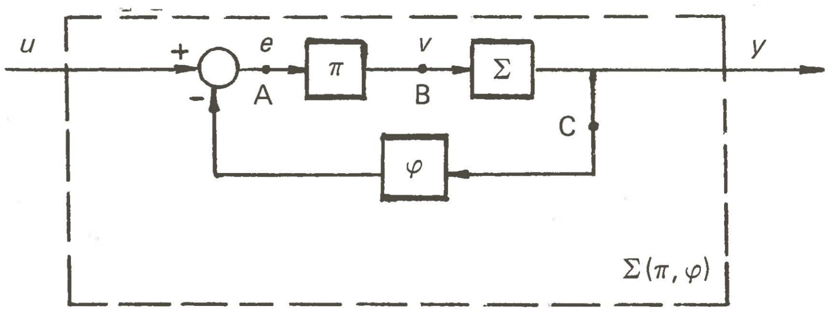

Figure 1.

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the finite-dimensional real space R^m and have their output values in the finite-dimensional real space R^p, and are recursive in the following sense. For each system Σ under consideration there exist integers η, $\mu \geqslant 0$ and a multi-variable vector-valued function $f: (R^p)^{\eta+1} \times (R^m)^{\mu+1} \rightarrow R^p$ such that each output sequence $\{y_j\} \subset R^p$ of Σ can be computed recursively from the input sequence $\{u_i\} \subset R^m$ generating it through the relationship

$$y_{k+\eta+1} = f(y_k, \ldots, y_{k+\eta} | u_k, \ldots, u_{k+\mu}), \quad k = \ldots, -1, 0, 1, \ldots$$

where the vertical line in the argument of f is used to separate the output and the input variables. Of course, the initial conditions $y_0, \ldots, y_\eta$ have to be specified. We call f the *recursion function* of Σ.

Our main objective is to study the conditions under which a given recursive system Σ can be stabilized through the configuration (1.1). ' Stabilized ' here is understood in a strong sense, usually referred to as ' internally stabilized ', meaning that the stability of the final system $\Sigma_{(\pi, \varphi)}$ is not destroyed by small disturbances or noise signals added at the points B and C in the diagram, nor is it destroyed by slight variations of the initial conditions of the composing systems Σ, π and φ. Our attention is focused on the connection between internal stabilization and the property of rationality. Roughly speaking, a recursive system Σ is rational if it can be represented as a quotient $\Sigma = PQ^{-1}$, where P and Q are stable recursive sytems (see § 3 for an exact definition). The connection between rationality and stabilization is most transparently demonstrated for the case when the given system Σ is injective (one-to-one). In case the system Σ is not injective, then one may look at the restriction of Σ to a set of input sequences over which it is injective, e.g., to a set which contains exactly one input sequence from each equivalence class in kernel Σ.

The main result of the present paper is given in § 4, where we show that any injective recursive system Σ that can be made internally stable by the application of output feedback has to be rational. Thus, when speaking about feedback stabilization, we can restrict our attention to rational systems alone, and the fraction representation $\Sigma = PQ^{-1}$ can be used to compute compensators π and φ which internally stabilize Σ. The importance of this observation stems from the fact that it indicates a close analogy between the theory of linear system stabilization and the general theory of stabilization of non-linear systems. In both cases, the key notions are rationality and fraction representations. A detailed study of rationality and of fraction representations of non-linear recursive systems was reported by Hammer (1983).

The underlying idea of our discussion in this paper, namely, that there is a connection between stabilization of non-linear systems and rationality, originates from the quest for an analogy to the well-known situation of the case of linear systems. It has been in the control theoretic background for quite a number of years, and in the past decade it has surfaced, sometimes implicitly, in numerous work such as that of Rosenbrock (1970), Desoer and Chan (1975), Desoer and Vidyasagar (1975) and Vidyasagar (1980), to mention a few. Indeed, that there is a connection between stabilization and rationality can be seen through the following simple argument. The system $\Sigma_{(\pi, \varphi)}$ (Fig. 1) can be expressed as

$$\Sigma_{(\pi, \varphi)} = \Sigma \Psi_{(\pi, \varphi)} \tag{1}$$

where

$$\Psi_{(\pi,\varphi)} = \pi[I + \varphi\Sigma\pi]^{-1} \tag{2}$$

is an invertible equivalent precompensator (see § 2 for details). Thus, we have that

$$\Sigma = \Sigma_{(\pi,\varphi)}\Psi_{(\pi,\varphi)}^{-1} \tag{3}$$

Now, the equivalent precompensator $\Psi_{(\pi,\varphi)}$ describes the relationship between the input sequence u and the intermediate sequence v in Fig. 1, through

$$v = \Psi_{(\pi,\varphi)}u \tag{4}$$

By the internal stability requirement, every intermediate signal sequence in Fig. 1 has to be bounded whenever the input sequence u is bounded. Whence, (4) implies that $\Psi_{(\pi,\varphi)}$ has, for every *bounded input* sequence u, a *bounded output* sequence v, that is, $\Psi_{(\pi,\varphi)}$ is *BIBO-stable*. Similarly, the input–output map $\Sigma_{(\pi,\varphi)}$ also is BIBO-stable. Thus, (3) actually states that the given system Σ is the quotient of two BIBO-stable systems, and we have that Σ is BIBO-rational. As we see, the connection between internal stabilization and rationality is basic.

In the discussion in the following sections we further explore the connection between rationality and stabilization with the aim of forming a mathematical framework for the investigation of non-linear system stabilization. We refine the concept of rationality, and we show that the connection between rationality and stabilization goes far beyond the above introductory remarks. The notion of stability that we employ in our discussion is the one due to Liapunov. Roughly speaking, a system is said to be stable if slight variations in its input sequence, or in its initial conditions, cause only slight variations in its output sequence. This notion of stability is substantially stronger than the notion of BIBO-stability mentioned above. We show that any injective system Σ that can be internally stabilized by the application of feedback must be rational under this notion of stability. Qualitatively, this means that Σ has a representation $\Sigma = PQ^{-1}$, where P and Q are stable recursive systems.

Investigations into the mathematical theory of non-linear feedback systems probably started with the classical work of Lurie (1951), where the so-called ' Lurie problem ' was introduced. Studies of the Lurie problem have captured the attention of the control theoretic community for several decades, and they lead to numerous classical works in the area. Among these are Popov (1961), Kalman (1963), Hale (1963), Sandberg (1964), Yacubovitz (1965), Lefshetz (1965), Zames (1966) and many others. In more recent years, a renewal of the interest in non-linear feedback control systems seems to have taken place, and some indication of the more recent trends can be found in the work of Desoer and Vidyasagar (1975), Utkin (1977), Sontag and Sussmann (1980), Baillieul *et al.* (1980), Sontag (1981) and the references cited in these works.

The present paper is organized as follows. In § 2 and § 3 we review and adapt to our present framework some classical facts related to causality and stability. The connection between rationality and stabilization is discussed in § 4, which is the main section of the paper.

2. Causality and feedback

We start with a review of the terminology, the notation, and the underlying framework that will accompany us throughout our discussion. This framework is reproduced from Hammer (1984), which contains a more detailed presentation of the various concepts. Let R be the set of real numbers, and let $m \geqslant 0$ be an integer. We denote by $S(R^m)$ the set of all two-sided infinite sequences u of the form $\ldots, 0, 0, \ldots, 0, u_{t(u)}, u_{t(u)+1}, \ldots$, where $u_j \in R^m$ for all integers j, where $u_j = 0$ for all $j < t(u)$, and where the integer $t(u) > -\infty$ depends on the sequence u. We denote by 0 the sequence in $S(R^m)$ which consists of only zero elements.

For each sequence $u \in S(R^m)$, we denote by u_j the jth element of the sequence, and by $u_i{}^j$, where $j \geqslant i$, the set of elements $u_i, u_{i+1}, \ldots, u_j$. In case $j < i$, then $u_i{}^j$ denotes the empty set. Given two sequences $u, v \in S(R^m)$, we define their sum $u+v$ coefficientwise by $(u+v)_i = u_i + v_i$ for all integers i.

An element $u \in S(R^m)$ can be regarded as input sequence to a discrete-time system. We define a system, admitting input values from R^m and having its output values in R^p, as a map $\Sigma : S(R^m) \rightarrow S(R^p)$. We require that every system under consideration should be time invariant (i.e. commute with the shift operator), and that it should possess at least one (possibly unstable) equilibrium point, corresponding, for example, to the ' off ' state of the system. For the sake of simplicity we assume that, for every system Σ under consideration, this equilibrium point is the zero point, namely

$$\Sigma 0 = 0 \tag{5}$$

Let $\Sigma : S(R^m) \rightarrow S(R^p)$ be a non-linear time-invariant system. We say that Σ is a *recursive system* if there exist integers $\eta, \mu \geqslant 0$ and a map $f : (R^p)^{\eta+1} + (R^m)^{\mu+1} \rightarrow R^p$ such that, for every input sequence $u \in S(R^m)$, the corresponding output sequences $y := \Sigma u$ satisfies

$$y_{k+\eta+1} = f(y_k^{k+\eta} | u_k^{k+\mu}) \tag{6}$$

for all integers k. The vertical line inside the argument of f is used to separate the output variables and the input variables. The function f is called a *recursion function* for Σ, and (6) is called a *recursive representation* of Σ. The integer η is called the *principal degree* of the representation.

We review now a few standard facts related to causality. A system $\Sigma : S(R^m) \rightarrow S(R^p)$ is *causal* (respectively, *strictly causal*) if, for every pair of input sequences $u, v \in S(R^m)$ and for every integer j, the equality $u_{-\infty}{}^j = v_{-\infty}{}^j$ implies that the corresponding output sequences satisfy $(\Sigma u)_{-\infty}{}^j = (\Sigma v)_{-\infty}{}^j$ (respectively, $(\Sigma u)_{-\infty}{}^{j+1} = (\Sigma v)_{-\infty}{}^{j+1}$). In the case where the system Σ is recursive, the following characterization of causality is an easy consequence.

Proposition 1

Let $\Sigma : S(R^m) \rightarrow S(R^p)$ be a recursive system. Then, Σ is causal (respectively, strictly causal) if and only if it has a recursive representation $y_{k+\eta+1} = f(y_k^{k+\eta} | u_k^{k+\mu})$ with $\mu \leqslant \eta + 1$ (respectively, with $\mu \leqslant \eta$).

The operation of addition in the space of sequences induces an operation of addition in the set of systems. Let $\Sigma_1, \Sigma_2 : S(R^m) \rightarrow S(R^p)$ be two systems. The sum $\Sigma := \Sigma_1 + \Sigma_2 : S(R^m) \rightarrow S(R^p)$ is defined pointwise for every element

$u \in S(R^m)$ by $\Sigma u = (\Sigma_1 u) + (\Sigma_2 u)$. Next, given two systems $\Sigma_1 : S(R^m) \to S(R^p)$ and $\Sigma_2 : S(R^p) \to S(R^q)$, we define their composition (or series combination) $\Sigma := \Sigma_2 \Sigma_1 : S(R^m) \to S(R^q)$ which is given, for every element $u \in S(R^m)$, by $\Sigma u := \Sigma_2(\Sigma_1 u)$, i.e. the usual composition of the maps. The following two statements are standard results in causality theory.

Proposition 2

The sum of two causal (respectively, of two strictly causal) time-invariant systems $\Sigma_1, \Sigma_2 : S(R^m) \to S(R^p)$ is a causal (respectively, a strictly causal) time-invariant system $\Sigma_1 + \Sigma_2 : S(R^m) \to S(R^p)$.

Proposition 3

Let $\Sigma_1 : S(R^m) \to S(R^p)$ and $\Sigma_2 : S(R^p) \to S(R^q)$ be causal time-invariant systems. Then, the series connection $\Sigma_2 \Sigma_1$ is a causal time-invariant system. If either Σ_1 or Σ_2 is strictly causal, then $\Sigma_2 \Sigma_1$ is strictly causal as well.

We turn now to a preliminary examination of Fig. 1. We assume throughout our discussion that the system $\Sigma : S(R^m) \to S(R^p)$ is strictly causal and time-invariant, and that $\pi : S(R^m) \to S(R^m)$ and $\varphi : S(R^p) \to S(R^m)$ are causal time-invariant systems. We denote by $\Sigma_{(\pi,\varphi)} : S(R^m) \to S(R^p)$ the overall composite system described by Fig. 1 (we show below that it is well defined). Let $u \in S(R^m)$ be an input sequence to $\Sigma_{(\pi,\varphi)}$, and let $y = \Sigma_{(\pi,\varphi)} u \in S(R^p)$ be a corresponding output sequence. Denoting by $e \in S(R^m)$ the sequence induced by u at the point A of Fig. 1, we observe that

$$\left.\begin{aligned} e &= u - \varphi y \\ y &= \Sigma \pi e \end{aligned}\right\} \tag{7}$$

Whence, using the definition of a sum of systems, and denoting by $I : S(R^m) \to S(R^m)$ the identity map, we obtain

$$(I + \varphi \Sigma \pi) e = u \tag{8}$$

In view of Propositions 2 and 3, the system

$$\Psi := (I + \varphi \Sigma \pi) : S(R^m) \to S(R^m) \tag{9}$$

is causal and time invariant and we have $u = \Psi e$. Below, we examine a few properties of Ψ.

Lemma 1

The system $\Psi : S(R^m) \to S(R^m)$ of (9) is injective.

Proof

Our proof is based on causality arguments. Let $x, z \in S(R^m)$ be input sequences, and assume that $\Psi x = \Psi z$. To prove injectivity of Ψ we have to show that the latter implies $x = z$. To this end we first note that, by (9), the equality $\Psi x = \Psi z$ implies

$$x - z = \varphi \Sigma \pi z - \varphi \Sigma \pi x \tag{10}$$

Assume further, by contradiction, that $x \neq z$. Then, by the definition of $S(R^m)$, the set of integers i for which $x_i \neq z_i$ is non-empty, and it contains a minimal element i_*. Now, since $\varphi\Sigma\pi$ is strictly causal (Proposition 3), we have $(\varphi\Sigma\pi x)_{-\infty}{}^{i_*} = (\varphi\Sigma\pi z)_{-\infty}{}^{i_*}$. But then, (10) implies that $x_{-\infty}{}^{i_*} = z_{-\infty}{}^{i_*}$, contradicting the existence of i_*. Therefore, $x = z$, so that $\Psi x = \Psi z$ implies $x = z$, and Ψ is injective.

Lemma 2

The map $\Psi : S(R^m) \to S(R^m)$ of (9) is surjective.

Proof

Let $x \in S(R^m)$ be any sequence. We construct recursively (element by element) a sequence $u \in S(R^m)$ for which $\Psi u = x$. Let j be an integer such that $x_i = 0$ for all $i \leqslant j$, and define $u_i = 0$ for all $i \leqslant j$. In preparation for recursion, assume that, for some integer k, the elements u_i, $i \leqslant k$, have been computed. To compute the next element u_{k+1} we proceed as follows. Let $v \in S(R^m)$ be any continuation of the partial sequence $u_{-\infty}{}^k$, that is, $v_i = u_i$ for all $i \leqslant k$. Then, define

$$u_{k+1} := x_{k+1} - (\varphi\Sigma\pi v)_{k+1} \in R^m$$

We note that, by the strict causality of $\varphi\Sigma\pi$, the element u_{k+1} is uniquely determined by x_{k+1} and $u_{-\infty}{}^k$. By repeating the same construction for each integer k, we obtain a sequence $u \in S(R^m)$ uniquely determined by x. It is then readily seen that $\Psi u = x$, so that $x \in \operatorname{Im} \Psi$, or $\operatorname{Im} \Psi = S(R^m)$, and our proof concludes. □

In view of Lemmas 1 and 2, the map $\Psi : S(R^m) \to S(R^m)$ is an isomorphism, and hence has a unique inverse $\Psi^{-1} : S(R^m) \to S(R^m)$. We have already noticed that Ψ is causal.

Lemma 3

The inverse $\Psi^{-1} : S(R^m) \to S(R^m)$ of the map Ψ of (9) is causal.

Proof

Let x, $z \in S(R^m)$ be any pair of sequences such that, for some integer k, we have $x_{-\infty}{}^k = z_{-\infty}{}^k$. We have to show that then also $(\Psi^{-1}x)_{-\infty}{}^k = (\Psi^{-1}z)_{-\infty}{}^k$. To this end, denote $x^* := \Psi^{-1}x$ and $z^* := \Psi^{-1}z$. Then, clearly, $\Psi x^* = x$ and $\Psi z^* = z$, and hence, by (9)

$$\left.\begin{aligned} x &= x^* + \varphi\Sigma\pi x^* \\ z &= z^* + \varphi\Sigma\pi z^* \end{aligned}\right\} \quad (11)$$

Now, either $x^* = z^*$ or $x^* \neq z^*$. If $x^* = z^*$ then evidently $(\Psi^{-1}x)_{-\infty}{}^k = (\Psi^{-1}z)_{-\infty}{}^k$, and our assertion holds. Otherwise, $x^* \neq z^*$. In such case, by the definition of $S(R^m)$, the set of integers i for which $x_i^* \neq z_i^*$ is non-empty, and it contains a minimal element i_*. We have to show that $i_* > k$. Indeed, since $\varphi\Sigma\pi$ is strictly causal, we have that $(\varphi\Sigma\pi x^*)_{-\infty}{}^{i_*} = (\varphi\Sigma\pi z^*)_{-\infty}{}^{i_*}$. Whence since $x_{i_*}{}^* \neq z_{i_*}{}^*$, it follows by (11) that $x_{i_*} \neq z_{i_*}$. But then, since by assumption $x_{-\infty}{}^k = z_{-\infty}{}^k$, we obtain that necessarily $i_* > k$. Thus, $x^*{}_{-\infty}{}^k = z^*{}_{-\infty}{}^k$, and Ψ^{-1} is causal. □

Continuing with our discussion of Fig. 1, and using the existence of the inverse Ψ^{-1} of Ψ, we obtain from (8) that $e=\Psi^{-1}u$, so that $y=\Sigma\pi e=\Sigma\Psi^{-1}u$. Thus, the overall composite system $\Sigma_{(\pi,\varphi)}$ is well defined, and it is given by

$$\Sigma_{(\pi,\varphi)}=\Sigma\pi\Psi^{-1} \tag{12}$$

We note that, since Ψ^{-1} is causal by Lemma 3, the overall system $\Sigma_{(\pi,\varphi)}$ is strictly casual.

Finally, we can now use $\Sigma_{(\pi,\varphi)}$ to obtain an alternative expression for Ψ^{-1}. Let $\Psi_1 := (I-\varphi\Sigma_{(\pi,\varphi)})$. Then $\Psi_1\Psi=(I+\varphi\Sigma\pi)-\varphi\Sigma\pi\Psi^{-1}\Psi=I$, where we have used the definition of a sum of systems. Whence, Ψ_1 is a left inverse of Ψ, so that, by the uniqueness of the inverse, $\Psi^{-1}=\Psi_1$. The intuitive meaning of this fact is simple—the additive feedback φ can be cancelled by the additive feedback $(-\varphi)$. We conclude with a summary of the main facts.

$$\left.\begin{aligned}\Sigma_{(\pi,\varphi)}&=\Sigma\pi\Psi^{-1}\\ \Psi^{-1}&=(I+\varphi\Sigma\pi)^{-1}=(I-\varphi\Sigma_{(\pi,\varphi)})\end{aligned}\right\} \tag{13}$$

3. Stability and internal stability: basic definitions

In this section we review the basic definitions of stability employed in our discussion. Broadly, we distinguish between two stability concepts—the stability of a single system, and the stability of a composite system. For a single system Σ, the notion of stability is essentially a version of the continuity notion, requiring that small changes in the input sequence or in the initial conditions of Σ cause only small changes in its output sequence. This notion of stability was first conceived by Liapunov. For a composite system, a stronger notion of stability is required, mainly due to the fact that noises may slightly distort the intermediate signals by which the subsystems communicate within the composite system. In this spirit, qualitatively, the composite system $\Sigma_{(\pi,\varphi)}$ of Fig. 1 is said to be internally stable if the input–output relationship $\Sigma_{(\pi,\varphi)}$ is stable, and if it remains so when slight distortions occur in the input signals to the composing systems Σ, π or φ. We start our discussion with a review of some standard notions related to continuity. (For a thorough and brief review of continuity properties, see Kuratowsky (1961).)

Let $u\in S(R^m)$ be a sequence, and, for each integer j, let $u_{1,j}, \ldots, u_{m,j}\in R$ be the components of the vector $u_j\in R^m$. We define

$$\rho(u_i^j) := \max\{|u_{\alpha,\beta}| : \alpha=1,\ldots,m,\ i\leqslant\beta\leqslant j\}$$

where $i\leqslant j$ are finite integers. For a sequence $u\in S(R^m)$ we define

$$\rho(u) := \sup_i 2^{-|i|}\rho(u_i)$$

The function ρ induces a metric on our spaces when defining $\rho(v_1, v_2) := \rho(v_1-v_2)$. For an element $(u|y)\in S(R^m)\times S(R^p)$, we define $\rho[(u|y)] := \max\{\rho(u), \rho(y)\}$. Whenever referring to continuity, we shall always mean continuity with respect to the topology induced by the metric ρ.

Next, we review the concept of *i/o* (input/output) *space* (Hammer (1984)). Let $\Sigma : S(R^m)\to S(R^p)$ be a recursive system, and let $\mathscr{S} : y_{k+\eta+1}=f(y_k^{k+\eta}|u_k^{k+\mu})$ be a recursive representation of Σ. Generally speaking, the i/o space is the set of all possible arguments of the function f, assuming that all sequences start from zero initial conditions. Formally, we define the set of all one-sided infinite sequences $S_0{}^{\mu}(R^m)$ to consist of all sequences, u_0, u_1, $\ldots \subset R^m$ for which $u_0 = u_1 = \ldots = u_{\mu} = 0$, i.e. the set of all sequences that start with μ zero elements. For each $u \in S_0{}^{\mu}(R^m)$ we denote by $\mathscr{S}(u)$ the sequence generated by the recursive representation $\mathscr{S}$ from the sequence u, when started from zero initial conditions. Explicitly, letting $y := \mathscr{S}(u)$, we have $y_{k+\eta+1} = f(y_k^{k+\eta} | u_k^{k+\mu})$, $k = 0, 1, 2, \ldots$, where $y_0 = y_1 = \ldots = y_{\eta} = 0$. The *i/o space* D_0 of $\mathscr{S}$ is then the subset of $(R^p)^{\eta+1} \times (R^m)^{\mu+1}$ given by

$$D_0 := \bigcup_{u \in S_0{}^{\mu}(R^m)} \bigcup_{k \geq 0} ([\mathscr{S}(u)]_k^{k+\eta} | u_k^{k+\mu}) \tag{14}$$

The interest in the i/o space D_0 stems from the fact that it is the subset over which the recursion function f is uniquely determined by Σ (Hammer 1984).

Further, we denote by $S_{\alpha}(R^m)$ the set of all one-sided infinite sequences of the form u_{α}, $u_{\alpha+1}$, $\ldots$, where $u_j \in R^m$ for all integers $j \geq \alpha$, and we identify the set $S_0(R^m)$ with the set of all elements $u \in S(R^m)$ for which $u_i = 0$ for all integers $i < 0$. Given an element $d := (v_0{}^{\eta} | w_0{}^{\mu}) \in (R^p)^{\eta+1} \times (R^m)^{\mu+1}$ and an element $u \in S_{\mu+1}(R^m)$, we denote by $\mathscr{S}(d,u)$ the output sequence generated by the recursive representation $\mathscr{S}$ when started from the initial conditions d and excited by the input sequence u. Explicitly, letting $y := \mathscr{S}(d, u)$, we have $y_{k+\eta+1} = f(y_k^{k+\eta} | u_k^{k+\mu})$, $k = 0, 1, 2, \ldots$, where $y_0{}^{\eta} = v_0{}^{\eta}$ and $u_0{}^{\mu} = w_0{}^{\mu}$.

Finally, we shall place considerable emphasis on the behaviour of systems under conditions where the input sequences are bounded. The interest in this case is motivated by practical as well as by theoretical considerations. From the practical point of view, every actual physical system is operated by bounded input sequences. From the theoretical point of view, restricting the attention to bounded input sequences may facilitate the use of properties related to compactness, which, in turn, may lead to simplification of mathematical argument. Thus, given a real number $\theta > 0$, we introduce the sets of bounded sequences $S(\theta^m)$ (respectively, $S_0{}^{\mu}(\theta^m)$, $S_{\alpha}(\theta^m)$), consisting of all elements $u \in S(R^m)$ (respectively, $u \in S_0{}^{\mu}(R^m)$, $u \in S_{\alpha}(R^m)$) satisfying $\rho(u_i) \leq \theta$ for all applicable integers i. We denote by $\mathscr{S}_{|\theta}$ the restriction of the recursive representation $\mathscr{S}$ to input sequences bounded by θ. We also define the *restricted i/o space* $D_0{}^{\theta}$ by

$$D_0{}^{\theta} := \bigcup_{u \in S_0{}^{\mu}(\theta^m)} \bigcup_{k \geq 0} ([\mathscr{S}(u)]_k^{k+\eta} | u_k^{k+\mu}) \tag{15}$$

that is, the part of the i/o space generated by input sequences which are bounded by θ.

We recall that the ζ-neighbourhood D_{ζ} of a set $D \subset R^q$ is the set of all elements $u \in R^q$ satisfying $\rho(u, D) < \zeta$. The following definition contains several concepts related to stability of a single system.

Definition 1

Let $\Sigma : S(R^m) \to S(R^p)$ be a recursive system, and let $\mathscr{S} : y_{k+\eta+1} = f(y_k^{k+\eta} | u_k^{k+\mu})$ be a recursive representation of Σ. Let $D \subset (R^p)^{\eta+1} \times (R^m)^{\mu+1}$ be a non-empty subset, and let $\theta > 0$ be a real number.

(i) $\mathscr{S}_{|\theta}$ is *stable* over D if, for every real $\epsilon > 0$, the following holds: for every $u \in S_{\mu+1}(\theta^m)$ and for every $d \in D$ there exists a real $\delta(u, d, \epsilon) > 0$ such that, whenever elements $u' \in S_{\mu+1}(\theta^m)$ and $d' \in D$ satisfy $\rho(u, u') < \delta$ and $\rho(d, d') < \delta$, then $\rho[\mathscr{S}(u, d), \mathscr{S}(u', d')] < \epsilon$.

(ii) $\mathscr{S}_{|\theta}$ is *i/o* (input/output) *stable* if it is stable over its restricted i/o space D_0^{θ}. The representation $\mathscr{S}$ is i/o stable (or, simply, stable) if $\mathscr{S}_{|\theta}$ is i/o stable for every real $\theta > 0$.

(iii) $\mathscr{S}_{|\theta}$ is *internally stable* if there exists a real $\zeta > 0$ such that $\mathscr{S}_{|\theta}$ is stable over the ζ-neighbourhood $D_{0,\zeta}^{\theta}$ of the restricted i/o space D_0^{θ}. The representation $\mathscr{S}$ is internally stable if $\mathscr{S}_{|\theta}$ is internally stable for every real $\theta > 0$.

(iv) The system Σ is *C-stable* if the restricted map $\Sigma: S_0(\theta^m) \to S(R^p)$ is continuous and bounded for every real $\theta > 0$.

(v) The system Σ is *BIBO* (bounded-input bounded-output) *stable* if, for every real $\theta > 0$, there exists a real $\alpha > 0$ such that $\Sigma[S_0(\theta^m)] \subset S(\alpha^p)$.

As we have mentioned in the introduction, most of our discussion in this paper evolves around the concept of rationality, which is defined as follows (Hammer 1984). Let $\Sigma: S(R^m) \to S(R^p)$ be a recursive system. When Σ is regarded as a map, it can always be factorized into a composition of maps $\Sigma = PQ$, where $Q: S(R^m) \to S$ is a surjective map, $P: S \to S(R^p)$ is an injective map, and S is an appropriate space (Maclane and Birkhoff (1979, Chap. 1)). The map Q, being surjective, possesses a right inverse $Q^*: S \to S(R^m)$, whereas the map P, being injective, possesses a left inverse $P^*: S(R^p) \to S$. We say that Σ is *right rational* (respectively, *right I-rational*, *right C-rational*, *right BIBO-rational*) if the above maps can be chosen in such a way that (i) P, Q and Q^* represent recursive systems, (ii) $S \subset S(R^q)$ for some integer q and (iii) P and Q^* are i/o stable (respectively, internally stable, C-stable, BIBO-stable). For a detailed discussion of rationality of recursive systems see Hammer (1984).

We now turn to a preliminary discussion of the stability of composite systems. Let $\Sigma_*: S(R^m) \to S(R^p)$ be a composite system consisting of n interconnected systems $\Sigma_i: S(R^{m_i}) \to S(R^{p_i})$, $i = 1, \ldots, n$, all of which are recursive. Let $\mathscr{S}: y_{k+\eta_i+1} = f_i(y_k^{k+\eta_i} | u_k^{k+\mu_i})$ be a recursive representation of the system Σ_i. An input sequence $u \in S(R^m)$ of the composite system Σ_* induces an input sequence $u(i)$ and an output sequence $y(i)$ for each one of the systems Σ_i, $i = 1, \ldots, n$. We assume that two types of disturbances may occur in Σ_*. First, due to noises and distortions caused by the environment, the actual input sequence $w(i) \in S(R^{m_i})$ of the system Σ_i will be slightly different from the sequence $u(i)$, say $\rho[w(i), u(i)] < \zeta$, where $\zeta > 0$ is a real parameter describing the magnitude of the noise or the distortion. Equivalently, we may say that the actual input sequence $w(i)$ of Σ_i is of the form $w(i) = u(i) + v(i)$, where $v(i) \in S(R^{m_i})$ is an additive disturbance signal satisfying $\rho[v(i)] < \zeta$. For the stability of the composite system Σ_*, we require that small disturbance signals $v(1), \ldots, v(n)$ cause only slight changes in the output sequence y of Σ_*.

Further, recalling that each one of the subsystems Σ_i is a recursive system, we have to account for the fact that the initial conditions of Σ_i cannot be set with absolute accuracy. Thus, if $d(i) \in (R^{p_i})^{\eta_i+1}$ are the prescribed initial conditions of Σ_i, we assume that the actual initial conditions of Σ_i are given by

$d'(i) \in (R^{p_i})^{\eta_i+1}$, where $\rho[d'(i), d(i)] < \zeta$, and $\zeta > 0$ is a parameter describing the uncertainty in setting the initial conditions. We require that the composite system Σ_* should not be significantly affected by slight uncertainties in the initial conditions, or by any combination of these and of small disturbances in the intermediate signals. We thus arrive at the following

Definition 2

Let $\Sigma_* : S(R^m) \to S(R^p)$ be a composite system consisting of n interconnected systems $\Sigma_1, \ldots, \Sigma_n$, and assume that Σ_* is i/o stable. For each $i = 1, \ldots, n$, let Σ_i map $S(R^{m_i}) \to S(R^{p_i})$, let η_i be the principal degree of Σ_i, and let $d_i \in (R^{m_i})^{\eta_i+1}$ be the prescribed initial conditions of Σ_i. Each input sequence $u \in S(R^m)$ of Σ_* induces an input sequence $u(i) \in S(R^{m_i})$ of Σ_i, $i = 1, \ldots, n$. We say that Σ_* is *internally stable* if the following hold.

(i) For every pair of real numbers $\theta, \alpha > 0$, and for every real $\epsilon > 0$, there exists a real $\delta > 0$ (depending on θ, α and ϵ) such that the system Σ'_* obtained from Σ_* through the operations (*a*) and (*b*) below satisfies $\rho[\Sigma'_* u, \Sigma_* u] < \epsilon$ for all $u \in S_0(\theta^m)$.

(*a*) For each $i = 1, \ldots, n$, add to the input sequence $u(i)$ of the system Σ_i an arbitrary element $v(i) \in S(\alpha^{m_i})$ satisfying $\rho[v(i)] < \delta$.

(*b*) For each $i = 1, \ldots, n$, replace the initial conditions $d(i)$ of Σ_i by an arbitrary element $d'(i) \in (R^{p_i})^{\eta_i+1}$ satisfying $\rho[d'(i), d(i)] < \delta$.

(ii) For every pair of real numbers θ, $\alpha > 0$ there exist real numbers N, $\xi > 0$ (depending only on θ and α) such that, whenever the input sequence u of Σ_* satisfies $u \in S(\theta^m)$, then the intermediate output sequences $y'(i)$ of Σ'_i, $i = 1, \ldots, n$, satisfy $y'(i) \in S(N^{p_i})$ for all $i = 1, \ldots, n$, where Σ'_i are the disturbed systems obtained through (*a*) and (*b*) with $\delta < \xi$.

Condition (ii) of Definition 2 just requires the boundedness of all signals in the configuration. We discuss the implications of Definition 2 in the next section.

4. Feedback and rationality

In this section we discuss the connection between internal stabilization of a given system Σ and the rationality of that system. Many of the underlying ideas are already apparent from the examination of the simple configuration of pure dynamic output feedback, so we concentrate on this configuration (Fig. 2). Here, $\Sigma : S(R^m) \to S(R^p)$ is a strictly causal recursive system, and $\varphi : S(R^p) \to S(R^m)$ is a causal recursive output feedback compensator. The overall system described by Fig. 2 is denoted by Σ_φ, and, recalling from (13), we have $\Sigma_\varphi = \Sigma \Psi_\varphi$, where $\Psi_\varphi = [I + \varphi\Sigma]^{-1} : S(R^m) \to S(R^m)$ is an equivalent bicausal precompensator.

The simplest situation from our present point of view arises when one adds to the requirements of internal stability the additional requirement that the signal e in Fig. 2 depend continuously on the input signal u, whenever u is bounded. This requirement seems very natural when one interprets the signal e as the error signal. We would like the error to be affected only slightly when a slight change in the input sequence occurs. Now, from § 2 we know that $e = \Psi_\varphi u$, so that, if e is to depend continuously on u for bounded u, then

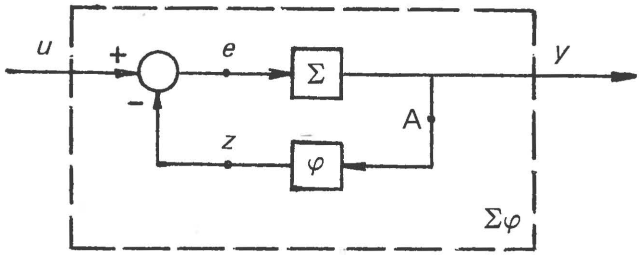

Figure 2.

the equivalent precompensator Ψ_φ has to be continuous over bounded inputs, namely C-stable. Combining this fact with the C-stability of Σ_φ and the equation $\Sigma=\Sigma_\varphi\Psi_\varphi^{-1}$, we obtain the following elementary statement.

Proposition 4

Let $\Sigma: S(R^m)\to S(R^p)$ be a strictly causal system. Assume that there exists a causal feedback compensator $\varphi: S(R^p)\to S(R^m)$ such that, in Fig. 2, the closed-loop system Σ_φ is C-stable and the error signal e depends continuously on the input signal u, for bounded $u\in S_0(R^m)$. Then, the equivalent precompensator Ψ_φ is C-stable, and $\Sigma=\Sigma_\varphi\Psi_\varphi^{-1}$ is a representation of the given system Σ as a quotient of C-stable systems.

The definition of internal stability, the way we stated it, does not directly require the continuous dependence of e on u. It only requires that the overall output signal y should depend continuously on (i) the input signal u (whenever u is bounded), and on (ii) the internal noises in the configuration. We now examine the implications of these requirements. As we have just mentioned, continuity of the equivalent precompensator Ψ_φ directly implies rationality of Σ. Thus, we have to consider only the possibility of discontinuities in Ψ_φ. To understand the origin of possible discontinuities in Ψ_φ, we recall from (13) that $\Psi_\varphi=I-\varphi\Sigma_\varphi$. Clearly, if the closed-loop system Σ_φ is stable, then all discontinuities of Ψ_φ originate from discontinuities of the feedback compensator φ. We state this fact as the following lemma.

Lemma 4

Let $\Sigma: S(R^m)\to S(R^p)$ be a strictly causal system, and assume that there exists a causal feedback compensator $\varphi: S(R^p)\to S(R^m)$ such that Σ_φ is C-stable. If, for some real $\theta>0$, a point $u\in S_0(\theta^m)$ is a discontinuity point of the restriction of Ψ_0 to $S_0(\theta^m)$, then the point $\Sigma_\varphi u\in S_0(R^p)$ is a discontinuity point of the restriction of the feedback compensator φ to $\Sigma_\varphi[S_0(\theta^m)]$.

In recent years, several authors (Sussmann 1979, Sontag and Sussman 1980) have discussed the problem of stabilizing a non-linear system by a continuous feedback compensator, concentrating on the connection between stabilization and controllability. For the case of a continuous feedback compensator φ, Lemma 4 directly implies that, if Σ can be stabilized by φ, then Σ is rational. Indeed, if φ and Σ_φ are both continuous, then so also is the equivalent precompensator $\Psi_\varphi(=I-\varphi\Sigma_\varphi)$, and $\Sigma=\Sigma_\varphi\Psi_\varphi^{-1}$ is a representation of Σ as a quotient of stable systems. To state these facts in more precise form, we need some notation. Let $A: S(R^m) \to S(R^p)$ be a map. For every real $\theta > 0$, we denote by $\mathrm{Im}_\theta\, A$ the set of all elements $y \in S(R^p)$ such that $y = Au$ for some $u \in S_0(\theta^m)$, namely the image of all bounded by θ elements. We set $\mathscr{I}m\, A := \bigcup_{\theta > 0} \mathrm{Im}_\theta\, A$, i.e. the image of all bounded elements.

Corollary 1

Let $\Sigma: S(R^m) \to S(R^p)$ be a strictly causal system, and assume that there exists a causal feedback compensator $\varphi: S(R^p) \to S(R^m)$ such that Σ_φ is C-stable. If the restriction of φ to $\mathscr{I}m\, \Sigma_\varphi$ is a C-stable map, then Ψ_φ is C-stable, and $\Sigma = \Sigma_\varphi \Psi_\varphi^{-1}$ is a representation of Σ as a quotient of C-stable systems.

In general, however, the feedback compensator φ may possess certain discontinuities without destroying the internal stability of Σ_φ. To understand the nature of these discontinuities, we may refer to the following qualitative argument. Assume that φ has a discontinuity which causes a discontinuity in the equivalent precompensator Ψ_φ. In view of the fact that Σ_φ is stable, this discontinuity cannot effect Σ_φ. Very qualitatively, if we think of the discontinuity in Ψ_φ as a jump, then, in view of the relation $\Sigma_\varphi = \Sigma \Psi_\varphi$, the system Σ has to produce the same output sequence for both ends of this jump, otherwise a jump in Σ_φ will occur, contradicting our assumption that Σ_φ is continuous. Thus a jump of Ψ_φ can occur only between points over which the original system Σ is constant, namely, between points contained in one and the same equivalence class of kernel Σ. The precise statement of this argument is somewhat more intricate than the present qualitative version, and it is given in the proof of the next lemma. For every element $u \in S(R^m)$, we denote by $[u]_\Sigma$ the equivalence class of u in kernel Σ consisting of all elements $v \in S(R^m)$ for which $\Sigma v = \Sigma u$. We also denote by $\{u^i\}$ a sequence of elements $u^0, u^1, u^2, \ldots$, where $u^i \in S(R^m)$ for all i.

Lemma 5

Let $\Sigma: S(R^m) \to S(R^p)$ be a strictly causal system and assume that there exists a causal feedback compensator $\varphi: S(R^p) \to S(R^m)$ for which Σ_φ is internally stable. For every discontinuity point $u \in S_0(\theta^m)$, $\theta > 0$, of the equivalent precompensator Ψ_φ, let S_u be the set of all sequences $\{u^i\} \subset S_0(\theta^m)$ converging to u, and let Δ_u be the set of all accumulation points of the sequences $\{\Psi_\varphi u^i\}$, where $\{u^i\} \in S_u$. Then, $\Delta_u \subset [\Psi_\varphi u]_\Sigma$.

Proof

Let $\{u^i\} \subset S_0(\theta^m)$ be any sequence in S_u, and let w be an accumulation point of the sequence $\{\Psi_\varphi u^i\}$. Notice that by condition (ii) of Definition 2 there is a real $N > 0$ such that $w \in S_0(N^m)$. Our proof will conclude upon showing that $\Sigma w = \Sigma \Psi_\varphi u$. To this end, let $\{a^i\}$ be a subsequence of $\{u^i\}$ such that the sequence $\{\Psi_\varphi a^i\}$ converges to w. Then, since $\Psi_\varphi a^i = a^i - \varphi \Sigma_\varphi a^i$, the sequence $\{\varphi \Sigma_\varphi a^i\}$ converges to the point $w_0 := u - w$. Let $e := \Psi_\varphi u$, and construct the sequences $v^i := \Sigma_\varphi a^i - \Sigma_\varphi u = \Sigma_\varphi a^i - \Sigma e$, and $\epsilon^i := \varphi \Sigma_\varphi u - \varphi \Sigma_\varphi a^i = \varphi \Sigma e - \varphi \Sigma_\varphi a^i$. Then, by the continuity of Σ_φ, the sequence $\{v^i\}$ converges to 0, and

$$\lim_{i \to \infty} \epsilon^i = \varphi \Sigma e - w_0 =: \epsilon_0$$

By condition (ii) of Definition 2, we also have that $\{v^i\}$, $\{\epsilon^i\} \subset (S(2N)^p)$. Apply now the elements of the sequence $\{v^i\}$ one at a time as noise input at the point A in Fig. 2, and denote by Σ_{φ, v^i} (respectively, Ψ_{φ, v^i}) the so disturbed system (respectively, the so disturbed equivalent precompensator). Then, $\Psi_{\varphi, v^i}{}^{-1}(e) = e + \varphi(v^i + \Sigma e) = e + \varphi \Sigma_\varphi a^i = e + \varphi \Sigma e + \epsilon^i = \Psi_\varphi^{-1} e + \epsilon^i = u + \epsilon^i$, so that

$$\Psi_{\varphi, v^i}(u + \epsilon^i) = e = \Psi_\varphi u \tag{*}$$

Consequently, since $\Sigma_{\varphi, v^i} = \Sigma \Psi_{\varphi, v^i}$ and $\Sigma_\varphi = \Sigma \Psi_\varphi$, we obtain that $\Sigma_{\varphi, v^i}(u + \epsilon^i) = \Sigma_\varphi(u)$ for all integers $i \geq 0$, so that

$$\lim_{i \to \infty} \Sigma_{\varphi, v^i}(u + \epsilon^i) = \Sigma_\varphi(u)$$

Next, by the internal stability of Σ_φ and in view of the convergences $v^i \to 0$ and $\epsilon^i \to \epsilon_0$, there exists, for every $\epsilon > 0$, an integer q_ϵ such that, for all $i \geqslant q_\epsilon$, we have $\rho[\Sigma_{\varphi, v^i}(u + \epsilon^i) - \Sigma_\varphi(u + \epsilon^i)] < \epsilon/2$ (condition (*a*) of Definition 2) and $\rho[\Sigma_\varphi(u + \epsilon^i) - \Sigma_\varphi(u + \epsilon_0)] < \epsilon/2$ (the C-stability of Σ_φ). Then, $\rho[\Sigma_{\varphi, v^i}(u + \epsilon^i) - \Sigma_\varphi(u + \epsilon_0)] < \epsilon$ for all $i \geqslant q_\epsilon$, and it follows that

$$\lim_{i \to \infty} \Sigma_{\varphi, v^i}(u + \epsilon^i) = \Sigma_\varphi(u + \epsilon_0)$$

Combining this result with our previous alternative computation of the same limit, we obtain that $\Sigma_\varphi(u + \epsilon_0) = \Sigma_\varphi(u)$. Now, $\Psi_\varphi(u + \epsilon_0) = u + \epsilon_0 - \varphi \Sigma_\varphi(u + \epsilon_0) = u + \epsilon_0 - \varphi \Sigma_\varphi u = \Psi_\varphi u + \epsilon_0 = e + \epsilon_0 = e + \varphi \Sigma e - w_0 = \Psi_\varphi^{-1} e - w_0 = u - (u - w) = w$. Hence, the equality $\Sigma_\varphi(u + \epsilon_0) = \Sigma_\varphi u$ implies that $\Sigma \Psi_\varphi(u + \epsilon_0) = \Sigma w = \Sigma \Psi_\varphi u$, so that $w \in [\Psi_\varphi u]_\Sigma$, and our proof concludes. □

Of course, a discontinuity of Ψ_φ can also be caused by a sequence $u^i \to u$ for which the sequence $\{\Psi_\varphi u^i\}$ has no accumulation points at all. In the next lemma we show that this situation cannot occur for input sequences in $S_0(\theta^m)$.

Lemma 6

Let $\Sigma : S(R^m) \to S(R^p)$ be a strictly causal system and assume that there exists a causal feedback compensator $\varphi : S(R^p) \to S(R^m)$ for which Σ_φ is internally stable. If the restriction of Ψ_φ to $S_0(\theta^m)$ has a discontinuity point $u \in S_0(\theta^m)$, then the set Δ_u of Lemma 5 contains more than one point.

Proof

Let $u \in S_0(\theta^m)$ be a discontinuity point of the restriction of Ψ_φ to $S_0(\theta^m)$, and let $\{u^i\} \subset S_0(\theta^m)$ be a sequence converging to u for which the sequence $\{\Psi_\varphi u^i\}$ does not converge to $\Psi_\varphi u$. Now, since Σ_φ is internally stable, there exists a real $N > 0$ such that $\{\Psi_\varphi u^i\} \subset S(N^m)$ (see (ii) of Definition 2), and since Ψ_φ is causal, we have $\{\Psi_\varphi u^i\} \subset S_0(N^m)$. By the compactness of $S_0(N^m)$, every sub-sequence of $\{\Psi_\varphi u^i\}$ has an accumulation point. In view of our assumption that $\{\Psi_\varphi u^i\}$ does not converge to $\Psi_\varphi u$, it follows then that the sequence $\{\Psi_\varphi u^i\}$ has at least one accumulation point $w \in S_0(N^m)$ different from $\Psi_\varphi u$. Thus, $w \in \Delta_u$, and, since always $\Psi_\varphi u \in \Delta_u$, our proof concludes. □

Consider now the case when the given system Σ is injective. Then, for every $u \in S(R^m)$, the equivalence class $[u]_\Sigma$ in kernel Σ contains exactly one point. In such case, Lemmas 5 and 6 directly imply that, if Σ_φ is internally stable, the equivalent precompensator Ψ_φ cannot have any discontinuity points when restricted to $S_0(\theta^m)$, for any real $\theta > 0$. This proves the following statement, which is a main result of this section.

Theorem 1

Let $\Sigma : S(R^m) \to S(R^p)$ be a strictly causal system, and assume that there exists a causal feedback compensator $\varphi : S(R^p) \to S(R^m)$ for which Σ_φ is internally stable. If Σ is injective, then the equivalent precompensator Ψ_φ is C-stable, and $\Sigma = \Sigma_\varphi \Psi_\varphi^{-1}$ is a representation of Σ as a quotient of C-stable systems.

We discuss next the environment in which the feedback compensator φ operates in the closed-loop system Σ_φ. It is clear from Fig. 2 that the set of input sequences to φ is exactly the set of output sequences y of the closed-loop system Σ_φ. Assume now that Σ_φ is internally stable and that Σ is injective. We next show that the input sequences fed by the system into φ are only such for which φ exhibits continuous behaviour. Thus, if φ contains in its construction any jumps, then none of them is activated during the operation of the closed-loop system.

Corollary 2

Let $\Sigma : S(R^m) \to S(R^p)$ be a strictly causal system, and assume that there is a causal feedback compensator $\varphi : S(R^p) \to S(R^m)$ for which Σ_φ is internally stable. If Σ is injective then, for every real $\theta > 0$, the restriction of the feedback compensator φ to the set $\Sigma_\varphi[S_0(\theta^m)]$ is a continuous map.

Proof

Let $\theta > 0$ be a real number, and let $\{v^i\} \subset \Sigma_\varphi[S_0(\theta^m)]$ be any sequence converging to a point $v \in \Sigma_\varphi[S_0(\theta^m)]$. We have to show that

$$\lim_{i \to \infty} \varphi v^i = \varphi v$$

Now, $\Sigma_\varphi = \Sigma \Psi_\varphi$, so that, since Σ is injective and Ψ_φ is bijective, Σ_φ is injective. Let $A : S_0(\theta^m) \to \Sigma_\varphi[S_0(\theta^m)]$ be the restriction of Σ_φ to $S_0(\theta^m)$. Then A is bijective and continuous, and hence, since $S_0(\theta^m)$ is compact, A is a homeomorphism (Kuratowski 1961). Letting $u^i := A^{-1} v^i$, $i = 1, 2, \ldots$, it follows that the sequence $\{u^i\} \subset S_0(\theta^m)$ converges to $u := A^{-1} v$. Clearly,

$$\lim_{i \to \infty} \Sigma_\varphi u^i = \lim_{i \to \infty} v^i = v = \Sigma_\varphi u$$

and since Ψ_φ is continuous on $S_0(\theta^m)$ by Theorem 1, then also

$$\lim_{i \to \infty} \Psi_\varphi u^i = \Psi_\varphi u$$

Consequently, recalling that $\Psi_\varphi = I - \varphi \Sigma_\varphi$, we have

$$\begin{aligned} \lim_{i \to \infty} \varphi v^i &= \lim_{i \to \infty} \varphi \Sigma_\varphi u^i \\ &= \lim_{i \to \infty} (u^i - \Psi_\varphi u^i) \\ &= u - \Psi_\varphi u \\ &= \varphi \Sigma_\varphi u = \varphi v \end{aligned}$$

and our proof concludes. □

Up to this point we have discussed the rationality properties of a stabilizable system $\Sigma : S(R^m)\rightarrow S(R^p)$, showing that, if Σ is injective (or if the error signal e depends continuously on the input signal u, or if the feedback compensator φ is continuous), then there exist C-stable maps P and Q such that $\Sigma = PQ^{-1}$. We turn now to a discussion of the recursivity properties of the systems P and Q. The next statement states that the systems P and Q can be chosen as recursive systems, so that if the injective system Σ can be stabilized, then it is right C-rational in the sense of § 2. Presently, we restrict our attention to the case of injective systems. Clearly, the same facts also apply to the restriction of a non-injective system to a set of inputs over which it is injective. The following is the main result of this section.

Theorem 2

Let $\Sigma : S(R^m)\rightarrow S(R^p)$ be a strictly causal, recursive, and injective system. If there exists a causal recursive feedback compensator $\varphi : S(R^p)\rightarrow S(R^m)$ such that Σ_φ is internally stable, then the given system Σ is right C-rational.

Proof

Let $\varphi : S(R^p)\rightarrow S(R^m)$ be a recursive causal feedback compensator for which Σ_φ is internally stable. Then, by Theorem 1, the equivalent precompensator Ψ_φ is C-stable. Let τ be the one-step shift to the left operator, so that for every $y \in S(R^p)$, the sequence $x := \tau y$ satisfies $x_i = y_{i+1}$ for all integers i. Define the maps

$$Q : S(R^m)\rightarrow \operatorname{Im} Q \subset S(R^{m+p}) : e \rightarrow \begin{pmatrix} u \\ x \end{pmatrix} := \begin{pmatrix} \Psi_\varphi^{-1} e \\ \tau\Sigma e \end{pmatrix}$$

$$P : \operatorname{Im} Q \rightarrow S(R^p) : \begin{pmatrix} u \\ x \end{pmatrix} \rightarrow y := \tau^{-1} x$$

$$Q^* : \operatorname{Im} Q \rightarrow S(R^m) : \begin{pmatrix} u \\ x \end{pmatrix} \rightarrow e := \Psi_\varphi u = u - \tau^{-1}\varphi x$$

Then, clearly, $\Sigma = PQ$, and Q^* is a right inverse of Q. Also P is evidently C-stable, and, since Ψ_φ is C-stable, so also is Q^*. By the injectivity of Σ_φ, it follows that P is injective. Thus, in order to prove that $\Sigma = PQ$ is a right C-rational presentation, it only remains to show that the maps P, Q and Q^* are recursive. Now P is evidently recursive ($y_k = x_{k-1}$). To show that Q and Q^* are recursive, let $\mathscr{S}_\Sigma : y_{k+\eta+1} = f(y_k^{k+\eta} | e_k^{k+\mu})$ and $\mathscr{S}_\varphi : z_{k+\alpha+1} = g(z_k^{k+\alpha} | y_k^{k+\beta})$, where $z = \varphi y$, be recursive representations of Σ and φ respectively. Then, since $e = u - \varphi y = u - z$, or $z = u - e$, we obtain after substitution into $\mathscr{S}_\varphi$ that $e_{k+\alpha+1} = u_{k+\alpha+1} - g([u-e]_k^{k+\alpha} | y_k^{k+\beta}) = u_{k+\alpha+1} - g([u-e]_k^{k+\alpha} | x_{k-1}^{k+\beta-1})$, which is a recursive representation of Q^*, and Q^* is recursive. Next, for Q, let $\gamma := \max\{\eta, \alpha\}$, denote $\epsilon_1 := \gamma - \eta$ and $\epsilon_2 := \gamma - \alpha$, and notice that, by the causality of φ, we have $\epsilon_2 + \beta \leqslant \alpha + 1 + \epsilon_2 = \gamma + 1$. Using the equation $z = \varphi y = \varphi\Sigma e$, we obtain

$$
\begin{aligned}
\begin{pmatrix} z \\ x \end{pmatrix}_{k+\gamma+1} &= \begin{pmatrix} z_{k+\gamma+1} \\ y_{k+\gamma+2} \end{pmatrix} \\
&= \begin{pmatrix} g(z_{k+\epsilon_2}^{k+\epsilon_2+\alpha} | y_{k+\epsilon_2}^{k+\epsilon_2+\beta}) \\ f(y_{k+\epsilon_1+1}^{k+\epsilon_1+1+\eta} | e_{k+\epsilon_1+1}^{k+\epsilon_1+1+\mu}) \end{pmatrix} \\
&= \begin{pmatrix} g(z_{k+\epsilon_2}^{k+\epsilon_2+\alpha} | x_{k+\epsilon_2-1}^{k+\epsilon_2-1+\beta}) \\ f(x_{k+\epsilon_1}^{k+\epsilon_1+\eta} | e_{k+\epsilon_1+1}^{k+\epsilon_1+1+\mu}) \end{pmatrix}
\end{aligned}
\tag{16}
$$

which, since $\beta+\epsilon_2-1 \leqslant \gamma$, is a recursive representation of $\begin{pmatrix} z \\ x \end{pmatrix}$ in terms of e.

Now, $u=e-z$, or $z=e-u$, so that, by substitution into (16), we obtain

$$
\begin{pmatrix} u \\ x \end{pmatrix}_{k+\gamma+1} = \begin{pmatrix} e_{k+\gamma+1} - g([e-u]_{k+\epsilon_2}^{k+\epsilon_2+\alpha} | x_{k+\epsilon_2-1}^{k+\epsilon_2-1+\beta}) \\ f(x_{k+\epsilon_1}^{k+\epsilon_1+\eta} | e_{k+\epsilon_1+1}^{k+\epsilon_1+1+\mu}) \end{pmatrix}
\tag{17}
$$

Since $\epsilon_2-1+\beta \leqslant \gamma$, eqn. (17) is a recursive representation of Q, and Q is recursive. □

As we recall, the definition of internal stability also requires continuous dependence of the output sequence y of Σ_φ on small variations of the initial conditions of Σ and of φ. For the sake of simplicity, we have ignored this requirement in our discussion up to this point, and we have restricted our attention to C-stability and to the effect of internal noises. When the consideration of variations of the initial conditions is added, then the following result can be proved along the lines of our previous discussion.

Theorem 3

Let $\Sigma : S(R^m) \to S(R^p)$ be a strictly causal, recursive, and injective system. If there exists a causal recursive feedback compensator $\varphi : S(R^p) \to S(R^m)$ such that Σ_φ is internally stable, then the given system Σ is right I-rational.

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