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ON DISTURBANCES AND STATE FEEDBACK IN NONLINEAR CONTROL

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ABSTRACT

The problem of controlling a nonlinear system in the presence of disturbances and modeling inaccuracies is considered. The objective is to design a static state feedback controller that controls the system so that the combined effect of all disturbances and inaccuracies on the response of the closed loop system is below a specified bound.

This note is an extended summary of HAMMER [1997].

1. INTRODUCTION

A common difficulty in the utilization of digital controllers for nonlinear systems is the extensive computational burden imposed by the controller. Even with modern computer systems, the implementation of controllers for nonlinear systems of moderate to high dimensionality is a daunting task.

An important parameter in the design of a discrete controller for a system with continuously valued signals is the size of the discretization step used in the analog-to-digital conversion process. For a fixed signal amplitude, a larger discretization step leads to fewer points in the discrete space over which the discrete controller operates, thus lowering the computational requirements of the controller. The present paper concentrates on the problem of finding the largest discretization step that is compatible with specified performance requirements. At the same time, it also describes the design of a controller appropriate for this discretization step.

The system being controlled is a nonlinear discrete-time system Σ, represented by a recursion of the form

(1.1) $$x_{k+1} = f(x_k,u_k) \qquad y_k = x_k,\ k = 0, 1, 2, \ldots$$

Here, the initial condition is x_0. The variable x_k is an n-dimensional real vector, usually called the "state" of Σ at the step k; The input value of Σ at the step k is given by u_k, an m-dimensional real vector. The function f is called the *recursion function* of Σ, and is required to be a continuous function. We concentrate here on the control of systems whose state is provided as output, so the output value y_k at the step k is equal to the state of the system at that step.

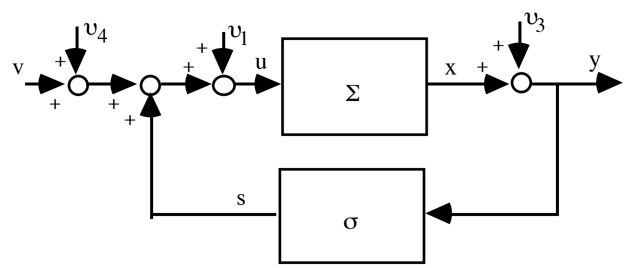
The system Σ is subject to a variety of disturbances and modeling inaccuracies. These consist of an input disturbance signal υ_1, an output disturbance signal υ_3, and a modeling error υ_2. When these disturbances and errors are taken into account, the recursive representation of the system becomes

(1.2) $$x_{k+1} = f(x_k,u_k+\upsilon_{1k}) + \upsilon_{2k} \qquad y_k = x_k+\upsilon_{3k},\ k = 0, 1, 2, \ldots$$

Here, υ_{ik} is the value of the disturbance signal υ_i at the step k. The only apriori information available about the disturbances υ_1, υ_2, and υ_3 is a bound on their largest amplitude. Thus, we assume that there is a specified real number $d_0 > 0$ such that the amplitudes of the disturbances υ_1, υ_2, and υ_3 do not exceed d_0. No other assumptions are made about the nature of the disturbances.

The class of controllers considered in the present paper is the class of static state feedback controllers, as represented by the following diagram.

(1.3)



Here, σ is a static state feedback function, v is an external reference input, and υ_4 is an input disturbance. The amplitude of the disturbance υ_4 is also bounded by $d_0 > 0$, as the other disturbance amplitudes are. The closed loop system described by the diagram is denoted by Σ_σ. The disturbance signals υ_1, υ_2, υ_3, and υ_4 are not assumed to be infinitesimal.

Generally speaking, the disturbances υ_1, υ_2, υ_3, and υ_4 may originate from a variety of different sources. When the feedback function σ is implemented on a digital computer, one important disturbance source is the discretization noise, caused by the analog-to-digital and digital-to-analog conversion processes. The maximal amplitude of this disturbance source is the size d_d of the discretization step. When this disturbance is added to the existing disturbances in the system, one obtains the bound

(1.4) $$d := d_0 + d_d$$

on the amplitudes of the combined disturbances.

The objective is to find the largest possible value of d that is compatible with given design specifications. Once the maximal value of d has been found, one can obtain the size of the largest permissible discretization step.

The specific design problem considered here is the calculation of a state feedback function σ that drives the system Σ along a prescribed nominal path, which we take as the zero path.

When active, the disturbances $\upsilon_1, \upsilon_2, \upsilon_3$, and υ_4 may cause the closed loop system Σ_σ to deviate from its specified nominal path. The largest magnitude of this deviation determines the performance accuracy of the closed loop system. The design requirement is that the deviation not exceed a prescribed bound $\Delta > 0$. Our aim is to find the largest disturbance amplitude bound d for which there is a feedback function σ that fulfills this requirement.

The bound d on the maximal permissible disturbance amplitude is characterized in Sections 3 and 4. In principle, the calculation of the bound d involves the solution of a set of algebraic inequalities that are derived from the given recursion function f of the system Σ. The construction of state feedback functions σ that permit disturbances of amplitudes up to the value d is also described. The calculation of permissible disturbance amplitudes, as well as the derivation of appropriate state feedback functions σ, is simplest when the system Σ satisfies certain reachability requirements.

As mentioned earlier, our main interest here is in the derivation of state feedback functions for digital computer implementation, as other implementations of nonlinear controllers are usually impractical. In such an implementation, the feedback function σ operates over a discrete grid, where the interval size of the grid is given by the discretization interval d_d. Thus, there is no need to require the function σ to be a continuous function, and we can permit σ to possess jumps.

The present note is an extended summary of HAMMER [1997], and it builds on results presented in HAMMER [1989b] and [1991]. Alternative studies on the global control of nonlinear systems are given in HAMMER [1984a and b, 1985, 1989a, 1994], DESOER and KABULI [1988], VERMA [1988], SONTAG [1989], CHEN and de FIGUEIREDO [1990], PAICE and MOORE [1990], VERMA and HUNT [1993], PAICE and van der SCHAFT [1994], BARAMOV and KIMURA [1995], the references cited in these papers, and others.

2. PRELIMINARIES

Let R^m be the set of all m-dimensional real vectors, where m is a positive integer. Denote by $S(R^m)$ the set of all sequences $u_0, u_1, u_2, \ldots$ of real vectors $u_i \in R^m$, $i = 0, 1, 2, \ldots$. It is convenient to use the letter u to denote the sequence $u_0, u_1, u_2, \ldots$. Then, u_i is the i-th element of the sequence u.

Our discussion relates to systems Σ that are given in terms of a state representation of the form (1.1). Here, $u = u_0, u_1, u_2, \ldots \in S(R^m)$ is the input sequence of Σ, and $x = x_0, x_1, x_2, \ldots \in S(R^n)$ is the sequence of states through which the system Σ passes. The initial condition of the system is then x_0. The recursion function $f : R^n \times R^m \to R^n$ of Σ is a continuous function. It will be convenient to denote by $\Sigma(x_0)$ the response of the system Σ from the initial condition x_0.

In order to describe the magnitude of disturbances or of their effects, we shall use the standard ℓ^∞-norm, which is defined as follows. Given a vector $v \in R^m$ with the components $(v^1, \ldots, v^m)$, denote by

$$|v| := \max_{i = 1, 2, \ldots, m} |v^i|$$

the maximal absolute value of a component. For a sequence $u = (u_0, u_1, \ldots) \in S(R^m)$, set

$$|u| := \sup_{i \geq 0} |u_i|,$$

the standard ℓ^∞-norm of the sequence u.

According to (1.3) and (1.2), there are four disturbance signals that affect our configuration, namely, the signals $\upsilon_1, \upsilon_2, \upsilon_3$, and υ_4. Of these, two disturbance signals (υ_2 and υ_3) affect the state value, and two (υ_1 and υ_4) affect the input value. It will be convenient to combine υ_1 and υ_4 into one total disturbance that acts on the system input, by defining

$$\upsilon_u := \upsilon_1 + \upsilon_4.$$

We then impose an amplitude bound on the total input disturbance by requiring $|\upsilon_u| \leq \delta$.

The disturbance signals υ_2 and υ_3 of (1.2) affect the state value of the system Σ. As it turns out, the disturbance υ_2 representing the modeling errors, has a double effect; intuitively speaking, this occurs since υ_2 affects both the value of the recursion (1.2) and the value of the state feedback function σ. In this sense, the total disturbance amplitude relating to the state x is in fact equivalent to $2|\upsilon_2| + |\upsilon_3|$. Allowing each of these two disturbance sources the same amplitude, and requiring the combined effect not to exceed δ, we restrict $|\upsilon_2| \leq \delta/3$ and $|\upsilon_3| \leq \delta/3$, so that $2|\upsilon_2| + |\upsilon_3| \leq \delta$. We can then summarize the combined disturbance restrictions in the form

$$(2.1) \quad \begin{aligned} &|\upsilon_1 + \upsilon_4| \leq \delta \\ &|\upsilon_2 + \upsilon_3| \leq 2\delta/3 \end{aligned}$$

Our objective is to find the largest value δ_m of the real number $\delta > 0$ that is compatible with the design specifications. The largest uniform bound on the amplitudes of the individual disturbances is then $d = \delta_m/3$ (although for the input disturbances υ_1 and υ_4, amplitudes up to $\delta_m/2$ are allowed). For the sake of simplifying the terminology, the term "disturbance amplitude" will refer in the sequel to the number δ, rather than to the individual amplitudes of the disturbances $\upsilon_1, \upsilon_2, \upsilon_3$, or υ_4.

The term "nominal" is used below to refer to the response of the configuration (1.3) when all disturbances are set to zero, and the external input sequence v is the zero sequence. We denote by $\Sigma_\sigma(x_0)$ the nominal response of the closed loop system when started at the initial condition $x_0 \in R^n$. When disturbances $\upsilon_1, \upsilon_2, \upsilon_3$ or υ_4 are present, it is sometimes convenient to denote the response of the closed loop system by $\Sigma_\sigma(x_0)*(\upsilon_1,\upsilon_2,\upsilon_3,\upsilon_4)$; the initial condition here is still x_0, and the external input sequence is still $v = 0$.

3. STATE FEEDBACK

We turn now to a more detailed examination of the control configuration (1.3). Here, Σ is the system that needs to be controlled. Its nominal model is given by (1.1), while its model with the disturbances active is represented by (1.2). The feedback is created by the feedback function $\sigma : R^n \to R^m : s_k = \sigma(y_k)$, k = 0, 1, 2, ... With no disturbances present, we have $y_k = x_k$, k = 0, 1, 2, ..., and the input sequence u of Σ satisfies $u_k = v_k + s_k$, so that $x_{k+1} = f(x_k,v_k+s_k)$. Thus, without disturbances, the closed loop system Σ_σ is described by the recursion

$$x_{k+1} = f(x_k, (\sigma(x_k)+v_k)), \; k = 0, 1, 2, ...$$

With all disturbances active, the recursive representation of the closed loop system Σ_σ takes the form

(3.0.1) $x_{k+1} = f(x_k, (\sigma(x_k+\upsilon_{3k})+v_k+\upsilon_{1k}+\upsilon_{4k})) + \upsilon_{2k}$,

k = 0, 1, 2, ... Recall that our objective is to find a feedback function σ that drives the system Σ so that the output of the closed loop system Σ_σ does not deviate by more than $\Delta > 0$ from the zero output sequence. This has to be achieved over a range of initial conditions and disturbance signals, while the nominal external reference sequence v is set to zero. In specific terms, the problem can be stated as follows.

Design Problem.

Given a pair of real numbers $\rho, \Delta > 0$, find the largest real number $\delta > 0$ (if one exists) for which there is a feedback function $\sigma : R^n \to R^m$ satisfying

(3.0.2)

$$|y_{k+1}| = |f(x_k, (\sigma(x_k+\upsilon_{3k})+\upsilon_{1k}+\upsilon_{4k})) + \upsilon_{2k} + \upsilon_{3k+1}| \le \Delta,$$

k = 0, 1, 2, ..., for all disturbance signals satisfying $|\upsilon_1| \le \delta/2$, $|\upsilon_4| \le \delta/2$, $|\upsilon_2| \le \delta/3$, and $|\upsilon_3| \le \delta/3$, and for all initial conditions x_0 with $|x_0| \le \rho$. For this value of δ, find an appropriate state feedback function σ. ♦

The number ρ represents here a permissible magnitude range for the initial condition x_0. To comply with the requirement $|y_0| \le \Delta$, while permitting a total discrepancy of magnitude δ between the initial condition x_0 and the initial output value y_0, we restrict $\rho \le \Delta - \delta$.

The main results of the present paper are the derivation of necessary and sufficient conditions for the existence of appropriate state feedback functions σ; the development of computational techniques for the calculation of appropriate state feedback functions σ; and the characterization of the maximal permissible disturbance amplitude bound δ.

3.1 Static state feedback and eigensets.

Given a real number $\delta > 0$ and a vector $x \in R^q$, denote by $B(\delta,x)$ the ball of radius δ in R^q that is centered at the point x, namely,

$$B(\delta,x) := \{z \in R^q : |z - x| \le \delta\}.$$

Note that since we are using the ℓ^∞-norm, a ball is in fact a rectangular cube.

Next, for a subset $S \subset R^q$, denote by $N_\delta(S)$ the "δ-neighborhood" of S in R^q, i.e., the set

$$N_\delta(S) := \bigcup_{x \in S} B(\delta,x).$$

The radius $|S|$ of a subset $S \subset R^q$ is the radius of the smallest ball around the origin that contains S, and is given by

$$|S| := \sup_{x \in S} |x|.$$

Given a pair of subsets $S_x \subset R^n$ and $S_u \subset R^m$, and a function $f : R^n \times R^m \to R^n$, we denote by $f[S_x,S_u]$ the image of the cross product set $S_x \times S_u$ through f, namely,

$$f[S_x,S_u] = \{f(x,u) : x \in S_x, u \in S_u\} \subset R^n$$

Finally, denote by $\Pi_x : R^n \times R^m \to R^n : (x,u) \mapsto x$ the standard projection onto the first n coordinates.

We can now define the basic concept on which our discussion is based. This concept is a slight variant of the concept of an eigenset introduced in HAMMER [1989b].

(3.1.1) DEFINITION. Let $f : R^n \times R^m \to R^n$ be a function, and let $\delta, \Delta > 0$ be a pair of real numbers, where $\delta \le \Delta$. A non-empty subset $S \subset R^n \times R^m$ is a (δ,Δ)-*eigenset* of the function f if it satisfies the following conditions:

(i) $|\Pi_x S| \le \Delta - \delta$, and

(ii) $f[N_\delta(S)] \subset \Pi_x S$. ♦

Property (ii) of the Definition means that S is a conditional invariant subset of the function f. Furthermore, deviations of magnitude not exceeding δ in the state x, as well as in the input value u, do not destroy this conditional invariance property. The notion of an eigenset is a refinement of the classical concepts of invariant subset and conditional invariant subset, which have played important roles in the evolution of nonlinear as well as linear system theory (e.g., LASALLE and LEFSCHETZ [1961], LEFSCHETZ [1965], WONHAM [1974]). Before discussing the calculation of (δ,Δ)-eigensets, we discuss their significance to the control problem at hand.

Let $S \subset R^n \times R^m$ be a (δ,Δ)-eigenset of the recursion function $f : R^n \times R^m \to R^n$. With each state $x \in N_\delta(\Pi_x S)$, we associate a set of input values $U(x,S) \subset R^m$ for which the next state of the system is in $\Pi_x S$, as follows.

(i) For a state $x \in \Pi_x S$, the set $U(x,S)$ consists of all vectors $u \in R^m$ for which $(x,u) \in S$.

(ii) For a state $x \notin \Pi_x S$, we distinguish among two cases:

(a) When $x \in N_{2\delta/3}(\Pi_x S)$, then $U(x,S)$ consists of all vectors $u \in R^m$ for which $(y,u) \in S$ for some $y \in R^n$ satisfying $|y - x| \le 2\delta/3$.

(b) When $x \notin N_{2\delta/3}(\Pi_x S)$, then $U(x,S)$ consists of all vectors $u \in R^m$ for which $(y,u) \in S$ for some vector $y \in R^n$ satisfying $|y - x| \le \delta$.

To see the significance of the set $U(x,S)$ to our discussion, consider a system $\Sigma : S(R^m) \to S(R^n)$ having the recursive representation $x_{k+1} = f(x_k,u_k)$. Assume f has a (δ,Δ)-eigenset S, and that the system is started from an initial condition $x_0 \in \Pi_x S$. We can construct an input sequence $u = (u_0, u_1, ...) \in S(R^m)$ for Σ that drives Σ so that all the states along the resulting path in state space belong to $\Pi_x S$. Indeed, using the fact that $x_0 \in \Pi_x S$, we construct the input sequence u recursively as follows: Whenever $x_i \in \Pi_x S$, take an input value

(3.1.2) $u_i \in U(x_i,S)$.

This yields $x_{i+1} = f(x_i,u_i) \in \Pi_x S$. Since $x_0 \in \Pi_x S$, it follows that $x_i \in \Pi_x S$ for all integers $i \geq 0$, and the required input sequence u is obtained. Note that since $|\Pi_x S| \leq \Delta - \delta$, the sequence x of states also satisfies $|x + \upsilon| \leq \Delta$ for any disturbance υ of amplitude not exceeding δ.

The input sequence u constructed according to (3.1.2) has the following critical property: At each step $i \geq 0$, the input value u_i is assigned based on the value x_i of the state at that step. In other words, the input value is assigned through a state feedback mechanism.

The assignment (3.1.2) yields a general methodology for the design of static state feedback controllers for nonlinear control systems, as follows.

(3.1.3) THEOREM. Let Σ be a system with the recursion function $f : R^n \times R^m \to R^n$, and assume f has a (δ,Δ)-eigenset S for some real numbers $\delta, \Delta > 0$. Define a state feedback function $\sigma : R^n \to R^m$ as follows:

(i) For a state $x \in N_\delta(\Pi_x S)$, set $\sigma(x) := u$, where u is any element of the set $U(x,S)$;

(ii) For all other states $x \in R^n$, set $\sigma(x) := 0$.

Then, for any initial condition $x_0 \in \Pi_x S$ and for any disturbances satisfying $|\upsilon_1| \leq \delta/2$, $|\upsilon_4| \leq \delta/2$, $|\upsilon_2| \leq \delta/3$, and $|\upsilon_3| \leq \delta/3$, the closed loop system satisfies $|\Sigma_\sigma(x_0)*(\upsilon_1,\upsilon_2,\upsilon_3,\upsilon_4)| \leq \Delta$. ♦

Thus we see that a (δ,Δ)-eigenset of the recursion function f gives rise to state feedback functions σ that satisfy our design requirements. Note that for this to be valid, the system Σ must start from an initial condition $x_0 \in \Pi_x S$. It is usually desirable in applications to have a range of permissible initial conditions, namely, that the set of permissible initial conditions contain a ball $B(\rho,0)$ of some radius $\rho > 0$. This requirement has been incorporated into the statement of Design Problem (3.0.2). From our present discussion, it leads to the condition $B(\rho,0) \subset \Pi_x S$.

4. REACHABILITY AND EIGENSETS

In the present section we show that the notion of reachability is instrumental for the calculation of eigensets in the general nonlinear case. First, of course, we have to clarify what is meant by the term "reachability" in the nonlinear case, as numerous definitions have been used in the literature. The definition employed here is a direct adaptation of the linear notion of reachability.

4.1 Local and global reachability.

Let Σ be a system described by the recursive representation

(4.1.1) $x_{k+1} = f(x_k,u_k)$, $k = 0, 1, 2, ...$,

where $x_k \in R^n$ and $u_k \in R^m$, and consider the behavior of Σ for the first i steps, where $i \geq 1$ is an integer. Assume the system is started from the initial condition $x_0 = x \in R^n$, and is driven by the input list $u_0, u_1, ..., u_{i-1}$. The states x_k, $k = 1, ..., i$, through which the system passes can be calculated recursively; we have $x_1 = f(x,u_0)$, $x_2 = f(x_1,u_1) = f(f(x,u_0),u_1)$, ..., and, in general,

$$x_i = f(f...f(f(x, u_0), u_1) ...., u_{i-1}),$$

where the recursion function f is iterated i times. It is convenient to use the following shorthand notation for this iteration

$$f^i(x, u_0, ..., u_{i-1}) := f(f...f(f(x, u_0), u_1) ...., u_{i-1}),$$

so that $x_i = f^i(x, u_0, ..., u_{i-1})$.

In some instances we shall be interested in properties of the function f^i for a fixed value of the initial state x. In such cases, we shall consider the partial function $f^i(x,\cdot) : (R^m)^i \to R^n : (u_0, ..., u_{i-1}) \mapsto f^i(x, u_0, ..., u_{i-1})$.

We say that a state $x' \in R^n$ is *reachable* from the state $x \in R^n$ in i steps if there is an input list $u_0, ..., u_{i-1}$ for which $f^i(x, u_0, ..., u_{i-1}) = x'$.

Recall that a function $g : R^q \to R^n$ is an *open function* if it maps every open subset of R^q onto an open subset of R^n.

(4.1.2) DEFINITION. The realization (4.1.1) is *everywhere locally reachable* if there is an integer $p \geq 1$ for which the function $f^p(x,\cdot)$ is an open function for all states $x \in R^n$. ♦

Local reachability can be verified by checking the rank of the Jacobian of the recursion function f with respect to the input variables; it reduces to the standard definition of reachability in the linear case (see HAMMER [1997]).

Consider a realization (4.1.1) that is everywhere locally reachable. Then, by definition, there is an integer p for which the partial function $f^p(x,\cdot)$ is an open function for all $x \in R^n$. We denote by η the smallest possible value of the positive integer p, and we call η the *reachability integer* of the recursion function f.

Assume then that the realization (4.1.1) is everywhere locally reachable, with the reachability integer η. We say that this realization is *globally reachable* if every state $x' \in R^n$ is reachable from every state $x \in R^n$ in η steps.

4.2 Reachability, disturbances, and eigensets.

Let Σ be a system having the nominal recursive representation $x_{k+1} = f(x_k,u_k)$, $k = 0, 1, 2, ...$. Throughout the ensuing discussion we assume that the recursion function f of Σ is a continuous function, that Σ is everywhere locally reachable as well as globally reachable, and we let η be the reachability integer. The objective of the present subsection is to present an effective computational technique for the calculation of eigensets of the recursion function f of Σ. As seen in Theorem 3.1.3, these eigensets can be used to construct state feedback functions that solve the design problem (3.0.2).

The fact that the system Σ is globally reachable implies that its state values can be assigned arbitrarily at steps that are integer multiples of the reachability integer η. In other words, for any sequence of vectors $\xi_0, \xi_1, \xi_2, ... \in R^n$, there is an input sequence u that drives Σ in such a way that the resulting trajectory x satisfies $x_0 = \xi_0, x_\eta = \xi_1, x_{2\eta} = \xi_2, x_{3\eta} = \xi_3, ...$, or $x_{i\eta} = \xi_i$ for all integers $i \geq 0$. Indeed, global reachability implies that for every integer $i = 0, 1, 2, ...$, there is an input list $u_0(i), ..., u_{\eta-1}(i) \in R^m$ such that

(4.2.1) $\xi_{i+1} = f^\eta(\xi_i, u_0(i), ..., u_{\eta-1}(i))$.

The concatenated input sequence

(4.2.2) $u = u_0(0), ..., u_{\eta-1}(0), u_0(1), ..., u_{\eta-1}(1), u_0(2), ..., u_{\eta-1}(2), ...$

clearly achieves the desired result.

Note that although the states can be assigned arbitrarily at steps that are integer multiples of η, there is usually little choice when it comes to selecting the states through which the trajectory passes at steps that are not integer multiples of η. These states are restricted by system characteristics, and cannot be assigned arbitrarily. In order to satisfy the design problem (3.0.2), one has to guaranty, among others, that the amplitudes of these states do not exceed Δ. Whether or not this requirement can be satisfied depends on the value of Δ and on the characteristics of the recursion function f of Σ.

Consider now the problem of finding a state feedback function σ that satisfies (3.0.2) for some real numbers $\delta, \Delta > 0$, where $\delta \leq \Delta$. According to Theorem 3.1.3, the existence of such a feedback function is guaranteed when the initial condition of the controlled system Σ is within a (δ,Δ)-eigenset of the recursion function f of Σ.

Assume now that the system Σ is started from an initial conditions $x_0 \in B(\rho,0)$. When Σ is globally reachable with reachability integer η, and no disturbances are present, it follows by (4.2.1) and (4.2.2) that an input sequence u for Σ can be found for which the resulting nominal state sequence x satisfies

(4.2.3) $x_{k\eta} \in B(\rho,0)$ for all integers $k \geq 0$.

In other words, for this input sequence, the nominal trajectory re-enters the ball $B(\rho,0)$ at least once every η steps.

Of course, since the values $x_{k\eta}, k = 1, 2, ...$, can be assigned arbitrarily (by choosing an appropriate input sequence u), one could restrict these values even further, and force them to be within a ball of radius smaller than ρ. In the present discussion, however, we use (4.2.3) as our guiding requirement.

To summarize, we require the nominal trajectory x of the closed loop system (1.3) to satisfy

$$|x_k| \leq \Delta\text{-}\delta \text{ and } |x_{k\eta}| \leq \rho, k = 0, 1, 2, ...,$$

where the first inequality takes into account the fact that a disturbance of amplitude not exceeding δ may be added to the nominal trajectory at each step.

In addition to the practical significance discussed so far, condition (4.2.3) also has important mathematical implications. This condition allows us to calculate a (δ,Δ)-eigenset of the recursion function f by examining only the first η steps of the recursion $x_{k+1} = f(x_k,u_k)$. It then yields a finite technique for the calculation of (δ,Δ)-eigensets, as follows.

Consider a system Σ having the recursive representation $x_{k+1} = f(x_k,u_k)$. For an initial condition $x \in R^n$, an integer $i \geq 0$, a list of input values $u_0, ..., u_{i-1} \in R^m$, and a real number $\delta > 0$, we construct recursively a subset $f^i_\delta(x,u_0, ...,u_{i-1}) \subset R^n$ as follows.

$f^0_\delta := N_\delta(x)$; and

$$f^k_\delta(x,u_0, ...,u_{k-1}) := f\Big[N_\delta\Big(f^{k-1}_\delta(x,u_0, ...,u_{k-2})\Big),N_\delta\Big(u_{k-1}\Big)\Big]$$

$k = 1, ..., i$. In intuitive terms, the set $f^i_\delta(x,u_0, ...,u_{i-1})$ consists of all states the system can reach at the step i under the following conditions: The system is started from the nominal initial condition x and is driven by the nominal input list $u_0, ..., u_{i-1}$, while at each step the state value as well as the input value are disturbed by a disturbance of amplitude not exceeding δ.

Now, let $\Delta > 0$ be the specified bound on the disturbance effects, as in (3.0.2). Also, let $\delta, \rho > 0$ be two real numbers satisfying $\rho + \delta \leq \Delta$. For a state $x \in B(\rho,0)$, consider the set of all input lists $u_0(x), ..., u_{\eta-1}(x) \in R^m$ for which the following hold.

(4.2.4) $|f^i_\delta(x,u_0(x), ...,u_{i-1}(x))| \leq \Delta\text{-}\delta$ for all $i = 1, ..., \eta\text{-}1$;

and

(4.2.5) $|f^\eta_\delta(x,u_0(x), ...,u_{\eta-1}(x))| \leq \rho$.

Note that these relations simply represent a finite set of inequalities based on the recursion function f of Σ. Using the solution of these inequalities we can build a (δ,Δ)-eigenset of the recursion function f as follows.

First, for each state $x \in B(\rho,0)$, let $U(x,\rho,\delta,\Delta) \subset (R^m)^\eta$ be the set of all input lists $(u_0(x), ...,u_{\eta-1}(x)) \in (R^m)^\eta$ that satisfy (4.2.4) and (4.2.5). In other words, $U(x,\rho,\delta,\Delta)$ is the solution set of the inequalities (4.2.4) and (4.2.5). Assume that $U(x,\rho,\delta,\Delta) \neq \emptyset$ for all $x \in B(\rho,0)$; conditions on the recursion function f of Σ under which this assumption is valid are discussed later (see also HAMMER [1997]).

Next, for a list $(u_0(x), ...,u_{\eta-1}(x)) \in U(x,\rho,\delta,\Delta)$ and an integer $i \in 0, ..., \eta\text{-}1$, denote by

$$\pi_i(u_0(x), ...,u_{\eta-1}(x)) := u_i(x)$$

the projection onto the i-th element of the list. Build the following subsets of $R^n \times R^m$, consisting of state-input pairs.

$S_0 := \{(y,u) : y \in N_\delta(x), x \in B(\rho,0), u \in \pi_0 U(x,\rho,\delta,\Delta)\}$,

$S_i := \{(y,u) : y \in f^i_\delta(x,u_0(x), ...,u_{i-1}(x)), x \in B(\rho,0), u \in \pi_i U(x,\rho,\delta,\Delta)\}$,

$i = 1, ..., \eta\text{-}1$. Finally, combine these subsets into the set

(4.2.6) $S_\rho(\delta,\Delta) := \bigcup_{i=0, ...,\eta-1} S_i$, which is a subset of $R^n \times R^m$. A slight reflection shows then that (4.2.4) and (4.2.5) imply the following

(4.2.7) THEOREM. Let Σ be a system with the recursive representation $x_{k+1} = f(x_k,u_k)$, and assume there are real numbers $\Delta > 0$, $\delta > 0$, and $\rho > 0$, where $\rho+\delta \le \Delta$, for which the solution set $U(x,\rho,\delta,\Delta)$ of (4.2.4) and (4.2.5) is not empty for all $x \in B(\rho,0)$. Then, the set $S_\rho(\delta,\Delta)$ of (4.2.6) is a (δ,Δ)-eigenset of the recursion function f, and $B(\rho,0) \subset \Pi_x S_\rho(\delta,\Delta)$. ♦

In view of the fact that $S_\rho(\delta,\Delta)$ is obtained from the solution of the finite set of inequalities (4.2.4) and (4.2.5), we have a finite procedure for the calculation of (δ,Δ)-eigensets of the recursion function f. Using the procedure of Theorem 3.1.3, the eigenset $S_\rho(\delta,\Delta)$ allows us to build a state feedback function σ for which the closed loop system Σ_σ permits disturbances of amplitude not exceeding δ, and may be started from any initial condition of magnitude not exceeding ρ, all without violating the bound Δ on the output sequence.

Of course, when solving the inequalities (4.2.4) and (4.2.5), one has to obtain the largest possible value of δ for which a solution exists. For the largest value of δ, the closed loop system Σ_σ permits the largest disturbance amplitudes possible within the framework of the present section. In this way, we have obtained a computable solution for the design of a state feedback function σ. We provide now an example of the process.

(4.2.8) EXAMPLE. Consider the system $\Sigma : S(R) \to S(R^2)$ with the following realization

$$x_{k+1} = \begin{pmatrix} 2x_{1,k} + [(x_{1,k})^2 + 1]u_k \\ x_{1,k} \end{pmatrix}.$$

In this case,

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

and the recursion function is

$$f(x_1,x_2,u) = \begin{pmatrix} 2x_1 + [(x_1)^2 + 1]u \\ x_1 \end{pmatrix}.$$

Iterating the recursion function we obtain

$$f^2(x_1,x_2,u_0,u_1) =$$

$$\begin{pmatrix} 4x_1 + 2[(x_1)^2 + 1]u_0 + \{[2x_1 + [(x_1)^2 + 1]u_0]^2 + 1\}u_1 \\ 2x_1 + [(x_1)^2 + 1]u_0 \end{pmatrix}.$$

Take $\Delta = 1$ and $\rho = 1/2$. Then, we must have $\delta \le \Delta-\rho = 1/2$, and (4.2.4) and (4.2.5) in this case lead to the following: Find u_0, u_1 for which the inequalities

(a) $|f(x_1,x_2,v_0)| \le 1 - \delta$

(b) $|f^2(x_1,x_2,v_0,v_1)| \le 1/2$

are valid for all $|x| \le 1/2+\delta$ and for all v_0, v_1 satisfying $|v_0 - u_0| \le \delta$ and $|v_1 - u_1| \le \delta$. A direct calculation then shows (see HAMMER [1997]) that the largest value of δ is $\delta = 0.16$ in this case, and an appropriate state feedback function is given by

$$\sigma(x_1,x_2) = \frac{-2x_1}{(x_1)^2 + 1}. \text{ ♦}$$

Finally, it is shown in HAMMER [1997] that for a reachable system, there are always numbers $\rho, \delta, \Delta > 0$ for which the inequalities (4.2.4) and (4.2.5) have a non-empty solution set.

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