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Non-linear systems, stabilization, and coprimeness†

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A theory of coprimeness is developed for non-linear systems, with the intention of constructing analytic tools for the solution of the problem of stabilizing a non-linear system through the application of additive non-linear feedback. The qualitative features of the resulting theory are strikingly similar to the situation in the linear case.

1. Introduction

Let Σ be a non-linear time-invariant discrete-time system, and consider the classical additive-feedback configuration shown in Fig. 1, where π is a non-linear time-invariant precompensator, φ is a non-linear time-invariant feedback compensator, and $\Sigma_{(\pi,\varphi)}$ denotes the overall system. This configuration has been widely used in engineering applications for the purpose of transforming the possibly unstable given system Σ into a stable system $\Sigma_{(\pi,\varphi)}$. We refer to such a transformation as the *stabilization* of Σ. Of course, the main problem here is how to find compensators π and φ for which the resulting system $\Sigma_{(\pi,\varphi)}$ is stable. In the present paper we study the solution of certain functional equations that arise in the computation of compensators π and φ needed to stabilize the given system Σ.

We shall require throughout our discussion that the system Σ be strictly causal, and that the compensators π and φ be causal. For the moment, we also require that the precompensator π be an isomorphism, i.e. that it possess an inverse π^{-1}. This requirement guarantees that the precompensator π does not destroy any degrees of freedom of the control variables, so that the final system $\Sigma_{(\pi,\varphi)}$ has the same control capabilities as the original system Σ. In our discussion of the configuration (1.1) we

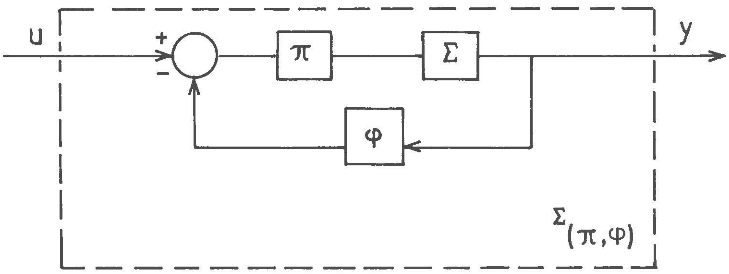

Figure 1

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distinguish between *input/output stability*, by which we mean that the input/output relationship induced by $\Sigma_{(\pi,\varphi)}$ is stable, and the stronger notion of *internal stability*, by which we mean that all the degrees of freedom in (1.1), including those not directly affecting the input/output relationship, are stable.

The point of departure for our present discussion is a fundamental analogy that exists between the general case, where Σ, π and φ are non-linear systems, and the well-understood particular case where Σ, π and φ are linear. The origin of this analogy is the property of *rationality*. Roughly speaking, a system Σ is said to be *right-rational* if there exist stable systems P and Q, where Q is invertible, such that $\Sigma = PQ^{-1}$; The system Σ is *left-rational* if there exist stable systems S and T, where S is invertible, such that $\Sigma = S^{-1}T$ (see § 2 for exact definitions). Now, a finite-dimensional linear time-invariant system Σ_L is well-known to be both right- and left-rational, and its representation as a quotient of stable systems $\Sigma_L = PQ^{-1}$ plays a fundamental role in the solution to the linear feedback-stabilization problem (Rosenbrock 1970, Desoer and Chan 1975, Desoer and Vidyasagar 1975, Hammer 1983 a, b). The situation in the general case when the system Σ is non-linear turns out to be strikingly similar. In Hammer (1984 b) it was shown that every (injective) non-linear system Σ that can be internally stabilized through the application of additive output feedback *must* be right-rational. Thus, when studying the problem of stabilizing a non-linear system Σ by additive feedback, we can assume *a priori* that Σ has a fraction representation $\Sigma = PQ^{-1}$, where P and Q are stable systems. In the present paper we consider the problem of how to use the systems P and Q of such a representation in order to compute compensators π and φ that stabilize the given system Σ. (The computation of fraction representations $\Sigma = PQ^{-1}$, with P and Q stable systems, was discussed in Hammer (1984 a).)

To be more specific, the systems Σ, π and φ map (input) sequences of finite-dimensional real vectors into (output) sequences of finite-dimensional real vectors. The sum $A + B$ of two systems A and B, having the same input and output spaces, is defined pointwise for every input sequence u by $(A + B)u = Au + Bu$. Then, under the previously mentioned causality requirements on Σ, π and φ, the overall system $\Sigma_{(\pi,\varphi)}$ can be expressed by

$$\Sigma_{(\pi,\varphi)} = \Sigma\psi_{(\pi,\varphi)} \tag{1.1}$$

where

$$\psi_{(\pi,\varphi)} := \pi[I + \varphi\Sigma\pi]^{-1}$$

is an equivalent precompensator (see Desoer and Lin (1983) and Hammer (1984 b). Assume now that the given system Σ has a rational representation $\Sigma = PQ^{-1}$, where P and Q are stable systems. The simplest case from our present point of view arises when the feedback compensator φ is stable and the precompensator π is of the form $\pi = R^{-1}$, where R is a stable invertible system. In this case we obtain

$$\begin{aligned}\Sigma_{(\pi,\varphi)} &= PQ^{-1}R^{-1}[I + \varphi PQ^{-1}R^{-1}]^{-1}\\ &= PQ^{-1}R^{-1}[(RQ + \varphi P)Q^{-1}R^{-1}]^{-1}\\ &= P[RQ + \varphi P]^{-1}\end{aligned}$$

Thus, *the input/output relationship represented by $\Sigma_{(\pi,\varphi)}$ is stable if the (stable) map $M := RQ + \varphi P$ has a stable inverse M^{-1}.* In other words, if we can find stable maps R and φ (where R is invertible and φ, R^{-1} are causal) for which the combination $RQ+\varphi P$ has a stable inverse, then, upon setting $\pi := R^{-1}$, we clearly obtain that $\Sigma_{(\pi,\varphi)}$ is input/output stable. (We remark that under these circumstances $\Sigma_{(\pi,\varphi)}$ will also be internally stable, but we do not elaborate on this point here.) This situation is clearly reminiscent of the case of linear systems. We thus arrive at the following.

Question 1

Given two stable systems P and Q, when do there exist stable systems A and B for which the (stable) system $M := AP + BQ$ has a stable inverse M^{-1}?

In the case of linear systems, the answer to question 1 is well known—maps A and B exist if and only if the transfer matrices representing P and Q are right-coprime. In the present paper we discuss question 1 for the case where P and Q are non-linear systems. Basically, we show that the situation in general is closely analogous to the situation in the linear case. In § § 3 and 4 we define coprimeness of two non-linear stable systems. Very qualitatively, two systems P and Q (with common input space) are right-coprime if, for every *un*bounded input sequence u, at least one of the output sequences Pu, Qu is unbounded. We then show that systems A and B solving question 1 exist if and only if P and Q are coprime in this sense.

In view of our discussion, we are mainly interested in question 1 for the case where the maps P and Q are derived from a fraction representation $\Sigma = PQ^{-1}$ of a right-rational system Σ. By definition, a right-rational system Σ always has a representation $\Sigma = PQ^{-1}$, where P and Q are stable systems. However, it is not clear whether or not the systems P and Q can be chosen so that they are also *coprime*. We therefore have to provide an answer to the following (in our present discussion we restrict our attention to injective systems).

Question 2

When does a right-rational injective system Σ possess a right-coprime fraction representation, namely, a representation $\Sigma = PQ^{-1}$ where P and Q are stable *coprime* systems?

In the case of finite-dimensional linear systems, it is well known that the answer to question 2 is 'always', namely, every such system has a right-coprime fraction representation. In the case of non-linear systems the situation is more delicate, and requires careful study. The answer to question 2 in the non-linear case depends, to a large extent, on what notion of stability one adopts. In the present paper we discuss two types of stability notions. Under the first, which we call *BIBO-stability*, a system Σ is regarded as stable if, for every *bounded input* sequence u, it produces a *bounded output* sequence Σu. Under the second, which we call *C-stability*, a system Σ is regarded as stable if it is BIBO-stable and if it represents a continuous map from the space of input sequences into the space of output sequences. This notion is essentially the classical stability notion due to Liapunov.

Now, for the case of BIBO-stability, the answer to question 2 is again 'always', namely, every injective system Σ possessing a representation $\Sigma = PQ^{-1}$ where P and Q are BIBO-stable, also has a representation $\Sigma = P_c Q_c^{-1}$ where P_c and Q_c are BIBO-stable and BIBO-coprime (§ 3). Thus, the case of BIBO-stability closely resembles the linear case. However, in the case of C-stability (§ 4), the situation is different, and not every right-rational system has a right-coprime fraction representation. In particular, if the system Σ possesses any finite-jump discontinuities, then it cannot be represented as a right-coprime fraction with C-stable numerator and denominator. The only systems possessing right-coprime fraction representations in the C-stability sense are the 'homogeneous' systems defined in § 4. Roughly speaking, a homogeneous system is a system which exhibits continuous behaviour whenever its output is bounded.

In summary, one can draw a close conceptual analogy between non-linear additive feedback systems and linear feedback systems. In both cases the notion of coprime fraction representations determines the underlying mathematical structure.

2. Notation and preliminaries

Our discussion throughout the present paper will be stated in terms of the framework developed in Hammer (1984 a, b), so we now review some parts of this framework. First, since we deal with discrete-time systems, we have to define their spaces of input sequences and of output sequences, which will simply consist of sequences of finite-dimensional real vectors. Formally, let R denote the set of real numbers. We denote by $S(R^m)$ the set of all two-sided infinite sequences $u := \{\ldots, u_n, u_{n+1} \ldots\}$, where $u_j \in R^m$ for all integers j, and where there exists an integer $t(u)$ (depending on u) such that $u_i = 0$ for all $i < t(u)$. The zero sequence in $S(R^m)$, i.e. the sequence having all its elements equal to the zero vector, will be denoted by 0. Given a sequence $u \in S(R^m)$, we denote by u_j the jth element of the sequence, and by u_i^j, where $i \leqslant j$, the set of all elements $u_i, u_{i+1}, \ldots, u_j$. If $i > j$, then u_i^j denotes the empty set $\varnothing$. The set of all sequences $u \in S(R^m)$ for which $u_j = 0$ for all $j < 0$ is denoted by $S_0(R^m)$.

An element $u \in S(R^m)$ is regarded as an input sequence to a system. More precisely, a system admitting input values from the real space R^m and having its output values in the real space R^p, is a map $\Sigma: S(R^m) \to S(R^p)$, transforming input sequences into output sequences. We assume that every system Σ under consideration has a (possibly unstable) equilibrium point at 0 (corresponding, for example, to the 'off' state of the system), so that $\Sigma 0 = 0$. Finally, for every pair of elements $u, v \in S(R^m)$, one defines the sum $u + v$ coefficientwise by $(u+v)_i := u_i + v_i$ for all integers i. For a pair of systems $\Sigma_1, \Sigma_2 : S(R^m) \to S(R^p)$, the sum $\Sigma_1 + \Sigma_2 : S(R^m) \to S(R^p)$ is defined pointwise, for every element $u \in S(R^m)$, by $(\Sigma_1 + \Sigma_2)u := \Sigma_1 u + \Sigma_2 u$.

The set $S(R^m)$ evidently contains all possible input sequences, whether bounded or unbounded. From the practical point of view, the most interesting input sequences are, of course, the bounded ones. To treat bounded input sequences we need the following notation. Adhering to the usual convention, we denote, for every real $\theta > 0$, by $[-\theta, \theta]^m$ the set of all real vectors in R^m with entries in the interval $[-\theta, \theta]$. Further, we denote by $S(\theta^m)$ the set of all sequences $u \in S(R^m)$ for which $u_i \in [-\theta, \theta]^m$ for all integers i. Thus, $S(\theta^m)$ is just the set of all input sequences 'bounded by θ'. Similarly, we denote by $S_0(\theta^m)$ the set of all elements $u \in S(\theta^m)$ for which $u_i = 0$ for all integers $i < 0$.

Most of our ensuing discussion involves considerations regarding the stability of non-linear systems. The literature of control theory has accepted, in different applications contexts, a variety of stability notions for non-linear systems. The general underlying framework of our discussion in the present paper can be adapted to accommodate most of these concepts of stability. We shall, however, mainly study two types of stability notions—one related to boundedness, and the other to continuity. Let $\Sigma: S(R^m) \to S(R^p)$ be a system. Given a set $S \subset S(R^m)$, let $\Sigma[S]$ denote the image of S through Σ. We say that Σ is *BIBO* (bounded-input bounded-output)-*stable* if, for every real $M > 0$, there exists a real $N > 0$ such that $\Sigma[S(M^m)] \subset S(N^p)$. Thus, a BIBO-stable system transforms bounded input sequences into bounded output sequences.

In order to review the classical notion of stability that originated with the pioneering work of Liapunov (1947), we need to discuss a few topics related to continuity. First, we introduce on the space $S(R^m)$ a norm ρ as follows. For an element $\alpha(\alpha^1, \ldots, \alpha^m) \in R^m$, we define $\rho(\alpha) := \max\ \{|\alpha^i|,\ i = 1, \ldots, m\}$, and for the finite set of elements u_i^j, $i \leqslant j$, we let $\rho(u_i^j) := \max\ \{\rho(u_\kappa),\ i \leqslant k \leqslant j\}$. For an element $u \in S(R^m)$, we define $\rho(u) := \sup_i 2^{-|i|}\rho(u_i)$. The norm ρ induces a metric ρ on $S(R^m)$ when, for any pair of elements $u, v \in S(R^m)$, one defines $\rho(u, v) := \rho(u - v)$. In its turn, the metric ρ induces a topology on our spaces, and all notions of continuity mentioned below refer to continuity with respect to this topology. A system Σ: $S(R^m) \to S(R^p)$ is *C-stable* if it is BIBO-stable, and if, for every real $\theta > 0$, its restriction $\Sigma: S_0(\theta^m) \to S(R^p)$ is a continuous map. Basically, C-stability requires that a 'small' change in the input sequence to Σ should cause only a 'small' change in the output sequence from Σ.

Finally, we review very briefly the concept of rationality. A detailed definition of the concept of rationality is given by Hammer (1984 a). For our present discussion, we are interested only in the particular case of injective systems, for which the definition can be simplified to the following. An injective system Σ: $S(R^m) \to S(R^p)$ is *right BIBO-rational* (respectively, *right C-rational*) if there exist BIBO-stable (respectively, C-stable) systems $P: S \to S(R^p)$ and $Q: S \to S(R^m)$, where $S \subset S(R^q)$ for some integer $q > 0$, and where Q is invertible, such that $\Sigma = PQ^{-1}$. We shall further discuss these fraction representations of systems in the following sections. A final remark on terminology. An element $u \in S(R^m)$ will be said to be *bounded* if there exists a real $\theta > 0$ such that $u \in S(\theta^m)$; if such a θ does not exist, then u is *un*bounded.

3. Coprimeness of non-linear systems: the case of BIBO-stability

Let $S \subset S(R^q)$ be a subspace, and let $P: S \to S(R^p)$ and $Q: S \to S(R^m)$ be BIBO-stable systems. The first question we study in the present section is the following. Under what conditions do there exist BIBO-stable maps A: $S(R^p) \to S$ and $B: S(R^m) \to S$ for which the (BIBO-stable) map

$$M := AP + BQ: S \to S \tag{3.1}$$

is an isomorphism having a BIBO-stable inverse M^{-1}. We start with a discussion of some necessary conditions. Probably the most evident condition necessary for the existence of A and B is that

(*) for every unbounded input sequence $u \in S$, at least one of the output sequences Pu or Qu must be unbounded

For the particular linear case where P and Q are single-variable polynomials, this condition reduces to the requirement that P and Q have no unstable zeros in common. To see the origin of (*) in the general non-linear case, let $u \in S$ be any unbounded element, and assume, by contradiction, that both of the output sequences Pu and Qu are bounded. Then, since A and B are BIBO-stable, it follows that the element $v := APu + BQu = Mu$ also is bounded. But then, since $M^{-1}v = u$, the map M^{-1} maps the bounded element v into the unbounded element u, contradicting the assumption that M^{-1} is *BIBO*-stable. Thus, at least one of the elements Pu or Qu must be unbounded, and (*) is necessary. In order to state (*) in somewhat more concise form, denote, for any integer n, by $U(n)$ the set of all unbounded elements in $S(R^n)$, and let C^* denote the set-theoretic inverse of a function $C: S(R^m) \to S(R^p)$. Then, (*) can be stated as

$$U(q) \cap S \subset P^*[U(p)] \cup Q^*[U(m)] \tag{3.2}$$

The argument used to derive (3.2) actually leads to a stronger condition, as follows. Let $u^1, u^2, \ldots$, be a sequence $\{u^i\}$ of elements of S. If there is a real $\theta > 0$ such that $\{u^i\} \subset S(\theta^q)$, then we say that $\{u^i\}$ is a *bounded* sequence; otherwise, we say that $\{u^i\}$ is an unbounded sequence. Now, assume again that there exist BIBO-stable maps A and B for which the map $(AP + BQ)$ has a BIBO-stable inverse, and let $\{u^i\} \subset S$ be an unbounded sequence. Then, a slight variation of the argument leading to (3.2) implies that at least one of the sequences $\{Qu^i\}$ or $\{Pu^i\}$ must be an unbounded sequence. Equivalently,

(**) if both of the sequences $\{Qu^i\}$ and $\{Pu^i\}$ are bounded then the sequence (of input sequences) $\{u^i\} \subset S$ must be a bounded sequence.

In more accurate terms (**) takes the following form:

$$\left.\begin{array}{l}\text{for every real } \tau > 0, \text{ there exists a real } \theta > 0 \text{ such that} \\ \qquad P^*[S(\tau^p)] \cap Q^*[S(\tau^m)] \subset S(\theta^q)\end{array}\right\} \tag{3.3}$$

Notice that (3.2) is included in (3.3). In summary, an elementary argument leads us to the conclusion that (3.3) is a necessary condition for the existence of BIBO-stable maps A and B for which the map $(AP + BQ)$ has a BIBO-stable inverse. This condition is reminiscent of the condition of coprimeness in the linear case. Moreover, one of the main results of the present section is that (3.3) is not only necessary, but also sufficient for the existence of such maps A and B. So motivated, we introduce the following.

Definition 1

Let $S \subset S(R^q)$ be a subspace. Two BIBO-stable maps $P: S \to S(R^p)$ and $Q: S \to S(R^m)$ are *right BIBO-coprime* if they satisfy (3.3).

Below, it will be convenient to employ the following terminology. Let $S_1 \subset S(R^q)$, $S_2 \subset S(R^p)$ be subspaces. A map $M: S_1 \to S_2$ is *BIBO-unimodular* if it is an isomorphism, and if both M and M^{-1} are BIBO-stable. Our next objective is to show that the property of coprimeness is a sufficient condition for the existence of BIBO-stable maps A, B for which $(AP + BQ)$ is BIBO-unimodular. Actually, the following somewhat stronger assertion is true (we limit our attention here to injective systems).

Theorem 1

Let $S \subset S(R^q)$ be a subspace, and let $P: S \to S(R^p)$ and $Q: S \to S(R^m)$ be BIBO-stable maps, where P is injective and Q is an isomorphism. If P and Q are right BIBO-coprime, then, for every BIBO-unimodular map $M: S \to S$, there exist BIBO-stable maps $A: S(R^p) \to S$ and $B: S(R^m) \to S$ satisfying $AP + BQ = M$.

Proof

Suppose, for a moment, that the space $S \subset S(R^q)$ has been decomposed into two disjoint sets Λ_A and Λ_B which satisfy the following conditions:

(i) $\Lambda_A \cup \Lambda_B = S$, $\quad \Lambda_A \cap \Lambda_B = \varnothing$.

(ii) For every real $\tau > 0$ there exists a real $\theta > 0$ such that $Q^*[S(\tau^m)] \cap \Lambda_B \subset S(\theta^q)$ and $P^*[S(\tau^p)] \cap \Lambda_A \subset S(\theta^q)$.

Let $M: S \to S$ be a BIBO-unimodular map. Using the above decomposition, we define maps $A: S(R^p) \to S$ and $B: S(R^m) \to S$ as follows:

$$Bu := \begin{cases} MQ^*u & \text{if } u \in Q[\Lambda_B] \\ 0 & \text{if } u \notin Q[\Lambda_B] \end{cases}$$

$$Au := \begin{cases} MP^*u & \text{if } u \in P[\Lambda_A] \\ 0 & \text{if } u \notin P[\Lambda_A] \end{cases}$$

Then, for every real $\tau > 0$, we obtain

$$\begin{aligned} B[S(\tau^m)] &= B[Q[\Lambda_B] \cap S(\tau^m)] \cup \{0\} \\ &= M[\Lambda_B \cap Q^*[S(\tau^m)]] \cup \{0\} \\ &\subset M[S(\theta^q) \cap S] \subset S(T^m) \end{aligned}$$

where $T > 0$ is a real number satisfying $M[S(\theta^q) \cap S] \subset S(T^m)$, which exists by virtue of the BIBO-stability of M. Similarly

$$\begin{aligned} A[S(\tau^p)] &= A[P[\Lambda_A] \cap S(\tau^p)] \cup \{0\} \\ &= M[\Lambda_A \cap P^*[S(\tau^p)]] \cup \{0\} \\ &\subset M[S(\theta^q) \cap S] \subset S(T^m) \end{aligned}$$

Thus, A and B are BIBO-stable. Finally, we show that $AP + BQ = M$. Indeed, by (i), every element u of S is either in Λ_A or in Λ_B, exclusively. When $u \in \Lambda_A$, we have $(AP + BQ)u = APu + BQu = MP^*Pu + 0 = Mu$ and when $u \in \Lambda_B$ we have $(AP + BQ)u = APu + BQu = 0 + MQ^*Qu = Mu$, where we have used the injectivity of P and of Q. Thus, $(AP + BQ)u = Mu$, for all $u \in S$, and our proof will conclude upon the construction of the sets Λ_A and Λ_B, which is what we do next.

For every real $\theta > 0$, we denote $U(\theta^m) := S(R^m) \backslash S(\theta^m)$ (the difference set), so that $U(\theta^m)$ and $S(\theta^m)$ are disjoint sets, and whence $Q^*[S(\theta^m)] \cap Q^*[U(\theta^m)] = \varnothing$. Now, let $0 < \tau_1 < \tau_2 < \ldots$ and $0 < \theta_1 < \theta_2 < \ldots$ be diverging sequences of real numbers. For every integer $i \geqslant 1$, we define the set

$$K_i := Q^*[S(\tau_i^m)] \cap U(\theta_i^q)$$

we let

$$K := \bigcup_{i=1}^{\infty} K_i$$

and

$$\Lambda_0 := S \backslash K \text{ (the difference set)}, \ \Lambda_0' := K$$

Then, for every integer $i \geqslant 1$, we have $Q^*[S(\tau_i^m)] \cap \Lambda_0 \subset S(\theta_i^q)$. However, the set $P^*[S(\tau_i^p)] \cap \Lambda_0'$ may be unbounded. The key to correcting this situation is the fact that if $u \in P^*[S(\tau_i^p)]$ is an unbounded element, then, by coprimeness, Qu must be unbounded, and so $u \in Q^*[U(\tau_i^m)]$. We define, for every integer $i \geqslant 1$, the sets

$$R_i := P^*[S(\tau_i^p)] \cap \Lambda_0' \cap Q^*[U(\tau_i^m)]$$
$$R_\infty := \{u \in U(q) \cap S : Qu \text{ is unbounded}\}$$

and we let

$$R := R_\infty \cup \left\{ \bigcup_{i=1}^{\infty} R_i \right\}$$

Consider now the sets

$$\Lambda_A := \Lambda_0' \backslash (R \cap \Lambda_0'), \quad \Lambda_B := \Lambda_0 \cup R$$

We claim that these sets satisfy conditions (i) and (ii). Evidently, $\Lambda_A \cup \Lambda_B = \Lambda_0' \cup \Lambda_0 = S$, and $\Lambda_A \cap \Lambda_B \subset \Lambda_0' \cap \Lambda_0 = \varnothing$, so that (i) holds. Further,

$$R \cap Q^*[S(\tau_i^m)] = \left\{ \bigcup_{j=1}^{i-1} R_j \right\} \cap Q^*[S(\tau_i^m)] \subset P^*[S(\tau_i^p)] \cap Q^*[S(\tau_i^m)]$$

so that, by the right-coprimeness of P and Q, there is a real $\alpha_i > 0$ such that $R \cap Q^*[S(\tau_i^m)] \subset S(\alpha_i^q)$. Consequently,

$$\begin{aligned}\Lambda_B \cap Q^*[S(\tau_i^m)] &= \{\Lambda_0 \cap Q^*[S(\tau_i^m)]\} \cup \{R \cap Q^*[S(\tau_i^m)]\} \subset S(\theta_i^q) \cup S(\alpha_i^q) \\ &\subset S(\beta_i^q)\end{aligned}$$

where $\beta_i := \max\{\theta_i, \alpha_i\}$. Also, recalling that $\Lambda_A = \Lambda_0' \backslash (R \cap \Lambda_0')$, we obtain

$$\begin{aligned}\Lambda_A \cap P^*[S(\tau_i^p)] &\subset P^*[S(\tau_i^p)] \cap \Lambda_0' \cap Q^*[S(\tau_i^m)] \subset P^*[S(\tau_i^p)] \cap Q^*[S(\tau_i^m)] \\ &\subset S(\alpha_i^q) \subset S(\beta_i^q)\end{aligned}$$

and (ii) holds. Thus, Λ_A and Λ_B satisfy conditions (i) and (ii), and this concludes the proof. □

As we have mentioned in the introduction to this paper, our interest in the equation $AP + BQ = M$ is principally for the case where the BIBO-stable maps P and Q originate from a fraction representation $\Sigma = PQ^{-1}$ of a BIBO-rational system Σ. In view of Theorem 1, we need the maps P and Q to be right BIBO-coprime. We therefore arrive at the following question: does every BIBO-rational system Σ have a fraction representation $\Sigma = PQ^{-1}$, where P and Q are right BIBO-coprime? An affirmative answer to this question is provided by the next statement.

Theorem 2.

Let $\Sigma : S(R^m) \to S(R^p)$ be an injective system having a fraction representation $\Sigma = PQ^{-1}$, where $P : S \to S(R^p)$ and $Q : S \to S(R^m)$ are BIBO-stable systems, and $S \subset S(R^q)$ for some integer $q > 0$. Then, Σ also has a fraction representation $\Sigma = P_c Q_c^{-1}$ where $P_c : S_c \to S(R^p)$ and $Q_c : S_c \to S(R^m)$ are BIBO-stable and right BIBO-coprime, and where $S_c \subset S(R^q)$ is a suitable subspace.

Proof

Let $0 = \theta_0 < \theta_1 < \theta_2 < \ldots$ be a divergent sequence of real numbers. For notational convenience we define, for every integer $n \geqslant 0$, the difference set

$$\Delta(n, i) := S(\theta^n_{i+1}) \backslash S(\theta^n_i) \quad \text{(the difference set)}$$

Now, for every integer $i \geqslant 0$, let

$$\mathscr{I}_i := \Sigma[\Delta(m, i)]$$

be the image of $\Delta(m, i)$ through Σ, and construct the sets

$$\alpha_i := \Sigma^*[\mathscr{I}_i \cap S(\theta^p_i)]$$

$$\beta_i := \Sigma^*[\mathscr{I}_i \cap U(p)]$$

$$\delta_k := \mathscr{I}_k \backslash \Sigma\{\alpha_k \cup \beta_k\}$$

$$\gamma_i := \bigcup_{k=0}^{i} \{\Sigma^*[\delta_k \cap \Delta(p, i)]\}$$

In view of the injectivity of Σ, we have that α_i is the set of all elements in $\Delta(m, i)$ which are mapped by Σ into $S(\theta^p_i)$; β_i is the set of all elements in $\Delta(m, i)$ having unbounded images through Σ; and, finally, γ_i is the set of all elements in $S(\theta^m_i)$ mapped by Σ into the 'interval' $\Delta(p, i)$. Further, let σ_i and ζ_i be two disjoint subsets of $\Delta(q, i)$ such that σ_i is isomorphic to α_i, and ζ_i is isomorphic to γ_i (without loss of generality, we can assume, after possibly using an evident embedding, that $q \geqslant m$, so that σ_i and ζ_i exist). Since Q is an isomorphism, there are isomorphisms

$$\psi^1_i : Q^{-1}[\alpha_i] \cong \sigma_i, \quad \psi^2_i : Q^{-1}[\gamma_i] \cong \zeta_i$$

For unbounded elements in Im Σ, we define

$$S_\infty := \sum_{i=0}^{\infty} \beta_i = \{\Sigma^*[U(p)]\} \cap \{S(R^m) \backslash U(m)\}$$

By the BIBO-stability of P, and since $\Sigma[S_\infty] = PQ^{-1}[S_\infty] \subset U(p)$, it follows that

$$Q^{-1}[S_\infty] \subset U(q) \tag{3.4}$$

Also, by the BIBO-stability of Q, we have $Q^{-1}[U(m)] \subset U(q)$ (otherwise, Q will map some bounded elements into unbounded elements), so the set $S_* := \{Q^{-1}[S_\infty]\} \cup \{Q^{-1}[U(m)]\}$ is a subset of $U(q)$. We now consider the set

$$S_1 := S_* \cup \left\{ \bigcup_{i=0}^{\infty} (\sigma_i \cup \zeta_i) \right\} \subset S(R^q)$$

and the isomorphism $A : \text{Im } Q^{-1} \to S_1$ given by

$$Ax := \psi^1_i x \quad \text{if} \quad x \in Q^{-1}[\alpha_i],\ i = 0, 1, 2, \ldots$$

$$Ax := \psi^2_i x \quad \text{if} \quad x \in Q^{-1}[\gamma_i],\ i = 0, 1, 2, \ldots$$

$$Ax := x \quad \text{if} \quad x \in S_*$$

We next show that the maps A, PA^{-1}, and QA^{-1} are BIBO-stable, and that PA^{-1} and QA^{-1} are right BIBO-coprime. Then, our proof will conclude, since the conditions of Theorem 2 can be met upon setting $P_c := PA^{-1}$, $Q_c := QA^{-1}$, and $S_c := S_1$. (In (1), (2), (3), and (4) below, i varies over all positive integers.)

(1) *The map $A:S \to S_1$ is BIBO-stable*

Recalling that $S := \operatorname{Im} Q^{-1}$, and using (3.4), we obtain

$$S(\theta_i^q) \cap S = S(\theta_i^q) \cap \left\{Q^{-1}\left[\bigcup_{j=0}^{\infty} \gamma_j\right] \cup Q^{-1}\left[\bigcup_{j=0}^{\infty} \alpha_j\right]\right\}$$

Now, since P is BIBO-stable, there is a real $\tau > 0$ for which $P[S(\theta_i^q) \cap S] \subset S(\tau^p)$. Let b be an integer such that $\theta_b \geqslant \tau$. Then, by the definition of γ_i and the injectivity of P and of Σ, it follows that $S(\theta_i^q) \cap S = S(\theta_i^q) \cap \{Q^{-1}\left[\bigcup_{j=0}^{b} \gamma_j\right] \cup Q^{-1}\left[\bigcup_{j=0}^{\infty} \alpha_j\right]\}$. Similarly, since Q is BIBO-stable, there exists a real $\tau_x > 0$ such that $Q[S(\theta_i^q) \cap S] \subset S(\tau_x^m)$. Let k be an integer for which $\theta_k \geqslant \tau_x$. Then, by the definition of α_i, we have

$$\begin{aligned} S(\theta_i^q) \cap S &= S(\theta_i^q) \cap \left\{Q^{-1}\left[\bigcup_{j=0}^{b} \gamma_j\right] \cup Q^{-1}\left[\bigcup_{j=0}^{k} \alpha_j\right]\right\} \\ &\subset Q^{-1}\left[\bigcup_{j=0}^{b} \gamma_j\right] \cup Q^{-1}\left[\bigcup_{j=0}^{k} \alpha_j\right] \end{aligned}$$

Whence, letting $a := \max\{b, k\}$, we obtain

$$A[S(\theta_i^q) \cap S] \subset AQ^{-1}\left[\bigcup_{j=0}^{a} \gamma_j\right] \cup AQ^{-1}\left[\bigcup_{j=0}^{a} \alpha_j\right] \subset S(\theta_{a+1}^q)$$

where the last inclusion is by the construction of A. Thus, A is BIBO-stable.

(2) *The map $PA^{-1}:S_1 \to S(R^p)$ is BIBO-stable*

By the construction of A, we have

$$A^{-1}[S(\theta_i^q) \cap S_1] \subset \left\{\bigcup_{j=0}^{i-1} Q^{-1}[\alpha_j]\right\} \cup \left\{\bigcup_{j=0}^{i-1} Q^{-1}[\gamma_j]\right\}$$

Whence,

$$PA^{-1}[S(\theta_i^q) \cap S_1] \subset \left\{\Sigma\left[\bigcup_{j=0}^{i-1} \alpha_j\right]\right\} \cup \left\{\Sigma\left[\bigcup_{j=0}^{i-1} \gamma_j\right]\right\} \subset S(\theta_{i+1}^p)$$

by the definition of $\{\alpha_j\}$ and $\{\gamma_j\}$. Thus, PA^{-1} is BIBO-stable.

(3) *The map $QA^{-1}:S_1 \to S(R^m)$ is BIBO-stable*

As before,

$$QA^{-1}[S(\theta_i^q) \cap S_1] \subset Q\left\{Q^{-1}\left[\bigcup_{j=0}^{i-1} (\alpha_j \cup \gamma_j)\right]\right\} \subset \bigcup_{j=0}^{i-1} (\alpha_j \cup \gamma_j) \subset S(\theta_i^m)$$

and QA^{-1} is BIBO-stable.

(4) *The maps PA^{-1} and QA^{-1} are right BIBO-coprime*

We have

$$\begin{aligned} (PA^{-1})^*[S(\theta_i^p)] = AQ^{-1}\{\Sigma^*[S(\theta_i^p)]\} &\subset AQ^{-1}\left\{\left(\bigcup_{j=0}^{\infty} \alpha_j\right) \cup \left(\bigcup_{j=0}^{i-1} \gamma_j\right)\right\} \\ &\subset \left(\bigcup_{j=0}^{\infty} \sigma_j\right) \cup \left(\bigcup_{j=0}^{i-1} \zeta_j\right) \end{aligned}$$

Also, since

$$S(\theta_i^m) \subset \left(\bigcup_{j=0}^{i-1} \alpha_j\right) \cup \left(\bigcup_{j=0}^{i-1} \beta_j\right) \cup \left(\bigcup_{j=0}^{\infty} \gamma_j\right)$$

we obtain

$$(QA^{-1})^*[S(\theta_i^m)] = AQ^{-1}[S(\theta_i^m)] \subset \left(\bigcup_{j=0}^{i-1} \sigma_j\right) \cup (U(q)) \cup \left(\bigcup_{j=0}^{\infty} \zeta_j\right)$$

Thus, since

$$\left(\bigcup_{j=0}^{\infty} \sigma_j\right) \cap \left(\bigcup_{j=0}^{\infty} \zeta_j\right) = \varnothing \quad \text{and} \quad \left(\bigcup_{j=0}^{\infty} (\sigma_j \cup \zeta_j)\right) \cap U(q) = \varnothing$$

we finally get

$$\{(PA^{-1})^*[S(\theta_i^p)]\} \cap \{(QA^{-1})^*[S(\theta_i^m)]\} \subset \left(\bigcup_{j=0}^{i-1} \sigma_j\right) \cup \left(\bigcup_{j=0}^{i-1} \zeta_j\right) \subset S(\theta_i^q)$$

which implies that (PA^{-1}) and (QA^{-1}) are right BIBO-coprime. This concludes our proof. □

Remark

The general existence of right BIBO-coprime fraction representations can be proved using a method similar to the one employed in the proof of Theorem 6 below. We preferred the present proof for Theorem 2 since it invokes the concept of 'cancellation', as evidenced in the relations $P_c = PA^{-1}$ and $Q_c = QA^{-1}$. □

One of the main questions that arises when considering coprime fraction representations of the form described in Theorem 2 is: to what extent are these representations determined by the system Σ? In other words, given two right BIBO-coprime representations of the same system $\Sigma = PQ^{-1}$ and $\Sigma = P_1 Q_1^{-1}$, what is the connection between the BIBO-stable maps P, Q and the BIBO-stable maps P_1, Q_1? Evidently, if $\Sigma = PQ^{-1}$ is a representation where $P: S \to S(R^p)$ and $Q: S \to S(R^m)$ are BIBO-stable and right BIBO-coprime maps, and if $M: S_1 \to S$ is any BIBO-unimodular map, then the fraction representation $\Sigma = P_1 Q_1^{-1}$, where $P_1 := PM: S_1 \to S(R^p)$ and $Q_1 := QM: S_1 \to S(R^m)$, is again a right BIBO-coprime fraction representation of Σ. In the next statement we show that, by varying the BIBO-unimodular map M (together with the space S_1), we can generate in this way *all* coprime fraction representations of Σ from one such representation $\Sigma = PQ^{-1}$.

Theorem 3

Let $\Sigma: S(R^m) \to S(R^p)$ be an injective system, and let $\Sigma = PQ^{-1}$ be a fraction representation where $P: S \to S(R^p)$ and $Q: S \to S(R^m)$ are BIBO-stable and right BIBO-coprime maps, and where Q is an isomorphism and $S \subset S(R^q)$ for some integer $q > 0$. If $\Sigma = P_1 Q_1^{-1}$ is any fraction representation where $P_1: S_1 \to S(R^p)$ and $Q_1: S_1 \to S(R^m)$ are BIBO-stable and BIBO-coprime maps, and where Q_1 is an isomorphism and $S_1 \subset S(R^q)$, then there exists a BIBO-unimodular map $M: S_1 \to S$ such that $P_1 = PM$ and $Q_1 = QM$.

The proof of Theorem 3 depends on the following result.

Lemma 1

Let $\Sigma : S(R^m) \to S(R^p)$ be an injective system, and let $\Sigma = PQ^{-1}$ be a fraction representation where $P : S \to S(R^p)$ and $Q : S \to S(R^m)$ are BIBO-stable and right BIBO-coprime maps with Q an isomorphism and $S \subset S(R^q)$ for some integer $q > 0$. Also, let $T : S_1 \to S(R^m)$ be a BIBO-stable map, where $S_1 \subset S(R^q)$. If the map ΣT is BIBO-stable, then so also is the map $Q^{-1}T$.

Proof (of Lemma 1)

Let $\theta > 0$ be a real number. In view of the BIBO-stability of the maps $T : S_1 \to S(R^m)$ and $\Sigma T : S_1 \to S(R^p)$, there exist positive numbers τ_1 and τ_2 such that $S_2 := T[S(\theta^q) \cap S_1] \subset S(\tau_1^m)$ and $S_3 := \Sigma T[S(\theta^q) \cap S_1] \subset S(\tau_2^p)$. Now, let $X := Q^{-1}[S_2]$. Then, clearly, $P[X] = S_3$, so that, by injectivity, $X = P^*[S_3] \cap Q^*[S_2]$. Letting $\tau := \max\{\tau_1, \tau_2\}$, and using the facts that $S_2 \subset S(\tau^m)$ and $S_3 \subset S(\tau^p)$, we obtain $X \subset P^*[S(\tau^p)] \cap Q^*[S(\tau^m)]$. By the right BIBO-coprimeness of P and Q, there exists a real $\xi > 0$ for which $P^*[S(\tau^p)] \cap Q^*[S(\tau^m)] \subset S(\xi^q)$, so $X \subset S(\xi^q)$. Thus, for every real $\theta > 0$ there exists a real $\xi > 0$ such that $Q^{-1}T[S(\theta^q) \cap S_1] = Q^{-1}[S_2] = X \subset S(\xi^q)$, and $Q^{-1}T : S_1 \to S$ is BIBO-stable. □

Proof (of Theorem 3)

We use the notation of the theorem. Clearly, the maps ΣQ_1 $(= P_1)$ and ΣQ $(= P)$ are BIBO-stable. Thus, by Lemma 3.9 and our assumption that $\Sigma = PQ^{-1}$ and $\Sigma = P_1 Q_1^{-1}$ are coprime fraction representations, it follows that the maps $Q^{-1}Q_1$ and $Q_1^{-1}Q$ are BIBO-stable. But, the map $M := Q^{-1}Q_1 : S_1 \to S$ is evidently an isomorphism, and $M^{-1} = Q_1^{-1}Q$. Consequently, both M and M^{-1} are BIBO-stable, and, since $Q_1 = QM$ and $P_1 = \Sigma Q_1 = PQ^{-1}QM = PM$, our proof is complete. □

Let $\Sigma : S(R^m) \to S(R^p)$ be an injective right BIBO-rational system, and consider a coprime fraction representation $\Sigma = PQ^{-1}$, where $P : S \to S(R^p)$ and $Q : S \to S(R^m)$ are BIBO-stable and right BIBO-coprime maps, and where $S \subset S(R^q)$ for some integer $q > 0$. In view of Theorem 3, the space S is uniquely determined by the system Σ up to a BIBO-unimodular isomorphism. We call S the *factorization space* of Σ, and we denote it by $F(\Sigma)$. Also, for brevity, we say that two sets $S_1 \subset S(R^m)$, $S_2 \subset S(R^n)$ are *BIBO-morphic* if there exists a BIBO-unimodular isomorphism $M : S_1 \cong S_2$. Now, in case the factorization space $F(\Sigma)$ is BIBO-morphic to $S(R^m)$, we can choose $S(R^m)$ as our factorization space. Indeed, if $M : S(R^m) \cong F(\Sigma)$ is a BIBO-unimodular map, then the BIBO-stable maps $P_c := PM$ and $Q_c := QM$ evidently induce a coprime fraction representation $\Sigma = P_c Q_c^{-1}$ having $S(R^m)$ as its factorization space. The situation where $F(\Sigma)$ is $S(R^m)$ is particularly convenient from the control-theoretic point of view. However, in some cases, $F(\Sigma)$ may not be BIBO-morphic to $S(R^m)$. Below, we provide a necessary and sufficient condition on the given system Σ for $F(\Sigma)$ to be BIBO-morphic to $S(R^m)$. We shall denote by $G(\Sigma)$ the set of all ordered pairs $(u, \Sigma u)$, where u varies over all of $S(R^m)$, namely, the set of all pairs consisting of input elements $u \in S(R^m)$ and their corresponding outputs Σu. Clearly, the set $G(\Sigma)$ traces the graph of Σ in the space $S(R^m) \times S(R^p)$, so we refer to $G(\Sigma)$ as the *graph* of Σ. As we show in the next statement, it is the structure of the graph $G(\Sigma)$ that determines the structure of the factorization space $F(\Sigma)$.

Theorem 4

Let $\Sigma: S(R^m) \to S(R^p)$ be an injective system. The factorization space $F(\Sigma)$ is BIBO-morphic to the graph $G(\Sigma)$.

Proof

Let $\Sigma = PQ^{-1}$, where $P: F(\Sigma) \to S(R^p)$ and $Q: F(\Sigma) \to S(R^m)$ are BIBO-stable and right BIBO-coprime maps. For every real $\tau > 0$, we denote by $S_*(\tau) := G(\Sigma) \cap \{S(\tau^m) \times S(\tau^p)\}$, i.e. the set of all pairs $(u, \Sigma u)$ for which both the input u and the output Σu are 'bounded' by τ. The map $R: F(\Sigma) \to G(\Sigma): Rx := (Qx, Px)$ induces an isomorphism $F(\Sigma) \simeq G(\Sigma)$, and its inverse $R^{-1}: G(\Sigma) \to F(\Sigma)$ can be expressed as $R^{-1}(\alpha, \beta) = Q^*\alpha \cap P^*\beta$ since we always have $Q^*\alpha = P^*\beta$ for every $(\alpha, \beta) \in G(\Sigma)$. Now, we claim that R is BIBO-unimodular. Indeed, since both P and Q are BIBO-stable, clearly R is BIBO-stable as well. Also, since P and Q are right BIBO-coprime, there exists, for every real $\theta > 0$, a real $\tau > 0$ satisfying $Q^*[S(\theta^m)] \cap P^*[S(\theta^p)] \subset S(\tau^q)$ (where q is such that $F(\Sigma) \subset S(R^q)$), so we obtain $R^{-1}[S_*(\theta)] \subset Q^*[S(\theta^m)] \cap P^*[S(\theta^p)] \subset S(\tau^q)$, and R^{-1} is BIBO-stable. Thus, R is BIBO-unimodular, and $G(\Sigma)$ is BIBO-morphic to $F(\Sigma)$. □

As a direct consequence of Theorem 4, we obtain the next result.

Corollary

Let $\Sigma: S(R^m) \to S(R^p)$ be an injective system. The factorization space $F(\Sigma)$ is BIBO-morphic to $S(R^m)$ if and only if the graph $G(\Sigma)$ is BIBO-morphic to $S(R^m)$.

4. Coprimeness of non-linear systems: the case of *C*-stability

In the present section we discuss the concept of coprimeness of non-linear systems, using C-stability as the underlying notion of stability. Basically, the situation here is similar to the one encountered in § 3, where the underlying notion of stability was BIBO-stability, but, naturally, some modifications are necessary. In view of our definition of C-stability, it will be convenient to restrict our attention in the present section to input sequences starting at $t = 0$, i.e. to input sequences from $S_0(R^m)$. For the case of causal time-invariant systems, such a restriction does not impair in any sense the generality of the discussion. We start with a statement of the problem. Let $S \subset S_0(R^q)$, where $q > 0$ is an integer, be a subspace, and let $P: S \to S_0(R^p)$ and $Q: S \to S_0(R^m)$ be C-stable systems, with Q an isomorphism. We wish to find under what conditions there exist C-stable maps $A: S_0(R^p) \to S$ and $\mathrm{B}: S_0(R^m) \to S$ for which the (C-stable) map

$$M := AP + BQ: S \to S \tag{4.1}$$

is an isomorphism having a C-stable inverse M^{-1}. An isomorphism $N: S_1 \to S_2$ for which both N and its inverse N^{-1} are C-stable will be referred to as a *C-unimodular* map.

Now, by definition, every C-stable map is BIBO-stable as well. Thus, the maps A, B, P and Q of (4.1) are, in particular, BIBO-stable, and M is BIBO-unimodular. The discussion in § 3 leads us therefore to the conclusion that, for A and B to exist, the maps P and Q have to be right BIBO-coprime. So motivated, we introduce the following definition.

Definition 2

Let $S \subset S_0(R^q)$ be a subspace. Two C-stable maps $P: S \to S_0(R^p)$ and $Q: S \to S_0(R^m)$ are *right C-coprime* if the following conditions hold:

(α) For every real $\tau > 0$ there exists a real $\theta > 0$ such that $P^*[S_0(\tau^p)] \cap Q^*[S_0(\tau^m)] \subset S_0(\theta^q)$, and

(β) For every real $\tau > 0$, the set $S \cap S_0(\tau^q)$ is complete (i.e. is a closed subset of $S_0(\tau^q)$).

As we can see, condition (α) is just the condition for right BIBO-coprimeness to which we have alluded before. Condition (β) is motivated by continuity considerations, as follows. Consider the equation $AP + BQ = M$, where we choose M as the restriction of a C-unimodular map $M: S_0(R^q) \to S_0(R^q)$ (for example, take $M = I$, the identity). Since all the maps in this equation are continuous and BIBO-stable, they possess unique continuous extensions to the closure $\overline{S \cap S_0(\tau^q)}$ in $S_0(\tau^q)$. Evidently, this extension will still satisfy the same equation, but this time over the larger space – the closure. Thus, the solution on $\overline{S \cap S_0(\tau^q)}$ actually provides a solution on the closure $\overline{S \cap S_0(\tau^q)}$, whence the coprimeness condition (α) must hold (by BIBO-coprimeness considerations) over the closure. Hence (β). In the next statement we show that right C-coprimeness ensures the existence of C-stable maps A and B satisfying (4.1). (We again restrict our attention to injective systems.)

Theorem 5

Let $S \subset S_0(R^q)$ be a subspace, and let $P: S \to S_0(R^p)$ and $Q: S \to S_0(R^m)$ be C-stable maps, where P is injective and Q is an isomorphism. If P and Q are right C-coprime, then, for every C-unimodular map $M: S \to S$, there exist C-stable maps $A: \mathrm{Im}\, P \to S_0(R^q)$ and $B: S_0(R^m) \to S_0(R^q)$ satisfying $AP + BQ = M$.

Proof

The present proof is a refinement of the proof of Theorem 1. We again propose to construct two subsets Λ_A and Λ_B of the space $S \subset S_0(R^q)$ which satisfy the following conditions:

(i) $\Lambda_A \cup \Lambda_B = S$.

(ii) For every real $\tau > 0$ there is a real $\theta > 0$ such that $Q^*[S_0(\tau^m)] \cap \Lambda_B \subset S_0(\theta^q)$ and $P^*[S_0(\tau^p)] \cap \Lambda_A \subset S_0(\theta^q)$.

(iii) For every real $\tau > 0$, the sets $Q^*[S_0(\tau^m)] \cap \Lambda_B$ and $P^*[S_0(\tau^p)] \cap \Lambda_A$ are closed subsets of $S_0(\theta^q)$ (where θ is from (ii)), and hence are compact.

We note that by condition (β) of Definition 2, every closed subset of $S \cap S_0(\tau^q)$ is also a closed subset of $S_0(\tau^q)$, and is therefore compact. Below, we tacitly make repeated use of this observation.

Suppose, for a moment, that a decomposition satisfying conditions (i), (ii), and (iii) has been achieved. Then, we can define a pair of maps $A: \mathrm{Im}\, P \to S_0(R^q)$ and $B: S_0(R^m) \to S_0(R^q)$ as follows. Let $0 < \theta_1 < \theta_2 < \ldots$ be a divergent sequence of real numbers. For every element $u \in \{Q[\Lambda_B]\} \cup U(m)$, let $Bu := MQ^*u$. Now, since $Q[\Lambda_B] \cap S_0(\theta_i^m) = Q\{Q^*[S_0(\theta_i^m)] \cap \Lambda_B\}$, since $Q^*[S_0(\theta_i^m)] \cap \Lambda_B$ is compact, and since Q is injective and continuous, it follows that the isomorphism $Q: Q^*[S_0(\theta_i^m)] \cap \Lambda_B \cong Q[\Lambda_B] \cap S_0(\theta_i^m)$ is a homeomorphism, and, thus, the restriction of Q^{-1} to the set $Q[\Lambda_B] \cap S_0(\theta_i^m)$ is continuous for every $i \geqslant 1$. Hence, the restriction of MQ^* to $Q[\Lambda_B]$ is C-stable, and B is C-stable on $Q[\Lambda_B]$. Consequently, for each integer $i \geqslant 1$, there is a real $\tau_i > 0$ such that $B\{Q[\Lambda_B] \cap S_0(\theta_i^m)\} \subset S_0(\tau_i^q)$. We now extend B inductively into a C-stable map $S_0(R^m) \to S_0(R^q)$. Let $S_0(\theta_0^m) := 0$, and assume that, for some integer $i \geqslant 0$, the map B has been extended into a continuous map $S_0(\theta_i^m) \to S_0(\tau_i^p)$. Then, since B was originally defined on $Q[\Lambda_B]$, we have B defined as a continuous map on the closed subset $\{Q[\Lambda_B] \cap S_0(\theta_{i+1}^m)\} \cup S_0(\theta_i^m)$ of $S_0(\theta_{i+1}^m)$, and its values there are in the complete set $S_0(\tau_{i+1}^p)$. By the extension theorem for continuous functions (for example, Kuratowski (1961)), there is a continuous extension $B: S_0(\theta_{i+1}^m) \to S_0(\tau_{i+1}^p)$. Repeating the same procedure for all integers $i \geqslant 0$, we obtain a C-stable map $B: S_0(R^m) \to S_0(R^q)$.

Next, we define the map $A: \operatorname{Im} P \to S$ by $Au := (M - BQ)P^*u$ for all $u \in P[\Lambda_A]$, and by $Au := 0$ for all $u \in P[\Lambda_B]$. To verify consistency of this definition, we note that, for every element $v \in \Lambda_A \cap \Lambda_B$, one has $APv = 0$ from $Pv \in P[\Lambda_B]$, and, from $Pv \in P[\Lambda_A]$, we obtain $APv = (M - BQ)P^*Pv = Mv - BQv = 0$ by the construction of B (since v is also in Λ_B), and it follows that our definition of the map A is consistent. We next show that A is C-stable. First, the restriction of A to $P[\Lambda_B]$, being constantly zero, is evidently C-stable. Regarding the restriction of A to $P[\Lambda_A]$, we note that, by conditions (ii) and (iii), the set $P^*[S_0(\tau^p)] \cap \Lambda_A$ is compact, so the isomorphism restriction $P: P^*[S_0(\tau^p)] \cap \Lambda_A \to P[\Lambda_A] \cap S_0(\tau^p)$ is a homeomorphism, since it is a continuous injective function on a compact domain. Thus by an argument similar to the one used to prove the C-stability of the restriction of B to $Q[\Lambda_B]$, we obtain that the restriction of A to $P[\Lambda_A]$ is C-stable. Recalling that $\operatorname{Im} P = P[\Lambda_A] \cup P[\Lambda_B]$, we further show that A is C-stable over all of $\operatorname{Im} P$. To this end, let $\{u_i'\} \subset P[\Lambda_A] \cap S_0(\theta^p)$ and $\{v_i'\} \subset P[\Lambda_B] \cap S_0(\theta^p)$ be two sequences converging to the same point

$$u' := \lim_{i \to \infty} u_i' = \lim_{i \to \infty} v_i'$$

Since clearly

$$\lim_{i \to \infty} Av_i' = \lim_{i \to \infty} 0 = 0$$

we need to show, in order to prove continuity, that also

$$\lim_{i \to \infty} Au_i' = 0$$

Let $u_i := P^*u_i'$, $i = 1, 2, \ldots$, and note that $u_i \in \Lambda_A \cap P^*[S_0(\theta^p)] \subset S_0(\tau^p)$. Since the restriction $P: \Lambda_A \cap P^*[S_0(\theta^p)] \to P[\Lambda_A] \cap S_0(\theta^p)$ is a homeomorphism, the sequence $\{u_i\}$ converges to a point $u = P^*u' \in \Lambda_A \cap P^*[S_0(\theta^p)] \subset S_0(\tau^p)$. Now, clearly, $S_0(\tau^p) \cap S = \{\Lambda_A \cap S_0(\tau^p)\} \cup \{\Lambda_B \cap S_0(\tau^p)\}$. Thus, the following two possibilities arise:

(α) u is an internal point of $\Lambda_A \cap S_0(\tau^p)$, or

(β) u is not an internal point of $\Lambda_A \cap S_0(\tau^p)$

(in both cases with respect to the topology on $S \cap S_0(\tau^p)$). Now, $\overline{\Lambda_A \cap S_0(\tau^p)}$ is a compact set, so the restriction of P to it is a homeomorphism. Therefore, in case (α), the point $u' = Pu$ is an internal point of $P[\Lambda_A \cap S_0(\tau^p)]$. Whence, since $v_i' \to u'$, there is a subsequence $\{w_i\}$ of $\{v_i'\}$ which is completely contained in $P[\Lambda_A \cap S_0(\tau^p)]$. But then, since we have already shown that A is C-stable on $P[\Lambda_A]$, we have $\lim Au_i' = \lim$

Aw_i, and, since $Aw_i = 0$ for all i (by same property of the sequence $\{v_i\}$), it follows that $\lim Au'_i = 0$. When (β) holds, there is a sequence $\{z_i\} \subset \Lambda_B \cap S_0(\tau^p)$ which converges to u. In such case, by the C-stability of the map $M - BQ$, we obtain $\lim (M - BQ)u_i = \lim (M - BQ)z_i = 0$, where the last equality holds by the definition of B since $z_i \in \Lambda_B$. Whence, $\lim Au'_i = \lim (M - BQ)P^*u'_i = \lim (M - BQ)u_i = 0$. Thus, in both cases $\lim Au'_i = 0$, and A is C-stable.

Finally, an explicit computation similar to the one given in proof of Theorem 1, shows that $(AP + BQ)u = Mu$ for all $u \in S$. Summarizing, starting from the sets Λ_A and Λ_B, we have constructed C-stable maps A, B satisfying $AP + BQ = M$. To conclude our proof, it only remains to construct the sets Λ_A and Λ_B, which is our next objective.

First, we use the construction described in proof of Theorem 1 to obtain a pair of sets Λ_1 and Λ_2 which satisfy conditions (i) and (ii) when setting Λ_1 for Λ_A and Λ_2 for Λ_B. Using the sets Λ_1 and Λ_2, we proceed now to construct the required sets Λ_A and Λ_B. Let $0 < \tau_1 < \tau_2 < \ldots$, and $0 < \theta_1 < \theta_2 < \ldots$ be the two divergent sequences of real numbers employed in the construction of the sets Λ_1 and Λ_2 (proof of Theorem 1). We define the closed 'intervals'

$$\delta_1 := S_0(\tau_1^m)$$

$$\delta_i := \overline{S_0(\tau_i^m)\backslash S_0(\tau_{i-1}^m)} \subset S_0(\tau_i^m), \quad i = 2, 3, \ldots$$

Then, by (ii), we clearly have $Q^*[\delta_i] \cap \Lambda_2 \subset S_0(\theta_i^q)$ for all integers $i \geqslant 1$. By the C-stability of Q, the set $Q^*[\delta_i] \cap S_0(\xi^q)$ is closed for all real $\xi > 0$. Thus, the closure $\overline{Q^*[\delta_i] \cap \Lambda_2} \subset S_0(\theta_i^q)$ is contained in $Q^*[\delta_i]$, so there is, for each $i \geqslant 1$, a subset $H_i \subset Q^*[\delta_i] \cap S_0(\theta_i^q)$ for which $Q^*[\delta_i] \cap \{\Lambda_2 \cup H_i\} = \overline{Q^*[\delta_i] \cap \Lambda_2}$. Define now

$$\Lambda_B := \Lambda_2 \cup \left\{\bigcup_{i=1}^{\infty} H_i\right\}$$

Then, since $Q^*[S_0(\tau_i^m)] = \bigcup_{j=1}^{i} Q^*[\delta_j]$, we obtain

$$\begin{aligned}
\Lambda_B \cap Q^*[S_0(\tau_i^m)] &= \{Q^*[S_0(\tau_i^m)] \cap \Lambda_2\} \cup \left\{Q^*[S_0(\tau_i^m)] \cap \left(\bigcup_{j=1}^{\infty} H_j\right)\right\} \\
&= \{Q^*[S_0(\tau_i^m)] \cap \Lambda_2\} \cup \left\{\left(\bigcup_{j=1}^{i} H_j\right) \cup \left(Q^*[\delta_i] \cap H_{i+1}\right)\right\} \\
&= \left\{\bigcup_{j=1}^{i} (Q^*[\delta_j] \cap (\Lambda_2 \cup H_j))\right\} \\
&\quad \cup \{Q^*[\delta_i] \cap (Q^*[\delta_{i+1}] \cap (\Lambda_2 \cup H_{i+1}))\}
\end{aligned}$$

where in the last term we used the fact $H_{i+1} \subset Q^*[\delta_{i+1}]$, which implies that $Q^*[\delta_i] \cap H_{i+1} = Q^*[\delta_i] \cap (Q^*[\delta_{i+1}) \cap H_{i+1})$. Thus, the set $Q^*[S_0(\tau_i^m)] \cap \Lambda_B$ is the union of a finite number of closed sets, and is thus closed. Consequently, the set Λ_B satisfies all its respective requirements in (i), (ii), and (iii). The construction of Λ_A from Λ_1 is done similarly, by replacing Q by P, Λ_2 by Λ_1, and Λ_B by Λ_A throughout the last paragraph. □

We turn now to an examination of the existence of right-coprime fraction representations in the C-stability sense. Let $\Sigma : S_0(R^m) \to S_0(R^p)$ be an injective system. We say that Σ has a *right C-coprime fraction representation* if there exists a pair of C-stable and right C-coprime maps $P:S\to S_0(R^p)$, $Q:S\to S_0(R^m)$, where $S\subset S_0(R^q)$ for some integer $q>0$, and where P is injective and Q is an isomorphism, such that $\Sigma=PQ^{-1}$. The set S is then called the *factorization space* of this representation. We first show that, in order to possess a right C-coprime fraction representation, the system Σ must exhibit continuous behaviour whenever its outputs are bounded.

Proposition

Let $\Sigma:S_0(R^m)\to S_0(R^p)$ be an injective system having a right C-coprime fraction representation. Let $\theta>0$ be a real number, and let $S_*\subset S_0(\theta^m)$ be a subset. Assume that there exists a real $\tau>0$ such that $\Sigma[S_*]\subset S_0(\tau^p)$. Then, the restriction of Σ to the closure $\bar{S}_*$ of S_* in $S_0(\theta^m)$ is a continuous map.

Proof

Let $\Sigma=PQ^{-1}$ be a right C-coprime fraction representation of Σ. Let $S_1:=Q^{-1}[S_*]$, and $S_2:=\Sigma[S_*]$. Then, denoting $\alpha:=\max\{\theta,\tau\}$, we have $S_1=Q^*[S_*]=P^*[S_2]=Q^*[S_*]\cap P^*[S_2]\subset Q^*[S(\alpha^m)]\cap P^*[S(\alpha^p)]$. Whence, by the right-coprimeness of P and Q, there is a real $\beta>0$ such that $S_1\subset S(\beta^q)$. Now, let $\bar{S}_1$ be the closure of S_1 in $S(\beta^q)$, so $\bar{S}_1$ is a compact set. By coprimeness, $S\cap S(\beta^q)$ is a closed subset of $S(\beta^q)$, so $\bar{S}_1$ is still a subset of S; by the C-stability of Q, the set $Q[\bar{S}_1]$ is well-defined, closed, and bounded, and we have $Q[\bar{S}_1]=\bar{S}_*$. Consequently, the continuous isomorphism $Q:\bar{S}_1\to\bar{S}_*$ is actually a homeomorphism (see for example Kuratowski (1961)), and the restriction of Q^{-1} to $\bar{S}_*$ is a continuous map. But then, by the continuity of P, the restriction of Σ $(=PQ^{-1})$ to $\bar{S}_*$ is a continuous map as well. □

The property of the system Σ described in the Proposition plays a fundamental role in our theory of C-rationality. It will be convenient to give it a special name.

Definition 3

Let $\Sigma:S_0(R^m)\to S_0(R^p)$ be a system. We say that Σ is *homogeneous* if for every real $\theta>0$ the following holds: for every subset $S_*\subset S(\theta^m)$ for which there exists a real $\tau>0$ such that $\Sigma[S_*]\subset S_0(\tau^p)$, the restriction of Σ to the closure $\bar{S}_*$ of S_* in $S_0(\theta^m)$ is a continuous map.

In view of the Proposition, homogeneity is a necessary condition for the existence of a right-coprime fraction representation in the C-stability sense. We next show that homogeneity is also a sufficient condition for the existence of such a fraction representation.

Theorem 6

An injective system $\Sigma:S_0(R^m)\to S_0(R^p)$ has a right C-coprime fraction representation if and only if it is homogeneous.

Proof

The 'only if' direction is stated in the Proposition. We prove now the 'if' direction by constructing a right C-coprime fraction representation $\Sigma=PQ^{-1}$.

Adhering to the notation of § 3, we let $G(\Sigma) := \{(u, \Sigma u) : u \in S_0(R^m)\}$ be the graph of Σ, and we denote, for brevity, $S := G(\Sigma)$, so S is a subset of $S_0(R^q)$ with $q = m + p$. Let $P_1 : S_0(R^m) \times S_0(R^p) \to S_0(R^m)$ and $P_2 : S_0(R^m) \times S_0(R^p) \to S_0(R^p)$ be the respective natural projections, and define the maps $P : S \to S_0(R^p) : Px := P_2 x$ for all $x \in S$, and $Q : S \to S_0(R^m) : Qx := P_1 x$ for all $x \in S$. In view of the fact that Σ is an injective map, we have that P is injective and Q is an isomorphism. Also, clearly, $\Sigma = PQ^{-1}$, and, since P and Q are just restrictions of the natural projections from the space product, the maps P and Q are C-stable. To show that P and Q are right C-coprime, we first note that, by the construction of P and Q, one evidently has $P^*[S_0(\theta^p)] \cap Q^*[S_0(\theta^m)] \subset S \cap S_0(\theta^q) \subset S_0(\theta^q)$, and condition ($\alpha$) of Definition 2 holds. Thus, we only have to show that $S \cap S_0(\theta^q)$ is a closed subset of $S_0(\theta^q)$, for all real $\theta > 0$. To this end, let $u_1, u_2, \ldots \subset S \cap S_0(\theta^q)$ be a sequence of elements converging to a point $u \in S_0(\theta^q)$. Our proof will conclude upon showing that $u \in S$. Now, the projected sequence $u_i' := P_1 u_i$, $i = 1, 2, \ldots$, clearly converges to the point $u' := P_1 u \in S_0(\theta^m)$, whereas the projected sequence $u_i'' := P_2 u_i$, $i = 1, 2, \ldots$, converges to the point $u'' := P_2 u \in S_0(\theta^p)$. Further, since $u_i \in S$, we have $u_i'' = \Sigma u_i'$ for all $i = 1, 2, \ldots$, and

$$\lim_{i \to \infty} \Sigma u_i' = \lim_{i \to \infty} u_i'' = u''$$

On the other hand, since $\{\Sigma u_i'\} \subset S_0(\theta^p)$ and $\{u_i'\} \subset S_0(\theta^m)$, it follows by homogeneity that Σ is continuous on the closure $\overline{\{u_i'\}}$, so $\lim \Sigma u_i' = \Sigma(\lim u_i') = \Sigma u'$. Thus $u'' = \Sigma u'$, or $u = (u', u'')$ belongs to S, and $S \cap S_0(\theta^q)$ is closed in $S_0(\theta^q)$. □

We now turn to a discussion of the uniqueness of coprime fraction representations in the C-stability sense. The situation here closely resembles the situation for the case of BIBO-rationality described in Theorem 3.

Theorem 7

Let $\Sigma : S_0(R^m) \to S_0(R^p)$ be an injective homogeneous system, and let $\Sigma = PQ^{-1}$ and $\Sigma = P_1 Q_1^{-1}$ be two right C-coprime fraction representations of Σ, with factorization spaces $S, S_1 \subset S_0(R^q)$, respectively. Then, there exists a C-unimodular map $M : S_1 \to S$ such that $P_1 = PM$ and $Q_1 = QM$.

Our proof of Theorem 7 depends on the following auxiliary result.

Lemma 2

Let $\Sigma : S_0(R^m) \to S_0(R^p)$ be an injective system having a right C-coprime fraction representation $\Sigma = PQ^{-1}$, and let $T : S_1 \to S_0(R^m)$ be a C-stable isomorphism, where S_1 is a subspace of $S_0(R^q)$ for which the set $S_1 \cap S_0(\theta^q)$ is a closed subset of $S_0(\theta^q)$, for any $\theta > 0$. If the map ΣT is C-stable, then so also is the map $Q^{-1}T$.

Proof (of Lemma 2)

Let $\theta > 0$ be a real number. In view of the C-stability of the maps $T : S_1 \to S_0(R^m)$ and $\Sigma T : S_1 \to S_0(R^p)$, there exist real positive numbers τ_1 and τ_2 for which $S_2 := T[S_0(\theta^q) \cap S_1] \subset S_0(\tau_1^m)$ and $S_3 := \Sigma T[S_0(\theta^q) \cap S_1] \subset S_0(\tau_2^p)$. Also, since $S_0(\theta^q) \cap S_1$ is closed in $S_0(\theta^q)$ it is compact, and, hence, the set S_2, being the continuous image of a compact set, is a closed subset of $\mathrm{Im}\, T \cap S_0(\tau_1^m) = S_0(\tau_1^m)$. Thus,

S_2 is compact. Now, let $X := Q^{-1}[S_2]$. Then, clearly, $P[X] = S_3$, so that, by injectivity, $X = P^*[S_3] \cap Q^*[S_2]$. Letting $\tau := \max\{\tau_1, \tau_2\}$, we obtain $X \subset P^*[S_0(\tau^p)] \cap Q^*[S_0(\tau^p)]$. By the right C-coprimeness of P and Q, there exists a real $\xi > 0$ for which $P^*[S_0(\tau^p)] \cap Q^*[S_0(\tau^m)] \subset S_0(\xi^q)$, and it follows that $X \subset S_0(\xi^q)$. Further, let S be the factorization space of the fraction representation $\Sigma = PQ^{-1}$. Then, since Q is continuous, the set $X = Q^{-1}[S_2]$ is a closed subset of $S \cap S_0(\xi^q)$, and whence, by the coprimeness requirement, a closed subset of $S_0(\xi^q)$. Consequently, X is compact, and the restriction of Q to X is a homeomorphism. Since $Q[X] = S_2$, it follows that the restriction of Q^{-1} to S_2 is a continuous function. Thus, by the continuity of T and the fact that $S_2 = T[S_1 \cap S_0(\theta^q)]$, the restriction of $Q^{-1}T$ to $S_1 \cap S_0(\theta^q)$ is continuous (and bounded), for every real $\theta > 0$. This proves our assertion. □

Proof (of Theorem)

We use the notation of the theorem. Clearly, the maps $\Sigma Q_1 (= P_1)$ and $\Sigma Q\ (= P)$ are C-stable. Thus, by Lemma 2 and our assumption that $\Sigma = PQ^{-1}$ and $\Sigma = P_1Q_1^{-1}$ are right C-coprime fraction representations, it follows that the maps $Q^{-1}Q_1$ and $Q_1^{-1}Q$ are both C-stable. But the map $M : Q^{-1}Q_1 : S_1 \to S$ is an isomorphism, and its inverse is $M^{-1} = Q_1^{-1}Q$. Consequently, both M and M^{-1} are C-stable, and, since $Q_1 = QM$ and $P_1 = \Sigma Q_1 = PQ^{-1}QM = PM$, our proof is concluded. □

Finally, combining Theorem 7 with the proof of Theorem 6, we obtain the following analogue of Theorem 4. (We say that two spaces S_1, S_2 are *C-morphic* if there exists a C-unimodular map $M : S_1 \cong S_2$.)

Theorem 8

Let $\Sigma : S_0(R^m) \to S_0(R^p)$ be an injective homogeneous system. The factorization space of any right C-coprime fraction representation of Σ is C-morphic to the graph $G(\Sigma)$.

We conclude with an example of a class of homogeneous systems.

Example†

A system $\Sigma : S_0(R^m) \to S_0(R^p)$ is called *recursive* if there exist integers η, $\mu > 0$ and a function $f : (R^p)^{\eta+1} \times (R^m)^{\mu+1} \to R^p$ such that, for every pair of sequences $u \in S(R^m)$ and $y := \Sigma u \in S(R^p)$, the following relationship holds for all integers k:

$$y_{k+\eta+1} = f(y_k, y_{k+1}, \ldots, y_{k+\eta}, u_k, \ldots, u_{k+\mu})$$

The function f is called the *recursion* function of Σ. Using our definition of the topologies on $S_0(R^m)$ and $S_0(R^p)$, one can readily show through an explicit computation that the following holds.

Let $\Sigma : S_0(R^m) \to S_0(R^p)$ be a recursive system with recursion function $f : (R^p)^{\eta+1} \times (R^m)^{\mu+1} \to R^p$. If the function f is continuous, then Σ is a homogeneous system.

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Thus, we see that a large class of commonly encountered systems is indeed homogeneous. (Of course, the continuity of f does not imply that Σ is stable in any sense.) In particular, we see from here that the rationality of linear recursive systems is 'caused' not by linearity, but rather by the evident continuity of their recursion functions.

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