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Non-linear systems : stability and rationality

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The problem of when a non-linear system can be represented as a quotient of two stable non-linear systems is considered. Attention is mainly directed toward non-linear discrete-time recursive systems, where recursive means that the relationship between an input sequence and the corresponding output sequence can be expressed in terms of a finite number of recursive equations. Necessary and sufficient conditions are derived for the existence of a fraction representation of a recursive system, where the numerator and the denominator are stable recursive systems. The explicit construction of such a fraction representation is described.

1. Introduction

Let Σ be a non-linear time-invariant dynamic system, admitting input values from the finite-dimensional real space R^m and having its output values in the finite-dimensional real space R^p. Our main attention in this paper is devoted to the following question. Under what conditions (on Σ) do there exist non-linear systems P and Q, both of which are *stable*, such that Σ can be represented as a quotient of the form $\Sigma = P^{-1}Q$ or $\Sigma = PQ^{-1}$. When such a representation is possible, we say that the system Σ is *rational*. Furthermore, for a rational system Σ, we also wish to find an explicit construction that yields systems P and Q for which $\Sigma = P^{-1}Q$ or $\Sigma = PQ^{-1}$, as the case may be.

The question of rationality seems to be pertinent to the problem of stabilizing a given non-linear dynamic system Σ. Roughly speaking, if one needs to stabilize a non-linear rational system $\Sigma = PQ^{-1}$, then one has to ' cancel ' the denominator Q which, by virtue of the stability of P, is the sole cause for instability in Σ. Of course, any such ' cancellation ' has to be done with due care, so that the resulting system would be not just input–output stable, but would be internally stable as well. Some insight into this situation can be gained from the case of linear systems. Though the theory of linear-system stabilization is not directly related to our discussion in this paper, familiarity with the works of Rosenbrock (1970), Wonham and Pearson (1974), Desoer and Chan (1975), Desoer and Vidyasagar (1975) and Hammer (1983 a, b) may be helpful.

The class of systems that we study in this paper consists of non-linear dynamic systems which are time-invariant and discrete-time, and which are *recursive* in the following sense. The relationship between an input sequence to the system Σ and the corresponding output sequence from Σ can be described in terms of a recursive equation involving only a finite number of input values and a finite number of output values. More explicitly, the system Σ is excited by an input sequence $\ldots, u_{-1}, u_0, u_1, \ldots$, where, for each integer j, the element u_j is an m-dimensional real vector. For each input sequence, the system Σ generates an output sequence $\ldots, y_{-1}, y_0, y_1, \ldots$, where, for each integer j, the

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element y_j is a p-dimensional real vector. The crucial point is that the output sequence $\{y_j\}$ can be computed recursively from the input sequence $\{u_j\}$ so that, for each integer k, there is a relationship of the form

$$y_{k+\eta+1} = f(y_k, \ldots, y_{k+\eta} | u_k, \ldots, u_{k+\mu}) \tag{1.1}$$

where η and μ are fixed integers, and where $f: (R^p)^{\eta+1} \times (R^m)^{\mu+1} \to R^p$ is a multivariable vector-valued function, which we call the *recursion function* of Σ. The vertical line in f is used to separate between the output variables and the input variables. Equation (1.1) is called a *recursive representation* of Σ. If a system has a recursive representation, then we say that it is a *recursive system*. Given some fixed initial time t_0, the output sequence $y_{t_0}, y_{t_0+1}, \ldots,$ of the recursive system Σ is uniquely determined once an input sequence $u_{t_0}, u_{t_0+1}, \ldots,$ is specified together with a set of *initial conditions* $y_{t_0}, y_{t_0+1}, \ldots, y_{t_0+\eta}$. We say that two recursive systems are *input/output equivalent* if, when started from zero initial conditions, they generate the same output sequence when excited by the same input sequence. The class of recursive systems includes, of course, the class of finite-dimensional linear systems, but it is much larger and includes most systems which are of engineering interest. We remark that the assumptions of time-invariance and discrete-time are of a technical nature, and our present theory can be extended to include time-varying or continuous-time systems as well. However, we will not discuss this point in the present paper.

For our discussion of stability we adopt a somewhat stronger version of the classical notion of stability due to Liapunov. In qualitative terms, we say that a recursive system Σ is stable if a slight change in the input sequence to Σ or in the initial conditions of Σ (or in both) causes only a slight change in the corresponding output sequence from Σ. Thus, stability implies that the system Σ, when interpreted as a map transforming input sequences into output sequences, is continuous. Stability, however, is a stronger notion than mere continuity of the map, since it also involves the initial conditions (continuity of Σ as a map is sometimes referred to as 'stability in the Liapunov sense', though Liapunov was, of course, aware of the effect of the initial conditions as well). Our discussion of stability inevitably employs certain notions related to continuity, all of which can be found in the excellent (and short) treatise by Kuratowski (1961).

In order to somewhat refine our previous notion of rationality, consider a system Σ. When Σ is interpreted as a map (transforming input sequences into output sequences), it can always be factored into a composition of maps $\Sigma = PQ$, where P is injective and Q is surjective. The map P, being injective, has a left inverse P^*, whereas Q, being surjective, has a right inverse Q^*. We say that Σ is *left rational* if such a factorization exists where P^* and Q are recursive and stable. Similarly, Σ is *right rational* if P and Q^* are recursive and stable. Then, a left rational system can be rendered input/output stable by connecting it in series with a non-singular stable postcompensator (as in $P^*\Sigma = Q$), whereas a right rational system can be made input/output stable by connecting it in series with a non-singular stable precompensator (as in $\Sigma Q^* = P$). Thus, rationality is equivalent to input/output stabilization through non-singular and stable compensation connected in series. From the practical point of view, we would like, of course, not only to make Σ input/output stable, but to transform it into an internally stable system, i.e. into a system for which not only the input–output behaviour is stable, but so also is the internal behaviour. However, input/output stabilization is necessary for internal stabilization, so that the problem of input/output stabilization (i.e. the problem of rationality) has to be studied first.

The main point of our discussion in this paper is that one can characterize rationality of a recursive system directly in terms of certain properties of its recursion function f. Thus, one can decide whether a recursive system Σ is left rational, or right rational, or neither, by checking its recursion function f in the recursive representation (1.1), which is usually given. More explicitly, we show in § 5 that the recursive system Σ is left rational if and only if its recursion function f can be decomposed into a sum of functions

$$f = f_1 + f_2 \tag{1.2}$$

where the functions f_1 and f_2 are required to satisfy certain algebraic and continuity conditions. In some cases (though, of course, not always), such a sum decomposition may be obtained by mere inspection. Moreover, once such functions f_1 and f_2 are known, one can directly obtain from them recursive representations of systems P and Q for which $\Sigma = PQ$ is a left rational representation. Thus, our characterization of left rationality is constructive. The case of right rationality is dual.

Interestingly, rational systems have many properties which have previously been associated with linearity. One such property is the following. A system Σ is called *BIBO* (bounded-input bounded-output)-*stable* if it responds with a bounded output sequence to every bounded input sequence. It is well known that, for a finite-dimensional time-invariant linear system, BIBO-stability implies stability in the Liapunov sense. Indeed, this fact plays a fundamental role in the theory of linear systems. As it turns out, this property is a direct consequence of rationality. Every non-linear left rational system which is BIBO-stable is also stable in the Liapunov sense (§ 5).

Much of our discussion in this paper depends on the notion of the *input/output space* which we associate, in § 2, with each recusive representation of the form (1.1). Roughly speaking, the input/output space is a subspace of $(R^p)^{\eta+1} \times (R^m)^{\mu+1}$ which is invariant under the evolution of the system Σ. It is the minimal subspace over which the recursion function f has to be defined, given that the system Σ is always started from zero initial conditions at some finite time in the past. The role of the input/output space in our present discussion is comparable to the role of the state-space in classical linear system theory, though the definitions are somewhat different. We introduce the input/output space in § 2, and we continue to study its significance in the later sections.

The paper is organized as follows. Section 2, after setting up our basic framework, is devoted to a discussion of the reduction of non-linear systems, which refers to the following context. The recursive representation (1.1) is not uniquely determined by the input–output behaviour of the system Σ. Different recursive representations, differing by the function f and by the integers η and μ, may represent input/output equivalent systems, similar to the situation that exists in the case of non-canonical realizations of linear systems. It is, of course, of particular interest to find a minimal recursive representation which is input/output equivalent to Σ, one for which the integers η and μ are as small as possible. Evidently, a minimal representation is the easiest one to implement. In § 2 we study the problem of constructing a minimal recursive representation input/output equivalent to a system Σ, when an arbitrary recursive representation of Σ is given.

In § 3 we study the series connection, the sum, and the inversion of non-linear recursive systems, and related properties. In § 4 we define our basic stability notions. As has been common practice in system theory for the past few decades, we distinguish between two notions of stability—*input/output stability* (or, simply, *stability*), and *internal stability*. The notion of internal stability is a stronger stability notion, indicating that not only is the system stable with respect to variations of the input sequence, but its ' hidden ' internal degrees of freedom, which do not affect the input–output relationship, are stable as well. We discuss several questions related to stability and internal stability. In particular, we consider the question of when an input/output stable system has an internally stable representation, i.e. when can a non-linear recursive system be physically implemented as a robust construction. Finally, the paper is concluded in § 5, where we discuss rationality of non-linear systems, as mentioned in the opening of this introduction.

Studies into the theory of stability of non-linear systems cover extended portions of the literature of numerous scientific disciplines, ranging from mathematics through engineering to economics and social sciences. It is, of course, outside the scope of this paper to provide a detailed account of the evolution of non-linear system theory. Much of this evolution has been inspired by the monumental work of Liapunov (1947), which still forms the conceptual framework of stability theory.

In the late fifties and the early sixties of the present century, most of the attention in the non-linear systems literature was directed toward the study of static non-linear output feedback applied to linear systems, in the context of the classical Lurie (1951) problem, and toward extensions and refinements of the Liapunov methods. These studies culminated in a large number of classical works in non-linear system theory, like those of Popov (1961), Lasalle and Lefschetz (1961), Kalman (1963), Hale (1963), Sandberg (1964), Yacubovich (1965), Lefschetz (1965), Zames (1966), the references mentioned in these works, and many, many others.

2. Recursive systems and their representations

In the present section we introduce the underlying framework and the notation for our discussion in this paper. Let R be the set of real numbers. We denote by $S(R^m)$ the set of all two-sided infinite sequences of the form $u := (\dots, 0, \dots, 0, u_{t(u)}, u_{t(u)+1}, \dots) \equiv \{u_i\}$, where $u_i \in R^m$ for all integers i, and where for each sequence u there exists an integer $t(u)$ (depending on u) such that $u_j = 0$ for all $j < t(u)$. Intuitively speaking, for the sequence $\{u_i\}$, the integer i can be regarded as the time marker, and $t(u)$ can be regarded as the starting time of the sequence. The symbol 0 will also be used for the zero sequence in $S(R^m)$—the sequence consisting of only zero elements. Given a sequence $u \in S(R^m)$, we denote by u_j its jth element, and by u_i^j, where $i \leqslant j$, the set of elements $u_i, u_{i+1}, \dots, u_j$. If $i > j$ then u_i^j denotes the empty set. For each pair of sequences u, v of $S(R^m)$, we define the sum $u + v$ coefficientwise by $(u+v)_i = u_i + v_i$ for all integers i. Clearly, $S(R^m)$ is closed under addition.

We regard a *system* as a map transforming input sequences into output sequences, and we require every system under consideration to possess at least one (possibly unstable) equilibrium point (corresponding, for example, to the ' off ' state of the system). Formally, by a non-linear system with input space R^m and output space R^p we mean a map $\Sigma : S(R^m) \to S(R^p)$ satisfying the condition $\Sigma(0)=0$, i.e. mapping the zero sequence in $S(R^m)$ into the zero sequence in $S(R^p)$. Given two systems $\Sigma_1 : S(R^m) \to S(R^p)$ and $\Sigma_2 : S(R^p) \to S(R^q)$, we denote by $\Sigma_2\Sigma_1 : S(R^m) \to S(R^q)$ the system represented by the composite map. As usual, we say that the system $\Sigma : S(R^m) \to S(R^p)$ is *invertible* if there exists a system $\Sigma' : S(R^p) \to S(R^m)$ satisfying $\Sigma\Sigma' = I$ (the identity $S(R^p) \to S(R^p)$) and $\Sigma'\Sigma = I$ (the identity $S(R^m) \to S(R^m)$). The sum $\Sigma_1 + \Sigma_2$ of two systems $\Sigma_1, \Sigma_2 : S(R^m) \to S(R^p)$ is defined pointwise by $(\Sigma_1 + \Sigma_2)u := \Sigma_1 u + \Sigma_2 u$ for all $u \in S(R^m)$. Evidently, the set of systems is closed under composition and addition.

Conceptually, our main assumption on the non-linear systems that we discuss in the present paper is the following recursivity assumption which, in our present framework, is the analogue of the finite-dimensionality assumption commonly imposed in linear system theory.

Definition 2.1

A system $\Sigma : S(R^m) \to S(R^p)$ is *recursive* if there exist integers $\eta, \mu \geq 0$ and a function $f : (R^p)^{\eta+1} \times (R^m)^{\mu+1} \to R$ such that, for every input sequence $u \in S(R^m)$ to Σ, the corresponding output sequence $y := \Sigma u \in S(R^p)$ from Σ satisfies

$$y_{k+\eta+1} = f(y_k^{k+\eta} | u_k^{k+\mu}) \tag{2.2}$$

for all integers k. The function f is then called a *recursion function* for the system Σ. Equation (2.2) is called a *recursive representation* of Σ. □

We remark that, in view of our assumption that $\Sigma 0 = 0$, it follows that the recursion function f of (2.2) satisfies the condition $f(0, \ldots, 0 | 0, \ldots, 0) = 0$. We also note that, by the definition of the spaces $S(R^m)$ and $S(R^p)$, each output sequence of Σ starts from zero initial conditions at some finite time in the past. In case the system Σ has a recursive representation of the form $y_{k+1} = f(u_{k+1})$, then we say that Σ is a *static system*. The integer η in (2.2) is called the *principal degree* of the recursive representation. When the system Σ is a linear system, then the function f is linear, and the minimal possible principal degree of a recursive representation of Σ is simply $(\nu_1 - 1)$, where ν_1 is the maximal observability index of a canonical realization of Σ.

Let $\mathscr{S} : y_{k+\eta+1} = f(y_k^{k+\eta} | u_k^{k+\mu})$ and $\mathscr{S}' : y_{k+\eta'+1} = f'(y_k^{k+\eta'} | u_k^{k+\mu'})$ be recursive representations of systems $\Sigma, \Sigma' : S(R^m) \to S(R^p)$, respectively. We say that the representations $\mathscr{S}$ and $\mathscr{S}'$ are *i/o* (input/output)-*equivalent* (notation $\mathscr{S} \sim \mathscr{S}'$) if $\Sigma = \Sigma'$. Evidently, i/o-equivalence is an equivalence relation, and thus it partitions the set of all recursive representations into *i/o-equivalence classes*, where each class consists of all the recursive representations which represent one and the same system $\Sigma : S(R^m) \to S(R^p)$.

It is easy to see that a recursive system $\Sigma : S(R^m) \to S(R^p)$ has infinitely many different i/o-equivalent recursive representations. Indeed, let $\mathscr{S} : y_{k+\eta+1} = f(y_k^{k+\eta} | u_k^{k+\mu})$ be a recursive representation of Σ. Then, another recursive representation of Σ, differing from $\mathscr{S}$ by the parameters η and μ, can be constructed as follows. By time invariance, we have that $y_{k+\eta+2}=f(y_{k+1}^{k+\eta+1}|u_{k+1}^{k+\mu+1})$, and, expressing $y_{k+\eta+1}$ through $\mathscr{S}$, we obtain

$$y_{k+\eta+2}=f(y_{k+1}, \ldots, y_{k+\eta}, f(y_k^{k+\eta}|u_k^{k+\mu})|u_{k+1}^{k+\mu+1}) =: f'(y_k^{k+\eta+1}|u_k^{k+\mu+1})$$

which is a new recursive representation, i/o-equivalent to $\mathscr{S}$. Our main interest in the present section is, in a sense, to reverse this construction, in order to obtain a minimal recursive representation of Σ, one for which the parameters η and μ are as low as possible. In addition, we also wish to find out what properties of a recursive representation are determined by the system Σ it represents, and what properties are arbitrary. This will allow us later to choose the properties which are not determined by Σ in a convenient way (in § 4).

Input/output-equivalence can be characterized in terms of the recursive representations alone. For this purpose we need the following concept. Let $\mathscr{S}: y_{k+\eta+1}=f(y_k^{k+\eta}|u_k^{k+\mu})$ be a recursive representation of a system $\Sigma: S(R^m)\to S(R^p)$. Let $S_0^\mu(R^m)$ denote the set of all one-sided infinite sequences $u_0, u_1, \ldots, u_n, \ldots$, with elements $u_j \in R^m$, $j=0, 1, 2, \ldots$, satisfying $u_0=u_1=\ldots=u_\mu=0$. For each element $u\in S_0^\mu(R^m)$, let $\mathscr{S}(u)$ denote the output sequence $y_0, y_1, \ldots$, computed recursively through $\mathscr{S}$ under the initial conditions $y_0=y_1=\ldots \ldots =y_\eta=0$. We define the *i/o-space* D_0 of $\mathscr{S}$ by

$$D_0 := \bigcup_{u\in S_0^\mu(R^m)} \bigcup_{k\geq 0} ([\mathscr{S}(u)]_k^{k+\eta}|u_k^{k+\mu}) \tag{2.3}$$

so that D_0 is the subset of $(R^p)^{\eta+1}\times(R^m)^{\mu+1}$ consisting of all the segments of 'length' $(\eta+1, \mu+1)$ of the (input|output) sequences. The i/o-space is the minimal domain over which the recursion function f has to be defined in order to characterize the input–output relationship induced by the system Σ. It plays a central role in our present framework serving, in a sense, as the analogue of the state-space in classical linear system theory. As a simple example, the i/o-space of $y_{k+1}=\frac{1}{2}(y_k)^2+u_k$ is R^2, whereas the i/o-space of $y_{k+1}=\frac{1}{2}(y_k)^2+(u_k)^2$ is $[0, \infty)\times R$.

Next, given a subset $A\subset(R^p)^{\alpha+1}\times(R^m)^{\beta+1}$, where $\alpha\geq\eta$ and $\beta\geq\mu$, we define for every integer $i\geq 0$ the i-*step extension* $\mathscr{S}^i[A]$ of A by

$$\left.\begin{aligned} \mathscr{S}^0[A] &:= A \\ \mathscr{S}[A] &:= \bigcup_{\substack{(z_0^\alpha|v_0^\beta)\in A\\ b\in R^m}} (z_0, \ldots, z_\alpha, f(z_{\alpha-\eta}^\alpha|v_{\beta-\mu}^\beta)|v_0, \ldots, v_\beta, b) \\ \mathscr{S}^i[A] &:= \mathscr{S}[\mathscr{S}^{i-1}[A]] \end{aligned}\right\} \tag{2.4}$$

As we see, for every $i\geq 0$, the set $\mathscr{S}^i[A]$ is a subset of $(R^p)^{\alpha+i+1}\times(R^m)^{\beta+i+1}$, and each element of it is an i-step extension along the trajectory of $\mathscr{S}$ of an element in A, with new input values covering R^m. When A is the i/o-space D_0 of $\mathscr{S}$, then simply

$$\mathscr{S}^i[D_0]=\bigcup_{u\in S_0^\mu(R^m)} \bigcup_{k\geq 0} ([\mathscr{S}(u)]_k^{k+\eta+i}|u_k^{k+\mu+i}), \quad i\geq 0$$

which is a subset of $(R^p)^{\eta+i+1}\times(R^m)^{\mu+i+1}$. For every element $(a_0^{\eta+i+1}|b_0^{\mu+i+1})$ in $\mathscr{S}^{i+1}[D_0]$, where $i\geq 0$, we have that $a_{\eta+i+1}=f(a_i^{\eta+i}|b_i^{\mu+i})$, and that $b_{\mu+i+1}$ is an arbitrary element of R^m. Thus, once D_0 is known, the sets $\mathscr{S}^i[D_0]$,

$i=1, 2, \ldots,$ can be constructed recursively step by step using the recursion function f. In these terms, i/o-equivalence can be characterized as follows.

Theorem 2.5

Let $\mathscr{S}: y_{k+\eta+1}=f(y_k^{k+\eta}|u_k^{k+\mu})$ and $\mathscr{S}': y_{k+\eta'+1}=f'(y_k^{k+\eta'}|u_k^{k+\mu'})$, where $\eta'-\eta=\mu'-\mu$, be recursive representations mapping $S(R^m)\to S(R^p)$. Let D_0 and D_0' be the i/o-spaces of $\mathscr{S}$ and $\mathscr{S}'$, respectively. Assume that $\eta'\geqslant\eta$, and denote $\gamma := \eta'-\eta$. Then $\mathscr{S}$ and $\mathscr{S}'$ are i/o-equivalent if and only if $\mathscr{S}^{\gamma+1}[D_0]=\mathscr{S}'[D_0']$.

In view of this theorem, i/o-equivalence of recursive representations can be verified by checking whether two subsets of $(R^p)^{\eta'+2}\times(R^m)^{\mu'+2}$ are equal.

Proof

If $\mathscr{S}\sim\mathscr{S}'$, then it follows by our construction of the respective sets that $\mathscr{S}^{\gamma+1}[D_0]=\mathscr{S}'[D_0']$. Conversely, assume that $\mathscr{S}^{\gamma+1}[D\ ]=\mathscr{S}'[D_0']$. By time invariance, our proof that $\mathscr{S}\sim\mathscr{S}'$ will conclude upon showing that $\mathscr{S}(u)=\mathscr{S}'(u)$ for all elements $u\in S_0^{\mu'}(R^m)$ (note that, since $\mu'\geqslant\mu$, we have $S_0^{\mu'}(R^m)\subset S_0^{\mu}(R^m)$). Now, let $u\in S_0^{\mu'}(R^m)$ be an arbitrary element. Then, clearly, $[\mathscr{S}'(u)]_0^{\eta'}=0_0^{\eta'}=[\mathscr{S}(u)]_0^{\eta'}$. Further, preparing for induction, assume that $[\mathscr{S}'(u)]_0^n=[\mathscr{S}(u)]_0^n$ for some integer $n\geqslant\eta'$. Then, using the fact that $([\mathscr{S}(u)]_{n-\eta'}^{n+1}|u_{n-\eta'}^{n-\eta'+\mu+1})$ belongs to $\mathscr{S}^{\gamma+1}[D_0]$—and whence also to $\mathscr{S}'[D_0']$—we obtain

$$[\mathscr{S}(u)]_{n+1}=f'([\mathscr{S}(u)]_{n-\eta'}^n|u_{n-\eta'}^{n-\eta'+\mu'})=[\mathscr{S}'(u)]_{n+1}$$

Thus, $[\mathscr{S}(u)]_0^{n+1}=[\mathscr{S}'(u)]_0^{n+1}$ so that, by induction, $[\mathscr{S}(u)]_0^\infty=[\mathscr{S}'(u)]_0^\infty$ for any $u\in S_0^{\mu'}(R^m)$, and our proof concludes. □

Theorem 2.5 can be slightly strengthened in the case when η' is strictly larger than η. This stronger version, which is stated below, is important to our ensuing discussion.

Corollary 2.6

Let $\mathscr{S}: y_{k+\eta+1}=f(y_k^{k+\eta}|u_k^{k+\mu})$ and $\mathscr{S}': y_{k+\eta'+1}=f'(y_k^{k+\eta'}|u_k^{k+\mu'})$, where $\eta'-\eta=\mu'-\mu$, be recursive representations mapping $S(R^m)\to S(R^p)$. Let D_0 and D_0' be the i/o-spaces of $\mathscr{S}$ and $\mathscr{S}'$, respectively. Assume that $\eta'>\eta$, and denote $\gamma := \eta'-\eta$. Then, $\mathscr{S}$ and $\mathscr{S}'$ are i/o-equivalent if and only if $\mathscr{S}^{\gamma}[D_0]=D_0'$.

Proof

If $\mathscr{S}$ and $\mathscr{S}'$ are i/o-equivalent, then the equality $\mathscr{S}^{\gamma}[D_0]=D_0'$ is a direct consequence of the definitions of the involved sets. Conversely, assume that $\mathscr{S}^{\gamma}[D_0]=D_0'$, and consider an arbitrary element $(z_0^{\eta'+1}|v_0^{\mu'+1})\in\mathscr{S}'[D_0']$. Then, clearly, $(z_1^{\eta'+1}|v_1^{\mu'+1})\in D_0'$, and, since $D_0'=\mathscr{S}^{\gamma}[D_0]$ with $\gamma\geqslant 1$, it follows that $z_{\eta'+1}=f(z_{\eta'-\eta}^{\eta'}|v_{\mu'-\mu}^{\mu'})$. Thus, $\mathscr{S}'[D_0']=\mathscr{S}[D_0']$, whence $\mathscr{S}'[D_0']=\mathscr{S}[\mathscr{S}^{\gamma}[D_0]]=\mathscr{S}^{\gamma+1}[D_0]$, and $\mathscr{S}\sim\mathscr{S}'$ by Theorem 2.5. □

Let $\Sigma: S(R^m)\to S(R^p)$ be a recursive system. Motivated by Corollary 2.6, we next construct a recursive representation of Σ which is of minimal principal degree in its i/o-equivalence class. First, we need some preliminary considerations. Let $\mathscr{S}: y_{k+\eta+1}=f(y_k^{k+\eta}|u_k^{k+\mu})$ be any recursive representation of Σ, and let D_0 be its i/o-space. We assume that $\mu \geqslant 1$. (Formally, the value of μ can always be increased without actually changing the function f by defining the new function $f'(y_k^{k+\eta}|u_k^{k+\mu+1}) := f(y_k^{k+\eta}|u_k^{k+\mu})$.) For each point $(z_0^{\eta-1}|v_0^{\mu-1}) \in (R^p)^\eta \times (R^m)^\mu$ we construct a (possibly empty) projection set $P(z_0^{\eta-1}|v_0^{\mu-1})$ consisting of all points $z_\eta \in R^p$ for which $(z_0^\eta|v_0^\mu) \in D_0$ for some $v_\mu \in R^m$ (if $\eta=0$, replace $(z_0^{\eta-1}|v_0^{\mu-1})$ by $(v_0^{\mu-1})$ throughout). We say that D_0 is *globally degenerate* if, for every point $(z_0^{\eta-1}|v_0^{\mu-1}) \in (R^p)^\eta \times (R^m)^\mu$, the set $P(z_0^{\eta-1}|v_0^{\mu-1})$ contains at most one point (that is, the one step continuation z_η of $(z_0^{\eta-1}|v_0^{\mu-1})$ is uniquely determined). We then have the following

Theorem 2.7

Let $\Sigma : S(R^m) \to S(R^p)$ be a recursive system. Let $\mathscr{S}$ be any recursive representation of Σ, let η be its principal degree, and let D_0 be its i/o-space. Then, Σ has a recursive representation with principal degree less than η if and only if D_0 is globally degenerate.

We start our proof of Theorem 2.7 by showing that if D_0 is globally degenerate, then one can construct a recursive representation of Σ having principal degree $(\eta-1)$. To this end, assume that D_0 is globally degenerate, and let $D_0^1 \subset (R^p)^\eta \times (R^m)^\mu$ be the set of all points $(z_0^{\eta-1}|v_0^{\mu-1})$ for which the set $P(z_0^{\eta-1}|v_0^{\mu-1})$ is non-empty. Then, for every point $(z_0^{\eta-1}|v_0^{\mu-1}) \in D_0^1$, the set $P(z_0^{\eta-1}|v_0^{\mu-1})$ contains exactly one point, so that we can define the function $f_1' : D_0^1 \to R^p$ by $f_1'(z_0^{\eta-1}|v_0^{\mu-1}) := P(z_0^{\eta-1}|v_0^{\mu-1})$ (as before, if $\eta=0$, replace $(z_0^{\eta-1}|v_0^{\mu-1})$ by $(v_0^{\mu-1})$, where we always assume that $\mu \geqslant 1$). Let $f_1 : (R^p)^\eta \times (R^m)^\mu \to R^p$ be any extension of f_1', and consider the recursive representation $\mathscr{S}_1$ of principal degree $(\eta-1)$ given by

$$\mathscr{S}_1 : \; y_{k+\eta} = f_1(y_k^{k+\eta-1}|u_k^{k+\mu-1}) \tag{2.8}$$

We next show that $\mathscr{S}_1$ is another recursive representation of the system Σ, thus proving the ' if ' direction of Theorem 2.7.

Lemma 2.9

The recursive representation $\mathscr{S}_1$ of (2.8) is i/o-equivalent to $\mathscr{S}$, and has D_0^1 as its i/o-space.

Proof of Lemma 2.9

Let $(z_0^\eta|v_0^\mu)$ be any point in D_0, and let $z_{\eta+1} := f(z_0^\eta|v_0^\mu)$. Then, clearly, $(z_1^{\eta+1}|v_1^{\mu+1}) \in D_0$ for every $v_{\mu+1} \in R^m$, and whence, by our construction of f_1, we obtain $z_{\eta+1} = f_1(z_1^\eta|v_1^\mu) = f(z_0^\eta|v_0^\mu)$ for all points $(z_0^\eta|v_0^\mu) \in D_0$. But this implies that $\mathscr{S}_1(u) = \mathscr{S}(u)$ for every $u \in S_0^\mu(R^m)$, so that (by time invariance) $\mathscr{S}_1$ and $\mathscr{S}$ are i/o-equivalent. Moreover, the i/o-space D^* of $\mathscr{S}_1$ is given by

$$\begin{aligned} D^* &= \bigcup_{u \in S_0^{\mu-1}(R^m)} \bigcup_{k \geqslant 0} ([\mathscr{S}_1(u)]_k^{k+\eta-1}|u_k^{k+\mu-1}) \\ &= \bigcup_{u \in S_0^{\mu}(R^m)} \bigcup_{k \geqslant 0} ([\mathscr{S}(u)]_k^{k+\eta-1}|u_k^{k+\mu-1}) \\ &= D_0^1 \end{aligned}$$

as asserted. □

In order to complete the proof of Theorem 2.7 it only remains to prove the necessity direction.

Proof of Theorem 2.7

Necessity. Assume that Σ has a recursive representation of principal degree $\eta' < \eta$, given by $y_{k+\eta'+1} = f'(y_k^{k+\eta'} | u_k^{k+\mu'})$, where we choose $\mu' - \mu = \eta' - \eta$. Let $(z_0^\eta | v_0^\mu)$ be any point of D_0. Then we clearly have that $z_\eta = f'(z_{\eta-1-\eta'}^{\eta-1} | v_{\mu-1-\mu'}^{\mu-1})$, and, since $\eta - 1 - \eta' = \mu - 1 - \mu' \geqslant 0$, it follows that z_η is uniquely determined by $(z_0^{\eta-1} |_0^{\mu-1})$ for every point $(z_0^\eta | v_0^\mu) \in D_0$. Thus, D_0 is globally degenerate, and our proof concludes. □

Returning now to our construction of (2.8), we have obtained from the given recursive representation $\mathscr{S}$ with principal degree η, the reduced recursive representations $\mathscr{S}_1$, having principal degree $(\eta - 1)$ and i/o-space D_0^1. Now, if D_0^1 is still globally degenerate, the same procedure can be applied to $\mathscr{S}_1$, leading to a new recursive representation $\mathscr{S}_2$ of Σ, having principal degree $(\eta - 2)$ and i/o-space D_0^2. After n such steps, where n is at most $\eta + 1$, we obtain a recursive representation $\mathscr{S}_n$ of Σ having principal degree $(\eta - n)$, for which either $\eta - n = -1$ or its i/o-space is no longer globally degenerate. If $\eta - n = -1$ then the recursion function of $\mathscr{S}_n$ does not depend on the output variables, and is of the form $y_k = f_n(u_k^{k+\mu-n})$. Thus, in view of Theorem 2.7, the principal degree of $\mathscr{S}_n$ cannot be further reduced in either case, and $\eta - n$ is the minimal principal degree possible for a recursive representation of Σ. As we see, the minimal recursive representation $\mathscr{S}_n$ can be obtained from an arbitrary given recursive representation $\mathscr{S}$ of Σ through a step by step reduction procedure. We summarize this point in the following.

Corollary 2.10

Let $\mathscr{S}$ be any recursive representation of a recursive system $\Sigma : S(R^m) \to S(R^p)$, and let $\mathscr{C}$ be its i/o-equivalence class. Then, a recursive representation having minimal principal degree in $\mathscr{C}$ can be derived from $\mathscr{S}$ in a finite number of successive reduction steps.

3. Interconnections of recursive systems

In this section we study algebraic properties of recursive systems, related to the series connection, inversion, and sum of such systems. We start with an examination of series connections of recursive systems. Let $\Sigma_1 : S(R^m) \to S(R^p)$ and $\Sigma_2 : S(R^p) \to S(R^q)$ be recursive systems represented by the recursive representations

$$\mathscr{S}_1 : \; v_{k+\eta_1+1} = f_1(v_k^{k+\eta_1} | u_k^{k+\mu_1}) \tag{3.1 a}$$

$$\mathscr{S}_2 : \; y_{k+\eta_2+1} = f_2(y_k^{k+\eta_2} | v_k^{k+\mu_2}), \quad \mu_2 \leqslant \eta_2 \tag{3.1 b}$$

respectively, and consider the series combination $\Sigma_3 := \Sigma_2 \Sigma_1 : S(R^m) \to S(R^q)$. Defining the augmented vector $z_i := \begin{pmatrix} y_i \\ v_i \end{pmatrix} \in R^q \times R^p$, we can directly obtain a recursive representation of the sequence z in terms of the input sequence u to Σ_1 as follows. Let $\eta := \max \{\eta_1, \eta_2\}$, and define the integers $e_1, e_2 \geqslant 0$ (one of which is zero) so that $\eta = \eta_1 + e_1 = \eta_2 + e_2$. Then, since $\mu_2 \leqslant \eta_2$, we have the recursive representation

$$z_{k+\eta+1} = \begin{pmatrix} f_2(y_{k+e_2}^{k+\eta_2+e_2} | v_{k+e_2}^{k+\mu_2+e_2}) \\ f_1(v_{k+e_1}^{k+\eta_1+e_1} | u_{k+e_1}^{k+\mu_1+e_1}) \end{pmatrix} =: f(z_k^{k+\eta} | u_k^{k+\mu_1+e_1}) \tag{3.2}$$

where $f: (R^q \times R^p)^{\eta+1} \times (R^m)^{\mu_1+e_1+1} \to R^q \times R^p$, and where the output value y_i of Σ_3 can be retrieved from the first q entries of z_i, whereas the intermediate variable v_i can be retrieved from the last p entries of z_i.

The above procedure is evidently not applicable in cases where there is no access to the intermediate output sequence $\{v_i\}$—the output sequence of the first system Σ_1 in the series connection. It is also not applicable in stabilization theory, as we shall discuss in a later section. In situations in which the above procedure is not applicable, one has to require that the series combination $\Sigma_3 = \Sigma_2\Sigma_1$ possesses a recursive representation involving only the output sequence $\{y_i\}$ of Σ_2 and the input sequence $\{u_i\}$ of Σ_1. Namely, that there exist integers ξ and ζ and a function $f_3: (R^q)^{\xi+1} \times (R^m)^{\zeta+1} \to R^q$ such that $y_{k+\xi+1} = f_3(y_k^{k+\xi} | u_k^{k+\zeta})$. If such a representation exists, then we say that the series combination $\Sigma_2\Sigma_1$ is *strictly recursive*. We next examine the conditions under which a series combination is strictly recursive. These conditions depend on the structure of certain invariant subspaces which we discuss first.

Consider the systems Σ_1 and Σ_2 of (3.1). For given input values $u_k, u_{k+1}, \ldots, u_{k+\mu_1+j}$ and for fixed initial conditions $v_k, v_{k+1}, \ldots, v_{k+\eta_1}$ of Σ_1, the output values $v_{k+\eta_1+1}, v_{k+\eta_1+2}, \ldots, v_{k+\eta_1+j+1}$ of Σ_1 are uniquely determined through the recursive representation (3.1). Thus, for each integer $i \geqslant 0$, there is a unique function F_i such that

$$\mathbf{F}_i(v_k^{k+\eta_1}) := F_i(v_k^{k+\eta_1} | u_k^{k+e+i}) := (v_{k+i}, v_{k+i+1}, \ldots, v_{k+i+\mu_2}) \tag{3.3}$$

(the augmented vector), where $e := \mu_1 - 1 + \mu_2 - \eta_1$, and where we shall sometimes suppress the variables u_k^{k+e+i} for notational convenience. The functions $\{F_i\}$ directly express the evolution of the output sequence of Σ_1 (taking μ_2 output values at a time) in terms of the initial conditions and the input sequence. We now choose some integer k and leave it fixed throughout our present discussion. Denote by D_∞ the subset of $S(R^p) \times S(R^m)$ consisting of all pairs of sequences $(y|u)$, where $u \in S(R^m)$ and $y = \Sigma_2\Sigma_1 u$. Also, let D_0 be the i/o-space of $\mathscr{S}_1$, and, for every $u \in S(R^m)$, let $D_{v,k}(u)$ be the set of all elements $\alpha_0, \ldots, \alpha_{\eta_1} \in R^p$ for which $(\alpha_0^{\eta_1} | u_k^{k+\mu_1}) \in D_0$. Qualitatively, $D_{v,k}(u)$ is the set of all initial conditions $v_k^{k+\eta_1}$ of $\mathscr{S}_1$ that can appear with the input values $u_k^{k+\mu_1}$. (Recall that Σ_1 is always started from zero initial conditions at some finite time in the past.) Now, for each element $v_k^{k+\eta_1} \in D_{v,k}(u)$, and for each integer $i \geqslant 0$, we define the equivalence class $\{v_k^{k+\eta_1}\}_{i,u}$ consisting of all elements $\beta_0^{\eta_1} \in D_{v,k}(u)$ for which

$$f_2(y_{k+i}^{k+i+\eta_2} | \mathbf{F}_i(\beta_0^{\eta_1})) = f_2(y_{k+i}^{k+i+\eta_2} | \mathbf{F}_i(v_k^{k+\eta_1}))$$

where $y := \Sigma_2\Sigma_1 u$. Then, $\{v_k^{k+\eta_1}\}_{i,u}$ consists of all the initial conditions of Σ_1 which lead to the same output value $y_{k+i+\eta_2+1}$, for the given partial sequences $y_k^{k+i+\eta_2}, u_k^{k+i+e}$. Finally, for each element $(y|u) \in D_\infty$, we define the following decreasing sequence of subsets $K_i(y/u)$ of $(R^p)^{\eta_1+1}$:

$$\left.\begin{aligned} K_i(y|u) &:= \bigcap_{j=0}^{i} \{v_k^{k+\eta_1}\}_{j,u}, \quad i=0,1,2,\ldots \\ K_\infty(y/u) &:= \lim_{i\to\infty} K_i(y|u) \end{aligned}\right\} \tag{3.4}$$

The set $K_\infty(y|u)$ consists of all initial conditions $v_k, \ldots, v_{k+\eta_1}$ for which the output sequences of Σ_1' (corresponding to the input sequence u_k^∞) generate one and the same output sequence y_k^∞ of Σ_2. Qualitatively, $K_\infty(y|u)$ determines a 'Σ_1 invariant subspace in kernel Σ_2'. The set $K_\infty(y|u)$ is non-empty, since $(y|u)\in D_\infty$ implies that y_k^∞ was generated by u_k^∞ for some initial conditions of Σ_1.

We say that the ordered pair (Σ_2, Σ_1) is *asymptotically observable* if the set $K_\infty(y|u)$ contains only one point for each pair $(y|u)\in D_\infty$. When asymptotic observability holds, the initial conditions $v_k, \ldots, v_{k+\eta_1}$ are uniquely determined by the sequences y_k^∞ and u_k^∞, and thus can be uniquely expressed in terms of elements of these sequences. However, this expression may depend on an infinite number of terms of the sequences. We say that the pair (Σ_2, Σ_1) (or that the pair (f_2, f_1)) is *compatible* if there exists an integer r^* such that $K_{r^*+1}(y|u)=K_{r^*}(y|u)$ for all $(y|u)\in D_\infty$. The minimal value of r^* is called the *compatibility degree* of (Σ_2, Σ_1) (or of (f_2, f_1)). For a compatible pair of systems (Σ_2, Σ_1) with compatibility degree r^*, one can readily show that $K_r(y|u)=K_{r^*}(y|u)$ for all integers $r\geq r^*$ and all elements $(y|u)\in D_\infty$. In case the pair (Σ_2, Σ_1') is both compatible and asymptotically observable, then the initial conditions $v_k, \ldots, v_{k+\eta_1}$ of Σ_1' are not only uniquely determined by the sequences y_k^∞ and u_k^∞, but they can be expressed in terms of a finite number of elements of these sequences (see (3.8) below). This fact is of crucial importance to our discussion.

Returning now to series combinations of systems, we can state the following characterization of strict recursivity.

Theorem 3.5

Let $\Sigma_1 : S(R^m)\to S(R^p)$ and $\Sigma_2 : S(R^p)\to S(R^q)$ be recursive systems. Then the series combination $\Sigma_2\Sigma_1'$ is strictly recursive if and only if the pair (Σ_2, Σ_1) is compatible.

Remark

In some applications of Theorem 3.5 given below, the system Σ_1 has as its domain not all of $S(R^m)$, but only a time invariant (i.e. shift-invariant) subset D of it. For such a case our discussion here remains unchanged, except for the obvious fact that the input sequences u to Σ_1 have to be restricted to D throughout the discussion (including the construction of the i/o-space D_0). □

Proof

Assume first that the pair (Σ_2, Σ_1) is compatible, and let r be its compatibility degree. Using time invariance, we obtain from (3.1) for every integer $\zeta\geq 0$ the set of equations

$$y_{k+\eta_2+i+1} - f_2(y_{k+i}^{k+i+\eta_2}|v_{k+i}^{k+i+\mu_2})=0, \quad i=0, \ldots, \zeta \tag{3.6}$$

Substituting the relationship (3.3) into (3.6), and defining the functions

$$h_i(y_{k+i}^{k+i+\eta_2}|u_k^{k+i+e}|v_k^{k+\eta_1}) := f_2(y_{k+i}^{k+i+\eta_2}|F_i(v_k^{k+\eta_1}|u_k^{k+i+e}))$$

we obtain the equations

$$y_{k+\eta_2+i+1} - h_i(y_{k+i}^{k+i+\eta_2}|u_k^{k+i+e}|v_k^{k+\eta_1}) = 0, \quad i = 0, \ldots, \zeta \tag{3.7}$$

Consider now (3.7) as a set of equations for $v_k, \ldots, v_{k+\eta_1}$ in terms of given sequences y_k^∞, u_k^∞, where $(y|u)\in D_\infty$. A slight reflection shows that the subspace $K_j(y|u)$ of (3.4) is the kernel of the set (3.7) for $\zeta = j$. Let

$$(v_k, \ldots, v_{k+\eta_1}) = G(y_k^{k+r+1}|u_k^{k+r+e}) \tag{3.8}$$

be any solution of the set (3.7) for $\zeta = r$, where r is the compatibility degree. Then, since $K_\infty(y|u) = K_r(y|u)$, it follows that (3.8) satisfies the set of equations (3.7) for every integer $\zeta \geqslant r$ as well. Finally substituting (3.8) into (3.7) for $i = r+1$, we obtain the equation

$$y_{k+\eta_2+r+2} = h_{r+1}(y_{k+r+1}^{k+r+1+\eta_2}|u_k^{k+r+1+e}|G(y_k^{k+r+1}|u_k^{k+r+e})) \tag{3.9}$$

which is a recursive representation of the series combination $\Sigma_2\Sigma_1$. Thus, $\Sigma_2\Sigma_1$ is strictly recursive.

Conversely, assume that $\Sigma_2\Sigma_1$ is strictly recursive, and let

$$y_{k+\eta+1} = f(y_k^{k+\eta}|u_k^{k+\mu})$$

be a recursive representation of it (which holds for every $(y|u)\in D_\infty$). Then, for every integer j, the sequence y_j^∞ is uniquely determined by $y_j^{j+\eta}$ and the input sequence u_j^∞. Whence, every set of elements $v_k^{k+\eta_1}$ which satisfies (3.7) for $\zeta = \eta$, also satisfies (3.7) for $\zeta > \eta$. Consequently, when (3.7) is solved for $v_k^{k+\eta_1}$ in terms of y_k^∞ and u_k^∞, the solution is determined by the first η equations, and thus $K_\infty(y|u) = K_\eta(y|u)$ for all $(y|u)\in D_\infty$, and (Σ_2, Σ_1) is compatible. □

We give now an example of a pair of recursive systems for which the series combination is not strictly recursive.

Example 3.10

Let $f(x, y) : R^2 \to R$ be the 'staircase' function defined by $f(x, y) := (n/y)\,\mathrm{sign}\, x$ for $(n-1)/y < |x| \leqslant n/y$, $n = 1, 2, \ldots, y > 1$, and by $f(x, y) := 0$ for $y \leqslant 1$. Notice that when $y \to \infty$ this function tends to the identity function for x, but that, for any finite value of y, it is non-injective in x. Now, let $\Sigma : S(R^2) \to S(R^2)$ be the system represented by

$$\Sigma : z_{k+1} := \begin{pmatrix} z'_{k+1} \\ z''_{k+1} \end{pmatrix} = \begin{pmatrix} f(v'_k, v''_k) \\ v''_k \end{pmatrix}$$

Combine this system in series with the system $\Sigma' : S(R) \to S(R^2)$ given by

$$\Sigma' : v_{k+1} := \begin{pmatrix} v'_{k+1} \\ v''_{k+1} \end{pmatrix} = \begin{pmatrix} v'_k + u_k \\ v''_k + 1 \end{pmatrix}$$

to obtain the system $\Sigma'' := \Sigma\Sigma' : S(R) \to S(R^2)$. From the definition of the function $f(x, y)$ it follows then that in this case the set of equations (3.7) for the intermediate variable v_k' becomes injective when $\zeta \to \infty$ (since then $v_k'' \to \infty$), but this set of equations is non-injective for all finite values of ζ. Therefore, v_k' cannot be expressed in terms of any finite number of elements of the sequences z_k^∞ and u_k^∞, so that a function of the form (3.8) does not exist here. Thus, $\Sigma\Sigma'$ is not strictly recursive. □

As a consequence of the proof of Theorem 3.5 we obtain the following corollary, which is directly implied by (3.9).

Corollary 3.11

Let $\Sigma_1 : S(R^m) \to S(R^p)$ and $\Sigma_2 : S(R^p) \to S(R^q)$ be recursive systems having recursive representations with principal degrees η_1 and η_2, respectively. Assume that the pair (Σ_2, Σ_1) is compatible, and let r be its compatibility degree. Then, the strictly recursive combination $\Sigma_2\Sigma_1$ has a recursive representation with principal degree $\eta \leqslant \eta_2 + r + 1$.

We turn now to a discussion of inverse systems. Let $\Sigma : S(R^m) \to S(R^p)$ be a non-linear recursive system. By restricting the range of Σ to the image of Σ, we obtain a map $\Sigma^r : S(R^m) \to \operatorname{Im} \Sigma$ which is evidently surjective, and whence possesses a right inverse $\Sigma^* : \operatorname{Im} \Sigma \to S(R^m)$. Let $\Sigma^g : S(R^p) \to S(R^m)$ be any extension of Σ^* from $\operatorname{Im} \Sigma$ to the whole space $S(R^p)$. Then, for every element $y \in \operatorname{Im} \Sigma$, we evidently have that $\Sigma\Sigma^g y = \Sigma\Sigma^* y = y$. We call Σ^g a *generalized right inverse* of Σ. As usual, if Σ is not an isomorphism, then a generalized right inverse of Σ is non-unique. The main question that interests us here is whether a recursive system has a *recursive* generalized right inverse. The following statement provides an affirmative answer to this question.

Theorem 3.12

A recursive system $\Sigma : S(R^m) \to S(R^p)$ has a recursive generalized right inverse $\Sigma^g : S(R^p) \to S(R^m)$.

Proof

We construct a recursive generalized right inverse for Σ. Assume that Σ is represented by $y_{k+\eta+1} = f(y_k^{k+\eta} | u_k^{k+\mu})$, and let $y \in \operatorname{Im} \Sigma$ be an output sequence. We now construct recursively for every integer k the sets $S_0^k, S_1^k, \ldots, S_\mu^k$ as follows. The set $S_0^k(y_{k+\mu}^{k+\mu+\eta+1})$ consists of all elements $x \in R^m$ for which there exist elements $z_1, \ldots, z_\mu \in R^m$ satisfying $y_{k+\mu+\eta+1} = f(y_{k+\mu}^{k+\mu+\eta} | x, z_1, \ldots, z_\mu)$; for every $j = 1, \ldots, \mu$ and for every combination of elements $u_{k+\mu-j} \in S_0^{k-j}$, $u_{k+\mu-j+1} \in S_1^{k-j+1}, \ldots, u_{k+\mu-1} \in S_{j-1}^{k-1}$, the set $S_j^k(y_{k+\mu-j}^{k+\mu-j+\eta+1} | u_{k+\mu-j}^{k+\mu-1})$ consists of all elements $x \in R^m$ for which there exist elements $z_1, \ldots, z_{\mu-j} \in R^m$ satisfying $y_{k+\mu-j+\eta+1} = f(y_{k+\mu-j}^{k+\mu-j+\eta} | u_{k+\mu-j}, \ldots, u_{k+\mu-1}, x, z_1, \ldots, z_{\mu-j})$; and, finally, for any combination of elements $u_k \in S_0^{k-\mu}$, $u_{k+1} \in S_1^{k-\mu+1}, \ldots, u_{k+\mu-1} \in S_{\mu-1}^{k-1}$, the set $S_\mu^k(y_k^{k+\eta+1} | u_k^{k+\mu-1})$ consists of all elements $x \in R^m$ for which $y_{k+\eta+1} = f(y_k^{k+\eta} | u_k, \ldots, u_{k+\mu-1}, x)$.

Define now the intersection set $S^k(y_k^{k+\mu+\eta+1} | u_k^{k+\mu-1}) := \bigcap_{i=0}^{\mu} S_i^k$, and note that whenever u is an input sequence generating y, then the set S^k is non-empty for every k. Let D denote the set of all points $(y_k^{k+\mu+\eta+1} | u_k^{k+\mu-1})$ for which S^k is non-empty. For each point in D we choose an element $g(u_k^{k+\mu-1} | y_k^{k+\mu+\eta+1}) \in S^k(y_k^{k+\mu+\eta+1} | u_k^{k+\mu-1})$, and we let $\Sigma^* : \operatorname{Im} \Sigma \to S(R^m)$ be the recursive system represented by

$$u_{k+\mu}=g(u_k^{k+\mu-1}|y_k^{k+\mu+\eta+1}) \tag{3.13}$$

We next show that Σ^* is a right inverse of the restriction Σ^r of Σ to its own image. To this end, let $u_k, \ldots, u_{k+\mu-1} \in R^m$ be fixed elements, and let Y_0 denote the set of all output sequences from Σ generated by input sequences which are continuations of the partial sequence $u_k, \ldots, u_{k+\mu-1}$. Using the $u_k^{k+\mu-1}$ as initial conditions, we next show that (3.13) yields for every sequence $y \in Y_0$ a sequence $\{u_i^*\}$, $i \geqslant k$, where $u_i^*=u_i$ for $i=k, \ldots, k+\mu-1$, which satisfies $y_{k+i+\eta+1}=f(y_{k+i}^{k+i+\eta}|u^{*k+i+\mu}_{k+i})$ for all integers $i \geqslant 0$. Indeed, by our construction of the function g of (3.13) and the fact that $y \in Y_0$, it follows that for each $i=0, 1, 2, \ldots$ and for each $j=1, \ldots, \mu$, there exist elements $z_1, \ldots, z_j \in R^m$ such that $y_{k+i+\eta+1}=f(y_{k+i}^{k+i+\eta}|u^*_{k+i}, \ldots, u^*_{k+i+\mu-j}, z_1, \ldots, z_j)$. Moreover, since $u^*_{k+i+\mu-j+n} \in S^{k+i-j+n}$, $n=1, \ldots, j$, one can actually choose $z_n=u^*_{k+i+\mu-j+n}$ for each n. Thus, $y_{k+i+\eta+1}=f(y_{k+i}^{k+i+\eta}|u^{*k+i+\mu}_{k+i})$ for all integers $i \geqslant 0$, so that $\Sigma\Sigma^* y=y$ for all $y \in \mathrm{Im}\ \Sigma$.

Finally, in order to extend the domain of Σ^* from Im Σ to all of $S(R^p)$, let $g^e : (R^m)^\mu \times (R^p)^{\eta+\mu+2} \to R^m$ be any extension of the function g. Then, the recursive system $\Sigma^g : S(R^p) \to S(R^m)$ represented by $u_{k+\mu}=g^e(u_k^{k+\mu-1}|y_k^{k+\mu+\eta+1})$ is a generalized right inverse of Σ. □

We illustrate the construction of the generalized right inverse described in the proof of Theorem 3.12 through the following example.

Example 3.14

Consider the single-input single-output recursive system $\Sigma : S(R) \to S(R)$ represented by

$$y_{k+2}=h(u_{k+1})+u_k$$

where $h : R \to R$ is a function. In this case, $\eta=\mu=1$, and the sets S_j^k are as follows : $S_0^k(y_{k+1}^{k+3})$ consists of all elements $x \in R$ for which there exists an element $z \in R$ satisfying $y_{k+3}=h(z)+x$. For each element $u \in S_0^{k-1}$, the set $S_1^k(y_k^{k+2}|u_k)$ consists of all elements $x \in R$ satisfying $y_{k+2}=h(x)+u_k$. Finally, $S^k(y_k^{k+3}|u_k)= S_0^k \cap S_1^k$. In particular, if $h=0$, then $S_0^k(y_{k+1}^{k+3})=y_{k+3}$, and $S_1^k(y_k^{k+2}|u_k)=R$ (all real numbers). Thus, for $h=0$, we have $S^k(y_k^{k+3}|u_k)=y_{k+3}$, so that the function of (3.13) is simply $u_{k+1}=g(u\ |y_k^{k+3})=y_{k+3}$. Whence, the right inverse $\Sigma^* : S(R) \to S(R)$ of Σ is given by $u_{k+1}=y_{k+3}$, which is indeed the expected solution for the case $h=0$. □

If the recursive system $\Sigma : S(R^m) \to S(R^p)$ is an isomorphism, then its inverse map $\Sigma^{-1} : S(R^p) \to S(R^m)$ is, of course, uniquely determined. But then Theorem 3.12 implies that Σ^{-1} is a recursive system. We state this fact in the following.

Corollary 3.15

Let $\Sigma : S(R^m) \to S(R^p)$ be a recursive isomorphism. Then, its inverse $\Sigma^{-1} : S(R^p) \to S(R^m)$ also is a recursive isomorphism.

Finally, we shall need a certain refinement of Theorem 3.12 for the case when the domain of Σ is not all of $S(R^m)$, but only a subset of it. Consider a recursive system $\Sigma: D \to S(R^p)$, where D is a subset of $S(R^m)$. It can be shown that, for some choices of D, the system Σ may not have a recursive right inverse $\Sigma^*: \operatorname{Im}\Sigma \to D$, so that Theorem 3.12 cannot be generalized to the case of an arbitrary domain D. Nevertheless, Theorem 3.12 can be generalized to the case when the domain D is of the following particular form, which is of main interest to us below. A subset $D \subset S(R^m)$ is *recursive* if there exists an integer ξ and a function σ assigning to each point $(\alpha_0, \ldots, \alpha_\xi) \in (R^m)^{\xi+1}$ a subset $\sigma(\alpha_0^\xi) \subset R^m$ such that D consists of exactly all sequences $u \in S(R^m)$ satisfying $u_{k+\xi+1} \in \sigma(u_k^{k+\xi})$ for all integers k. The function σ is called the *generating function* of D. For instance, if $\Pi: S(R^q) \to S(R^m)$ is a recursive system, then $\operatorname{Im}\Pi$ is a recursive subset of $S(R^m)$. Indeed, letting $u_{k+\xi+1} = f(u_k^{k+\xi} | v_k^{k+\zeta})$ be a recursive representation of Π, we clearly have for $\operatorname{Im}\Pi$ the generating function $\sigma(u_k^{k+\xi}) := \{u \in R^m : u = f(u_k^{k+\xi} | v_k^{k+\zeta})$ for some $v_k^{k+\zeta}$ for which $(u_k^{k+\xi} | v_k^{k+\zeta}) \in D_0\}$, where D_0 is the i/o-space of Π.

Corollary 3.16

A recursive system $\Sigma: D \to S(R^p)$, where D is a recursive subset of $S(R^m)$, has a recursive right inverse $\Sigma^*: \operatorname{Im}\Sigma \to D$.

Proof

We only have to modify the function g of (3.13) so as to guarantee that the sequences it generates are in D. To this end, let $\sigma(u_k^{k+\xi})$ be a generating function of D, and, using the notation of the proof of Theorem 3.12, let $e := \max\{\xi - (\mu - 1), 0\}$. Then, we choose an element $g'(u_{k-e}^{k+\mu-1} | y_k^{k+\mu+\eta+1}) \in S^k(y_k^{k+\mu+\eta+1} | u_k^{k+\mu-1}) \cap \sigma(u_{k+\mu-1-\xi}^{k+\mu-1})$, and (3.13) will be replaced by $u_{k+\mu} = g'(u_{k-e}^{k+\mu-1} | y_k^{k+\mu+\eta+1})$ (so that there is an increase by e of the principal degree of the inverse). □

We conclude this section with a brief discussion of the sum of two systems, which we defined earlier in the section. Let $\Sigma_1, \Sigma_2: S(R^m) \to S(R^p)$ be recursive systems represented by

$$\Sigma_1: v_{k+\eta_1+1} = f_1(v_k^{k+\eta_1} | u_k^{k+\mu_1})$$
$$\Sigma_2: z_{k+\eta_2+1} = f_2(z_k^{k+\eta_2} | u_k^{k+\mu_2})$$

and let $\Sigma := \Sigma_1 + \Sigma_2$. By definition, the output sequence $y \in S(R^p)$ of Σ is given by $y = v + z$. As before, we are interested in the recursivity properties of the system Σ. Using an approach similar to the one employed in the derivation of (3.2), we can obtain a recursive representation as follows. Define the augmented vector $w_i := \begin{pmatrix} v_i \\ y_i \end{pmatrix}$; let $\eta := \max\{\eta_1, \eta_2\}$; let $e_1, e_2 \geqslant 0$ be integers satisfying $\eta = \eta_1 + e_1 = \eta_2 + e_2$; and let $\delta := \max\{e_1 + \mu_1, e_2 + \mu_2\}$. Then

$$w_{k+\eta+1} = \begin{pmatrix} f_1(v_{k+e_1}^{k+\eta} | u_{k+e_1}^{k+e_1+\mu_1}) \\ f_2([y-v]_{k+e_2}^{k+\eta} | u_{k+e_2}^{k+e_2+\mu_2}) + f_1(v_{k+e_1}^{k+\eta} | u_{k+e_1}^{k+e_1+\mu_1}) \end{pmatrix} =: f(w_k^{k+\eta} | u_k^{k+\delta}) \quad (3.17)$$

As in the case of the series connection, this representation has the disadvantage of not eliminating the intermediate variable v. We say that the sum Σ is *strictly recursive* if it has a recursive representation of the form $y_{k+\xi+1} = h(y_k^{k+\xi} | u_k^{k+\zeta})$, involving only the input sequence u and the output sequence y.

Explicit conditions for strict recursivity of a sum of recursive systems can be derived from (3.17) and Theorem 3.5. Here, we omit a general discussion of this point. We consider only the following particular case, which is encountered in our discussion in the later sections.

Proposition 3.18

Let $\Sigma_1, \Sigma_2 : S(R^m) \to S(R^p)$ be recursive systems, and assume that Σ_2 has a recursive representation of the particular form $z_{k+\eta_2+1} = f_2(u_k^{k+\mu_2})$. Then, the sum $\Sigma_1 + \Sigma_2$ is strictly recursive.

Proof

Let $v_{k+\eta_1+1} = f_1(v_k^{k+\eta_1} | u_k^{k+\mu_1})$ be a recursive representation of Σ_1, and let $y \in S(R^p)$ be an output sequence of $\Sigma_1 + \Sigma_2$ corresponding to the input sequence $u \in S(R^m)$. Then, denoting $e := \eta_2 - \eta_1$, we obtain

$$\begin{aligned} y_{k+\eta_1+1} &= v_{k+\eta_1+1} + z_{k+\eta_1+1} \\ &= f_1(v_k^{k+\eta_1} | u_k^{k+\mu_1}) + f_2(u_{k-e}^{k-e+\mu_2}) \\ &= f_1\{[y_k - f_2(u_{k-e-\eta_1-1}^{k-e-\eta_1-1+\mu_2})], \ldots, [y_{k+\eta_1} - f_2(u_{k-e-1}^{k-e-1+\mu_2})] | u_k^{k+\mu_1}\} \\ &\qquad + f_2(u_{k-e}^{k-e+\mu_2}) \end{aligned}$$

which is a (strictly) recursive representation of $\Sigma_1 + \Sigma_2$. □

4. Stability and internal stability

As was pointed out in the classical work of Liapunov, our intuitive notions of system stability can be accommodated in a formal mathematical framework through the concept of continuity. Continuity here is understood in a strong sense, referring to continuous dependence of the system's output sequence on the input sequence and on the initial conditions. Intuitively, a system is stable if ' small ' changes in the input sequence and in the initial conditions cause only ' small ' changes in the output sequence. Before stating the formal definition of stability that we shall use in our present discussion, we set up the necessary notation.

First, we define a conventional metric ρ on our spaces of sequences, starting with a metric of R^m. For any pair of elements $\alpha := (\alpha^1, \ldots, \alpha^m)$, $\beta := (\beta^1, \ldots, \beta^m)$ of R^m, we define $\rho(\alpha, \beta) := \max_{i=1,\ldots,m} |\alpha^i - \beta^i|$. Next, given two sets of elements $\gamma_1^n := (\gamma_1, \ldots, \gamma_n)$ and $\delta_1^n := (\delta_1, \ldots, \delta_n)$, where $\gamma_i, \delta_i \in R^m$ for all $i = 1, \ldots, n$, we let $\rho(\gamma_1^n, \delta_1^n) := \max\{\rho(\gamma_i, \delta_i), i = 1, \ldots, n\}$. Also for two elements $(z_0^\eta | v_0^\mu), (z'^\eta_0 | v'^\mu_0) \in (R^p)^{\eta+1} \times (R^m)^{\mu+1}$, we define $\rho[(z_0^\eta | v_0^\mu), (z'^\eta_0 | v'^\mu_0)] := \max\{\rho(z_0^\eta | z'^\eta_0), \rho(v_0^\mu, v'^\mu_0)\}$. Given two sequences $u, v \in S(R^m)$, we let $\rho_0(u, v) := \sup_i \rho(u_i, v_i)$, $\rho(u, v) := \sup_i 2^{-|i|} \rho(u_i, v_i)$, and $\rho(u) := \rho(u, 0)$. Summarizing, ρ is the conventional metric used in stability considerations. Our discussion can be easily adapted to alternative definitions of metrics.

Much of our discussion in the present section is related to studies of continuity properties of functions. When talking about continuity, we shall always refer to continuity with respect to the topology induced by the metric ρ. As is usually the case when studying properties related to continuity, it is convenient to restrict attention to bounded sets (of inputs). From the practical point of view, boundedness of the inputs is not a severe limitation, since all signals in a physical system are necessarily bounded. From the theoretical point of view, bounded sets offer some advantages, mainly because they frequently allow the use of the notion of uniform continuity, which is technically easier to treat than the more general notion of continuity. So motivated, we introduce, for every real $\theta \geqslant 0$, the set $S(\theta^m)$, which consists of all sequences $u \in S(R^m)$ for which $\rho_0(u) \leqslant \theta$. We denote by $S_0^\mu(\theta^m)$ the set of all sequences $u \in S_0^\mu(R^m)$ satisfying $\rho_0(u) \leqslant \theta$. Thus, we have sets of bounded (by θ) input sequences. Given a recursive representation $\mathscr{S}: y_{k+\eta+1} = f(y_k^{k+\eta}|u_k^{k+\mu})$ of a system $\Sigma: S(R^m) \to S(R^p)$, the set $S(\theta^m)$ induces the restricted i/o-space $D_0^\theta \subset (R^p)^{\eta+1} \times (R^m)^{\mu+1}$, defined by

$$D_0^\theta := \bigcup_{u \in S_0{}^\mu(\theta^m)} \bigcup_{k \geqslant 0} ([\mathscr{S}(u)]_k^{k+\eta}|u_k^{k+\mu})$$

which is a subset of the i/o-space D_0. As we see, D_0^θ is the i/o-space obtained from $\mathscr{S}$ when the input sequences are bounded by θ. The set $S_0^\mu(\theta^m)$, being a closed and bounded subset of the space of (one-sided) infinite sequences $S_0^\mu(R^m)$, is compact. However, the set D_0^θ is in general neither closed nor bounded. We shall denote by $\mathscr{S}_{|\theta}$ the restriction of $\mathscr{S}$ to $S(\theta^m)$. For example, consider the single-variable system $y_{k+1} = \frac{1}{2}(y_k)^2 + (u_k)^2$. Here, for $\theta = \sqrt{\frac{1}{2}}$, a simple computation shows that $D_0^{\sqrt{\frac{1}{2}}} = [0, 1) \times [-\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}]$, whereas for $\theta > \sqrt{\frac{1}{2}}$, the set D_0^θ is not bounded.

Finally, let $D \subset (R^p)^{\eta+1} \times (R^m)^{\mu+1}$ be a non-empty subset. For every element $d := (z_0^\eta|v_0^\mu) \in D$ and for every sequence $u := u_{k+\mu+1}^\infty$ of elements of R^m, we denote by $\mathscr{S}(d, u) := y_{k+\eta+1}^\infty$ the output sequence generated by $\mathscr{S}$ from the initial conditions d and the input sequence u, that is, $y_{k+j+\eta+1} = f(y_{k+j}^{k+j+\eta}|u_{k+j}^{k+j+\mu})$ for all integers $j \geqslant 0$, where $u_{k+i} := v_i$ for $i = 0, \ldots, \mu$, and $y_{k+i} := z_i$ for $i = 0, \ldots, \eta$. We now define the notion of stability.

Definition 4.1

Let $\mathscr{S}: y_{k+\eta+1} = f(y_k^{k+\eta}|_k^{k+\mu})$ be a recursive representation of a system $\Sigma: S(R^m) \to S(R^p)$, let D be a non-empty subset of $(R^p)^{\eta+1} \times (R^m)^{\mu+1}$, and let $\theta \geqslant 0$ be a real number. Then, $\mathscr{S}_{|\theta}$ is *stable over* D if, for every element $d \in D$ and for every sequence $u := u_{k+\mu+1}^\infty \subset R^m$ with $\rho_0(u) \leqslant \theta$, the following holds :

For every $\epsilon > 0$ there exists a $\delta(d, u, \epsilon) > 0$ such that, for all elements $d' \in D$ and $u' := u'^\infty_{k+\mu+1} \subset R^m$, where $\rho_0(u') \leqslant \theta$, which satisfy (i) $\rho(d, d') < \delta(d, u, \epsilon)$, and (ii) $\rho(u, u') < \delta(d, u, \epsilon)$, one has that $\rho(\mathscr{S}(d, u), \mathscr{S}(d', u')) < \epsilon$.

If the number $\delta(d, u, \epsilon)$ can be chosen independently of d and u, i.e. if $\delta(d, u, \epsilon) = \delta(\epsilon)$, then $\mathscr{S}_{|\theta}$ is *uniformly stable* over D. If $\mathscr{S}_{|\theta}$ is stable over D for any $\theta \geqslant 0$, then we say that $\mathscr{S}$ *is stable over* D. □

Qualitatively speaking, $\mathscr{S}_{|\theta}$ is stable over D if, when started from initial conditions within D, it is continuous with respect to (i) variations of initial conditions within D and with respect to (ii) variations of the input sequence (as long as it remains bounded by θ). The present definition is in the spirit of the classical Liapunov definition of stability.

When considering control problems, it is common practice to adopt the so called ' input/output point of view ', under which one assumes that the system was started (at some finite time in the past) from zero initial conditions. In such case, the domain of $\mathscr{S}$ will always remain confined to the i/o-space D_0. Then, we say that $\mathscr{S}_{|\theta}$ is *i/o-stable* if it is stable over its restricted i/o-space D_0^θ. We say that $\mathscr{S}$ (or Σ) is *i/o-stable* if $\mathscr{S}_{|\theta}$ is i/o-stable for all $\theta \geqslant 0$.

When conditions (i) and (ii) of Definition 4.1 are taken separately, they respectively imply the following two statements (where Proposition 4.3 also depends on the fact that $0 \in D_0^\theta$).

Proposition 4.2

Let $\mathscr{S}: y_{k+\eta+1} = f(y_k^{k+\eta} | u_k^{k+\mu})$ be a recursive representation of a system $\Sigma: S(R^m) \to S(R^p)$, let $D \subset (R^p)^{\eta+1} \times (R^m)^{\mu+1}$ be a subset, and let $\theta \geqslant 0$ be a real number. If $\mathscr{S}_{|\theta}$ is stable over D, then the function f is continuous over D.

Proposition 4.3

Let $\mathscr{S}: y_{k+\eta+1} = f(y_k^{k+\eta} | u_k^{k+\mu})$ be a recursive representation of a system $\Sigma: S(R^m) \to S(R^p)$, and let $\theta > 0$ be a real number. If $\mathscr{S}_{|\theta}$ is i/o-stable, then Σ represents a continuous map $S_0^\mu(\theta^m) \to S(R^p)$.

We remark that the converse directions of Propositions 4.2 and 4.3 are not true, namely, the continuity of f is not a sufficient condition for the stability of $\mathscr{S}_{|\theta}$, and neither is the continuity of Σ as a map $S_0^\mu(\theta^m) \to S(R^p)$.

Proposition 4.4

Let $\mathscr{S}: y_{k+\eta+1} = f(y_k^{k+\eta} | u_k^{k+\mu})$ be a recursive representation of a system $\Sigma: S(R^m) \to S(R^p)$, let $\theta > 0$ be a real number. If $\mathscr{S}_{|\theta}$ is i/o-stable, then $\Sigma[S_0^\mu(\theta^m)]$ is a ρ-bounded set.

Proof

Assume that $\mathscr{S}_{|\theta}$ is i/o-stable. Then, by Proposition 4.3, the map $\Sigma: S(\theta^m) \to S(R^p)$ is continuous. Whence, since $S_0^\mu(\theta^m)$ is a compact subset of $S(\theta^m)$, it follows that the image $\Sigma[S_0^\mu(\theta^m)]$ is compact (see, for example, Kuratowski (1961, ch. 15)). Consequently, $\Sigma[S_0^\mu(\theta^m)]$ is bounded. □

Regarding the notion of uniform stability, we have the following analogue of a classical theorem on continuous functions.

Proposition 4.5

Let $\mathscr{S}: y_{k+\eta+1} = f(y_k^{k+\eta} | u_k^{k+\mu})$ be a recursive representation of a system $\Sigma: S(R^m) \to S(R^p)$, let $D \subset (R^p)^{\eta+1} \times (R^m)^{\mu+1}$ be a bounded set, and let $\theta > 0$ be a real number. If $\mathscr{S}_{|\theta}$ is stable over the closure $\bar{D}$, then it is uniformly stable there.

Proof

Let $S_x(\theta^m)$ denote the set of all sequences $u := u_{k+\mu+1}^\infty \subset R^m$ satisfying $\rho_0(u) \leqslant \theta$. Then, the recursive representation $\mathscr{S}$ induces a map $\mathscr{S}^x$ from $\bar{D} \times S_x(\theta^m)$ to the set of sequences, given by $(d, u) \mapsto \mathscr{S}(d, u)$ for all $d \in \bar{D}$ and $u \in S_x(\theta^m)$. It is easy to see that $\mathscr{S}_{|\theta}$ is uniformly stable over $\bar{D}$ if and only if $\mathscr{S}^x$ is uniformly continuous over $\bar{D} \times S_x(\theta^m)$. Now, assume that $\mathscr{S}_{|\theta}$ is stable over

$\bar{D}$. Then, $\mathscr{S}^x$ is clearly continuous over $\bar{D}\times S_x(\theta^m)$. But, since $\bar{D}\times S_x(\theta^m)$ is compact, the latter implies that $\mathscr{S}^x$ is uniformly continuous over $\bar{D}\times S_x(\theta^m)$ (see, for example, Kuratowski (1961)), so that, by our opening remarks, $\mathscr{S}_{|\theta}$ is uniformly stable over $\bar{D}$. □

As we have discussed before, the notion of i/o-stability refers to the stability of the system in a situation where it was started from zero initial conditions at some finite time in the past. Though this situation is the most common one encountered in the practice of control engineering, it is nevertheless well established that the notion of i/o-stability is too weak to have any direct practical implications. The reason for this fact is that inevitable errors in measurement, and noises in the systems environment, actually preclude any possibility of fixing the initial conditions at any prescribed values with an absolute degree of accuracy. What is really known is that the initial conditions are ' close ' to their prescribed value. Thus, due to noise and measurement error, a system can never be confined exclusively to its i/o-space, and this inevitable deviation from the idealized input/output approach has to be taken into account. Such considerations have lead to the introduction of the already classical concept of internal stability, which, in our present framework, is defined as follows.

Let $\mathscr{S}: y_{k+\eta+1}=f(y_k^{k+\eta}|u_k^{k+\mu})$ be a recursive representation of a system $\Sigma: S(R^m)\to S(R^p)$, and let D_0^θ, where $\theta>0$, be the restricted i/o-space of $\mathscr{S}$. For a real number $\zeta>0$, denote by $D_{0,\zeta}^\theta$ the set of all elements $d\in(R^p)^{\eta+1}\times(R^m)^{\mu+1}$ satisfying $\rho(d, D_0^\theta)<\zeta$ (i.e. a ζ-neighbourhood of D_0^θ). Then, we say that $\mathscr{S}_{|\theta}$ is *internally stable* if there exists a $\zeta>0$ such that $\mathscr{S}_{|\theta}$ is stable over $D_{0,\zeta}^\theta$. Qualitatively speaking, $\mathscr{S}_{|\theta}$ is internally stable if any small deviations (possibly outside the i/o-space) in its initial conditions do not destroy stability. We say that $\mathscr{S}$ is *internally stable* if $\mathscr{S}_{|\theta}$ is internally stable for all $\theta>0$. Clearly, $D_0^\theta\subset D_{0,\zeta}^\theta$ for any $\zeta>0$, so that an internally stable system is i/o-stable as well.

As we have seen in § 3, a recursive system $\Sigma: S(R^m)\to S(R^p)$ determines a class $C(\Sigma)$ of i/o-equivalent recursive representations of itself. When the system Σ is stable, then each one of the recursive representations in $C(\Sigma)$ will be i/o-stable. The main question in this context is whether $C(\Sigma)$ also contains an internally stable representation (when Σ is stable). The interest in this question stems from our above observation that only internally stable representations are ' really ' stable from the engineering point of view. To state things in somewhat more exact terms, we are interested in the following question.

Problem 4.6

Given an i/o-stable recursive representation $\mathscr{S}$, when does there exist an internally stable recursive representation $\mathscr{S}_*$ which is i/o-equivalent to $\mathscr{S}$. And, if $\mathscr{S}_*$ exists, how does one construct it from the given representation $\mathscr{S}$.

In the particular case of time-invariant finite-dimensional linear systems, the answer to Problem 4.6 is always in the affirmative, namely, every stable linear system has an internally stable recursive representation. Indeed, it is well known that the minimal (i.e. reachable and observable) representation of a stable linear system exhibits a stable response for any set of initial conditions. Thus, the minimal representation of a stable linear system also is internally stable. However, in the general case of non-linear recursive systems, the situation is somewhat more delicate, and it deserves a detailed discussion. The statement that we have in our mind in this context is that, if a recursive representation $\mathscr{S}$ (of a system $\Sigma : S(R^m) \to S(R^p)$) is stable over the closure $\bar{D}_0$ of its i/o-space D_0, then it can be extended into an internally stable representation. (Recall from § 2 that $\mathscr{S}$ is uniquely determined by Σ on D_0, but it is arbitrary outside D_0.) That there is a connection between the system Σ, the stability of $\mathscr{S}$ over $\bar{D}_0$, and the existence of an internally stable representation of Σ, is seen as follows.

Consider a recursive representation $\mathscr{S} : y_{k+\eta+1} = f(y_k^{k+\eta} | u_k^{k+\mu})$ of a system $\Sigma : S(R^m) \to S(R^p)$. Let $\theta > 0$ be a real number, and let D_0^θ be the restricted i/o-space of $\mathscr{S}$. If $\mathscr{S}_{|\theta}$ is internally stable then, since evidently $\bar{D}_0^\theta \subset D_{0,\zeta}^\theta$ for any $\zeta > 0$, it follows that $\mathscr{S}_{|\theta}$ is stable over $\bar{D}_0^\theta$. Thus, the condition that $\mathscr{S}_{|\theta}$ is stable over $\bar{D}_0^\theta$ is a necessary condition for the internal stability of $\mathscr{S}$. We now claim that this condition depends directly on the system Σ, and not on the particular representation $\mathscr{S}$ that we consider. Indeed, in view of Theorem 2.5, we have that, for every point $d \in D_0^\theta$, the value $y := f(d)$ is uniquely determined by the input/output map Σ, so that the function f is uniquely determined by Σ on D_0^θ. Furthermore, if $\mathscr{S}_{|\theta}$ is stable over $\bar{D}_0^\theta$, then, by Proposition 4.2, the function f is continuous over $\bar{D}_0^\theta$. In such case, the values of f on $\bar{D}_0^\theta$ are uniquely determined by its values on D_0^θ through the continuity requirement that, for any point $d \in \bar{D}_0^\theta$

$$f(d) = \lim_{n\to\infty} f(x_n) \tag{4.7}$$

where $\{x_n\} \subset D_0^\theta$ is any sequence converging to d. Thus, if f has a continuous extension to $\bar{D}_0^\theta$, then its values there are uniquely determined by Σ. In summary, the condition that $\mathscr{S}_{|\theta}$ be stable over $\bar{D}_0^\theta$ is a necessary condition for internal stability, and it depends directly on the system Σ, and not on the particular representation $\mathscr{S}$. The question is, of course, whether this necessary condition also is sufficient. Below, we give an affirmative answer to this question under a certain assumption on the system Σ, which we use in order to simplify our discussion.

We start with an examination of the restricted i/o-space D_0^θ, showing that, for a stable system, the restricted i/o-space is a connected set.

Proposition 4.8

Let $\mathscr{S} : y_{k+\eta+1} = f(y_k^{k+\eta} | u_k^{k+\mu})$ be a recursive representation of a system $\Sigma : S(R^m) \to S(R^p)$, let $\theta > 0$ be a real number, and let D_0^θ be the restricted i/o-space of $\mathscr{S}$. If $\mathscr{S}_{|\theta}$ is i/o-stable, then D_0^θ is a connected set.

Proof

Assume that $\mathscr{S}_{|\theta}$ is i/o-stable, and recall that then the function f is continuous over D_0^θ by Proposition 4.2. Define the set

$$A_j := \bigcup_{u \in S_0^\mu(\theta^m)} \bigcup_{i=0}^{j} ([\mathscr{S}(u)]_i^{i+\eta} | u_i^{i+\mu})$$

so that $D_0^\theta = \bigcup_{j=0}^{\infty} A_j$. Evidently, $A_j \subset A_{j+1}$ for all integers $j \geq 0$. Assume now, for a moment, that $(*) A_j$ is a connected set for all integers $j \geq 0$. Then it follows by the previously mentioned facts that D_0^θ is a connected set as well. Thus, our proof will conclude upon proving (*), which we now do by induction. First, we note that $A_0=(0_0^\eta|0_0^\mu)$ consists of only one point, and whence is connected. Further, assume that, for some $n\geq 0$, the set A_n is connected, and define the function $F:(R^p)^{\eta+1}\times(R^m)^{\mu+1}\to(R^p)^{\eta+1}\times(R^m)^\mu$ by $F(y_0^\eta|u_0^\mu):=(y_1,\ldots,y_\eta,f(y_0^\eta|u_0^\mu)|u_1,\ldots,u_\mu)$. Then since f is continuous, so is also F. Let $B_{j+1}:=F[A_j]$, and notice that $A_{j+1}=B_{j+1}\times[-\theta,\theta]^m$ (i.e. adjoining $u_{\mu+1}$ to the input coordinates). By the continuity of F and the fact that A_n is connected by the induction assumption, it follows that B_{n+1} is connected (since it is the continuous image of a connected set (see Kuratowski 1961)). But then, since $A_{n+1}=B_{n+1}\times[-\theta,\theta]^m$ is the cross product of two connected sets, we obtain that A_{n+1} is connected, so that (*) holds by induction, and proof concludes. □

As we see from Proposition 4.8, the restricted i/o-space D_0^θ of a stable system is always a connected set. The proof of our following statement becomes simpler if we assume that D_0^θ is not only connected, but also convex. Recall that a set $S\subset R^n$ is convex if, for any pair of points $s_1, s_2\in S$, the straight line segment connecting s_1 and s_2 is in S. For example, consider the single-input single-output system $y_{k+1}=\frac{1}{2}(y_k)^2+u_k$ with $\theta=\frac{1}{2}$. Here, a simple computation shows that $D_0^{1/2}=[-\frac{1}{2},1)\times[-\frac{1}{2},\frac{1}{2}]$, which clearly is a convex set. The closure of the restricted i/o-space here is $\bar{D}_0^{1/2}=[-\frac{1}{2},1]\times[-\frac{1}{2},\frac{1}{2}]$. Below, we denote by $\bar{\mathcal{S}}_{|\theta}$ the recursive representation obtained from $\mathcal{S}_{|\theta}$ by extending the recursion function f from D_0^θ to $\bar{D}_0^\theta$ through (4.7), whenever such an extension exists.

Theorem 4.9

Let $\mathcal{S}: y_{k+\eta+1}=f(y_k^{k+\eta}|u_k^{k+\mu})$ be a recursive representation of a system $\Sigma: S(R^m)\to S(R^p)$, and assume that, for some $\theta>0$, the restricted i/o-space D_0^θ of $\mathcal{S}$ is a bounded convex set. If the extension $\bar{\mathcal{S}}_{|\theta}$ of $\mathcal{S}_{|\theta}$ exists and is stable over $\bar{D}_0^\theta$, then there is a recursive representation $\mathcal{S}'$ for which $\mathcal{S}'_{|\theta}\sim\mathcal{S}_{|\theta}$ and $\mathcal{S}'_{|\theta}$ is internally stable.

Proof

We assume that D_0^θ is bounded convex, and that $\bar{\mathcal{S}}_{|\theta}$ is stable over $\bar{D}_0^\theta$. Now, choose some $\zeta>0$. We construct a recursive representation $\mathcal{S}': y_{k+\eta+1}=f'(y_k^{k+\eta}|u_k^{k+\mu})$ defined over $\bar{D}_{0,\zeta}^\theta$, which satisfies (i) for all elements $d\in\bar{D}_0^\theta$ and $u\in S_x(\theta^m)$ (where $S_x(\theta^m)$ was defined in the proof of Proposition 4.5), one has $\mathcal{S}'(d,u)=\mathcal{S}(d,u)$; and (ii) there is a fixed integer $J\geq 0$ such that, for all elements $d\in\bar{D}_{0,\zeta}^\theta$ and $u\in S_x(\theta^m)$, one has that $([\mathcal{S}'(d,u)]_j^{j+\eta}|u_j^{j+\mu})\in\bar{D}_0^\theta$ for all $j\geq\kappa+J$, where κ is the time when the system is started from the initial conditions d. Namely, $\mathcal{S}'$ coincides with $\bar{\mathcal{S}}$ after at most J steps. Assume, for a moment, that we have constructed a representation $\mathcal{S}'$ satisfying conditions (i) and (ii), for which the function f' is continuous over $\bar{D}_{0,\zeta}^\theta$. Then, using the facts that $\bar{D}_{0,\zeta}^\theta$ is compact; that any finite iteration of f' is continuous (since f' is continuous) over $\bar{D}_{0,\zeta}^\theta$; and that $\mathcal{S}_{|\theta}$ is stable over $\bar{D}_0^\theta$, it can be readily shown that (i) and (ii) imply that the representation $\mathcal{S}'_{|\theta}$ is stable over $\bar{D}_{0,\zeta}^\theta$. Thus, our proof will conclude upon constructing a continuous function f' satisfying (i) and (ii).

In order to make the construction of f' more transparent, we divide it into two stages, where in the first stage we extend only the domain of the initial conditions y_0^η to a ζ-neighbourhood, and in the second stage we extend the domain of the initial input values u_0^μ to a ζ-neighbourhood (for all points of the previous stage). For the first stage, we construct an extension D_1 of $\bar{D}_0^\theta$ given by

$$D_1 := \{(s_0^\eta | v_0^\mu) : \rho(s_0^\eta, z_0^\eta) < \zeta \text{ and } (z_0^\eta | v_0^\mu) \in \bar{D}_0^\theta\}$$

that is, including neighbourhoods of radius ζ of all the initial conditions z_0^η. Let $d := (s_0^\eta | v_0^\mu)$ be any point of $\bar{D}_1$. We now assign to d a point $d^* \in \bar{D}_0^\theta$ as follows: Let $a_1, a_2, \ldots, a_{p(\eta+1)}$ be the real numbers such that $a_{ip+1}, a_{ip+2}, \ldots, a_{ip+p}$ are the entries of the vector $s_{(\eta-i)} \in R^p$, $i = 0, \ldots, \eta$ (note the reverse ordering). For notational convenience, we identify $(a_1, \ldots, a_{p(\eta+1)})$ with s_0^η through the above ordering. Now, for each $j = 1, 2, \ldots, p(\eta+1)$, let a_j^* be the closest real number to a_j for which there exist numbers $x_{j+1}, \ldots, x_{p(\eta+1)}$ satisfying

$$(a_1^*, a_2^*, \ldots, a_j^*, x_{j+1}, \ldots, x_{p(\eta+1)} | v_0^\mu) \in \bar{D}_0^\theta$$

Let $s^{*\eta}_0 := (a_1^*, a_2^*, \ldots, a_{p(\eta+1)}^*)$, and denote $d^* := (s^{*\eta}_0 | v_0^\mu)$. Since $\bar{D}_0^\theta$ is non-empty, d^* always exists. We next show that d^* is unique.

Let σ_j, $j = 1, \ldots, p(\eta+1) - 1$, be the set of all elements $(x_{j+1}, \ldots, x_{p(\eta+1)})$ for which $(a_1^*, \ldots, a_j^*, x_{j+1}, \ldots, x_{p(\eta+1)}) \in \bar{D}_0^\theta$, and let $\sigma_0 := \bar{D}_0^\theta$. Then, σ_j is non-empty for all j. In view of our assumption that D_0^θ is convex, it follows that all of the sets σ_j, $j = 0, \ldots, p(\eta+1) - 1$, are connected. Whence, for each $j = 0, \ldots, p(\eta+1) - 1$, the set γ_j of all elements $x \in R$ such that $(x, x_{j+2}, \ldots, x_{p(\eta+1)}) \in \sigma_j$ for some $x_{j+2}, \ldots, x_{p(\eta+1)}$ is an interval $[\alpha_j, \beta_j]$ in R (we note that $[\alpha_j, \beta_j]$ is the projection of σ_j on its first coordinate). But then, for each $j = 0, \ldots, p(\eta+1) - 1$, either $a_{j+1} \in [\alpha_j, \beta_j]$, or $a_{j+1} > \beta_j$, or $a_{j+1} < \alpha_j$, in which case the unique value for a_{j+1}^* is $a_{j+1}^* = a_{j+1}$, or $a_{j+1}^* = \beta_j$, or $a_{j+1}^* = \alpha_j$, respectively. Thus, d^* is uniquely determined by d, and we have obtained a function $\phi : \bar{D}_1' \to \bar{D}_0^\theta : d \to d^*$. Moreover, it can be readily seen that ϕ is continuous.

Still using the above notation, we define the function $f_1 : \bar{D}_1 \to R^p$ by

$$\mathscr{S}_1: \qquad f_1(s_0^\eta | v_0^\mu) := f(s^{*\eta}_0 | v_0^\mu) \quad (= f \circ \phi) \tag{4.10}$$

for all points $(s_0^\eta | v_0^\mu) \in \bar{D}_1$. Then, by the continuity of f (see Proposition 4.2) and the continuity of ϕ, the function f_1 is continuous over $\bar{D}_1$. Also, by our construction of f_1 and of $s^{*\eta}_0$, we obtain that, for every $d \in \bar{D}_1$ and $u \in S_x(\theta^m)$, the output sequence $y^{*\infty}_{\kappa+\eta+1} := \mathscr{S}_1(d, u)$ (where, as before, κ is the starting time from the initial conditions d) satisfies $y^*_{\kappa+\eta+1+i} = y_{\kappa+\eta+1+i}$ for all $i \geq \eta$. Thus, conditions (i) and (ii) hold for $\mathscr{S}_1$ over $\bar{D}_1$. In view of the continuity of f_1, this completes the first stage of our extension (going from $\bar{D}_0^\theta$ to $\bar{D}_1$).

The second and final stage of our extension of $\mathscr{S}$ (extending from $\bar{D}_1$ to $\bar{D}_{0,\zeta}^\theta$) is done similarly to the first stage, by interchanging the roles of z_0^η and v_0^μ; by replacing $\bar{D}_0^\theta$ by $\bar{D}_1$; and by replacing $\bar{D}_1$ by $\bar{D}_{0,\zeta}^\theta$. □

5. Rational systems

In the present section we study the representation of a given non-linear system as a quotient of two stable systems. More specifically, let $\Sigma : S(R^m) \to S(R^p)$ be a recursive system. Regarding Σ as a map, it is known (for example, Maclane and Birkhoff (1979, ch. 1)) that it can be factored into $\Sigma = PQ$, where $P: S \to S(R^p)$ is an injective map, $Q: S(R^m) \to S$ is a surjective map, and where S is a suitable space. Now, every injective map has a left inverse, and every surjective map has a right inverse. Thus, there are maps $P^*: S(R^p) \to S$ and $Q^*: S \to S(R^m)$ such that $P^*\Sigma = Q$ and $\Sigma Q^* = P$. From the control theoretic point of view, the cases of interest are those where either P^* and Q are both stable, or Q^* and P are both stable. In the first case, the system Σ can be stabilized by non-singular (on Im Σ) stable postcompensation, whereas in the latter case, the system Σ can be stabilized by non-singular stable precompensation (here, by non-singular we mean injective). The solution to the problem of stabilizing through non-singular stable compensation forms the first stage of the solution to the problem of internally stabilizing a non-linear system, in close analogy to the situation in the case of linear systems (see Desoer and Chan 1975, Hammer 1983 a, c). Of course, we are particularly interested in cases where P^* and Q^* are recursive systems. Formally, we devote the present section to the construction of so called 'stability representations', which are defined as follows.

Definition 5.1

Let $\Sigma: S(R^m) \to S(R^p)$ be a non-linear recursive system. Assume that there is a factorization $\Sigma = PQ$ where, for some integer $q \geqslant 0$, the system $Q: S(R^m) \to S$, $S \subset S(R^q)$, is a surjective recursive system, and $P: S \to S(R^p)$ is an injective recursive system. Then, $\Sigma = PQ$ is a *left stability representation* of Σ if $q = p$, and if Q is stable and P has a recursive stable left inverse P^* : Im $\Sigma \to S$; $\Sigma = PQ$ is a *right stability representation* of Σ if $q = m$ and if P is stable and Q has a recursive stable right inverse $Q^*: S \to S(R^m)$. The system Σ is *left* (respectively, *right*) *rational* if it has a left (respectively, right) stability representation. □

As is well known, a finite-dimensional time-invariant linear system always has both right and left stability representations. One such representation is induced by the usual polynomial matrix fraction representation of the transfer matrix of the system. However, in the non-linear case there are recursive systems which do not possess stability representations. In our discussion below we give necessary and sufficient conditions for the existence of left and of right stability representations. We also describe the construction of such representations whenever they exist. We start with a brief investigation of the discontinuities of a rational system. In the following statement we show that a left rational system cannot have finite jump discontinuities. The only type of discontinuity that it can have is divergence, namely, it may transform a bounded input sequence into an unbounded output sequence. Thus we see that, in contrast to the case of linear systems where rationality is a mild requirement, for non-linear systems rationality is a rather strong condition, and the class of non-linear rational systems is substantially smaller than the whole class of non-linear recursive systems. Let $\Sigma: S(R^m) \to S(R^p)$ be a recursive system. Adhering to classical terminology, we say that Σ is *BIBO* (bounded-input bounded-output)-*stable* if, for every $\theta > 0$, there exists an $M(\theta) > 0$ such that, whenever an input sequence $u \in S(R^m)$ satisfies $\rho_0(u) \leqslant \theta$, then the output sequence satisfies $\rho_0(\Sigma u) \leqslant M(\theta)$ (where $M(\theta)$ and θ are both finite).

Theorem 5.2

Let $\Sigma: S(R^m) \to S(R^p)$ be a recursive left rational system. If Σ is BIBO-stable, then $\Sigma: S_0^\mu(R^m) \to S(R^p)$ is a continuous map.

Proof

Let $\mathscr{S}: y_{k+\eta+1} = f(y_k^{k+\eta} | u_k^{k+\mu})$ be a recursive representation of Σ, and let $\Sigma = PQ$ be a left stability representation. Then, $P: \text{Im}\, Q \to \text{Im}\, \Sigma$ is both injective and surjective, and, since its inverse P^{-1} is stable, it follows by Proposition 4.3 that P^{-1} is continuous on $S_0^\mu(\theta^p)$. For an arbitrary $\theta > 0$, define the sets $A := Q[S_0^\mu(\theta^m)]$ and $B := \Sigma[S_0^\mu(\theta^m)]$, so that $B = P[A]$. Now, since $S_0^\mu(\theta^m)$ is compact and since Q is continuous by Proposition 4.3, it follows that the set A is compact. Whence, A is closed and, since P^{-1} is continuous and $B = (P^{-1})^{-1}[A]$, we have that B is closed as well. Also, by our assumption that Σ is BIBO-stable, B is a bounded set. Thus B, being bounded and closed, is compact. But then, since $P^{-1}[B] = A$, since P^{-1} is continuous, and since every continuous injective and surjective function over a compact domain is a homeomorphism (for example, Kuratowski (1961, ch. 15)), it follows that P is continuous over A, and whence $\Sigma = PQ$ is continuous over $S_0^\mu(\theta^m)$. Finally, since our argument holds for any $\theta > 0$, the assertion follows. □

Theorem 5.2 is a manifestation of a very interesting analogy between non-linear rational systems and finite-dimensional time-invariant linear systems. It is well known that such a linear system $\Sigma_L: S(R^m) \to S(R^p)$ has the important property that it is BIBO-stable if and only if it is continuous as a map (Kwakernaak and Sivan 1972). As we now see, this property is a direct consequence of left rationality, and it is shared by any recursive non-linear left rational system. The proof of Theorem 5.2 can be used to prove the following slightly stronger statement.

Corollary 5.3

Let $\Sigma: S(R^m) \to S(R^p)$ be a recursive left rational system, with recursive representation $\mathscr{S}: y_{k+\eta+1} = f(y_k^{k+\eta} | u_k^{k+\mu})$. Let $C \subset S_0^\mu(R^m)$ be a compact set. If $\Sigma[C]$ is a bounded set, then Σ is continuous over C.

Our next objective is to obtain a characterization of rationality in terms of the recursive representation of the system. This characterization will elucidate the connection between rationality and certain properties of the (given) recursion function of the system. To state things somewhat more precisely, let $\Sigma: S(R^m) \to S(R^p)$ be a recursive system, and let $\mathscr{S}: y_{k+\eta+1} = f(y_k^{k+\eta} | u_k^{k+\mu})$ be a recursive representation of it. We show that Σ is left rational if and only if the function f can be decomposed into a sum of functions $f = f_1 + f_2$, where f_1 and f_2 are required to satisfy certain conditions discussed below. Once the functions f_1 and f_2 are computed, they directly determine recursive representations for systems P and Q in a left stability representation $\Sigma = PQ$. One interesting feature of this characterization is the fact that its basic ingredients are valid for numerous definitions of the notion of stability, not just for the one adopted in Definition 4.1. Thus, it will also be suitable for application in situations in which a different notion of stability is used, for example, in cases where one is specifically interested in BIBO-stability, or in other notions of stability, as we discuss after stating the next theorem. In the meanwhile, we proceed with an introductory discussion, explaining some features of this characterization and giving an indication as to its origin.

Consider again the system Σ and its recursive representation $\mathscr{S}$. We denote by D_∞ the subset of $(\operatorname{Im}\Sigma)\times S_0^\mu(R^m)$ consisting of all elements $(y|u)$ such that $y=\Sigma u$. Let $\Sigma = PQ$ be any factorization of Σ, where $Q: S(R^m)\to S$, $S\subset S(R^p)$, and $P: S\to S(R^p)$ are recursive systems, and where P is injective and Q is surjective. Then, clearly, the kernel of Q is equal to the kernel of Σ. Now, by Corollary 3.16, the system $P^{-1}: \operatorname{Im}\Sigma\to S$ also is recursive, so let

$$\left.\begin{aligned}\mathscr{S}_1: \ z_{k+\alpha+1} &= h_1(z_k^{k+\alpha}|u_k^{k+\beta})\\ \mathscr{S}_2: \ z_{k+\alpha+1} &= h_2(z_k^{k+\alpha}|y_k^{k+\beta})\end{aligned}\right\} \tag{5.4}$$

be recursive representations of Q and P^{-1}, respectively. (Note that, since the principal degree can always be increased by shifting, we can assume that $\mathscr{S}_1$ and $\mathscr{S}_2$ have the same principal degree.) Let

$$\mathscr{S}_3: \ y_{k+\gamma+1} = h_2^*(y_k^{k+\gamma}|z_k^{k+\delta})$$

be a recursive representation of P. Now, since by assumption $\Sigma = PQ$ and Σ is recursive in u and y, it follows by Theorem 3.5 that the pair (h_2^*, h_1) is compatible. Whence, using (3.8) and the fact that P is injective, one can express $z_k = G_0(y_k^{k+c}|u_k^{k+d})$, $k=\dots,-1,0,1,\dots$, where c and d are suitable integers. Substituting now the function $G_0(\cdot)$ for $z_k,\dots,z_{k+\alpha}$ into the right hand sides of (5.4), we obtain the systems

$$A: \ D_\infty\to S(R^p): \ z_{k+\alpha+1} = f_3(y_k^{k+a}|u_k^{k+b}) \tag{5.5 a}$$

$$B: \ D_\infty\to S(R^p): \ z_{k+\alpha+1} = f_4'(y_k^{k+a}|u_k^{k+b}) \tag{5.5 b}$$

where the functions f_3 and f_4' are the respective compositions of h_1 and of h_2 with G_0, and where a, b are suitable integers. We note that the systems A and B are *trivially recursive* in the sense that the functions f_3 and f_4' do not depend on z. Equating (5.5 a) and (5.5 b), we obtain the equation

$$-f_3(y_k^{k+a}|u_k^{k+b}) + f_4'(y_k^{k+a}|u_k^{k+b}) = 0 \tag{5.6}$$

which holds for every element $(y|u)\in D_\infty$. From the recursive representation of Σ we also have $y_{k+\eta+1} - f(y_k^{k+\eta}|u_k^{k+\mu}) = 0$ for every element $(y|u)\in D_\infty$. Defining $f_4(y_k^{k+e}|u_k^{k+b}) := y_{k+\eta+1} - f_4'(y_k^{k+a}|u_k^{k+b})$, where $e := \max\{a, \eta+1\}$, and equating the latter two zero-expressions, we finally obtain the sum decomposition

$$f(y_k^{k+\eta}|u_k^{k+\mu}) = f_4(y_k^{k+e}|u_k^{k+b}) + f_3(y_k^{k+a}|u_k^{k+b}) \tag{5.7}$$

which is valid for all elements $(y|u)\in D_\infty$.

Some further properties of the sum decomposition (5.7) are of interest to us. First, let $h: (R^p)^{\eta'+1}\times(R^m)^{\mu'+1}\to R^p$ be a function. We say that the pair (h, f) is *adapted* if, for all pairs of elements $u, u'\in S(R^m)$ for which $\Sigma u = \Sigma u' =: y$, one has $h(y_k^{k+\eta'}|u_k^{k+\mu'}) = h(y_k^{k+\eta'}|u'^{k+\mu'}_k)$ for all integers k (that is, a kernel containment condition). Then, in view of the fact that $\operatorname{Ker} Q = \operatorname{Ker}\Sigma$, it follows that the pair (f_3, f) is adapted. and, using (5.7), this implies that also (i) the pair (f_4, f) is adapted.

Further, let $\mathscr{S}^*: u_{k+\mu}=g(u_k^{k+\mu-1}|y_k^{k+\eta+\mu+1})$ be a recursive representation of a generalized right inverse Σ^* of Σ (see Theorem 3.12), and consider the augmented system $[I_Y, \Sigma^*]$: Im $\Sigma \to (\text{Im } \Sigma) \times S(R^m)$ given by

$$\begin{pmatrix} y_k \\ u_{k+\mu} \end{pmatrix} = \begin{pmatrix} y_k \\ g(u_k^{k+\mu-1}|y_k^{k+\mu+\eta+1}) \end{pmatrix}$$

where, for brevity and clarity, we denote the right-hand side function by $[I_Y, g]$. Then, $P^{-1}=B[I_Y, \Sigma^*]$ (the series combination, which, by (i), is not affected by the non-uniqueness of Σ^*), so that, by Theorems 3.5 and 3.18, the fact that P^{-1} is recursive in z, y implies that (ii) the pair $(f_4, [I_Y, g])$ is compatible. Next, let the system $[\Sigma, I_U]: S(R^m) \to (\text{Im } \Sigma) \times S(R^m)$ be given by

$$\begin{pmatrix} y_{k+\eta+1} \\ u_k \end{pmatrix} = \begin{pmatrix} f(y_k^{k+\eta}|u_k^{k+\mu}) \\ u_k \end{pmatrix}$$

and denote the right-hand side function by $[f, I_U]$. Then, $Q=A[\Sigma, I_U]$, and whence, as before, it follows that (iii) the pair $(f_3, [f, I_U])$ is compatible. Thus we conclude that the existence of a sum decomposition (5.7) satisfying (i), (ii) and (iii) is necessary for any factorization of the system Σ into a composition of recursive systems $\Sigma=PQ$, where P is injective and Q is surjective.

Of course, in our present discussion we are interested not just in plain factorizations $\Sigma=PQ$, but in such factorizations where the systems P^{-1} and Q are stable. Adding the latter requirement to our previous considerations leads to the following.

Theorem 5.8

Let $\Sigma: S(R^m) \to S(R^p)$ be a recursive system with a recursive representation $\mathscr{S}: y_{k+\eta+1}=f(y_k^{k+\eta}|u_k^{k+\mu})$. Let $u_{k+\mu}=g(u_k^{k+\mu-1}|y_k^{k+\mu+\eta+1})$ be a recursive representation of a generalized right inverse Σ^* of Σ. Then, Σ is left rational if and only if the recursion function f can be decomposed into a sum

$$f(y_k^{k+\eta}|u_k^{k+\mu})=f_1(y_k^{k+\eta}|u_k^{k+\mu'})+f_2(y_k^{k+\eta}|u_k^{k+\mu'}), \quad (y|u)\in D_\infty \tag{5.9}$$

where $\mu' \geqslant 0$ is an integer, and where the functions f_1 and f_2 satisfy the conditions :

(α) The pair (f_2, f) is adapted, and the pairs $(f_1, [f, I_U])$ and $(f_2, [I_Y, g])$ are compatible.

(β) For every pair of elements $(y|u)$, $(y'|u')\in D_\infty$, the equality

$$y_{k+\eta+1}-f_2(y_k^{k+\eta}|u_k^{k+\mu'})=y'_{k+\eta+1}-f_2(y'^{k+\eta}_k|u'^{k+\mu'}_k)$$

for all integers k, implies that $y=y'$.

(γ) The series combination $A[\Sigma, I_U]: S(R^m)\to S(R^p)$, where A is the trivially-recursive system $w_k=f_1(y_k^{k+\eta}|u_k^{k+\mu'})$, is i/o-stable.

(δ) The series combination $B[I_Y, \Sigma^*]$: Im $\Sigma \to S(R^p)$, where B is the trivially-recursive system $v_k=f_2(y_k^{k+\eta}|u_k^{k+\mu'})$, is i/o-stable.

The proof of Theorem 5.8 will be stated later in this section. As we can see, condition (α) of the theorem is a restatement of the algebraic conditions (i), (ii) and (iii) of our previous discussion, while condition (β) originates from the invertibility of the system P : Im $Q \to$ Im Σ in the above used factorization $\Sigma = PQ$. The topology-related conditions (γ) and (δ) come to secure the stability of the systems P^{-1} and Q. It is worthwhile to note that the systems A and B are trivially-recursive, a fact that facilitates the verification of conditions (γ) and (δ).

There are numerous occasions of practical interest in control engineering for which Theorem 5.8 provides a convenient tool for the verification of rationality as well as for the construction of left stability representations. We now demonstrate a few such occasions. Let $\Sigma : S(R^m) \to S(R^p)$ be a recursive system. We say that Σ is *separable* if it has a recursive representation of the form $y_{k+\eta+1} = a(y_k^{k+\eta}) + b(u_k^{k+\mu})$, where $a : (R^p)^{\eta+1} \to R^p$ and $b : (R^m)^{\mu+1} \to R^p$ are continuous functions. For example, the system $S(R^2) \to S(R^2)$ described by the representation

$$\begin{pmatrix} y'_{k+3} \\ \\ y''_{k+3} \end{pmatrix} = \begin{pmatrix} 2(y'_{k+2})^3 y''_{k+1} + \sin y''_k + 3 + (u'_{k+4})^2 u''_k \\ \\ \cos (y'_k y''_{k+1}) + 5 \sin (u'_{k+1})^3 + 1 \end{pmatrix}$$

where $y_k = \begin{pmatrix} y'_k \\ y''_k \end{pmatrix}$ and $u_k = \begin{pmatrix} u'_k \\ u''_k \end{pmatrix}$, is a separable system. The class of separable systems includes such common classes of systems as the linear time-invariant systems, systems described by Ricatti equations with time-invariant coefficients, and, of course, many others. It is an easy consequence of Theorem 5.8 that every separable system is left rational, as we next show.

Corollary 5.10

A separable recursive system $\Sigma : S(R^m) \to S(R^p)$ is left rational.

Proof

In view of our separability assumption, let $y_{k+\eta+1} = a(y_k^{k+\eta}) + b(u_k^{k+\mu})$, where a and b are continuous functions, be a recursive representation of Σ. In the notation of Theorem 5.8, let $f_1 := b(u_k^{k+\mu})$ and $f_2 := a(y_k^{k+\eta})$. Then, since f_1 does not depend on y and since f_2 does not depend on u, it follows that the pair (f_2, f) is adapted, and that the pairs $(f_1, [f, I_U])$ and $(f_2, [I_Y, g])$ are compatible (since every function is evidently compatible with the identity). Thus, (α) holds. To show that (β) holds, we note that the unique inverse of $z_k = y_{k+\eta+1} - a(y_k^{k+\eta})$ is clearly given by $y_{k+\eta+1} = a(y_k^{k+\eta}) + z_k$, whence we have injectivity. Finally, turning to (γ) and (δ), we have that $v_k = a(y_k^{k+\eta})$ is a recursive representation of the system $B[I_Y, \Sigma^*]$, and that $w_k = b(u_k^{k+\mu})$ is a recursive representation of the system $A[\Sigma, I_U]$. But then, using our assumption that a and b are continuous functions, and noting that (by compactness) the functions a and b are uniformly continuous over the sets $S_0^\eta(\theta^p)$ and $S_0^\mu(\theta^m)$, respectively, for all $\theta > 0$, a direct verification of Definition 4.1 shows that conditions (γ) and (δ) are satisfied. Thus, all conditions of Theorem 5.8 hold, and Σ is left rational. □

As we have already mentioned before, it is easy to see that Theorem 5.8 continues to hold under a variety of different notions of stability, not only under the one of Definition 4.1. Actually, the only stability related property that we use in the proof of Theorem 5.8 is the following.

Property 5.11

If $T: v_{k+c+1}=h(v_k^{k+c}|y_k^{k+d})$ is an i/o-stable recursive representation, then the system F having input sequence y and output sequence z given, for all integers k, by $z_k=y_{k+\eta+1}-v_k$, also is i/o-stable.

We show in Lemma 5.16 that Property 5.11 holds for the notion of stability defined in Definition 4.1. For the other notions of stability that we discuss below, it is readily seen that it holds as well.

A common notion of stability extensively employed in the control theoretic literature is the notion of BIBO-stability that we have mentioned earlier, where the class of BIBO-stable systems consists of all recursive systems for which every bounded input sequence generates a bounded output sequence. We say that a recursive system $\Sigma: S(R^m)\rightarrow S(R^p)$ is *left BIBO-rational* if there exists a factorization $\Sigma=PQ$, where P and Q are recursive, P is injective, Q is surjective, and P^{-1} and Q are both BIBO-stable systems. Theorem 5.8 yields the following characterization of BIBO-rationality.

Corollary 5.12

Let $\Sigma: S(R^m)\rightarrow S(R^p)$ be a recursive system with a recursive representation $\mathscr{S}: y_{k+\eta+1}=f(y_k^{k+\eta}|u_k^{k+\mu})$. Then, Σ is left BIBO-rational if and only if the recursion function f can be decomposed into a sum

$$f(y_k^{k+\eta}|u_k^{k+\mu})=f_1(y_k^{k+\eta}|u_k^{k+\mu'})+f_2(y_k^{k+\eta}|u_k^{k+\mu'}),\quad (y|u)\in D_\infty$$

where the functions f_1 and f_2 satisfy conditions (α) and (β) of Theorem 5.8 together with the following conditions :

(γ') For every real $\theta>0$ there exists an $M(\theta)>0$ such that $\rho(f_1(y_k^{k+\eta}|u_k^{k+\mu'}))\leqslant M(\theta)$ whenever $\rho_0(u)\leqslant\theta$, for all $(y|u)\in D_\infty$, and all integers k.

(δ') For every real $\theta>0$ there exists an $N(\theta)>0$ such that $\rho(f_2(y_k^{k+\eta}|u_k^{k+\mu'}))\leqslant N(\theta)$ whenever $\rho_0(y)\leqslant\theta$, for all $(y|u)\in D_\infty$, and all integers k.

Thus we see that, for BIBO-rationality, conditions (γ) and (δ) of Theorem 5.8 reduce to the simple requirement that the functions f_1 and f_2 be bounded over respective regions in their domains. We shall state the proof of Corollary 5.12 later in this section. In the meanwhile, we give an example of its application.

Example 5.13

Consider the single-input single-output system given by

$$\Sigma: \ y_{k+1}=y_k^2+(u_k^2+1)\,[\exp(-y_k^2)+1]-2$$

(here the superscript 2 indicates square). This system is evidently not BIBO-stable (its response to $u_k=1$, $k\geqslant 0$, is unbounded). We now choose the functions

$$f_1:=(u_k^2+1)\,[\exp(-y_k^2)+1]-2\,;\quad f_2:=y_k^2$$

Then, conditions (γ') and (δ') are evidently satisfied, and, since the system $z_k=y_{k+1}-y_k^2$ has the unique inverse $y_{k+1}=z_k+y_k^2$, condition (β) holds too.

Also, since f_2 does not depend on u, the pair (f_2, f) is adapted, and the pair $(f_2, [I_Y, g])$ is compatible. Finally, in order to show that the pair $(f_1, [f, I_U])$ is compatible, we directly express the sequence $z_k := f_1(y_k|u_k)$, $k = \ldots, -1, 0, 1, \ldots$, recursively in terms of u and z. Such an expression can be obtained (in this case) simply by eliminating y_k^2 in terms of u_k and z_k, computing y_{k+1} from this expression by using the recursive representation of Σ, and substituting the result into z_{k+1}. One thus obtains

$$z_{k+1} = (u_{k+1}^2 + 1)\left\{\exp\left[-\left(z_k - \operatorname{Lan}\left(\frac{z_k+2}{u_k^2+1} - 1\right)\right)^2\right] + 1\right\} - 2 \qquad (5.14)$$

(This expression is, of course, defined only over its i/o-space.) Whence, all conditions of Corollary 5.11 are satisfied, and Σ is BIBO-rational. □

An additional common notion of stability is continuity. We say that a system $\Sigma : S(R^m) \to S(R^p)$ is *C-stable* if it constitutes a continuous map $S_0^\mu(R^m) \to S(R^p)$. Again, C-stability is a weaker notion of stability than the one used in Definition 4.1. We say that a recursive system $\Sigma : S(R^m) \to S(R^p)$ is *left C-rational* if it has a factorization $\Sigma = PQ$, where P and Q are recursive systems, P is injective, Q is surjective, and P^{-1} and Q are both C-stable systems. For C-rationality, conditions (γ) and (δ) of Theorem 5.8 can be reduced to certain continuity requirements on the functions f_1 and f_2, as follows. Let $\Sigma : S(R^m) \to S(R^p)$ be a recursive system, and, as before, let $D_\infty \subset (\operatorname{Im} \Sigma) \times S_0^\mu(R^m)$ be the set of all pairs $(y|u)$ where $y = \Sigma u$. We now induce on D_∞ two different topologies. First, let C_U be the class of all subsets $C_U(\theta) \subset D_\infty$, where θ varies over all positive reals, and where $C_U(\theta)$ consists of all elements $(y|u) \in D_\infty$ satisfying $\rho_0(u) < \theta$. We regard C_U as a base of a topology on D_∞, which we call the *U-topology*. Symmetrically, let C_U be the class of all subsets $C_U(\theta) \subset D_\infty$, where θ varies over all positive reals, and where $C_Y(\theta)$ consists of all elements $(y|u) \in D_\infty$ satisfying $\rho_0(y) < \theta$. Again, we regard C_Y as a base of a topology on D_∞, which we call the *Y-topology*. Using this terminology, we obtain the following characterization of left C-rationality, the proof of which will be stated later in this section.

Corollary 5.15

Let $\Sigma : S(R^m) \to S(R^p)$ be a recursive system with a recursive representation $\mathscr{S} : y_{k+\eta+1} = f(y_k^{k+\eta}|u_k^{k+\mu})$. Then, Σ is left C-rational if and only if the recursion function f can be decomposed into a sum

$$f(y_k^{k+\eta}|u_k^{k+\mu}) = f_1(y_k^{k+\eta}|u_k^{k+\mu'}) + f_2(y_k^{k+\eta}|u_k^{k+\mu'}), \quad (y|u) \in D_\infty$$

where the functions f_1 and f_2 satisfy conditions (α) and (β) of Theorem 5.8 together with the conditions

(γ'') The trivially recursive map $F_1 : D_\infty \to S(R^p) : w_k = f_1(y_k^{k+\eta}|u_k^{k+\mu'})$, $k = \ldots, -1, 0, 1, \ldots$, is continuous with respect to the U-topology on D_∞.

(δ'') The trivially recursive map $F_2 : D_\infty \to S(R^p) : v_k = f_2(y_k^{k+\eta}|u_k^{k+\mu'})$, $k = \ldots, -1, 0, 1, \ldots$, is continuous with respect to the Y-topology on D_∞.

Thus we see that, in the case of left C-rationality, conditions (γ) and (δ) of Theorem 5.8 reduce to suitable continuity requirements (γ'') and (δ'') on the functions f_1 and f_2. The verification of these requirements is simplified by the fact that the maps F_1 and F_2 are trivially recursive.

Theorem 5.8 also provides a method of actually constructing a left stability representation of a system, whenever one exists. The functions f_1 and f_2 defined in the theorem play a crucial role in this construction. The explicit construction of a left stability representation is described in the proof of Theorem 5.8 stated below. We shall demonstrate this construction on some examples immediately following the proof. The first step of our proof of Theorem 5.8 consists of the following auxiliary result.

Lemma 5.16

Property 5.11 holds for the notion of stability of Definition 4.1.

Proof

We use the notations of Property 5.11 and of Definition 4.1. By Proposition 3.18, the system F is recursive, so let $\mathscr{S}: z_{k+\xi+1} = h(z_k^{k+\xi} | y_k^{k+\zeta})$ be a recursive representation of F, where we choose $\xi \geqslant c$, and let D_0 be the i/o-space of $\mathscr{S}$. Then, by the definition of F, we obtain for any points $d := (z_k^{k+\xi} | y_k^{k+\zeta})$ and $d' := (z'^{k+\xi}_k | y'^{k+\zeta}_k)$ of D_0 and $y, y' \in S_0^\mu(\theta^p)$

$$\rho[\mathscr{S}(d, y), \mathscr{S}(d', y')] \leqslant \rho(y, y') + \rho\{T[(v_{k+\xi-c}^{k+\xi} | y_{k+\xi-c}^{k+\xi-c+d}), y], \\ T[(v'^{k+\xi}_{k+\xi-c} | y'^{k+\xi-c+d}_{k+\xi-c}), y']\}$$

where $v_k := y_{k+\eta+1} - z_k$ and $v'_k := y'_{k+\eta+1} - z'_k$. Also

$$\rho(v_{k+\xi-c}^{k+\xi}, v'^{k+\xi}_{k+\xi-c}) \leqslant \rho(y, y') + \rho(z_{k+\xi-c}^{k+\xi}, z'^{k+\xi}_{k+\xi-c})$$

Combining these facts with the stability of T, the stability of $\mathscr{S}$ follows through a standard '$\epsilon - \delta$' argument. □

Proof of Theorem 5.8

Necessity. We have already shown that the existence of a factorization $\Sigma = PQ$, where P and Q are recursive, P is injective and Q is surjective, implies the existence of the sum decomposition (5.7) which satisfies conditions (α) and (β) (where (β) is implied by the invertibility of P). In order to reconciliate the slight difference between (5.7) and (5.9), we recall that over D_∞ one has $y_{k+\eta+1} = f(y_k^{k+\eta} | u_k^{k+\mu})$. Whence, we can express in (5.7) the variables $y_{k+\eta+1}, \ldots, y_{k+e}$ in terms of $y_k, \ldots, y_{k+\eta}$ and $u_k, \ldots, u_{k+\mu+e-1}$. When these expressions are substituted in (5.7), the equivalent form (5.9) follows. Thus, it only remains to consider conditions (γ) and (δ). Recalling the left stability representation $\Sigma = PQ$ from which (5.7) was derived, it follows by our construction of f_1 that $Q = A[\Sigma, I_U]$, so that (γ) follows by the i/o-stability of Q. Finally, let $E: S(R^p) \rightarrow S(R^p)$ be the system represented by $z_k = y_{k+\eta+1}$. Then, by our construction of the function f_2, we have $P^{-1} = E - B[I_Y, \Sigma^*]$, so that $B[I_Y, \Sigma^*]$ $(= E - P^{-1})$ is stable by the stability of P^{-1} and Lemma 5.16, and (δ) is necessary.

Sufficiency. Assume that the functions f_1 and f_2 satisfy the conditions stated in the theorem. We show that in such case one can construct from f_1 and f_2 a pair of systems P and Q so that $\Sigma = PQ$ is a left stability representation. To this end, we first induce a slight transformation on the functions f_1 and f_2 to obtain the following functions f_1^* and f_2^* (for the purpose of guaranteeing that $P0=0$ and $Q0=0$).

$$f_i^*(y_k^{k+\eta}|u_k^{k+\mu'}) := f_i(y_k^{k+\eta}|u_k^{k+\mu'}) - f_i(0_k^{k+\eta}|0_k^{k+\mu'}), \quad i=1,2$$

In view of the fact that $f(0_k^{k+\eta}|0_k^{k+\mu})=0$, it follows that we still have a sum decomposition

$$f(y_k^{k+\eta}|u_k^{k+\mu}) = f_1^*(y_k^{k+\eta}|u_k^{k+\mu'}) + f_2^*(y_k^{k+\eta}|u_k^{k+\mu'}) \tag{5.17}$$

and, since f_1^* and f_2^* differ from f_1 and f_2, respectively, only by an additive constant, it is evident that f_1^* and f_2^* also satisfy conditions (α), (β), (γ) and (δ) of the theorem. We define now the systems

$$A^* : \ D_\infty \to S(R^p) : \ v_k = f_1^*(y_k^{k+\eta}|u_k^{k+\mu'})$$

$$B^* : \ D_\infty \to S(R^p) : \ w_k = y_{k+\eta+1} - f_2^*(y_k^{k+\eta}|u_k^{k+\mu'})$$

and we combine them into

$$Q : \ A^*[\Sigma, I_U] : \ S(R^m) \to S(R^p), \qquad W := B^*[I_Y, \Sigma^*] : \ \mathrm{Im}\,\Sigma \to S(R^p) \tag{5.18}$$

In view of (α), Theorem 3.5 and Proposition 3.18, the systems Q and W are strictly recursive. By (β), the system $W: \ \mathrm{Im}\,\Sigma \to \mathrm{Im}\,W$ is injective (and, whence, invertible), by (γ) the system Q is stable, and by (δ) and Lemma 5.16, the system W is stable. Furthermore, since by (5.17) we have

$$y_{k+\eta+1} - f_2^*(y_k^{k+\eta}|u_k^{k+\mu'}) = f_1^*(y_k^{k+\eta}|u_k^{k+\mu'}) \text{ over } D_\infty$$

it follows in particular that $\mathrm{Im}\,A^* = \mathrm{Im}\,B^*$, so that $\mathrm{Im}\,W = \mathrm{Im}\,Q$. Whence, letting $P := W^{-1}$, we have that $\mathrm{Im}\,Q = \mathrm{Domain}\,P$, and we can define the system $\Sigma' := PQ$ (where $P: \ \mathrm{Im}\,W \to \mathrm{Im}\,\Sigma$). We next show that Σ' is i/o-equivalent to Σ. To this end, let $u \in S(R^m)$ be any element, let $y := \Sigma u$, and assume for a moment that $Wy = Qu$ for all $u \in S(R^m)$. Then, clearly, $y = W^{-1}Qu = PQu = \Sigma' u$, so that Σ and Σ' are i/o-equivalent. Thus, our proof will conclude upon showing that $Wy = Qu$, which we now do. Let $u' := \Sigma^* y$, so that $\Sigma u' = y = \Sigma u$. Then, denoting $v := Qu$ and $w := Wy$, we have $v_k = f_1^*(y_k^{k+\eta}|u_k^{k+\mu'})$ and $w_k = y_{k+\eta+1} - f_2^*(y_k^{k+\eta}|u'^{k+\mu'}_k)$ for all integers k. Since the pair (f_2^*, f) is adapted and $\Sigma u = \Sigma u'$, we have that $f_2^*(y_k^{k+\eta}|u'^{k+\mu'}_k) = f_2^*(y_k^{k+\eta}|u_k^{k+\mu'})$ for all integers k, so that $w_k = y_{k+\eta+1} - f_2^*(y_k^{k+\eta}|u_k^{k+\mu'})$. But then, recalling that $y_{k+\eta+1} = f(y_k^{k+\eta}|u_k^{k+\mu})$, it follows by (5.17) that $w_k = v_k$ for all integers k. Thus, $Wy = Qu$, and our proof concludes. □

We now demonstrate the explicit construction of left stability representations using (5.18) for the examples that we have considered earlier in the section. First, consider a separable system $\Sigma : \ S(R^m) \to S(R^p)$ represented by $y_{k+\eta+1} = a(y_k^{k+\eta}) + b(u_k^{k+\mu})$, where a and b are continuous functions. In view of the proof of Corollary 5.10, we can choose the functions f_1 and f_2 of Theorem 5.8 as

$$f_1 := b(u_k^{k+\mu}), \quad f_2 := a(y_k^{k+\eta})$$

Following the proof of Theorem 5.8, and denoting $b_0 := b(0_k^{k+\mu})$ and $a_0 := a(0_k^{k+\eta})$ (note that $a_0 = -b_0$), we define

$$f_1^* := b(u_k^{k+\mu}) - b_0, \quad f_2^* := a(y_k^{k+\eta}) - a_0$$

In this case, f_1^* is independent of y and f_2^* is independent of u, so we immediately obtain the recursive representations of W and Q as

$$W : w_k = y_{k+\eta+1} - a(y_k^{k+\eta}) + a_0, \quad Q : v_k = b(u_k^{k+\mu}) - b_0$$

Then, $y_{k+\eta+1} = a(y_k^{k+\eta}) - a_0 + w_k$ is a recursive representation of P $(= W^{-1})$, and $\Sigma = PQ$ is a left stability representation of Σ. We note that in the case of separable systems, the systems P^{-1} and Q are trivially recursive. This is, of course, not necessarily the case in general, as we can see from the next example.

Example 5.19

Consider the BIBO-rational system of Example 5.13. We have already found there the functions f_1 and f_2 for this case, and, since here $f_1(0|0) = 0$ and $f_2(0) = 0$, we have that $f_1^* = f_1$ and $f_2^* = f_2$. The recursive representation for the system $Q = A^*[\Sigma, I_U]$ is given by (5.14). The recursive representation of the system W of (5.18) is, in view of the fact that f_2^* does not depend on u, given by $v_k = y_{k+1} - y_k^2$. For P $(= W^{-1})$ we have the recursive representation $y_{k+1} = y_k^2 + v_k$. Then, $\Sigma = PQ$ is a left BIBO-stability representation of Σ. □

We conclude our discussion of Theorem 5.8 with the proofs of its corollaries.

Proof of Corollary 5.12

We have to show that, for BIBO-rationality, conditions (γ) and (δ) of Theorem 5.8 are equivalent to conditions (γ') and (δ') of Corollary 5.12. We note the following facts :

(i) $[\Sigma, I_U]\{S(R^m)\} = D_\infty$ and $[I_Y, \Sigma^*]\{\text{Im } \Sigma\} \subset D_\infty$;

(ii) The system $A[\Sigma, I_U]$ is BIBO-stable if and only if for every $\theta > 0$ there exists an $M(\theta) > 0$ such that for all $(y|u) \in D_\infty$ for which $\rho_0(u) \leqslant \theta$ one has $\rho_0(A(y|u)) \leqslant M(\theta)$; and

(iii) The system $B[I_Y, \Sigma^*] : \text{Im } \Sigma \to S(R^p)$ is BIBO-stable if and only if for every $\theta > 0$ there exists an $N(\theta) > 0$ such that for all elements $(y|\Sigma^* y) \in D_\infty$ for which $\rho_0(y) \leqslant \theta$ one has $\rho_0(B(y|\Sigma^* y) \leqslant N(\theta)$.

Now, in view of the fact that the pair (f_2, f) is adapted, we can replace in (iii) the phrase ' all $(y|\Sigma^* y)$ ' by ' all $(y|u) \in D_\infty$ '. Then, (γ') and (δ') are just a rewording of (ii) and (iii), respectively, and our proof concludes. □

The proof of Corollary 5.15 is analogous to the proof of Corollary 5.12.

Up to this point we have concentrated on the characterization of left rationality. A theory of right rationality can be obtained by, in a sense, dualizing our previous discussion. We conclude this paper with a brief study of right rationality. Let $\Sigma : S(R^m) \to S(R^p)$ be a recursive system, and, as before, let D_∞ be the set of all pairs $(y|u)$ where $u \in S_0^\mu(R^m)$ and $y := \Sigma u$. Let $\mathcal{S} : y_{k+\eta+1} = f(y_k^{k+\eta} | u_k^{k+\mu})$ be a recursive representation of Σ. Assume that Σ

has a right stability representation $\Sigma = PQ$, where $Q : S(R^m) \to S(R^m)$, $P : \text{Im } Q \to S(R^p)$, and let

$$\mathscr{S}_1 : y_{k+\alpha+1} = f_1(y_k^{k+\alpha} | z_k^{k+\beta})$$
$$\mathscr{S}_2 : u_{k+\alpha+1} = f_2(u_k^{k+\alpha} | z_k^{k+\beta})$$

be recursive representations of P and of a recursive right inverse $Q^* : \text{Im } Q \to S(R^m)$ of Q, respectively. (Since the principal degree of a recursive representation can always be increased by shifting, we assumed that $\mathscr{S}_1$ and $\mathscr{S}_2$ have the same principal degree.) In view of the facts that P and Q are recursive, that PQ $(=\Sigma)$ is clearly strictly recursive, and that P is injective, it follows by (3.8) that there exists a function h such that, for some integers $\gamma, \delta \geqslant 0$, we can express $z_k = h(y_k^{k+\gamma} | u_k^{k+\delta})$ for all elements $(y|u) \in D_\infty$. The fact that P is injective implies that Ker $Q =$ Ker Σ, whence, for any pair of elements $(y|u)$, $(y'|u') \in D_\infty$, the equality $h(y'^{k+\gamma}_k | u'^{k+\delta}_k) = h(y_k^{k+\gamma} | u_k^{k+\delta})$ for all integers k implies that $\Sigma u = \Sigma u'$. Define now the trivially-recursive system $\Sigma_0 : D_\infty \to S(R^m) : z_k = h(y_k^{k+\gamma} | u_k^{k+\delta})$. Then, using our earlier notation, we have that $Q = \Sigma_0[\Sigma, I_U]$, and, since Q is recursive in u, z, we obtain that the pair $(h, [f, I_U])$ is compatible. Define now the augmented vector $w_k := \begin{pmatrix} y_k \\ u_k \end{pmatrix}$ and the recursive representation

$$\mathscr{S}^* : w_{k+\alpha+1} = F(w_k^{k+\alpha} | z_k^{k+\beta}) := \begin{pmatrix} f_1(y_k^{k+\alpha} | z_k^{k+\beta}) \\ \cdots\cdots\cdots\cdots \\ f_2(u_k^{k+\alpha} | z_k^{k+\beta}) \end{pmatrix} : \text{Im } \Sigma_0 \to D_\infty \quad (5.20)$$

Then, we clearly have that $\mathscr{S}^*$ is a right inverse of the trivially-recursive system Σ_0. Of crucial importance is the particular form of the recursion function F of $\mathscr{S}^*$, namely, that its first p entries depend only on the first p entries of the vector w (i.e. y), whereas its other m entries depend only on the last m coordinates of w (i.e. u). We call a recursive representation having this particular form *(p, m)-divided.* These arguments prove the necessity direction of the following characterization of right rationality.

Theorem 5.21

Let $\Sigma : S(R^m) \to S(R^p)$ be a recursive system with a recursive representation $\mathscr{S} : y_{k+\eta+1} = f(y_k^{k+\eta} | u_k^{k+\mu})$. Then, Σ is right rational if and only if, for some integers $\gamma, \delta \geqslant 0$, there exists a function $h : (R^p)^{\gamma+1} \times (R^m)^{\delta+1} \to R^m$ such that the trivially-recursive system $\Sigma_0 : D_\infty \to S(R^m) : z_k = h(y_k^{k+\gamma} | u_k^{k+\delta})$ satisfies the following conditions :

(α) The pair $(h, [f, I_U])$ is compatible, and, for any pair of elements $(y|u)$, $(y'|u') \in D_\infty$, the equality $h(y'^{k+\gamma}_k | u'^{k+\delta}_k) = h(y_k^{k+\gamma} | u_k^{k+\delta})$ for all integers k implies that $\Sigma u' = \Sigma u$.

(β) The system Σ_0 has a right inverse $\Sigma_0^* : \text{Im } \Sigma_0 \to D_\infty$ having a (p, m)-divided recursive representation $\mathscr{S}^*$ of the form (5.20).

(γ) The system $P : \text{Im } \Sigma_0 \to \text{Im } \Sigma$ (derived from $\mathscr{S}^*$) having the recursive representation $y_{k+\alpha+1} = f_1(y_k^{k+\alpha} | z_k^{k+\beta})$ is i/o-stable.

(δ) The system $T : \text{Im } \Sigma_0 \to S(R^m)$ (derived from $\mathscr{S}^*$) having the recursive representation $u_{k+\alpha+1} = f_2(u_k^{k+\alpha} | z_k^{k+\beta})$ is i/o-stable.

As in the case of Theorem 5.8, the conditions of Theorem 5.21 consist of algebraic conditions ((α), (β)), and of topological conditions ((γ), (δ)).

Proof

We have already shown that conditions (α), (β), (γ) and (δ) are necessary conditions for right rationality. In order to show that they are also sufficient, assume that (α), (β), (γ) and (δ) hold. After possibly substructing a constant, we can assume without loss of generality that $\Sigma_0 0=0$ (see proof of Theorem 5.8). Using the (m, p)-divided representation of (β), we construct the recursive systems P and T of (γ) and (δ), and the system $Q := \Sigma_0[\Sigma, I_U] : S(R^m)\to \mathrm{Im}\,\Sigma_0$. Then, by ($\alpha$), Q is recursive, Ker $Q=$ Ker Σ, and Q is surjective by its definition. Also, by (β), T is a right inverse of Q. The systems P and T are i/o-stable by (γ) and (δ). Further, the image of Q is the domain of P, so we can define the system $\Sigma' := PQ$. Now, let $u\in S(R^m)$ be an arbitrary element, let $y := \Sigma u$, and let $z := \Sigma_0(y|u)=\Sigma_0[\Sigma, I_U]u=Qu$. Also, let $(y'|u') := \Sigma_0^* z$, where Σ_0^* is the right inverse of Σ_0 defined in (β). Then, $\Sigma_0(y'|u')=z=\Sigma_0(y|u)$, so that, by ($\alpha$), $\Sigma u=\Sigma u'$, and consequently $y'=\Sigma u'=\Sigma u=y$. Thus, recalling the definition of P, we obtain $\Sigma' u=PQu=Pz=y'=y=\Sigma u$ for all $u\in S(R^m)$, so that Σ' and Σ are i/o-equivalent. Whence, $\Sigma=PQ$, and, since Ker $Q=$ Ker Σ, it follows that P is injective. Finally, having already shown that P is stable and that T is a recursive stable right inverse of Q, we obtain that $\Sigma=PQ$ is a right stability representation, and Σ is right rational. □

For the case of BIBO-rationality, Theorem 5.21 takes the following form.

Corollary 5.22

Let $\Sigma : S(R^m)\to S(R^p)$ be a recursive system with a recursive representation $\mathscr{S} : y_{k=\eta+1}=f(y_k^{k+\eta}|u_k^{k+\mu})$. Then, Σ is right BIBO-rational if and only if, for some integers $\gamma, \delta\geqslant 0$, there exists a function $h : (R^p)^{\gamma+1}\times(R^m)^{\delta+1}\to R^m$ such that the trivially-recursive system $\Sigma_0 : D_\infty\to S(R^m) : z_k=h(y_k^{k+\gamma}|u_k^{k+\delta})$ satisfies conditions (α) and (β) of Theorem 5.21 together with the condition :

(γ') The system Σ_0^* of (β) is BIBO-stable.

Corollary 5.22 is a direct consequence of the proof of Theorem 5.21 and the definition of BIBO-stability.

Remark 5.23

In case the system Σ of Corollary 5.22 is injective, then it can readily be seen that condition (γ') can be stated directly in terms of the function h as follows :

(γ_1') For every real $\theta>0$ there exists a real $M(\theta)>0$ such that, whenever $\rho(h(y_k^{k+\gamma}|u_k^{k+\delta}))\leqslant\theta$ for all integers k, then $\rho_0(y)\leqslant M(\theta)$ and $\rho_0(u)\leqslant M(\theta)$, for all $(y|u)\in D_\infty$. □

We demonstrate the application of Corollary 5.22 (and Remark 5.23) by the following simple numerical case.

Example

Consider the recursive system $\Sigma : S(R)\to S(R)$ represented by

$$y_{k+1}=\exp\,[u_k+y_k]-1$$

(defined on its i/o-space). Here, we can choose

$$h(y_k|u_k) := [u_k+y_k] =: z_k.$$

An easy computation then gives the inverse $\Sigma_0^* : h[D_\infty] \to D_\infty$ as

$$\begin{pmatrix} y_{k+1} \\ u_{k+1} \end{pmatrix} = \begin{pmatrix} \exp(z_k) - 1 \\ z_{k+1} - \exp(z_k) + 1 \end{pmatrix}$$

which is BIBO-stable. Whence, the system P of (γ) is given by $P : y_{k+1} = \exp(z_k) - 1$, and the system T of (δ) is given by $T : u_{k+1} = z_{k+1} - \exp(z_k) + 1$. The inverse of T is $Q := T^{-1} : z_{k+1} = u_{k+1} + \exp(z_k) - 1$, and $\Sigma = PQ$ is a right BIBO-stability representation. □

Finally, turning to the case of C-stability, we obtain the following consequence of the proof of Theorem 5.21.

Corollary 5.24

Let $\Sigma : S(R^m) \to S(R^p)$ be a recursive system with a recursive representation $\mathscr{S} : y_{k+\eta+1} = f(y_k^{k+\eta} | u_k^{k+\mu})$. Then, Σ is right C-rational if and only if, for some integers $\gamma, \delta \geqslant 0$, there exists a function $h : (R^p)^{\gamma+1} \times (R^m)^{\delta+1} \to R^m$ such that the trivially-recursive system $\Sigma_0 : D_\infty \to S(R^m) : z_k = h(y_k^{k+\gamma} | u_k^{k+\delta})$ satisfies conditions (α) and (β) of Theorem 5.21 together with the condition :

(γ'') The system Σ_0^* of (β) is C-stable.

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