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ON INTERNALLY STABLE LINEAR CONTROL

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Abstract

The problem of representing a precompensator as an internally stable configuration of linear dynamic output feedback and a minimal "remainder precompensator" is considered. The discussion is confined to the case of single-input single-output systems, and is a simplified version of the general case considered in HAMMER [1981c].

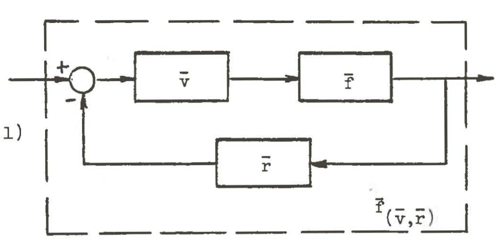
1. INTRODUCTION

In the present note we outline some features of an algebraic theory for the design of internally stable linear time invariant feedback control systems. We shall concentrate here on the case of single-input single-output systems, and will tacitly assume that all mentioned systems are of this kind. The general case is treated in detail in HAMMER [1981c], and our present note is meant to be an introduction to the more general setup given there. Therefore, we shall outline only the main features, and shall omit most proofs. Nevertheless, in the single-input single-output case, most statements can be verified through a direct calculation.

Before turning to the statement of the problem, we need some terminology. Let $\bar{f}$ be a rational transfer function, and let $\bar{f} = n/d$ be a coprime polynomial fraction representation of $\bar{f}$. We denote $\operatorname{ord} \bar{f} := \deg d - \deg n$, and say that $\bar{f}$ is <u>causal</u> (respectively, <u>strictly causal</u>) if $\operatorname{ord} \bar{f} \geq 0$ (respectively, $\operatorname{ord} \bar{f} \geq 1$). Also, we say that $\bar{f}$ is <u>input/output stable</u> if all roots of d are "stable". Equivalently, $\bar{f}$ is input/output stable if and only if every mode of $\bar{f}$, which is both reachable and observable, is stable. A stronger notion of stability is <u>internal stability</u>, by which we mean that <u>all</u> modes of $\bar{f}$, including the unreachable and the unobservable ones, are stable.

Now, let $\bar{f}$ be a strictly causal transfer function, and suppose that one is required to transform $\bar{f}$ into a specified transfer function $\bar{f}'$, through the employment of certain compensators. Due to practical limitations, the employment of postcompensators (i. e. dynamical transformations of the system outputs) is not allowed in most cases, and the preferred compensation scheme is of the following form.

(1.1)



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In the diagram, $\bar{v}$ is a causal precompensator, $\bar{r}$ is a causal dynamic output feedback, and $\bar{f}_{(\bar{v},\bar{r})}$ denotes the resulting composite system. As we mentioned, we assume that all systems are single-input single-output.

In order that diagram (1.1) be implementable, we have to add the requirement that the composite system $\bar{f}_{(\bar{v},\bar{r})}$ be internally stable. Thus, our problem resides in finding, if possible, causal systems $\bar{v}$ and $\bar{r}$ such that the transfer function of $\bar{f}_{(\bar{v},\bar{r})}$ is equal to the specified transfer function $\bar{f}'$, subject to the condition that $\bar{f}_{(\bar{v},\bar{r})}$ is internally stable. Further, in order to exploit the benefits of output feedback, we also require that the dynamical order of the precompensator $\bar{v}$ be reduced as much as possible. In this way, "as much as possible of the compensation dynamics is located in the feedback $\bar{r}$".

Through a routine calculation, one obtains that

$$\bar{f}_{(\bar{v},\bar{r})} = \bar{f}\bar{\ell}_{(\bar{v},\bar{r})},$$

where

$$\bar{\ell}_{(\bar{v},\bar{r})} = \bar{v}[I + \bar{r}\bar{f}\bar{v}]^{-1}$$

is an equivalent causal precompensator, which, evidently, satisfies $\bar{\ell}_{(\bar{v},\bar{r})} = \bar{f}_{(\bar{v},\bar{r})}/\bar{f}$. Further, as we shall show, the internal stability of $\bar{f}_{(\bar{v},\bar{r})}$ implies that the equivalent precompensator $\bar{\ell}_{(\bar{v},\bar{r})}$ is input/output stable. Conversely, assume that $\bar{\ell}$ is a causal and input/output stable precompensator for which $\bar{f}\bar{\ell}$ is input/output stable. Then, we also show that (in the single-input single-output case) there exist causal systems $\bar{v}$ and $\bar{r}$ such that $\bar{f}\bar{\ell} = \bar{f}_{(\bar{v},\bar{r})}$ and $\bar{f}_{(\bar{v},\bar{r})}$ is internally stable. Thus, our problem reduces to the following equivalent

<u>Feedback representation problem</u>: Given a causal and input/output stable precompensator $\bar{\ell}$ for which $\bar{f}\bar{\ell}$ is input/output stable, find causal systems $\bar{v}$ and $\bar{r}$ such that $\bar{f}\bar{\ell} = \bar{f}_{(\bar{v},\bar{r})}$, where $\bar{f}_{(\bar{v},\bar{r})}$ is internally stable, and $\bar{v}$ is of minimal possible dynamical order.

The feedback representation problem forms the topic of our following section.

Most of the literature in the area of internally stable control is relatively recent. Regulation with internal stability was considered by WONHAM [1974], WONHAM and PEARSON [1974], DAVISON [1976], and CHENG and PEARSON [1978]. The so called "output regulation problem" was treated by WOLOVICH and FERREIRA [1979]. More recently, questions related to internal stability were considered by DESOER, LIU, MURRAY and SAES [1980], and HAMMER [1981b and c].

2. FEEDBACK REPRESENTATION OF PRECOMPENSATORS

We start with some preliminary considerations. Let K be a field. We denote by Ω^+K the set of all polynomials in z with coefficients in the field K, and by Ω^-K the set of all power series in z^{-1} with coefficients in K, that is, $\Omega^-K = \{\sum_{t=0}^{\infty} k_t z^{-t} \mid k_0, k_1, \ldots \in K\}$. Further, let $\sigma \subset \Omega^+K$ be a multiplicative set of polynomials (i. e. for every pair of elements $k_1, k_2 \in \sigma$, also $k_1k_2 \in \sigma$). We say that σ is a <u>stability set</u> if it satisfies (i) $0 \notin \sigma$, and (ii) σ contains a polynomial of degree one. We now choose a stability set σ, and leave it fixed throughout our discussion.

Next, we denote by Ω_σ^+K the set of all rational functions of z which can be expressed as a polynomial fraction β/γ, with γ in the stability set σ (i. e. Ω_σ^+K is the localization of the ring Ω^+K at σ). Then, given a single-input single-output transfer function $\bar{f}$, we say that $\bar{f}$ is <u>input/output stable</u> if $\bar{f} \in \Omega_\sigma^+K$. Finally, we denote $\Omega_\sigma^-K := \Omega_\sigma^+K \cap \Omega^-K$ (the ring Ω_σ^-K was first employed by MORSE [1975]). Then, a transfer function $\bar{f}$ is both causal and input/output stable if and only if $\bar{f} \in \Omega_\sigma^-K$.

Let $\bar{f}$ be a rational single-input single-output transfer function. A fraction representation $\bar{f} = pq^{-1}$, where both p and q belong to Ω_σ^+K (i. e. are input/output stable) is called a <u>stability representation</u> (in the sense of σ). In the next statement, we describe a particular type of stability representation, which is essentially a characterization of the unstable zeros of $\bar{f}$ (for proof, in the multivariable case, see HAMMER [1981a]).

(2.1) THEOREM. Let $\bar{f}$ be a rational single-input single-output transfer function. Then, there exists a stability representation $\bar{f} = \underset{\sim}{\varrho} d^{-1}$ with $\underset{\sim}{\varrho}$ a polynomial, satisfying the following: If $\bar{f} = pq^{-1}$ is any stability representation with p a polynomial, then $\underset{\sim}{\varrho}$ is a divisor of p.

The stability representation $\bar{f} = \underset{\sim}{\varrho} d^{-1}$ described in Theorem 2.1 is called a canonical zero representation of $\bar{f}$ (in the sense of σ). Further, assume that $\bar{f} \neq 0$, and let $\bar{f}^{-1} = pn^{-1}$, where p is a polynomial, be a canonical zero representation of $\bar{f}^{-1}$. Then, $\bar{f} = np^{-1}$ is called a canonical pole representation of $\bar{f}$ (for the multivariable case, see HAMMER [1981a]).

Next, we define several integers related to the system structure. Let $\bar{f}$ be a nonzero single-input single-output rational transfer function. The latency degree λ of $\bar{f}$ is $\lambda := (\text{ord } \bar{f}) - 1$ (HAMMER and HEYMANN [1979]). Also, let $\bar{f} = \underset{\sim}{\varrho} d^{-1}$ and $\bar{f} = np^{-1}$ be canonical zero and pole representations of $\bar{f}$, respectively. Then, the zero degree η of $\bar{f}$ is $\eta := \deg \underset{\sim}{\varrho}$, and the pole degree ρ of $\bar{f}$ is $\rho := \deg p$ (both in the sense of σ). Finally, the σ-latency degree ν of $\bar{f}$ is defined as $\nu := \lambda + \eta$ (HAMMER [1981c]). The role of these integers in our discussion will become clear as we proceed.

Our next objective is to define a certain notion of invertibility. Let $\bar{f}$ be a single-input single-output transfer function. We say that $\bar{f}$ is σ^--invertible if its inverse $\bar{f}^{-1}$ exists and is both causal and input/output stable. If $\bar{f}$ is not itself σ^--invertible, then it can be made to be so by composing it with another system. Indeed, this can be done in a "minimal" way, as follows. Let $\bar{f}$ and d_σ be nonzero transfer functions. We say that d_σ is a σ^--annihilator of $\bar{f}$ if it satisfies the following: (i) $\bar{f} d_\sigma^{-1}$ is σ^--invertible, (ii) d_σ is input/output stable, and (iii) if d is any nonzero and input/output stable transfer function for which $\bar{f} d^{-1}$ is σ^--invertible, then $d d_\sigma^{-1} \in \Omega_\sigma^- K$. (for an algebraic definition of σ^--annihilators in the multivariable case see HAMMER [1981c]). We next describe the structure of σ^--annihilators (for proof (in the multivariable case), see HAMMER [1981c]).

(2.2) LEMMA. Let $\bar{f}$ be a nonzero single-input single-output strictly causal and rational transfer function, of σ-latency degree ν. Also, let $\bar{f} = \underset{\sim}{\varrho} d^{-1}$ be a canonical zero representation of $\bar{f}$, and let $(z + \alpha)$ be a polynomial of degree one in σ. Then, (i) $\bar{f}$ possesses a σ^--annihilator d_σ, and (ii) d_σ can be chosen in the form $d_\sigma = (z + \alpha)^{-(\nu+1)} \underset{\sim}{\varrho}$.

Moreover, if d is any σ^--annihilator of $\bar{f}$, then the MacMillan degree μ of d satisfies $\mu \geq \nu + 1$.

We note that $\text{ord } d_\sigma = \text{ord } \bar{f}$.

Next, we turn to a consideration of explicit conditions for the internal stability of the configuration (1.1). We tacitly assume that each one of the systems $\bar{f}$, $\bar{v}$ and $\bar{r}$ is completely described by its canonical realization (i. e. has no hidden modes). As a preliminary to our examination of (1.1), we need the following notion, which is related to series composition of systems. Let $\bar{f}$ and $\bar{v}$ be rational transfer functions with pole degrees $\rho_{\bar{f}}$ and $\rho_{\bar{v}}$, respectively. Then, as can be readily seen, the pole degree $\rho_{\bar{f}\bar{v}}$ of $\bar{f}\bar{v}$ always satisfies $\rho_{\bar{f}\bar{v}} \leq \rho_{\bar{f}} + \rho_{\bar{v}}$. In particular, if $\rho_{\bar{f}\bar{v}} = \rho_{\bar{f}} + \rho_{\bar{v}}$, then we say that the combination $\bar{f}\bar{v}$ is σ-detectable (HAMMER [1981c]). More qualitatively stated, $\bar{f}\bar{v}$ is σ-detectable if and only if all its unstable modes are both reachable and observable.

Finally, we recall that $\bar{f}_{(\bar{v},\bar{r})}$ denotes the configuration (1.1), and also denote

$$\bar{\ell}_{\bar{r}} := [I + \bar{r}\bar{f}\bar{v}]^{-1}.$$

We can now state the following conditions for internal stability of (1.1) (for proof, see HAMMER [1981c]).

(2.3) THEOREM. Let $\bar{f}$, $\bar{v}$ and $\bar{r}$ be rational transfer functions, where $\bar{f}$ is strictly causal, and $\bar{v}$ and $\bar{r}$ are causal. Then, $\bar{f}_{(\bar{v},\bar{r})}$ is internally stable if and only if the following hold: (i) $\bar{f}\bar{v}$ is σ-detectable, and (ii) all of

$\bar{f}_{(\bar{v},\bar{r})}$, $\bar{\ell}_{\bar{r}}$, $\bar{f}_{(\bar{v},\bar{r})}\bar{r}$ and $\bar{\ell}_{\bar{r}}\bar{r}$ are input/output stable.

Noting that $\bar{\ell}_{(\bar{v},\bar{r})} = \bar{v}\bar{\ell}_{\bar{r}}$, the following can be proved as a consequence of Theorem 2.3

(2.4) COROLLARY. *If $\bar{f}_{(\bar{v},\bar{r})}$ is internally stable, then $\bar{\ell}_{(\bar{v},\bar{r})}$ is input/output stable.*

Next, we need certain truncation operators. Let $\bar{f}$ be a rational transfer function, and let $\bar{f} = p'(z) + c'$ be a decomposition of $\bar{f}$, where $p'(z)$ is a polynomial and c' is causal. Also, let $(z + \alpha)$, where $\alpha \in K$, be a polynomial of degree one in σ. Then, evidently, there exists an element $a \in K$ such that $p'(\alpha) + a = 0$. Defining $p := p'(z) + a$ and $c := c' - a$, we have that p is a polynomial divisible by $(z + \alpha)$, c is still causal, and $\bar{f} = p + c$. Moreover, p and c are uniquely determined. We define now the truncation operator

$$L_{\alpha}^{+}(\bar{f}) := p,$$

so that $L_{\alpha}^{+}(\bar{f})$ is a polynomial divisible by $(z + \alpha)$, and $\bar{f} - L_{\sigma}^{+}(\bar{f})$ is causal. By a similar argument applied to a partial fraction decomposition of $\bar{f}$, one can prove the following

(2.5) LEMMA. *Let $\bar{f}$ be a rational single-input single-output transfer function, and let $\bar{f} = np^{-1}$ be a canonical pole representation. Also, let $(z + \alpha)$ be a polynomial of degree one in σ. There exists a decomposition $\bar{f} = \bar{f}_{+}^{\sigma} + \bar{f}_{-}^{\sigma}$ satisfying the following: (i) $\bar{f}_{-}^{\sigma}$ is both causal and input/output stable, and (ii) $\bar{f}_{+}^{\sigma}$ has a coprime polynomial fraction representation $\bar{f}_{+}^{\sigma} = n'p^{-1}$, where the polynomial n' is divisible by $(z + \alpha)$, (and p is from $\bar{f} = np^{-1}$).*

Given a rational transfer function $\bar{f}$, we shall denote

$$L_{\alpha}^{\sigma}(\bar{f}) := \bar{f}_{+}^{\sigma},$$

where $\bar{f}_{+}^{\sigma}$ is described in Lemma 2.5.

We now turn to the main topic of our discussion - the feedback representation problem. Let $\bar{f}$ be a nonzero strictly causal rational transfer function, and let $\bar{\ell}$ be a nonzero causal and input/output stable precompensator, for which $\bar{f}\bar{\ell}$ is input/output stable. We need to find causal functions $\bar{v}$ and $\bar{r}$ such that $\bar{f}\bar{\ell} = \bar{f}_{(\bar{v},\bar{r})}$, and $\bar{f}_{(\bar{v},\bar{r})}$ is internally stable. We note that according to (1.1), $\bar{v}$ and $\bar{r}$ satisfy

$$\bar{\ell}^{-1} = \bar{v}^{-1} + \bar{r}\bar{f},$$

so that we have to decompose $\bar{\ell}^{-1}$ into a suitable sum of functions. Now, let ν be the σ-latency degree of $\bar{f}$, and let $\bar{f} = \varrho d^{-1}$ and $\bar{f} = np^{-1}$ be canonical zero and pole representations, respectively. (It is important to note that ϱ and p are coprime polynomials.) Also, let $(z + \alpha)$ be a polynomial of degree one in σ. Then, according to Lemma 2.2, $\bar{f}$ has a σ^{-}-annihilator d_{σ} of the form $d_{\sigma} = (z + \alpha)^{-(\nu+1)}\varrho$. Further, letting $\bar{\ell} = \varrho' q^{-1}$ be a canonical zero representation, it follows by the input/output stability of $\bar{f}\bar{\ell}$ that p is a divisor of ϱ'.

Consider now the transfer function $\bar{\ell}^{-1}d_{\sigma}^{-1}$, and let $\bar{\ell}^{-1}d_{\sigma}^{-1} = ab^{-1}$ be a coprime polynomial fraction representation. Then, since both $\bar{\ell}$ and d_{σ} are input/output stable, it follows that both ϱ and ϱ' are divisors of the polynomial b. Consequently, since p is a divisor of ϱ', and since ϱ and p are coprime, there exists a polynomial factorization $b = b_1b_2$, where b_1 and b_2 are coprime polynomials, ϱ divides b_1, and p divides b_2. Hence, by the coprimeness of b_1 and b_2, there exists a partial fraction decomposition

$$\bar{\ell}^{-1}d_{\sigma}^{-1} = a_1/b_1 + a_2/b_2,$$

where a_1 and a_2 are polynomials. Also, since a and b are coprime, it follows that a_1 and b_1 as well as a_2 and b_2 are coprime. Thus, since ϱ divides b_1 and p divides b_2, we obtain that ϱ and a_1 as well as p and a_2 are coprime, and so are also p and b_1. We denote

$$\bar{g} := L_{\alpha}^{\sigma}(a_1/b_1) + L_{\alpha}^{+}(a_2/b_2),$$

$$\bar{h} := \bar{\ell}^{-1}d_{\sigma}^{-1} - \bar{g},$$

and note that $\bar{h}$ is causal.

Now, by the causality of the nonzero $\bar{\ell}$, ord $\bar{\ell}^{-1} \leq 0$, and, by the strict causality of the nonzero $\bar{f}$, ord $d_\sigma^{-1}(= -\text{ ord } \bar{f}) \leq -1$. Hence, ord $(\bar{\ell}^{-1}d_\sigma^{-1}) \leq -1$, and thus, since ord $h \geq 0$, we have that necessarily $\bar{g} \neq 0$, so that $\bar{g}^{-1}$ exists. We define now

$$
(2.6) \qquad \begin{aligned} \bar{v} &:= (\bar{g}d_\sigma)^{-1} \\ \bar{r} &:= \bar{h}(\bar{f}d_\sigma^{-1})^{-1} \end{aligned}
$$

Then, it can be readily seen that both of $\bar{v}$ and $\bar{r}$ are causal. Moreover, the following holds.

(2.7) THEOREM. <u>Let $\bar{f}$ be a nonzero strictly causal rational transfer function, and let $\bar{\ell}$ be a causal and input/output stable transfer function, both of which are single-input single-output. Assume that $\bar{f}\bar{\ell}$ is input/output stable, and let ν' be the σ-latency degree of $\bar{f}\bar{\ell}$. Then, for $\bar{v}$ and $\bar{r}$ given in</u> (2.6), <u>the following is true:</u> (i) $\bar{f}\bar{\ell} = \bar{f}_{(\bar{v},\bar{r})}$, (ii) $\bar{f}_{(\bar{v},\bar{r})}$ <u>is internally stable, and</u> (iii) <u>the MacMillan degree μ of $\bar{v}$ satisfies $\mu \leq \nu'$.</u>

The proof of Theorem 2.7 is by a direct verification, employing the conditions for internal stability of Theorem 2.3. A detailed proof (in the multivariable case) is given in HAMMER [1981c], where we also prove the next observation. The bound on the MacMillan degree of $\bar{v}$, as given in Theorem 2.7 (iii), is tight in the sense that there exists a causal and input/output stable precompensator $\bar{\ell}$, for which $\bar{f}\bar{\ell}$ is input/output stable, and which satisfies the following: If $\bar{v}$ and $\bar{r}$ are any causal transfer functions such that $\bar{f}\bar{\ell} = \bar{f}_{(\bar{v},\bar{r})}$ and $\bar{f}_{(\bar{v},\bar{r})}$ is internally stable, then the MacMillan degree of $\bar{v}$ is not less than the σ-latency degree of $\bar{f}\bar{\ell}$. Some sharper results are also given in <u>op</u>. <u>cit</u>.

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