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ON CAUSALITY, INVERSES, AND FEEDBACK

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1. INTRODUCTION

In this short note we describe, in a simplified way, some features of an algebraic theory dealing with feedback representation of precompensators. Due to space limitations, we shall completely ignore the stability aspect (which is, of course, crucial in concrete situations), and shall also restrict the class of systems under our consideration. Nevertheless, some ingredients of the approach will be exposed, and the present note can be used as an introduction to the more general setup.

We start with a statement of the problem. Let $\overline{f}$ be the transfer matrix of a strictly causal linear time invariant system. Suppose that one is required to change $\overline{f}$ and transform it into a specified transfer matrix $\overline{f}'$, through the employment of certain compensators. Due to practical limitations, the employment of postcompensators is not allowed, and the possible compensation scheme is of the following form.

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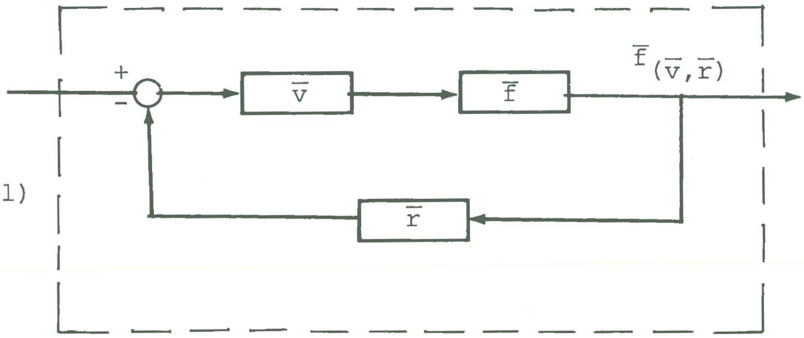
(1.1)



In the diagram, $\bar{v}$ is a causal precompensator, $\bar{r}$ is a causal dynamic output feedback, and $\bar{f}_{(\bar{v},\bar{r})}$ is the resulting system. Through a direct block diagram manipulation, one obtains

(1.2) $$\bar{f}_{(\bar{v},\bar{r})} = \bar{f}\bar{w}_{(\bar{v},\bar{r})},$$

where

(1.3) $$\bar{w}_{(\bar{v},\bar{r})} = \bar{v}[I + \bar{r}\bar{f}\bar{v}]^{-1}.$$

The system $w_{(\bar{v},\bar{r})}$ is an equivalent causal precompensator.

Our problem consists of finding causal transfer matrices $\bar{v}$ and $\bar{r}$ such that $\bar{f}' = \bar{f}_{(\bar{v},\bar{r})}$, whenever this is possible. Evidently, a representation of $\bar{f}'$ in the form (1.1) is possible if and only if there exists a causal precompensator $\bar{w}$ such that $\bar{f}' = \bar{f}\bar{w}$. A further essential requirement in our discussion is that the precompensator $\bar{v}$ be of minimal possible dynamical order, that is, as much as possible of the compensation dynamics should be located in the feedback system $\bar{r}$.

Assume now that there exists a causal precompensator $\bar{w}$ such that $\bar{f}' = \bar{f}\bar{w}$. (The existence of $\bar{w}$ can be verified through the latency conditions given in Hammer and Heymann [1].) In such case, the calculation of the class C of all causal precompensators $\bar{w}$ satisfying $\bar{f}' = \bar{f}\bar{w}$ is relatively simple, and can be done e.g., through the employment of a suitable generalized left inverse of $\bar{f}$. In particular, in case $\bar{f}$ is injective (one to one), C contains exactly one element. Henceforth we shall assume that $\bar{f}$ is injective. We can then assume that the precompensator $\bar{w}$ is given, and our problem can be equivalently restated as follows. <u>Given a causal precompensator</u> $\bar{w}$, <u>find a pair</u> $(\bar{v},\bar{r})$ <u>of transfer functions such that</u> $\bar{w} = \bar{v}[I + \bar{r}\bar{f}\bar{v}]^{-1}$, <u>and where</u> v <u>is of minimal possible dynamical order</u>.

In Hammer and Heymann [1] section 7, a solution to this problem was given in the particular case when the precompensator $\bar{w}$ is bicausal. It was shown that the solution is closely related to the theory of latency, introduced there. In the present note we extend this solution to the case where $\bar{w}$ is (causal, but) not necessarily bicausal.

2. REPRESENTATION OF PRECOMPENSATORS

This section is a direct continuation of section 7 in Hammer and Heymann [1], and we use the terminology established there. We recall that, for a ΛK-linear map $\bar{f}: \Lambda U \to \Lambda Y$, the latency module of $\bar{f}$ is Ker $\pi^{-}\bar{f}$. In case $\bar{f}$ is injective, Ker $\pi^{-}\bar{f}$ has an ordered proper basis $d_1,\ldots,d_m$, and the latency indices $\nu_1,\ldots,\nu_m$ of $\bar{f}$ are defined as $\nu_i := -\text{ord } d_i - 1$, $i = 1,\ldots,m$ (<u>op</u>. <u>cit</u>.). We start with the following:

<u>Definition 2.1</u>. *Let* $\bar{f}$: $\Lambda U \to \Lambda Y$ *be an injective* ΛK-*linear map, with latency indices* $\nu_1, \ldots, \nu_m$. *The latency degree* $\nu(\bar{f})$ *of* $\bar{f}$ *is* $\nu(\bar{f}) := \sum_{i=1}^{m} \nu_i$.

The explicit calculation of the latency indices is described in <u>op. cit</u>. In case of a nonsingular transfer matrix $\bar{v}$: $\Lambda U \to \Lambda U$, the latency degree is given by $\nu(\bar{v}) = \text{ord}\ (\det \bar{v}) - m$, where $m := \dim_K U$. Also, given an injective ΛK-linear map $\bar{f}$: $\Lambda U \to \Lambda Y$, the latency degree of the combination $\bar{f}\bar{v}$ is given by $\nu(\bar{f}\bar{v}) = \nu(\bar{v}) + \nu(\bar{f}) + m$.

The main result of the present note is the following:

<u>Theorem 2.2</u>. *Let* $\bar{f}$: $\Lambda U \to \Lambda Y$ *be an injective linear input/output map, and let* $\bar{w}$: $\Lambda U \to \Lambda U$ *be a nonsingular causal precompensator. Denote by* $\nu(\bar{f}')$ *the latency degree of the combination* $\bar{f}' := \bar{f}\bar{w}$. *Then, there exists a causal precompensator* $\bar{v}$: $\Lambda U \to \Lambda U$ *and a causal feedback* $\bar{r}$: $\Lambda Y \to \Lambda U$ *such that* $\bar{f}' = \bar{f}_{(\bar{v},\bar{r})}$ *and the MacMillan degree* $\mu(\bar{v})$ *of* $\bar{v}$ *satisfies* $\mu(\bar{v}) \leq \nu(\bar{f}')$.

Our proof depends on the following:

<u>Lemma 2.3</u>. *Let* $\bar{w}$: $\Lambda U \to \Lambda U$ *be a strictly causal nonsingular* ΛK-*linear map, and let* $\bar{g}$: $\Lambda U \to \Lambda U$ *be causal. Denote* $\bar{h} := \bar{w}^{-1} + \bar{g}$. *Then,* $\bar{h}$ *is nonsingular,* $\bar{h}^{-1}$ *is strictly causal, and the latency degrees satisfy* $\nu(\bar{h}^{-1}) = \nu(\bar{w})$.

<u>Proof</u>. Evidently, $\text{Ker}\ \pi^-\bar{w} = \bar{w}^{-1}[\Omega^- U]$. By Hammer and Heymann [1] Theorem 6.11, there exists a bicausal ΛK-linear map $\bar{\ell}$: $\Lambda U \to \Lambda U$ such that the columns of $\bar{w}^{-1}\bar{\ell}$ form an ordered proper basis of $\text{Ker}\ \pi^-\bar{w}$. Let $d_1, \ldots, d_m$ be the columns of the matrix $\bar{w}^{-1}\bar{\ell}$, and let

$\nu'_1, \ldots, \nu'_m$ be the latency indices of $\bar{w}$. Then, by definition, ord $d_i = -\nu'_i - 1$, $i = 1, \ldots, m$. Also, since $\bar{w}$ is strictly causal, we have $z\Omega^- U \subset \text{Ker } \pi^- \bar{w}$, so that, by op. cit. Proposition 6.9, ord $d_i \leq -1$ for all $i = 1, \ldots, m$. Now, let $g_1, \ldots, g_m$ be the columns of $\bar{g}\bar{\ell}$. Then, since $\bar{g}\bar{\ell}$ is still causal, we have, for all $i = 1, \ldots, m$, that ord $g_i \geq 0$. Hence, $(d_1 + g_1), \ldots, (d_m + g_m)$ have the same leading coefficients as $d_1, \ldots, d_m$, respectively. Thus, $(d_1 + g_1), \ldots, (d_m + g_m)$ are still properly independent, and ord $(d_i + g_i) =$ ord d_i for all $i = 1, \ldots, m$. Consequently, by op. cit. Lemma 4.2, $\bar{h}\bar{\ell}$ $(= \bar{w}^{-1}\bar{\ell} + \bar{g}\bar{\ell})$ is nonsingular. Moreover, since ord $(d_i + g_i) \leq -1$ for all $i = 1, \ldots, m$, it follows that Ker $\pi^-(\bar{h}\bar{\ell})^{-1}$ $(= \bar{h}\bar{\ell}[\Omega^- U]) \supset z\Omega^- U$, so that $(\bar{h}\bar{\ell})^{-1}$ is strictly causal, and since $\bar{\ell}$ is bicausal, $\bar{h}^{-1}$ is strictly causal as well. Also, since, as we showed, $(\bar{h}\bar{\ell})^{-1}$ and $\bar{w}$ have the same latency indices, the latency degrees satisfy $\nu[(\bar{h}\bar{\ell})^{-1}] = \nu(\bar{w})$. Finally, by op. cit. Corollary 6.22, $\nu[(\bar{h}\bar{\ell})^{-1}] = \nu(\bar{h}^{-1})$, so that $\nu(\bar{h}^{-1}) = \nu(\bar{w})$, and our proof is complete. □

Proof (of Theorem 2.2). We need to construct a representation $\bar{w} = \bar{v}[I + \bar{r}\bar{f}\bar{v}]^{-1}$, or, equivalently,

$$\bar{w}^{-1} = \bar{v}^{-1} + \bar{r}\bar{f},$$

where $\mu(\bar{v}) \leq \nu$. Now, by Hammer and Heymann [1] Theorem 6.19, there exists a strictly polynomial matrix $D: \Lambda U \to \Lambda U$ such that Ker $\pi^- \bar{f} = D[\Omega^- U]$. Let $\underset{\sim}{L}^-: \Lambda U \to \Omega^- U: \ \Sigma u_t z^{-t} = \sum_{t \geq 0} u_t z^{-t}$ be the truncation operator, and denote by $\psi := \underset{\sim}{L}^{-1}(\bar{w}^{-1} D)$ the truncated matrix. Then, $\psi: \Lambda U \to \Lambda U$ is causal, and, defining $\bar{\varphi} := \psi D^{-1}$, we have that Ker $\pi^- \bar{\varphi} = D[\text{Ker } \pi^- \psi] \subset D[\Omega^- U] =$ Ker $\pi^- \bar{f}$. Hence, by op. cit. Theorem 5.2, there exists a causal ΛK-linear map $\bar{r}: \Lambda Y \to \Lambda U$ such that $\bar{\varphi} = \overline{r}\bar{f}$.

Next, since $\bar{f}\bar{w}$ is strictly causal, and since $\operatorname{Ker} \pi^{-}\bar{f}\bar{w} = \operatorname{Ker} \pi^{-}D^{-1}\bar{w}$, the map $D^{-1}\bar{w}$ is strictly causal as well. Also, the latency degree of $D^{-1}\bar{w}$ is equal to the latency degree ν of $\bar{f}\bar{w}$. Hence, since $\bar{\varphi}D$ is causal, it follows by Lemma 2.3 that the map $P := \bar{w}^{-1}D - \varphi D$ is nonsingular, has a causal inverse, and its latency degree is ν. Also, by construction, both P and D are strictly polynomial. Thus, the ΛK-linear map $\bar{v} := DP^{-1}$ is nonsingular, its MacMillan degree μ satisfies $\mu \leq \deg\,(\deg P) - m = \nu$ (where $m = \dim_K U$), and $\bar{w}^{-1} = \bar{v}^{-1} + \bar{r}\bar{f}$. Finally, we show that $\bar{v}$ is causal as well. Indeed, $z\bar{v}^{-1} = z\bar{w}^{-1} - z\bar{r}\bar{f}$, so that, since $z\bar{r}\bar{f}$ is causal and $z^{-1}\bar{w}$ is strictly causal, Lemma 2.3 implies that $z^{-1}\bar{v}$ is strictly causal. Hence, $\bar{v}$ is causal, and our proof is complete. □

The bound on the MacMillan degree of $\bar{v}$ as given in Theorem 2.2 is tight in the following sense. (We note that the latency degree of a causal map is always greater or equal to $-m$, where $m = \dim_K U$.)

<u>Theorem 2.4</u>. *Let $\bar{f}: \Lambda U \to \Lambda Y$ be an injective ΛK-linear map, and let $\nu \geq -m$ be an integer. Then, there exists a causal nonsingular ΛK-linear map $\bar{w}: \Lambda U \to \Lambda U$ of latency degree ν such that the following holds: For every representation $\bar{f}\bar{w} = \bar{f}_{(\bar{v},\bar{r})}$, the MacMillan degree μ of $\bar{v}$ satisfies $\mu \geq \nu(\bar{f}\bar{w})$, where $\nu(\bar{f}\bar{w})$ is the latency degree of $\bar{f}\bar{w}$.*

The proof of Theorem 2.4 is similar to the proof of Theorem 7.9 in Hammer and Heymann [1].

REFERENCE

[1] J. Hammer and M. Heymann, *Causal factorization and linear feedback*, Preprint, Department of Electrical Engineering, Technion-Israel Institute of Technology, Haifa, Israel. To appear in SIAM Journal on Control and Optimization, July 1981.

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