Proceedings of the American Control Conference, San Diego, CA, June 1999

ON APPROXIMATE MODEL MATCHING IN NONLINEAR CONTROL

Jacob Hammer
Department of Electrical and Computer Engineering
University of Florida
Gainesville, FL 32611-6130, USA
hammer@leaf.ee.ufl.edu

ABSTRACT

Approximate model matching is the problem of controlling a nonlinear system to achieve a response resembling that of a desired model. This note presents a family of recursive output feedback controllers that achieve approximate model matching in all cases where it is possible. The design of these controllers depends on the solution of a set of algebraic inequalities. This presentation is an extended summary of HAMMER [1998b].

1. INTRODUCTION

Perhaps one of the most effective ways of specifying the desired behavior of a nonlinear system is by requiring the system to resemble a specified model. Consider a nonlinear plant Σ that needs to be controlled to perform a specific task. Let M be a model suitable for the required task; M can be selected through computer simulation or by qualitative considerations. Our objective is to design a controller C which, when combined with Σ, yields a controlled system Σ_c that 'resembles' the model M. The 'resemblance' must be preserved under disturbances and parameter uncertainties. We refer to this objective as *approximate model matching*.

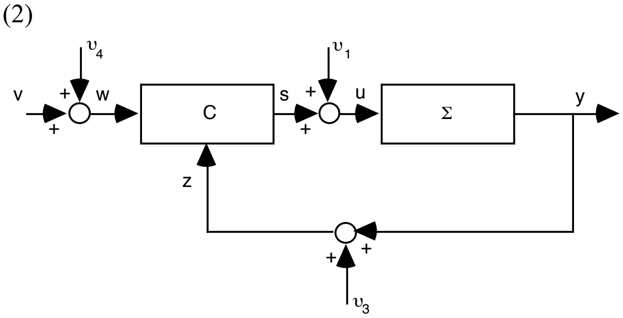
Let $\Im$ be the class of input signals for which Σ_c needs to 'resemble' the model M. Usually, $\Im$ consists of all input signals of amplitude not exceeding a specified bound $\theta > 0$. Our objective is to design the controller C so that, for every input signal $v \in \Im$, the response $\Sigma_c v$ of the controlled plant is 'close' to the response Mv of the desired model. Let $|u|$ denote the amplitude of a signal u, and let $\Delta > 0$ be a real number. Then, Σ_c is a Δ-*approximant* of M if

(1) $|\Sigma_c v - Mv| \leq \Delta$

for every input signal $v \in \Im$. Thus, for Σ_c to be a Δ-approximant of M, the amplitude of the discrepancy among the responses of Σ_c and of M may not exceed Δ for any input sequence of interest.

We consider here several aspects of the approximate model matching problem: necessary and sufficient conditions for the existence of a controller C that achieves approximate model matching; the derivation of a template for such controllers; and the development of computational design techniques. The basic control configuration is:

(2)



Here, Σ is the plant to be controlled, and C is the causal feedback controller we need to design. The signals υ_1, υ_3, and υ_4 represent disturbance signals. The only a-priori information available about each disturbance signal υ_i is that its amplitude does not exceed a specified bound $n_i > 0$, not considered infinitesimal. The closed loop system is denoted by Σ_c. Our objective is to find, if possible, a controller C for which Σ_c is a Δ-approximant of M in the presence of the disturbance signals υ_1, υ_3, and υ_4.

We consider discrete time nonlinear plants, including plants whose state is provided as output and plants whose output is not a state. The first case consists of plants Σ described by a nominal *state representation* of the form

(3) $y_{k+1} = f(y_k,u_k)$, $k = 0, 1, 2, ...$

Here, f is called the *recursion function* of Σ. The initial condition of Σ is y_0. A system described by (3) is called an *input/state* system. In this case the model M that needs to be approximately matched is also an input/state system, given by the nominal representation

(4) $\xi_{k+1} = \varphi(\xi_k,v_k)$, $k = 0, 1, 2, ...,$

where the vector ξ_k is of the same dimension as y_k. The initial condition ξ_0 of M is not assumed identical to y_0. In this notation, condition (1) takes the form

$$|y_k - \xi_k| \leq \Delta,\ k = 0, 1, 2, \ldots$$

for all input sequences $v \in \Im$, for all permissible disturbance signals, and for all permissible initial conditions.

To take into account possible inaccuracies of the recursion function f, we incorporate an additive disturbance υ_2 into the recursive representation of Σ:

$$y_{k+1} = f(y_k,u_k) + \upsilon_{2,k},\ k = 0, 1, 2, ...,$$

where the amplitude of υ_2 does not exceed a given bound $n_2 > 0$. The closed loop system Σ_c is still required to be a Δ-approximant of M for all disturbances $\upsilon_1, \upsilon_2, \upsilon_3$, and υ_4, as long as the disturbance amplitude bounds are not exceeded. Necessary and sufficient conditions for the existence of a controller C that satisfies this design objective are discussed in section 2; a technique for the design of C is described in section 3. The latter depends upon the solution of a set of algebraic inequalities derived from the available recursion functions f and φ.

We also derive in HAMMER [1998b] a template for controllers that solve the approximate model matching problem. In the notation of (2), this template is

(5) $$C : \begin{cases} \zeta_{k+1} = \varphi(\zeta_k,w_k) + \upsilon_{5,k}, \\ s_k = \sigma(z_k,\zeta_k,w_k),\ k = 0, 1, 2, ... \end{cases}$$

Here, φ is the recursion function of the model M, and σ is a feedback function that is derived as part of the controller design. The term υ_5 represents a disturbance signal that originates from inaccuracies in the implementation of the function φ within the controller. The amplitude of υ_5 does not exceed a given bound $n_5 \geq 0$. This template can be used in all cases where a controller solving the approximate model matching problem exists. The template has practical implications, as it vastly reduces the number of controller candidates that need to be examined in experimental design.

1.1 Recursive systems and formal realizations.

We turn now to the more general case where the output of Σ is not a state, and Σ is nominally described by

(6) $y_{k+\eta+1} = f(y_{k+\eta}, y_{k+\eta-1}, ..., y_k, u_{k+\mu}, u_{k+\mu-1}, ..., u_k)$,

$k = 0, 1, 2, \ldots$ Here, f is the *recursion function* of Σ, the output value is y_k, the input value is u_k, and η and μ are two non-negative integers satisfying $\mu \le \eta$. To distinguish from (3), we use $\eta \ge 1$. The initial conditions of Σ are $y_0, ..., y_{k+\eta}$; it is not assumed that the initial conditions are known accurately. The condition $\mu \le \eta$ implies that the system Σ is strictly causal.

The model M to be approximately matched is given by

(7) $\xi_{k+\eta+1} = \varphi(\xi_{k+\eta}, \xi_{k+\eta-1}, ..., \xi_k, v_{k+\mu}, v_{k+\mu-1}, ..., v_k)$,

$k = 0, 1, 2, ...$ Here, ξ_k is of the same dimension as y_k. For simplicity we assume that M and Σ share the same value of η, but this assumption can be released (see HAMMER [1998b]).

Disturbances and inaccuracies of the recursive representation of Σ are incorporated via a disturbance signal υ_2:

$$y_{k+\eta+1} = f(y_{k+\eta}, y_{k+\eta-1}, ..., y_k, u_{k+\mu}, u_{k+\mu-1}, ..., u_k) + \upsilon_{2,k+\eta},$$

$k = 0, 1, 2, ...$ The amplitude of the disturbance υ_2 does not exceed a specified bound $n_2 \ge 0$.

The present more general situation can be reduced to the case of systems with state output through the notion of a "formal realization," as follows. For a nominal recursive representation of the form (6), define the *formal state* x_k of Σ at the step k by the vector

$$x_k := (y_k, y_{k-1}, ..., y_{k-\eta}, u_{k-\eta+\mu-1}, ..., u_{k-\eta})^T,$$

where ^{T} indicates the transpose. The inequality $\mu \le \eta$ implies that the present value x_k is determined by present and past output values $y_k, y_{k-1}, ..., y_{k-\eta}$ and by past input values $u_{k-\eta+\mu-1}, ..., u_{k-\eta}$ of Σ. The formal state x_k contains all the input and output data necessary for the computation of the next step y_{k+1} of Σ, except for the latest relevant input value $u_{k+\mu-\eta}$. Combining the formal state with (6), we obtain the recursive representation

$$x_{k+1} = F(x_k, \lambda_k),\ k = 0, 1, 2, ...,$$

called a *formal realization* of the system Σ. Here $\lambda_k := u_{k+\mu-\eta}$, and the function F is given in terms of the components $y_k, y_{k-1}, ..., y_{k-\eta}, u_{k-\eta+\mu-1}, ..., u_{k-\eta}$ of x_k by (8) below. The requirement $\mu \le \eta$ implies that the sequence λ is either equal to the input sequence u, or is a delay of u by $(\eta - \mu)$ steps, guaranteeing causality.

$$(8)\quad x_{k+1} = \begin{pmatrix} f(y_k, y_{k-1}, ..., y_{k-\eta}, \lambda_k, u_{k-\eta+\mu-1}, ..., u_{k-\eta}) \\ y_k \\ \cdots \\ y_{k-\eta+1} \\ \lambda_k \\ u_{k-\eta+\mu-1} \\ \cdots \\ u_{k-\eta+1} \end{pmatrix}$$

A formal realization is a realization of the system in the usual sense, although it may not be a minimal realization.

Formal realizations provide a simple mechanism for deriving a realization under very general conditions. Furthermore, since the formal state is a combination of present and past output and input values of the system, feedback involving the formal state can be directly implemented without the need for an observer.

Formal realizations can be used to solve the approximate model matching problem for the system Σ of (6) and of the model M of (7). After building formal realizations for Σ and M, we can use the controller (5). Noting that $u = s + \upsilon_1$ by diagram (2), we obtain a controller C of the form

(9) C :

$$\begin{cases} \zeta_{k+\eta+1} = \varphi(\zeta_{k+\eta}, \zeta_{k+\eta-1}, ..., \zeta_k, w_{k+\mu}, w_{k+\mu-1}, ..., w_k) + \upsilon_{5,k}, \\ s_k = \sigma(z_k, ..., z_{k-\eta}, (s+\upsilon_1)_{k-\eta+\mu-1}, ..., (s+\upsilon_1)_{k-\eta}, \zeta_k, ..., \\ \qquad \zeta_{k-\eta}, w_{k-\eta+\mu-1}, ..., w_{k-\eta}). \end{cases}$$

In this way, formal realizations provide us with simple means to generalize results from the theory of state feedback to the more general case of input/output control of nonlinear recursive systems. The calculation of the function σ involves the solution of a set of algebraic inequalities derived from the given functions f and φ (see HAMMER [1998b] for details).

Examining (9), we can distinguish among two constituents of the controller C: one, represented by the first row of (9), is simply a simulation of the model M that needs to be approximately matched; the other, represented by the second row of (9), is a dynamic output feedback controller induced by the function σ. The inequality $\mu \le \eta$ implies that C is strictly causal.

The forms (5) and (9) form universal templates of controllers that solve the approximate model matching problem for a rather general class of nonlinear systems. Such templates are valuable in practice, where they can be used as a basis for numerical experimentation with design parameters.

The material of the present note is a continuation of the work presented in HAMMER [1989b and 1998a] on the global theory of nonlinear robust control. Alternative treatments of the global control of nonlinear systems are described in HAMMER [1984a and b, 1985, 1989a, 1991, 1994], DESOER and KABULI [1988], VERMA [1988], SONTAG [1989], CHEN and de FIGUEIREDO [1990], PAICE and MOORE [1990], VERMA and HUNT [1993], PAICE and van der SCHAFT [1994], BARAMOV and KIMURA [1995], the references cited in these papers, and others.

2. APPROXIMATE MODEL MATCHING

We start with some notation. Let R^m be the set of all m-dimensional real vectors, and let $S(R^m)$ be the set of all sequences $u_0, u_1, u_2, ...$ of real vectors $u_i \in R^m$, $i = 0, 1, 2, ...$. A discrete-time system Σ that accepts input sequences of m dimensional real vectors and generates output sequences of p dimensional real vectors induces a map $\Sigma : S(R^m) \to S(R^p)$.

We shall use the ℓ^∞-norm to characterize disturbances and their effects. For a vector $v = (v^1, v^2, ..., v^m) \in R^m$, denote by $|v| := \max\ \{|v^1|, |v^2|, ..., |v^m|\}$ the maximal absolute value of a component. For a sequence $u \in S(R^m)$,

$$|u| := \sup_{i\ge 0} |u_i|$$

is the ℓ^∞-norm of u. We denote by $S(\theta^m)$ the set of all sequences $u \in S(R^m)$ satisfying $|u| \le \theta$, where $\theta > 0$ is a real number; then $S(\theta^m)$ is the set of all sequences of amplitude not exceeding θ.

We turn now to the issue of approximate model matching for the case where the systems Σ and M are given in terms of state representations. The results can be applied to the general case of recursive systems through the notion of formal realization of subsection 1.1. Our discussion is based on the notion of "relative eigenset," discussed next.

2.1 Relative eigensets.

Let the system Σ be given by $y_{k+1} = f(y_k,u_k)$ and let the model M be given by $\zeta_{k+1} = \varphi(\zeta_k,v_k)$. Combine the two into the function

(10) $(f,\varphi) : (R^p\times R^m)^2 \to (R^p)^2 : (y,s,\zeta,w) \mapsto (f(y,s),\varphi(\zeta,w))$.

On the domain of (f,φ), define the following projections.

$\Pi_{y\zeta} : (R^p\times R^m)^2 \to (R^p)^2 : (y,s,\zeta,w) \mapsto (y,\zeta)$;

$\Pi_{y-\zeta} : (R^p\times R^m)^2 \to R^p : (y,s,\zeta,w) \mapsto y - \zeta$;

$\Pi_{y\zeta w} : (R^p\times R^m)^2 \to R^p\times R^p\times R^m : (y,s,\zeta,w) \mapsto (y,\zeta,w)$;

$\Pi_{ys\zeta} : (R^p\times R^m)^2 \to R^p\times R^m\times R^p : (y,s,\zeta,w) \mapsto (y,s,\zeta)$.

Next, let $\delta > 0$ be a real number, let $q > 0$ be an integer, and let z be a point in R^q. Denote by

$$N_\delta(z) := \{\zeta \in R^q : |\zeta - z| \le \delta\}$$

the closed neighborhood of radius δ around the point z. For a subset $S \subset R^q$, denote by

$$N_\delta(S) := \cup_{z\in S} N_\delta(z).$$

The following is the basic concept of our discussion.

(11) DEFINITION. Let $f, \varphi : R^p\times R^m \to R^p$ be two functions, let $\delta, \Delta > 0$ be real numbers satisfying $\Delta > 2\delta$, and let (f,φ) be as in (10). Then, a non-empty subset $S \subset (R^p\times R^m)^2$ is a (δ,Δ)-*eigenset of* f *relative to* φ if the following conditions hold.

(i) $|\Pi_{y-\zeta}[S]| \le \Delta - 2\delta$, and

(ii) $(f,\varphi)[N_\delta(S)] \subset \Pi_{y\zeta}[S]$.

The number δ is called the *contraction radius* of the relative eigenset S. ♦

To discuss the intuitive meaning of Definition 11, consider a (δ,Δ)-eigenset S of f relative to φ. Since S is a subset of $(R^p\times R^m)^2$, each point of S can be regarded as a twosome $((y,s),(\zeta,w))$ of state-input pairs, where (y,s) is a state-input pair of Σ and (ζ,w) is a state-input pair of M. In this way, S induces a correspondence among such pairs, where (y,s) corresponds to (ζ,w) if $(y,s,\zeta,w) \in S$. Now, let (y,s) and (ζ,w) be such a corresponding pair. Then, condition (i) of Definition 11 means that the discrepancy between y (the state of Σ) and ζ (the corresponding state of M) does not exceed Δ, even if independent disturbances of amplitude not exceeding δ are added to y and to ζ. Condition (ii) of Definition 11 indicates that S is a conditional invariant subset of the function (f,φ) (compare to LASALLE and LEFSCHETZ [1961], LEFSCHETZ [1965], WONHAM [1974]).

To further discuss the significance of condition (ii) of Definition 11, consider a pair $(y,\zeta) \in \Pi_{y\zeta}S$; recall that y is a state value of Σ and ζ is a state value of M. Since $(y,\zeta) \in \Pi_{y\zeta}S$, there are input values s and w such that $(y,s,\zeta,w) \in S$. Assume now that Σ is at the state y and M is at the state ζ; apply the input value s to Σ and the input value w to M. Let y^+ and ζ^+ denote the next states of Σ and M, respectively. Then, condition (ii) of Definition 11 implies that $(y^+,\zeta^+) \in \Pi_{y\zeta}S$ (which is the invariance property of the set). In view of condition (i) of Definition 11, this implies that the discrepancy between y^+ and ζ^+ likewise does not exceed Δ. In other words, by using the input values s and w indicated by S, we can maintain a discrepancy not exceeding Δ for the next step.

When repeated step after step, the process of the previous paragraph allows us to construct corresponding input sequences of Σ and of M that maintain a discrepancy of Δ or less among the trajectories of the two systems. We shall need the following.

(12) DEFINITION. Let $f, \varphi : R^p\times R^m \to R^p$ be two functions, and let $\theta, \delta, \Delta > 0$ be real numbers, where $\Delta > 2\delta$. A (δ,Δ)-eigenset S of f relative to φ is *input complete* with amplitude θ if $S \supset (\Pi_{ys\zeta}[S])\times[-(\theta+\delta),(\theta+\delta)]^m$.

An input complete eigenset S of f relative to φ has the special property that it includes all input values of φ of amplitudes up to $\theta+\delta$.

2.2 Deriving controllers.

In this subsection we describe the process that leads from an input-complete relative eigenset to an approximate model matching controller. The derivation of input-complete relative eigensets is described in HAMMER [1998b]. Recall that Σ and M are given by (3) and (4), respectively. The controllers we derive are of the form (5), so that the only quantity that needs to be calculated is the feedback function σ.

Let $S \subset (R^p\times R^m)^2$ be a (δ,Δ)-eigenset of f relative to φ, input complete with amplitude $\theta > 0$. For each point $(y,\zeta,w) \in R^p\times R^p\times R^m$, we construct now a subset $U_S(y,\zeta,w) \subset R^m$, called the *feedback value set* of S. We show later that $U_S(y,\zeta,w)$ consists of values the feedback function σ can take when the system Σ is at the state y, the model M is at the state ζ, and the external input value is w.

Construction of the feedback value set U_S:

(i) When $(y,\zeta,w) \in \Pi_{y\zeta w}[S]$:
The set $U_S(y,\zeta,w)$ consists of all points $s \in R^m$ satisfying $(y,s,\zeta,w) \in S$.

(ii) When $(y,\zeta,w) \notin \Pi_{y\zeta w}[S]$:
Let $A(y,\zeta,w)$ be the set of all points $(a,b,c) \in \Pi_{y\zeta w}[S]$ satisfying $|y - a| \le 2\delta/3$, $|\zeta - b| \le \delta$, and $|w - c| \le \delta$. Then, we distinguish among three cases:

a) If $A(y,\zeta,w) \neq \emptyset$, the set $U_S(y,\zeta,w)$ consists of all points $s \in R^m$ satisfying $(a,s,b,c) \in S$ for some vector $(a,b,c) \in A(y,\zeta,w)$.

b) If $(y,\zeta,w) \in N_\delta(\Pi_{y\zeta w}[S])$ and $A(y,\zeta,w) = \emptyset$, the set $U_S(y,\zeta,w)$ consists of all vectors $s \in R^m$ such that $(a,s,b,c) \in S$ for some vector $(a,b,c) \in \Pi_{y\zeta w}[S]$ satisfying $|(a,b,c) - (y,\zeta,w)| \le \delta$.

c) If $(y,\zeta,w) \notin N_\delta(\Pi_{y\zeta w}[S])$, then $U_S(y,\zeta,w) := 0$ is the set consisting of the zero vector alone.

To examine the qualitative significance of the feedback value set $U_S(y,\zeta,w)$, consider the closed loop system (2) with the controller C of (5). Assume for a moment that all disturbances and errors are zero. Let y_0 be the initial condition of Σ, let ξ_0 be the initial condition of M, let ζ_0 be the initial condition of the controller C of (5), and let $v \in S(\theta^m)$ be the external input sequence of the closed loop system. Since all disturbances are zero, the initial conditions satisfy $\xi_0 = \zeta_0$, and the input sequence of (2) satisfies $w = v$. This implies that the output sequence ξ of M is equal to the sequence ζ generated within the controller C of (5).

Assume that Σ and M start from initial conditions within the eigenset S, so that $(y_0,\zeta_0) \in \Pi_{y\zeta}[S]$. Let the controller C of (2) be constructed so that the sequence s it generates satisfies

(13) $s_k \in U_S(y_k,\zeta_k,w_k)$, k = 0, 1, 2, …

It can then be shown (HAMMER [1998b]) that

$$(y_k,\zeta_k) \in \Pi_{y\zeta}[S] \text{ for all integers } k \geq 0,$$

which, by Definition 11(i), implies that

$$|y_k - \zeta_k| \leq \Delta - 2\delta, k \geq 0.$$

Thus, the sequence s generated by (13) drives Σ so as to maintain a discrepancy of less than Δ among the output sequences of Σ and of M, and the elements of s are in the feedback value set U_S.

Most importantly, (13) represents a feedback rule for generating the sequence s. Indeed, define a function $\sigma : R^p\times R^p\times R^m \to R^m : (y,\zeta,w) \mapsto \sigma(y,\zeta,w)$ by setting

$$\sigma(y,\zeta,w) := s,$$

where s is a point of $U_S(y,\zeta,w)$. Our earlier discussion suggests that with this σ the controller C of (5) achieves approximate model matching. The next statement, which is one of the main results of this theory, shows that this is indeed the case.

(14) THEOREM. Let $\Sigma, M : S(R^m) \to S(R^p)$ be input/state systems having the recursion functions $f, \varphi : S(R^p)\times S(R^m) \to S(R^p)$ and the initial conditions y_0 and ξ_0, respectively. Let $\delta, \Delta > 0$ be two real numbers satisfying $\Delta > 2\delta$. Assume that there is a (δ,Δ)-eigenset S of f relative to φ which is input complete with amplitude $\theta > 0$, and that the initial conditions satisfy $(y_0,\xi_0) \in \Pi_{y\zeta}[S]$. Let $U_S(\cdot)$ be the feedback value set induced by S, and build a function $\sigma : R^p\times R^p\times R^m \to R^m$ by setting $\sigma(y,\zeta,w) := s$, where s is a point of $U_S(y,\zeta,w)$. Then, with this choice of σ, the controller C of (5) solves the approximate model matching problem, as long as all disturbance amplitudes are bounded by $\delta/3$.

Thus, a controller C that solves the approximate model matching problem can be derived from a relative eigenset S. The calculation of relative eigensets is discussed in HAMMER [1998b]. As we can see, the permissible disturbance amplitudes depend on the contraction radius δ of S.

3. DERIVATION OF CONTROLLERS

Due to space limitations, we only provide here a qualitative description of the derivation of controllers for approximate model matching; see HAMMER [1998b] for complete details.

Consider a system Σ with the nominal representation (3), starting from the initial condition $y \in R^p$ and driven by the input sequence $(u_0, u_1, u_2, ...) \in S(R^m)$. Using the shorthand notation

$$f^i(y, u_0, ..., u_{i-1}) := f(f...f(f(y, u_0), u_1) ..., u_{i-1}),$$

the i-th element of the output sequence of Σ is

$$y_i = f^i(y, u_0, ..., u_{i-1}), i = 1, 2, ...$$

A state $y' \in R^p$ is *reachable* from the state $y \in R^p$ in i steps if there is an input list $u_0, ..., u_{i-1}$ for which $f^i(y, u_0, ..., u_{i-1}) = y'$. The set of all states that are reachable from y in i steps is given by

$$\text{Im } f^i(y,\cdot) := \{f^i(y, u_0, ..., u_{i-1}) : u_0, ..., u_{i-1} \in R^m\}.$$

The realization (3) is *globally reachable* if there is an integer $n > 0$ for which the following is true: every state $y' \in R^p$ is reachable from every state $y \in R^p$ in n steps; i.e., if $\text{Im } f^n(y,\cdot) = R^p$ for all $y \in R^p$.

The realization (3) is *everywhere locally reachable* if there is an integer $q \geq 1$ for which the iterated function $f^q(y,\cdot)$ is an open function for all states $y \in R^p$.

Let Σ be a system with the realization (3), and assume that Σ is globally reachable as well as everywhere locally reachable. Let n be the smallest integer satisfying (i) $\text{Im } f^n(y,\cdot) = R^p$ for all $y \in R^p$, and (ii) $f^n(y,\cdot)$ is an open function for all states $y \in R^p$. Then, we call n the *reachability integer* of the system (see HAMMER [1998a] for a discussion).

Let Σ be globally reachable with the reachability integer n, and consider for a moment the nominal case (where all disturbances are set to zero). Let ξ_0 be the initial condition of the model M and let v be the input sequence of M. The output sequence $\xi_0, \xi_1, \xi_2, ... \in R^p$ of M is then given by the recursion $\xi_{k+1} = \varphi(\xi_k,v_k)$.

Assume that Σ starts from the initial condition $y_0 = \xi_0$, as does M. The global reachability of Σ implies that there is an input list $u_0(0), ..., u_{n-1}(0)$ such that

$$y_n = f^n(y_0, u_0(0), ..., u_{n-1}(0)) = \xi_n.$$

Repeating this process every n steps, we obtain for every integer $j \geq 0$ an input list $u_0(j), ..., u_{n-1}(j)$ for which

$$y_{(j+1)n} = f^n(y_{jn}, u_0(j), ..., u_{n-1}(j)) = \xi_{(j+1)n}$$
$$= \varphi^n(\xi_{jn}, v_{jn}, ..., v_{(j+1)n-1}).$$

Concatenating these lists into one sequence

(15) $u = u_0(0), ..., u_{n-1}(0), u_0(1), ..., u_{n-1}(1), ...,$

yields an input sequence that drives Σ so that its response sequence y satisfies

(16) $y_{jn} = \xi_{jn}$

for all integers $j \geq 0$. In other words, with this input sequence, the output values of Σ and of M are identical at all steps that are integer multiples of the reachability integer n. This sequence u can be generated by a feedback controller (see HAMMER [1998b]).

It is important to emphasize that although (16) can be satisfied in all cases when Σ is globally reachable, it is still possible that there is no input sequence u of Σ for which

(17) $|y_k - \xi_k| \leq \Delta - 2\delta$

for all integers $k \geq 0$. Substantial divergence between the two trajectories y and ξ may occur at steps that are not integer multiples of the reachability integer n. To prevent that, we impose the first line of (18) below. In this way, the notion of reachability helps transform the problem of finding the infinite sequence u into a problem of solving the n inequalities

$$(18)\quad \begin{cases} |f^i(y_0, u_0, ..., u_{i-1}) - \xi_i| \leq \Delta - 2\delta, i = 1, 2, ..., n-1, \\ |f^n(y_0, u_0, ..., u_{n-1}) - \xi_n| = 0. \end{cases}$$

Assume there is a solution $u_0(y_0,\xi_1), u_1(y_0, u_0, \xi_2), ..., u_{n-1}(y_0, u_0, ..., u_{n-2}, \xi_n)$ of (18), where we have made explicit the dependence of u_j on all relevant variables. Then, in line with (15), the entire sequence u becomes

$$(19)\quad \begin{cases} u_0(j) = u_0(y_{jn},\xi_{jn+1}) = u_0(y_{jn},\varphi(\xi_{jn},v_{jn})) \\ u_i(j) := u_i(y_{jn}, u_{jn}, ..., u_{jn+i-1}, \xi_{jn+i+1}) \\ \qquad = u_i(y_{jn}, u_{jn}, ..., u_{jn+i-1}, \varphi(\xi_{jn+i},v_{jn+i})) \end{cases}$$

$i = 1, ..., n-1$, $j = 0, 1, 2, ...$. This induces a finite computational process of n steps at a time.

To allow for the effects of disturbances, (16) needs to be weakened to permit some discrepancy among the values of y_{jn} and ξ_{jn}. To this end, we introduce a design parameter given by the real number $\rho > 0$, and replace (16) by the requirement

$$(20)\qquad |y_{jn} - \xi_{jn}| \le \rho,\ j = 0, 1, 2, ...$$

Since disturbances of amplitude δ can be added to y and to ξ, the restriction

$$\rho + 2\delta \le \Delta$$

is needed to guarantee that the total discrepancy never exceeds Δ. With (20), inequalities (18) take the form

$$(21)\quad \begin{cases} |f^i(y_0, u_0, ..., u_{i-1}) - \xi_i| \le \Delta - 2\delta,\ i = 1, 2, ..., n-1, \\ |f^n(y_0, u_0, ..., u_{n-1}) - \xi_n| \le \rho. \end{cases}$$

In combination with (19), inequalities (21) can be used to derive an appropriate input sequence u of Σ that achieves approximate model matching. The variables listed in (19) indicate that u can be generated by a causal feedback controller. This controller solves the approximate model matching problem for Σ and M. Due to space limitations, our discussion here is incomplete, and ignores the effects of some of the disturbances. A complete and detailed discussion is provided in HAMMER [1998b]. The main conclusion though remains valid: a controller for approximate model matching can be derived from the solution of a set of inequalities. The following example reproduced from HAMMER [1998b] demonstrates the form of the exact inequalities for a specific case.

EXAMPLE. Consider the model M given by the system

$$\xi_{k+1} = 0.5\xi_k + w_k,$$

where the initial condition of M satisfies $|\xi_0| \le 1$, and the input amplitude bound is $\theta = 1$. The system Σ that needs to be controlled is given by

$$y_{k+1} = [(y_k)^2 + 1]s_k.$$

Let the discrepancy bound be $\Delta = 1$, and take $\rho = 1/2$. Then, it is shown in HAMMER [1998b] that a controller that solves the approximate model matching problem in this case can be derived from the solution of the following set of inequalities:

$$[(y+\alpha)^2+1](s+\beta) - [0.5(\zeta+\gamma) + (w+\varepsilon)] \le 1/2,$$

$$|\alpha| \le \delta,\ |\beta| \le \delta,\ |\gamma| \le \delta,\ |\varepsilon| \le \delta,\ |\zeta| \le 2,\ |w| \le 1,\ |y - \zeta| \le 1/2.$$

From these inequalities and the controller template (5), one can obtain the controller

$$C : \begin{cases} \zeta_{k+1} = 0.5\zeta_k + v_k, \\ s_k = \sigma(y_k,\zeta_k,v_k) = \dfrac{0.5\zeta_k + v_k}{(y_k)^2 + 1} \end{cases} .$$

The maximal contraction radius is approximately $\delta \approx 0.048$ in this case (see HAMMER [1998b] for details). ♦

To conclude, we demonstrated a rather general theory for the design of controllers for nonlinear recursive systems. Complete details are provided in HAMMER [1998b].

6. REFERENCES

L. BARAMOV and H. KIMURA, [1995], "Nonlinear coprime factorizations and parametrization of a class of stabilizing controllers", Proceedings of the 1995 IEEE conference on Decision and Control, pp. 981-986.

G. CHEN and R.J.P. de FIGUEIREDO, [1990], "Construction of the left coprime fractional representations for a class of nonlinear control systems", Systems and Control Letters, Vol. 14, pp. 353-361.

C.A. DESOER and M.G. KABULI, [1988], "Right factorization of a class of nonlinear systems", Trans. IEEE, Vol. AC-33, pp. 755-756.

J. HAMMER, [1984a], "Nonlinear systems: stability and rationality", Int. J. Control, Vol. 40, pp. 1-35.

[1984b] "On nonlinear systems, additive feedback, and rationality", Int. J. Control, Vol. 40, pp. 953-969.

[1985] "Nonlinear systems, stabilization, and coprimeness", Int. J. Control, Vol. 42, pp. 1-20.

[1989a] "Robust stabilization of nonlinear systems", Int. J. Control, Vol. 49 No. 2, pp. 629-653.

[1989b] "State feedback for nonlinear control systems", International Journal of Control, Vol. 50, pp. 1961-1980.

[1991] "State feedback, confinement, and stabilization for nonlinear continuous time systems", International J. Control, Vol. 53, pp. 799-821.

[1994] "Internally stable nonlinear systems with disturbances: a parametrization", IEEE Transactions on Automatic Control, Vol. AC-39, pp. 300 - 314.

[1998a] "Handling disturbances in nonlinear control: the use of state feedback", International J. Control, Vol. 70, No. 5, pp. 761-787.

[1998b] "Approximate model matching for nonlinear control systems", submitted for publication.

J.F. LASALLE and S. LEFSCHETZ, [1961], "Stability by Liapunov's direct method with applications", Academic Press, N.Y.

S. LEFSCHETZ, [1965], "Stability of nonlinear control systems", Academic Press, N.Y.

A.D.B. PAICE and J.B. MOORE, [1990], "Robust stabilization of nonlinear plants via left coprime factorizations", Systems and Control Letters, Vol. 15, pp. 121-129.

A.D.B. PAICE and A.J. van der SCHAFT, [1994], "Stable kernel representation and the Youla parametrization for nonlinear systems", Proc. of the 33rd IEEE Conference on Decision and Control, Lake Buena Vista, FL, pp. 781-786.

E.D. SONTAG, [1989], "Smooth stabilization implies coprime factorization", Trans. IEEE, Vol. AC-34, pp. 435-444.

M.S. VERMA, [1988], "Coprime fractional representations and stability of nonlinear feedback systems", Int. J. Control, Vol. 48, pp. 897-918.

M.S. VERMA and L.R. HUNT, [1993], "Right coprime factorization and stabilization for nonlinear systems", IEEE Trans. on Automatic Control, AC-38, pp. 222-231.

W.M. WONHAM, [1974], "Linear multivariable control: a geometric approach", Lecture Notes in Economics and Mathematical Systems, No. 101, Springer Verlag, Berlin.