Proceedings of the European Control Conference, Grenoble, France, July 1991

NONLINEAR CONTROL AND FRACTION REPRESENTATIONS

Jacob Hammer
Center for Mathematical System Theory
Department of Electrical Engineering
University of Florida
Gainesville, FL 32611, USA

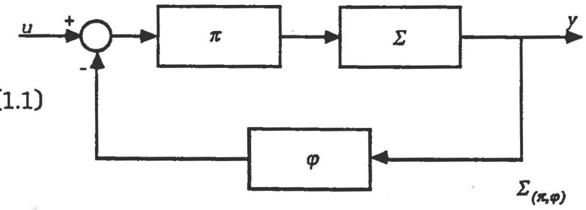
A fraction representation of a nonlinear system Σ is a factorization of the system into a composition of two nonlinear systems, one of which is stable and the other is the inverse of a stable system. There are two main kinds of fraction representations: a right fraction representation, which is of the form $\Sigma = PQ^{-1}$, and a left fraction representation, which is of the form $\Sigma = T^{-1}G$. As it turns out, fraction representations provide effective means for the solution of nonlinear control problems. The present note is a brief overview of the theory of fraction representations of nonlinear systems and its applications to nonlinear control.

Keywords: Nonlinear control; Robustness; Fraction representations.

1. INTRODUCTION

There seems to be an intimate connection between feedback theory and fraction representations, originating at the most fundamental level. Indeed, consider the following basic additive feedback configuration used in many practical applications of control theory.

(1.1)



Here, Σ is a nonlinear system that needs to be controlled; π is a nonlinear dynamic precompensator; and φ is a nonlinear dynamic feedback compensator. In order to prevent the loss of degrees of freedom, the precompensator π is required to be nonsingular. The overall system described by the configuration is denoted by $\Sigma_{(\pi,\varphi)}$. A simple calculation shows that

(1.2) $\Sigma_{(\pi,\varphi)} = \Sigma\pi[I + \varphi\Sigma\pi]^{-1}$.

Defining the equivalent precompensator

(1.3) $\ell_{(\pi,\varphi)} := \pi[I + \varphi\Sigma\pi]^{-1}$,

we obtain that

(1.4) $\Sigma_{(\pi,\varphi)} = \Sigma\ell_{(\pi,\varphi)}$.

Now, it is a quite simple matter to show (HAMMER [1984b]) that when the closed loop system $\Sigma_{(\pi,\varphi)}$ is well posed and internally stable, the equivalent precompensator $\ell_{(\pi,\varphi)}$ has the following two properties: (i) it is an invertible system, and (ii) it is stable. Combining these facts with (1.4), we obtain a representation of the given system Σ in the form

(1.5) $\Sigma = \Sigma_{(\pi,\varphi)}(\ell_{(\pi,\varphi)})^{-1}$,

which is simply a right fraction representation of the system, having the 'numerator' $\Sigma_{(\pi,\varphi)}$ and the denominator $\ell_{(\pi,\varphi)}$. Thus, we see that the process of feedback stabilization inherently gives rise to a (right) fraction representation of the system being stabilized. A similar phenomenon also occurs when the stabilizing feedback is non-additive (HAMMER [1989c]). Whence, considerations on a very basic level directly reveal a close association between fraction representations and feedback stabilization.

These observations give rise to the premonition that fraction representations form an important key to the development of a general theory of feedback stabilization for nonlinear control systems. Extensive work performed over the last few years by numerous investigators seems to bear out this premonition. The remaining portions of this note provide a brief overview of the theory of fraction representations of nonlinear systems and of its applications to the solution of the nonlinear feedback stabilization problem. The material presented is mostly a review and re-interpretation of the author's own work in this area (HAMMER [1984a, b, 1987, 1988, 1989a, b, c, and 1990]). Alternative points of view as well as further aspects of the theory are considered in DESOER and KABULI [1988], VERMA [1988], SONTAG [1989 and 1990], PAICE and MOORE [1990], CHEN and de FIGUEIREDO [1990], the references cited in these papers, and others.

2. NOTATION AND BASICS

We adopt the input/output point of view, and restrict the presentation to discrete-time systems. Thus, a system is regarded as a map that transforms input sequences into output sequences. Denote by $S(R^m)$ the set of all sequences of vectors $(u_0, u_1, u_2, \ldots)$, where $u_j \in R^m$ for all integers $j = 0, 1, \ldots$. Given a sequence $u \in S(R^m)$, we denote by $u_i \in R^m$ the i-th element of the sequence, $i = 0, 1, \ldots$, so that $u = (u_0, u_1, u_2, \ldots)$.

A system Σ that accepts sequences of m dimensional vectors as input and generates sequences of p dimensional vectors as output is simply a map $\Sigma : S(R^m) \rightarrow S(R^p)$. The image of a set $S \subset S(R^m)$ of input sequences through Σ is denoted by $\Sigma[S]$.

For our stability studies, we will need to consider bounded sequences of vectors. Let $S(\theta^m)$ denote the set of all sequences in $S(R^m)$ of vectors whose individual coordinates are bounded by the number $\theta > 0$. To be more precise, let $[-\theta, \theta]^m$ be the set of all vectors $(x_1, x_2, \ldots, x_m) \in R^m$ satisfying $-\theta \leq x_j \leq \theta$ for all $j = 1, \ldots, n$. Then, $S(\theta^m)$ consists of all elements $u \in S(R^m)$ with $u_j \in [-\theta, \theta]^m$ for all integers $j \geq 0$. In this notation, a system $\Sigma : S(R^m) \rightarrow S(R^p)$ is *BIBO* (Bounded-Input Bounded-Output) *-stable* if for every real number $\theta > 0$ there is a real number $M > 0$ such that $\Sigma[S(\theta^m] \subset S(M^p)$.

Two norms on the space $S(R^m)$ play an important role in our discussion - the usual ℓ^∞-norm and a weighted ℓ^∞-norm. First, for a vector $x = (x_1, \ldots, x_m) \in R^m$, let $|x| := \max\{|x_1|, \ldots, |x_n|\}$. Then, the standard ℓ^∞-norm on $S(R^m)$ is given, for every element $u \in S(R^m)$, by $|u| := \sup_{j \geq 0} |u_j|$. We also define the weighted ℓ^∞-norm ρ by

$$\rho(u) := \sup_{j \geq 0} 2^{-j} |u_j|.$$

Unless otherwise stated, all notions of continuity of systems are with respect to the norm ρ. From the mathematical standpoint, the basic advantage of the norm ρ is the fact that the domain $S(\theta^m)$ is compact under it, for any real number $\theta > 0$. When the norm ρ is combined with a separate boundedness requirement, it conforms with the intuitive interpretation of a norm in control theory (see HAMMER [1987], [1989a] for discussion).

A system $\Sigma : S(R^m) \rightarrow S(R^p)$ is *stable* if it is BIBO-stable, and if for every real number $\theta > 0$, the restriction $\Sigma : S(\theta^m) \rightarrow S(R^p)$ is a continuous map. This notion of stability is in the spirit of the input/output notions of stability stemming from Liapunov theory.

3. FRACTION REPRESENTATIONS : BACKGROUND

A fraction representation of a nonlinear system is characterized by three quantities - a numerator system, a denominator system, and a factorization space. We distinguish between two main types of fraction representations - right fraction representations and left fraction representations. In accurate terms, a right fraction representation of a nonlinear system $\Sigma : S(R^m) \rightarrow S(R^p)$ is characterized by a subset $S \subset S(R^q)$, where $q > 0$ is some integer, and by two stable systems $P : S \rightarrow S(R^p)$ and $Q : S \rightarrow S(R^m)$, with Q being invertible, so that $\Sigma = PQ^{-1}$. The subspace S is then called the *factorization space* of the fraction representation. In the expression PQ^{-1}, the systems P and Q^{-1} are combined by composition. In analogy, a left fraction representation of the system $\Sigma : S(R^m) \rightarrow S(R^p)$ is determined by a subset $S_L \subset S(R^r)$, where $r > 0$ is some integer, and by two stable systems $G : S(R^m) \rightarrow S_L$ and $T : S(R^p) \rightarrow S_L$, with T being invertible, so that $\Sigma = T^{-1}G$.

Apriori, there are no restrictions on the structure of the factorization space, but, as we shall discuss later, this structure determines to what extent a specific fraction representation aids in the solution of the stabilization problem.

We provide next some indications on the significance of fraction representations to the solution of the stabilization problem for nonlinear control systems. Consider the classical control configuration depicted in (1.1); A particularly simple situation results when the compensators φ and π are chosen in the form

$$\text{(3.1)} \qquad \begin{aligned} \varphi &= A, \\ \pi &= B^{-1}, \end{aligned}$$

where A and B are stable systems, with B being invertible. Of course, A and B^{-1} must be causal. Assuming that the system Σ that needs to be controlled has a right fraction representation $\Sigma = PQ^{-1}$, we obtain from (1.2)

$$\text{(3.2)} \qquad \begin{aligned} \Sigma_{(\pi,\varphi)} &= PQ^{-1}B^{-1}[I + APQ^{-1}B^{-1}]^{-1} \\ &= P[AP + BQ]^{-1}. \end{aligned}$$

Defining the system

$$\text{(3.3)} \qquad M := AP + BQ,$$

the input/output relation of the closed loop system becomes

$$\text{(3.4)} \qquad \Sigma_{(\pi,\varphi)} = PM^{-1}.$$

Thus, if the systems A and B can be chosen in such a way that the (stable) system M has a stable inverse M^{-1}, the closed loop system of (1.1) will be input/output stable. Furthermore, it can be shown that with some additional mild restrictions on the stable systems A and B, the closed loop system will actually be internally stable (HAMMER [1986]). A stable system M that possesses a stable inverse M^{-1} is called a *unimodular system*.

Another fundamental implication of (3.4) is that the dynamical behavior of the closed loop system $\Sigma_{(\pi,\varphi)}$ can be influenced by appropriately choosing the unimodular system M. This observation leads to the nonlinear analog of the linear theory of pole assignment, and was discussed in detail in HAMMER [1988].

We discuss now briefly some implications on the structure of the factorization space. From (3.4) it follows that the domain of input sequences for the closed loop system $\Sigma_{(\pi,\varphi)}$ is identical to the domain of input sequences of the system M^{-1}, namely, to the domain of *output* sequences of the system M. Normally, the domain of input sequences of a control system is required to contain all sequences of amplitude not exceeding a specified bound, so the input domain of $\Sigma_{(\pi,\varphi)}$ must contain the set $S(\theta^m)$ for some real number $\theta > 0$. This directly implies then that the image of M must contain the same, i.e.,

(3.5) $\quad S(\theta^m) \subset Im\ M.$

Now, let $S \subset S(R^q)$ be the factorization space of the fraction representation $\Sigma = PQ^{-1}$. Then, considering (3.3), the domain of M is S, and

(3.6) $\quad M : S \to S(R^m);\ M[S] \supset S(\theta^m).$

The design procedure that emerges from the theory we describe here proceeds roughly as follows.

(i) Choose a desired unimodular system M. As indicated earlier, the choice of M determines the dynamical behavior of the closed loop system $\Sigma_{(\pi,\varphi)}$.

(ii) Find an appropriate pair of stable systems A and B satisfying (3.3). The conditions under which such A and B exist and the means for computing them are discussed in section 4.

From (i) it follows that an important step in the design of stabilizing compensators for the system Σ is the construction of a unimodular system $M : S \to S(R^m)$ which contains a domain of the form $S(\theta^m)$ in its image. This implies that we must find a subspace $S' \subset S$ that is homeomorphic to $S(\theta^m)$ together with a homeomorphism $M : S' \equiv S(\theta^m)$ (which is a restriction of M, and is actually the only interesting part of M; by the way, this also shows that S must contain a subset homeomorphic to $S(\theta^m)$.). Now, the construction of such a homeomorphism is not an easy task in general. Nevertheless, if a subspace of the form $S(\alpha^m)$ is contained within the factorization space S, then one can take $S' = S(\alpha^m)$, and construct a desirable homeomorphism $M : S(\alpha^m) \to S(\theta^m)$. The construction of such homeomorphism is easy - take for instance $M = (\theta/\alpha)I$. Thus, we conclude that it is extremely important to choose the fraction representation $\Sigma = PQ^{-1}$ so that its factorization space contains $S(\alpha^m)$ for some real $\alpha > 0$. In fact, as discussed later, it is possible to construct a fraction representation whose factorization space is equal to $S(\alpha^m)$. Such fraction representation is then particularly instrumental for stabilizing the system Σ.

We consider next an application of left fraction representations. In general terms, left fraction representations enable us to find all the solutions A, B of the basic stabilization equation (3.3), when given one such solution; They yield a remarkably simple and transparent parametrization of all pairs A, B of stable systems satisfying (3.3). This parametrization can then be employed with optimization techniques to find the 'best' compensators π and φ, under a suitable optimization criterion.

Assume that the given system Σ has a left fraction representation $\Sigma = T^{-1}G$ with the factorization space S_L, and recall that S is the factorization space of the right fraction representation $\Sigma = PQ^{-1}$. Obviously, $PQ^{-1} = T^{-1}G$, or

(3.7) $\quad TP = GQ.$

Suppose now that one pair of stable systems A_0, B_0 satisfying the equation $A_0P + B_0Q = M$ has been found. Let $h : S_L \to S$ be any stable system, and consider the pair of stable systems A, B given by

$$(3.8)\qquad \begin{aligned} A &= A_0 + hT, \\ B &= B_0 - hG. \end{aligned}$$

Then,

$$(3.9)\qquad \begin{aligned} AP + BQ &= (A_0 + hT)P + (B_0 - hG)Q \\ &= (A_0P + B_0Q) + (hTP - hGQ) \\ &= A_0P + B_0Q = M, \end{aligned}$$

where the third equality is implied by (3.7). Thus, we see that for any stable system h, the systems A and B of (3.8) satisfy our basic stabilization equation. In this way, by varying h, we can generate infinitely many solutions of our equation; Furthermore, for appropriate (i.e., 'coprime') left fraction representations $\Sigma = T^{-1}G$, (3.8) generates *all* solutions of (3.3) (HAMMER [1987]). This leads to a simple and transparent parametrization of the set of all compensators (of the form (3.1)) that stabilize the given system Σ, with closed loop dynamics assigned by the unimodular system M. Whence, left fraction representations also play an important role in control theory.

4. FRACTION REPRESENTATIONS: BASIC RESULTS

Having provided some background on the utilization of fraction representations in nonlinear control, we turn now to a brief survey of certain basic aspects of their theory. A simplified exposition of the theory of fraction representation of nonlinear systems was presented in HAMMER [1987], and the remaining part of the present section is based on this reference. The theory incorporates the realistic premise that every system permits only bounded input values. Explicitly, we write $\Sigma : S(\alpha^m) \to S(R^p)$ to indicate that the input amplitudes must be bounded by $\alpha > 0$.

Vital to the theory of fraction representations of nonlinear systems is the notion of right coprimeness. Indeed, it is only when the systems P and Q are right coprime that a solution A, B to (3.3) exists. Intuitively, two stable systems P and Q with common input domain are right coprime if, for every unbounded input sequence u, at least one of the output sequences Pu, Qu is unbounded. When specialized to the case of linear systems, this reduces to the requirement that P and Q have no unstable zeros in common. The accurate definition in the nonlinear case is as follows (HAMMER [1985, 1987]). (For a system $P : S_1 \to S_2$ and a subset $S_3 \subset S_2$, denote by $P^*[S_3]$ the inverse image of the set S_3 through P, namely, the set of all input sequences $u \in S_1$ for which $Pu \in S_3$.)

(4.1) DEFINITION. Let $S \subset S(R^q)$ be a subset. Two stable systems $P : S \to S(R^p)$ and $Q : S \to S(R^m)$ are *right coprime* if the following conditions hold:

(i) For every real number $\tau > 0$ there exists a real number $\theta > 0$ such that

$$P^*[S(\tau^p)] \cap Q^*[S(\tau^m)] \subset S(\theta^q).$$

(ii) For every real number $\tau > 0$, the set $S \cap S(\tau^q)$ is a closed subset of $S(\tau^q)$ (with respect to the topology induced by the norm ρ).♦

A *right coprime fraction representation* $\Sigma = PQ^{-1}$ is a right fraction representation in which the systems P and Q are right coprime. The next statement, which is one of the most fundamental results in the theory of fraction representations, indicates that whenever the systems P and Q are right coprime, one can always find a solution A, B for the basic stabilization equation (3.3), for *any* M. The Theorem is restricted to the case where the given system Σ that needs to be controlled is injective (one-to-one), but, as discussed in HAMMER [1987], this does not constitute a substantial limitation from a control theoretic point of view.

(4.2) THEOREM. Let $\Sigma : S(\alpha^m) \to S(R^p)$ be an injective system, and let $\Sigma = PQ^{-1}$ be a right coprime factorization with factorization space $S \subset S(R^q)$. Then, for every stable system $M : S \to S$ there exists a pair of stable systems $A : Im\ \Sigma \to S(R^q)$ and $B : S(\alpha^m) \to S(R^q)$ satisfying $AP + BQ = M$.♦

Methods for finding appropriate systems A and B that satisfy the conditions of the Theorem are described in HAMMER [1986, 1987, 1988, and 1989a]. A preliminary topic in this context is, of course, the characterization of the class of nonlinear systems that possess right coprime fraction representations. For this purpose, we need the following

(4.3) DEFINITION. A system $\Sigma : S(\alpha^m) \to S(R^p)$ is a *homogeneous system* if, for every subset $S \subset S(\alpha^m)$ for which there exists a real number $\theta > 0$ such that $\Sigma[S] \subset S(\theta^p)$, the restriction of Σ to the closure $\overline{S}$ of S is a continuous map $\Sigma : \overline{S} \to S(\theta^p)$.♦

Roughly speaking, a homogeneous system is characterized by the property that it is continuous over any set of input sequences that yield bounded output sequences. Homogeneous systems are quite common; In fact, most systems of practical interest are homogeneous, as follows (HAMMER [1987]).

(4.4) PROPOSITION. Let $\Sigma : S(R^m) \to S(R^p)$ be a system described by the equations

$$z_{k+\eta+1} = f(z_k, ..., z_{k+\eta}, u_k, ..., u_{k+\mu}),$$
$$y_k = h(z_k),$$

$k = 0, 1, 2, ...,$ where $u \in S(R^m)$ is the input sequence, $y \in S(R^p)$ is the output sequence, and $z \in S(R^q)$ is an intermediate sequence. If the functions f and h are continuous, then Σ is a homogeneous system.♦

The significance of the class of homogeneous systems stems from the fact that it consists exactly of all systems possessing right coprime fraction representations, as follows (HAMMER [1985, 1987]).

(4.5) THEOREM. An injective system $\Sigma : S(\alpha^m) \to S(R^p)$ has a right coprime fraction representation if and only if it is a homogeneous system.♦

The computation of right coprime fraction representations is reviewed in the next section. Presently we consider the existence of left fraction representations (HAMMER [1987]).

(4.6) THEOREM. An injective homogeneous system $\Sigma : S(\alpha^m) \to S(R^p)$ has a left fraction representation.♦

The significance of Theorem (4.6) goes beyond its statement; as shown in the reference, the left fraction representations that arise in this framework are 'coprime' in the sense that when used in the scheme (3.8), they yield all possible solutions A, B of the equation $AP + BQ = M$.

To conclude, the class of homogeneous systems is a broad class containing most systems of practical interest; it consists of all systems possessing a right coprime fraction representation; and all its members possess left fraction representations as well. It thus forms a natural environment for the development of the theory of nonlinear control.

5. FRACTION REPRESENTATIONS: COMPUTATION

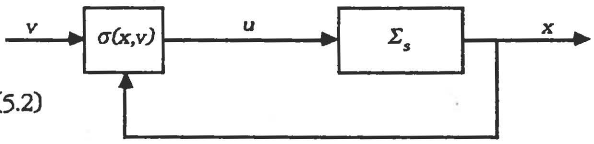
As we have seen, the derivation of a right coprime fraction representation is a critical step in the stabilization process. Generally speaking, a right coprime fraction representation is relatively simple to derive (HAMMER [1987]). However, if due care is not taken in the construction to obtain a 'nice' right coprime fraction representation, then the computation of the stabilizing compensators might turn out to be quite difficult. On account of the cogitation following (3.6), a 'nice' right coprime fraction representation is one that has a simple factorization space S; ideally, one for which the factorization space is of the form $S = S(\alpha^m)$ for some $\alpha > 0$. The construction of such fraction representations is described in HAMMER [1989b and c], and is based on the theory of reversible nonlinear state feedback. The present section is concerned with a review of this topic.

We restrict our attention to nonlinear systems Σ that can be described by equations of the form

$$x_{k+1} = f(x_k, u_k),$$
(5.1)
$$y_k = h(x_k),$$

where $u \in S(R^m)$ is the input sequence; $y \in S(R^p)$ is the output sequence; $x \in S(R^q)$ is an intermediate sequence of states; and the initial condition x_0 is specified. The functions $f : R^q \times R^m \to R^q$ and $h : R^q \to R^p$ are continuous. A system Σ that can be described in this form is said to have a *continuous realization*. The system represented by $x_{k+1} = f(x_k, u_k)$ is called the *input/state part* of Σ, and is denoted by Σ_s. In view of Proposition (4.4), the systems Σ and Σ_s are both homogeneous, and whence possess right coprime fraction representations.

We compute a 'nice' right coprime fraction of the system Σ by deriving first a right coprime fraction representation of the input/state part Σ_s. A critical tool for the latter is the theory of reversible state feedback developed in HAMMER [1989b], from where we review some basic terminology next. Consider the following static state feedback loop around the input/state system Σ_s



(5.2)

Here, $\sigma : R^q \times R^m \to R^m$ is a continuous function representing the static state feedback; the closed loop system is denoted by $\Sigma_{s\sigma}$, and is given by

(5.3) $$x_{k+1} = f(x_k, \sigma(x_k, v_k)).$$

The feedback induced by the function σ is said to be *reversible* if the system Σ_s can be recovered from the closed loop system $\Sigma_{s\sigma}$ through another feedback operation. It can be shown that σ induces a reversible feedback operation if and only if the function $\sigma(x,v)$ is injective (one-to-one) in v for any possible state x (HAMMER [1989b]).

A detailed theory of reversible feedback has been developed in HAMMER [1989b]. The theory includes necessary and sufficient conditions for the existence of a reversible feedback function σ that stabilizes a given input/state system Σ_s; a complete characterization of the set of all reversible feedback functions that stabilize the system; as well as explicit and implementable constructions of the stabilizing feedback functions σ. Basically, a system can be stabilized by reversible state feedback whenever one would intuitively expect the system to be at all stabilizable. These results are then applied in HAMMER [1989c] to the construction of 'nice' right coprime fraction representations for systems possessing continuous realizations. Here, we only provide the following brief summary of the situation. (The phrase 'σ stabilizes Σ_s over the input domain $S(\theta^m)$' means that the closed loop system $\Sigma_{s\sigma}$ is internally stable for all inputs of amplitude not exceeding $\theta > 0$. Also, a system is *bicausal* if it is causal and possesses a causal inverse.)

(5.4) THEOREM. Let $\Sigma : S(R^m) \to S(R^p)$ be a system having a continuous realization, and let $\Sigma_s : S(R^m) \to S(R^q)$ be the input/state part of Σ. Assume there is a reversible feedback function $\sigma : R^q \times R^m \to R^m$ that stabilizes the input/state part Σ_s over the input domain $S(\theta^m)$, for some $\theta > 0$. Then, the system Σ has a right coprime fraction representation $\Sigma = PQ^{-1}$ with the factorization space $S(\theta^m)$. Furthermore, the (stable) systems $P : S(\theta^m) \to S(R^p)$ and $Q : S(\theta^m) \to S(R^m)$ both have continuous realizations, and the restriction $Q : S(\theta^m) \to Q[S(\theta^m)]$ is bicausal.♦

Briefly, this means that every stabilizable nonlinear system has a 'nice' right coprime fraction representation. This fraction representation can then be used to derive compensators that internally and robustly stabilize the system (without access to the state), while assigning to the closed loop a desirable dynamical behavior, as discussed in an earlier section. Detailed implementable constructions of all relevant quantities are described in the references quoted before.

6. PRESERVATION OF STABILITY

The computation of stabilizing compensators for a system Σ depends, of course, on a specific given mathematical model, which provides (only) an approximate description of the system. One needs to address then the question of whether the closed loop will remain stable when the true system is inserted into it, rather than the model for which it was designed. We refer to the model as the 'nominal' description of the system.

Let Σ_n be a nominal model of the system to be controlled, and let Σ denote the actual system. Configuration (1.1) is used with compensators π and φ of the form (3.1). The compensators are computed using the nominal description Σ_n of the system, while the system actually inserted into the loop is Σ. We would like to characterize the class of all systems Σ for which the closed loop remains stable.

At the outset, it is obviously necessary to assume that the nominal system Σ_n as well as the real system Σ are stabilizable. To be more specific, we shall assume that both systems admit continuous realizations of the form (5.1), with input/state parts being stabilizable by reversible state feedback. This implies the existence of right coprime fraction representations $\Sigma_n = P_n Q_n^{-1}$ and $\Sigma = PQ^{-1}$ in accordance with Theorem (5.4). We then compute the fraction representation $\Sigma_n = P_n Q_n^{-1}$ from the given nominal model Σ_n; While not assuming that the systems P and Q of the fraction representation of Σ are known, we shall characterize the set of all P and Q for which the closed loop remains stable. This will yield a characterization of the set of all systems Σ stabilized by the closed loop, with fixed compensators π, φ computed for Σ_n. For brevity, only input/output stability is considered here; see HAMMER [1990] for internal stability.

Having derived the fraction representations $\Sigma_n = P_n Q_n^{-1}$ and $\Sigma = PQ^{-1}$ in accordance with Theorem (5.4), the denominator systems Q_n and Q are both bicausal, and the factorization space for both fraction representations can be taken in the form $S(\beta^m)$ for some real number $\beta > 0$ (see HAMMER [1990] for details).

Assume then that compensators π and φ of the form (3.1) have been designed to stabilize the nominal system Σ_n by choosing A and B to satisfy

(6.1) $\quad M_n := AP_n + BQ_n,$

with M_n unimodular (section 3). Furthermore, by (3.5), there is a real number $\theta > 0$ such that

(6.2) $\quad S(\theta^m) \subset M_n[S(\beta^m)].$

Now, when the system Σ is inserted into the closed loop instead of the nominal system Σ_n for which it was designed, equation (3.3) takes the form

(6.3)
$$\begin{aligned} &AP + BQ = \\ &= AP_n + BQ_n + [(AP - AP_n) + (BQ - BQ_n)] \\ &= M_n + \Delta, \end{aligned}$$

where

(6.4) $\quad \Delta := (AP - AP_n) + (BQ - BQ_n).$

As can be seen, the quantity Δ describes the deviation from nominality within the critical stabilization equation, as caused by the deviation of Σ from Σ_n. Preservation of stability can then be characterized as follows (HAMMER [1990]). (Recall that $S(\beta^m)$ is the factorization space of our fraction representations.)

(6.5) THEOREM. The closed loop system $\Sigma_{(\pi,\varphi)}$ is input/output stable if and only if the deviation Δ satisfies the following condition: there is a real number $\alpha > 0$ such that $S(\alpha^m) \subset (M_n + \Delta)[S(\beta^m)]$. When the latter holds, the closed loop system $\Sigma_{(\pi,\varphi)}$ is input/output stable over the domain of input sequences $S(\alpha^m)$.♦

Interpreting the Theorem in intuitive terms, we can view the effect of the disturbance Δ as a 'shift' of the image of the unimodular system M_n. The only requirement for the preservation of the stability of the closed loop system is that a subset of the form $S(\alpha^m)$, $\alpha > 0$, remain contained within the image of $(M_n + \Delta)$; all other conditions would then be automatically satisfied. (Recall that M_n contains the subset $S(\theta^m)$ in its image by (6.2).) This necessary and sufficient condition for the preservation of stability under system variations is very simple in nature, and is purely algebraic. It provides yet another manifestation of the power of the fraction representation approach to nonlinear control. Somewhat philosophically, we may say that the fraction representation approach has the advantage of automatically incorporating the topological considerations of the theory of stabilization of nonlinear systems, leaving us to verify only relatively simple algebraic conditions.

Finally, we comment that Theorem (6.5) leads to some rather simple sufficient conditions for the verification of stability under system perturbations (HAMMER [1990]). Preservation of internal stability under system perturbations is discussed in HAMMER [1990].

REFERENCES

G. CHEN and R.J.P. de FIGUEIREDO
[1990] "Construction of the left coprime fractional representation for a class of nonlinear control systems", Systems and Control Letter (to appear).

C.A. DESOER and M.G. KABULI
[1988] "Right factorization of a class of nonlinear systems", Trans. IEEE, Vol. AC-33, pp. 755-756.

J. HAMMER
[1984a] "Nonlinear systems: stability and rationality", Int. J. Control, Vol. 40, pp. 1-35.
[1984b] "On nonlinear systems, additive feedback, and rationality", Int. J. Control, Vol. 40, pp. 953-969.
[1985] "Nonlinear systems, stabilization, and coprimeness", Int. J. Control, Vol. 42, pp. 1- 20.
[1986] "Stabilization of nonlinear systems", Int. J. Control, Vol. 44, pp. 1349-1381.
[1987] "Fraction representations of nonlinear systems: a simplified approach", Int. J. Control, Vol. 46, pp. 455-472.
[1988] "Assignment of dynamics for nonlinear recursive feedback systems", Int. J. Control, Vol. 48, pp. 1183-1212.
[1989a] "Robust stabilization of nonlinear systems", Int. J. Control, Vol 49, pp. 629-653.
[1989b] "State feedback for nonlinear control systems", Int. J. Control, Vol. 50, pp. 1961-1980.
[1989c] "Fraction representations of nonlinear systems and non-additive state feedback", Int. J. Control, Vol. 50, pp. 1981-1990.
[1990] "Preservation of stability under system perturbations in nonlinear control: a simple characterization", Systems and Control Letters (to appear).

A.D.B. PAICE and J.B. MOORE
[1990] "Robust stabilization of nonlinear plants via left coprime factorization", Systems and Control Letters (to appear).

E.D. SONTAG
[1989] "Smooth stabilization implies coprime factorization", Trans. IEEE, Vol. AC-34, pp. 435-443.
[1990] "Further facts about input to state stabilization", Trans. IEEE, Vol. 35, pp. 473-476.

M.S. VERMA
[1988] "Coprime fractional representations and stability of nonlinear feedback systems", Int. J. Control, Vol. 48, pp. 897-918.

This research was supported in part by the National Science Foundation, USA, Grant number 8913424.