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LINEAR SYSTEM FACTORIZATION

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1. Introduction

In HAUTUS and HEYMANN [1978], an investigation was initiated of the algebraic structure of discrete time, time invariant, finite dimensional linear systems (or, simply, linear systems) with particular emphasis on static state feedback. This investigation was extended to the study of dynamic as well as static output feedback in HAMMER and HEYMANN [1981]. Pivotal in the extended theory was the problem of causal factorization, i.e., the problem of factoring two system maps over each other through a causal factor. The theory was further extended in HAMMER and HEYMANN [1980] where the structural invariants of precompensation orbits and the concept of strict observability were studied in detail. Algebraically, the theory of strict observability hinges on the problem of polynomial factorization, i.e., the problem of factoring two system maps over each other through a polynomial factor.

It has since become increasingly clear, that the theory of linear systems can be formulated in a very general algebraic setup in which the central concepts of causality (and hence of feedback), of stability and of realization are investigated in a unified framework. In the present paper we present some of the essentials of this theory with particular emphasis on the issue of system stability. Proofs of theorems are omitted because of space limitations and will appear in a future expanded paper HAMMER and HEYMANN [1982].

2. The Mathematical Setup

We assume that the reader has basic familiarity with the setup and terminology of HAUTUS and HEYMANN [1978], HAMMER and HEYMANN [1981] as well as HAMMER and HEYMANN [1980]. We review the principal aspects of this setup very briefly.

For a field K and a K-linear space S, we denote by ΛS the set of all formal Laurent series in z^{-1} with coefficients in S, i.e., series of the form

$$s = \sum_{t=t_0}^{\infty} s_t z^{-t} \quad ; \quad s_t \in S . \tag{2.1}$$

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In ΛS, the set of polynomial elements of the form $\sum_{t\leq 0} s_t z^{-t}$, is denoted by $\Omega^+ S$, and the set of causal elements, that is, the set of power series of the form $\sum_{t\geq 0} s_t z^{-t}$, is denoted by $\Omega^- S$.

The set ΛK is a field under coefficientwise addition and convolutional multiplication and, under similar operations, the set ΛS becomes a ΛK-linear space. The polynomial subset $\Omega^+ K$ of ΛK and the set of causal elements $\Omega^- K$ are subrings (principal ideal domains) of ΛK. The field ΛK is then an $\Omega^+ K$-module and an $\Omega^- K$-module as well.

The $\Omega^- K$-order of an element $s = \sum s_t z^{-t} \in \Lambda S$ is defined by

$$(2.2) \qquad \mathrm{ord}_{\Omega^- K}\, s := \begin{cases} \min t\in Z \mid s_t \neq 0 & \text{if } s\neq 0 \\ \infty & \text{if } s=0 \end{cases}$$

where Z denotes the integers.

Let the K-linear spaces U and Y be given. A ΛK-linear map $\bar{f}:\Lambda U\to\Lambda Y$ represents a linear time invariant system, having U as the input value space and Y as the output value space. It is assumed throughout the paper that all underlying K-linear (value) spaces, and, in particular, U and Y are finite dimensional. The $\Omega^- K$-order (or, simply, order) of a ΛK-linear map $\bar{f}:\Lambda U\to\Lambda Y$ is defined as

$$(2.3) \qquad \mathrm{ord}\,\bar{f} := \inf\{\mathrm{ord}\,\bar{f}(u) - \mathrm{ord}\,u \mid 0\neq u\in\Lambda U\}.$$

The map $\bar{f}$ is said to be of finite order if $\mathrm{ord}\,\bar{f} > -\infty$.

If $\bar{f}$ is a ΛK-linear map of finite order t_o, we associate with it its transfer function, i.e., an element

$$T = \sum_{t=t_o}^{\infty} T_t z^{-t} \in \Lambda L ,$$

where L is the K-linear space of K-linear maps $U\to Y$ as follows. We define the K-linea maps p_t and i_u by

$$(2.4) \qquad \begin{cases} i_u: U\to\Lambda U: u \mapsto u \text{ (canonical injection)} \\ p_k: \Lambda Y\to Y: \quad y_t z^{-t} \mapsto y_k \end{cases} .$$

and then for all integers $t\geq t_o$ we let $T_t := T_t(\bar{f}) := p_t\cdot\bar{f}\cdot i_u$. Conversely, with each element $T = \sum T_t z^{-t}\in\Lambda L$ we associate a ΛK-linear map $\bar{f}=\bar{f}_T$ of finite order whose action on elements $u = \sum u_t z^{-t}\in\Lambda U$ is defined through the convolution formula

$$\bar{f}_T\cdot u := \sum_t (\sum_k T_k u_{t-k}) z^{-t} .$$

For a map $\bar{f}: \Lambda U\to\Lambda Y$ and a subset $A\subset\Lambda U$, we denote by $\bar{f}[A]$ the image of A under $\bar{f}$, i.e., $\bar{f}[A] = \{\bar{f}(u) \mid u\in A\}$. A ΛK-linear map $\bar{f}: \Lambda U\to\Lambda Y$ is called causal if $\mathrm{ord}\,\bar{f}\geq 0$ or, equivalently, if $\bar{f}[\Omega^- U]\subset\Omega^- Y$. Similarly, $\bar{f}$ is called strictly causal if $\mathrm{ord}\,\bar{f}\geq 1$ or, equivalently, if $\bar{f}[\Omega^- U]\subset z^{-1}\Omega^- Y$. We have the following

(2.5) DEFINITION. A ΛK-linear map $\bar{f}: \Lambda U \to \Lambda Y$ is called a linear input/output (or i/o) map if it is strictly causal and of finite order.

Associated with a linear i/o map $\bar{f}: \Lambda U \to \Lambda Y$ are two further maps as follows. First, we restrict the inputs to the Ω^+K-module Ω^+U, and consider the projection of the corresponding outputs on the quotient Ω^+K-module $\Gamma^+Y := \Lambda Y/\Omega^+Y$. Then we obtain the restricted linear i/o map $\tilde{f}: \Omega^+U \to \Gamma^+Y$ associated with $\bar{f}$ through

$$\tilde{f} = \pi^+ \cdot \bar{f} \cdot j^+ \quad ,$$

where $j^+: \Omega^+U \to \Lambda U$ is the canonical injection and $\pi^+: \Lambda Y \to \Gamma^+Y$ is the canonical projection. It is readily seen that $\tilde{f}$ is an Ω^+K-homomorphism. Next, we associate with $\bar{f}$ the output response map $f: \Omega^+U \to Y$ given by $f := p_1 \cdot \bar{f} \cdot j^+$ or, more explicitly,

$$f: \Omega^+U \to Y: u \mapsto f(u) := p_1 \bar{f}(u) \quad .$$

Since the map p_1 is K-linear, so is also the output response map f. The case in which f is an Ω^+K-homomorphism as well, is of particular importance and we have

(2.6) DEFINITION. A linear i/o map $\bar{f}: \Lambda U \to \Lambda Y$ is called an input/state (or i/s) map if there exists an Ω^+K-module structure on Y, compatible with its K-linear structure, such that the output response map $f = p_1 \cdot \bar{f} \cdot j^+$ is an Ω^+K-homomorphism.

3. Rationality and Stability: General Considerations

An element $s \in \Lambda S$ is called Ω^+K-rational (or sometimes simply rational) if there exists a nonzero polynomial $\psi \in \Omega^+K$ such that $\psi s \in \Omega^+S$.† The set of Ω^+K-rationals in ΛS is denoted $Q_{\Omega^+K}S$. For an element $s \in Q_{\Omega^+K}S$, the set of polynomials $\psi \in \Omega^+K$ for which $\psi s \in \Omega^+S$ is easily seen to be an ideal in Ω^+K. Since Ω^+K is a principal ideal domain, this ideal is generated by a monic polynomial ψ_s, which we call the least denominator of s. The zeros of ψ_s are called the poles of s. (In case $K=\mathbb{R}$, the field of real numbers, it is customary to consider not only poles in $\mathbb{R}$ but also in $\mathbb{C}$, the field of complex numbers). The definition of Ω^+K-rationality applies, in particular, also to transfer functions of ΛK-linear maps and we call a ΛK-linear map $\bar{f}: \Lambda U \to \Lambda Y$ Ω^+K-rational (or, simply, rational) if so is its transfer function.

We turn now to the concept of stability. If $\mathcal{D}$ is a set of polynomials, we say that an Ω^+K-rational map is $\mathcal{D}$-stable if its least denominator is in $\mathcal{D}$. We impose a number of restrictions on the set $\mathcal{D}$ of stable denominators (see MORSE [1976]) as follows :

(3.1) DEFINITION. A set $\mathcal{D}$ of (monic) polynomials over K is called a denominator set if it satisfies the following conditions :

(i) $\mathcal{D}$ is multiplicatively closed, i.e., $p \in \mathcal{D}$, $q \in \mathcal{D}$ imply $p \cdot q \in \mathcal{D}$.

(ii) The unit polynomial 1 belongs to $\mathcal{D}$ but the zero polynomial does not belong to $\mathcal{D}$.

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† Throughout the paper S denotes a finite dimensional K-linear space.

(iii) $\mathcal{D}$ contains at least one polynomial of degree one, i.e., there exists $\alpha \in K$ such that $z-\alpha \in \mathcal{D}$.

(iv) $\mathcal{D}$ is saturated, i.e., if $p \in \mathcal{D}$ and q is a monic divisor of p, then $q \in \mathcal{D}$.

Conditions (i) and (ii) say that $\mathcal{D}$ is a multiplicative set so that one can define the set $\Omega_{\mathcal{D}}K$ as the set of fractions p/q, where $p \in \Omega^+K$ and $q \in \mathcal{D}$. Conditions (iii) and (iv) are motivated by considerations that are discussed shortly. We now introduce the following

(3.2) DEFINITION. Let $\mathcal{D}$ be a denominator set. Then an element $s \in Q_{\Omega^+K}S$ is called stable (or, explicitly, $\mathcal{D}$-stable) if there exists $\psi \in \mathcal{D}$ such that $\psi s \in \Omega^+S$, or, equivalently, if the least denominator $\psi_s \in \mathcal{D}$. The set of stable elements in $Q_{\Omega^+K}S$ is denoted by $\Omega_{\mathcal{D}}S$. The set of stable and causal elements is denoted by $\Omega^-_{\mathcal{D}}S$, i.e.,

$$\Omega^-_{\mathcal{D}}S = \Omega_{\mathcal{D}}S \cap \Omega^-S \quad . \tag{3.3}$$

The above definition of stability is easily seen to be a generalization to arbitrary fields of the usual concept of stability in system theory defined in an algebraic framework.

Definition 3.2 applies, in particular, to the case $S = L$, the space of all linear maps $U \to Y$ and we have a definition of stable transfer functions and stable ΛK-linear maps. In particular, we have the following

(3.4) PROPOSITION. The map $\bar{f} \in \Omega_{\mathcal{D}}L$ if and only if $\bar{f}[\Omega_{\mathcal{D}}U] \subset \Omega_{\mathcal{D}}Y$.

The set $\Omega_{\mathcal{D}}K$ is easily seen by direct computation to be a subring (with identity) of the rational field Q_{Ω^+K} $(=Q_{\Omega^+K}K)$, and is actually a principal ideal domain. In fact, we have even more :

(3.5) PROPOSITION. The ring $\Omega_{\mathcal{D}}K$ is a Euclidean domain.

Since we are interested in causal systems, we shall be interested in the ring $\Omega^-_{\mathcal{D}}K$ which, as was proved in MORSE [1976] is also a principal ideal domain and, in fact, just as $\Omega_{\mathcal{D}}K$, is also a Euclidean domain. We generalize now our framework of consideration so as to include the preceding examples as special cases. In particular, since we encountered as substructures of ΛK the rings Ω^+K, Ω^-K, $\Omega_{\mathcal{D}}K$ and $\Omega^-_{\mathcal{D}}K$ all of which are Euclidean domains or, more generally, principal ideal domains, we consider now a more general framework as follows :

Let ΩK be a principal ideal domain (P.I.D.) properly contained as a subring in ΛK. The ΛK-linear space ΛS is then also an ΩK-module. Define ΩS to be the ΩK-submodule of ΛS generated by S, i.e., if $s_1,\ldots,s_n$ is a basis for S then

$$\Omega S := \{s \in \Lambda S \mid s = \sum_{i=1}^{n} \alpha_i s_i \ , \quad \alpha_i \in \Omega K, \quad i = 1,\ldots,n\} \ . \tag{3.6}$$

We now extend some basic concepts and terminology to the P.I.D. ΩK. An element $s \in \Lambda S$ is called ΩK-rational if there exists a nonzero element $\psi \in \Omega K$ such that $\psi s \in \Omega S$. The set of ΩK-<u>rationals</u> in ΛS is denoted $Q_{\Omega K}S$. Just as in the case Ω^+K, the definition of ΩK-rationality also applies to transfer functions of ΛK-linear maps and we call a ΛK-linear map ΩK-<u>rational</u> if so is its transfer function. It is readily seen that $\bar{f}: \Lambda U \to \Lambda Y$ is an ΩK-rational map if and only if $\bar{f}[Q_{\Omega K}U] \subset Q_{\Omega K}Y$. (The sufficiency of this condition depends on the finite dimensionality of U). An element $s \in \Lambda S$ is called an ΩK-<u>element</u> if $s \in \Omega S$. Thus, a ΛK-liner map $\bar{f}: \Lambda U \to \Lambda Y$ is an ΩK-<u>map</u> in case its transfer function is an ΩK-element of ΛL. $\bar{f}$ is called ΩK-<u>unimodular</u> if it is an invertible ΩK-map and its inverse is also an ΩK-map.

We shall make use of the following notation :

$$(3.7) \qquad \begin{cases} j_{\Omega K}: \Omega S \to \Lambda S: s \mapsto s & \text{(natural injection)} \\ \\ \pi_{\Omega K}: \Lambda S \to \Lambda S/_{\Omega S} =: \Gamma_{\Omega K}S & \text{(canonical injection)} \end{cases}$$

We can write the following

(3.8) THEOREM. <u>Let</u> $\bar{f}: \Lambda U \to \Lambda Y$ <u>be a</u> ΛK-<u>linear map. Then</u> $\bar{f}$ <u>is an</u> ΩK-<u>map if and only if</u> $\bar{f}[\Omega U] \subset \Omega Y$ <u>(or, equivalently, if and only if</u> $\Omega U \subset \ker \pi_{\Omega K}\bar{f}$).

The following corollary to Theorem 3.8 is very useful

(3.9) COROLLARY. <u>A</u> ΛK-<u>linear map</u> $\bar{\ell}: \Lambda U \to \Lambda U$ <u>is</u> ΩK-<u>unimodular if and only if</u> $\bar{\ell}[\Omega U] = \Omega U$ (<u>equivalently,</u> $\ker \pi_{\Omega K}\bar{\ell} = \Omega U$).

4. <u>The Order and Adapted Bases</u>

Our main objective in this section is to obtain finitary characterizations of ΩK-submodules of ΛK-linear spaces and of related properties of ΛK-linear maps. As before, we let ΩK be a principal ideal domain properly contained as a subring in ΛK and let $Q_{\Omega K}$ ($=Q_{\Omega K}K$) denote the field of quotients generated by ΩK.

For an element $s \in \Lambda S$ we define the <u>order</u> of s, denoted $\mathrm{ord}_{\Omega K}s$ (or, simply, ord s when the underlying ring is clear) as the set of all elements $\alpha \in Q_{\Omega K}$ for which $\alpha s \in \Omega S$. When $s=0$ we obviously have that ord s $= Q_{\Omega K}$, i.e., the whole quotient field generated by ΩK. In general, it is an easy exercise to verify that ord s is an ΩK-module (submodule of $Q_{\Omega K}$). In fact, we have the following :

(4.1) THEOREM. <u>If</u> $s \in \Lambda S$ <u>is nonzero, then</u> ord s <u>is a cyclic</u> ΩK-<u>module</u>.

Let $0 \neq s \in \Lambda S$ be any element and let $\alpha \in Q_{\Omega K}$ be any generator of ord s (possibly zero). If $\alpha' \in Q_{\Omega K}$ is another generator of ord s, then it is clearly an associate of α with respect to ΩK, i.e. $\alpha' = \mu\alpha$ where $\mu \in \Omega K$ is a unit (i.e., an invertible). It follows that α is uniquely defined modulo units in ΩK, and it will sometimes be convenient to identify ord s with one of its generators.

Before we proceed with our discussion, let us consider some examples of special interest.

First, let ΩK be the ring Ω^-K of causal elements. It is easily seen that $Q_{\Omega^-K} = \Lambda K$ since for every $\alpha \in \Lambda K$, either α or α^{-1} is in ΩK (or both). Further, for every element

$0 \neq \alpha \in \Lambda K$ there is a unique integer k such that $\alpha = \mu z^{-k}$ for some unit $\mu \in \Omega^- K$. Thus, for each $0 \neq s \in \Lambda S$, there exists a unique integer k such that $\mathrm{ord}_{\Omega^- K} s = (z^{-k})_{\Omega^- K}$ and we may identify $\mathrm{ord}_{\Omega^- K} s$ with the integer k associated with it. This definition of order of an element as an integer is precisely the (standard) definition of order as given in (2.2) above. (See also HAUTUS and HEYMANN [1978] and HAMMER and HEYMANN [1980],[1981])

As the second example let ΩK be the ring $\Omega^+ K$ of polynomials. In this case $Q_{\Omega^+ K}$ is the usual field of rationals. For an element $s \in \Lambda S$, $\mathrm{ord}_{\Omega^+ K} s \neq 0$ if and only if $s \in Q_{\Omega^+ K} S$, i.e., if and only if s is rational (in the classical sense). Let $0 \neq s \in Q_{\Omega^+ K} S$ be given as $s = (s_1, \ldots, s_m)$ with $s_i = \frac{p_i}{q_i}$, $p_i, q_i \in \Omega^+ K$ being coprime for all $i=1,\ldots,m$. Then $\mathrm{ord}_{\Omega^+ K} s$ is generated by the rational element q/p where q and p are the monic polynomials $q = \mathrm{l.c.m.}(q_1, \ldots, q_m)$ and $p = \mathrm{g.c.d.}(p_1, \ldots, p_m)$ (l.c.m. and g.c.d. denoting, respectively, the <u>least common multiple</u> and the <u>greatest common divisor</u>). To see this, write $p_i = p\bar{p}_i$ and $q = q_i \bar{q}_i$ for polynomials $\bar{p}_i$, $\bar{q}_i$, $i = 1, \ldots, m$. Then $\frac{q}{p} \cdot s = (\frac{q}{p} s_1, \ldots, \frac{q}{p} s_m) = (\bar{q}_1 \bar{p}_1, \ldots, \bar{q}_m \bar{p}_m) \in \Omega^+ S$ so that $(\frac{q}{p})_{\Omega^+ K} \subset \mathrm{ord}_{\Omega^+ K} s$. Conversely, let $\frac{r}{t}$ be any element in $\mathrm{ord}_{\Omega^+ K} s$ where r and t are coprime polynomials. Then for each $i=1,\ldots,m$, $\frac{r}{t} \frac{p_i}{q_i} \in \Omega^+ K$. Thus, q_i is a divisor of r for each i, and since q is the l.c.m. of the q_i's it follows that q is a divisor of r as well, that is, $r = q\bar{r}$ for some $\bar{r} \in \Omega^+ K$. Similarly, t is a divisor of each of the p_i's and hence also of p, so that $p = t\bar{p}$ for some $\bar{p} \in \Omega^+ K$. Thus, $\frac{r}{t} = \frac{q\bar{r}}{t} = \frac{q\bar{r}\bar{p}}{t\bar{p}} = \frac{q}{p}(\bar{r}\bar{p})$ and it follows that $\frac{r}{t} \in (\frac{q}{p})_{\Omega^+ K}$, and combining with our previous observations, we have that $\mathrm{ord}_{\Omega^+ K} s = (\frac{q}{p})_{\Omega^+ K}$.

Finally, we consider the case when ΩK is the ring $\Omega^-_{\mathcal{D}} K$ of causal and stable elements. The quotient field $Q_{\Omega^-_{\mathcal{D}} K}$ again coincides with the usual field of rationals $Q_{\Omega^+ K}$ and an element $s \in \Lambda S$ has nonzero $\Omega^-_{\mathcal{D}} K$-order if and only if $s \in Q_{\Omega^+ K} S$. Let $s = (s_1, \ldots, s_m) \in Q_{\Omega^+ K} S$ be a nonzero element and write each entry s_i, $i = 1, \ldots, m$ as $s_i = \frac{p_i r_i}{q_i}$ where $r_i, q_i \in \mathcal{D}$ are coprime (with respect to $\Omega^+ K$) and where $(0 \neq) p_i \in \Omega^+ K$ is coprime with every element of $\mathcal{D}$. Then it can be verified by direct computation that $\mathrm{ord}_{\Omega^-_{\mathcal{D}} K} s$ is generated by an element $\frac{q}{rp} \in Q_{\Omega^+ K}$ as follows: $p = \mathrm{g.c.d.}(p_1, \ldots, p_m)$ and q and r are any coprime elements of $\mathcal{D}$ such that $\mathrm{ord}_{\Omega^- K}(\frac{q}{pr}) = -\,\mathrm{ord}_{\Omega^- K} s$.

We proceed now with the discussion of some general properties of the order.

(4.2) THEOREM. <u>Let</u> $s \in \Lambda S$ <u>be any element. Then</u> ord $s \neq 0$ <u>if and only if</u> $s \in Q_{\Omega K} S$.

Next, we have the following simple characterization of elements in ΩS.

(4.3) PROPOSITION. <u>Let</u> $s \in \Lambda S$ <u>be any element. Then</u> $s \in \Omega S$ <u>if and only if</u> $\Omega K \subset$ ord s.

Let $s_1, \ldots, s_m \in Q_{\Omega K} S$ be a set of elements with orders ord $s_i = (\gamma_i)_{\Omega K}$, $i=1,\ldots,m$. Then the intersection ord $s_1 \cap \ldots \cap$ ord s_m is also a cyclic ΩK-module, and hence there is a generator $\gamma \in Q_{\Omega K}$ such that ord $s_1 \cap \ldots \cap$ ord $s_m = (\gamma)_{\Omega K}$. It is easily seen that γ is a least common multiple over ΩK of $\gamma_1, \ldots, \gamma_m$, (i.e., γ divides every element $\gamma' \in Q_{\Omega K}$ satisfying the condition that there exists for each i an element $\bar{\gamma}_i \in \Omega K$ such that $\gamma' = \gamma_i \bar{\gamma}_i$). If $s \in Q_{\Omega K} S$ and $\alpha \in Q_{\Omega K}$ are any elements, then ord $\alpha s = \alpha^{-1}$ ord s so that if ord $s = (\gamma)_{\Omega K}$, then ord $\alpha s = (\alpha^{-1}\gamma)_{\Omega K}$. In particular if $\alpha \in \Omega K$, then ord $s \subset$ ord αs. Furthermore, if $s_1, \ldots, s_m \in Q_{\Omega K} S$ is any set of elements, then

$$\text{(4.4)} \qquad \text{ord } s_1 \cap \ldots \cap \text{ord } s_m \subset \text{ord } (s_1 + \ldots + s_m) \ .$$

Finally, we shall say that a set of elements $s_1, \ldots, s_m \in \Lambda S$ is ΩK-<u>ordered</u> (or simply <u>ordered</u>) if ord $s_1 \subset \ldots \subset$ ord s_m.

We turn now to characterization of when a ΛK-linear map $\bar{f}: \Lambda U \to \Lambda Y$ is an ΩK-map. Recall that $\bar{f}$ is an ΩK-map if $\bar{f}[\Omega U] \subset \Omega Y$ and let $0 \neq u \in Q_{\Omega K} U$ be any element. Then ord $u = (\gamma)_{\Omega K}$ for some $\gamma \in Q_{\Omega K}$ and $\gamma u \in \Omega U$. If $\bar{f}$ is an ΩK-map, then $\bar{f}(\gamma u) = \gamma \bar{f}(u) \in \Omega Y$ so that $\Omega K \subset$ ord $\bar{f}(\gamma u)$ (see Proposition 4.3), or, equivalently, $\Omega K \subset$ ord $\gamma \bar{f}(u) = \gamma^{-1}$ ord $\bar{f}(u)$. Thus we conclude that $(\gamma)_{\Omega K} \subset$ ord $\bar{f}(u)$, and a necessary condition for $\bar{f}$ to be an ΩK-map is that ord $u \subset$ ord $\bar{f}(u)$. This condition is actually also sufficient and we have the following

(4.5) THEOREM. <u>Let</u> $\bar{f}: \Lambda U \to \Lambda Y$ <u>be a</u> ΛK-<u>linear map. Then</u> $\bar{f}$ <u>is an</u> ΩK-<u>map if and only if</u> ord $u \subset$ ord $\bar{f}(u)$ <u>for each</u> $u \in Q_{\Omega K} U$.

The condition of Theorem 4.5 is, of course, not easily tested directly and we would like to find a finite "test set" of elements in $Q_{\Omega K} U$ which is sufficient for verification that a ΛK-linear map is an ΩK-map. That a basis for $Q_{\Omega K} U$ may not be appropriate for this purpose is seen in the following simple example.

(4.6) EXAMPLE. Let $\Omega K = \Omega^- K$ and let $Y = U = K^2$. Take as basis for $Q_{\Omega K} K^2$ the elements $u_1 = \binom{z^{-1}}{1}$ and $u_2 = \binom{z^{-2}}{1}$ and define $\bar{f}: \Lambda K^2 \to \Lambda K^2$

$$\bar{f}(u_1) = u_1 + u_2$$
$$\bar{f}(u_2) = u_2 \quad .$$

Obviously, $\Omega^- K = \text{ord}_{\Omega^- K}\, u_1 = \text{ord}_{\Omega^- K}\, \bar{f}(u_1) = \text{ord}_{\Omega^- K}\, u_2 = \text{ord}_{\Omega^- K}\, \bar{f}(u_2)$. Thus, $\bar{f}$ satisfie the condition of Theorem 4.5 for the basis u_1, u_2, yet it is not an $\Omega^- K$-map (that is, no causal). Indeed, since $\bar{f}(u_1 - u_2) = u_1$ and since $u_1 - u_2 = \binom{z^{-1} - z^{-2}}{0}$, we have

$$\text{ord}_{\Omega^- K}(u_1 - u_2) = z\Omega^- K \not\subset \text{ord}_{\Omega^- K}\, u_1 = \Omega^- K \ .$$

Let us explore now the cause of difficulty encountered in the above example. If $s_1, \ldots, s_m \in Q_{\Omega K} S$ is a given set of elements and $\alpha_1, \ldots, \alpha_m \in Q_{\Omega K}$ is any set of scalars, then by formula (4.4),

$$\bigcap_{i=1}^{m} \text{ord } \alpha_i s_i \subset \text{ord } \sum_{i=1}^{m} \alpha_i s_i \quad .$$

But, the above inclusion, in general, need not hold with equality (even when the s_i are $Q_{\Omega K}$-linearly independent). This order "deficiency" also occurs in the example and therefore the basis selected there failed as a test set for causality. Indeed, we have there

$$\bigcap_{i=1}^{2} \text{ord}_{\Omega^- K}\, u_i = \Omega^- K \neq \text{ord}_{\Omega^- K}(u_1 - u_2) = z\Omega^- K \quad .$$

Thus, we are motivated to introduce the following

(4.7) DEFINITION. A set of nonzero elements $s_1,\ldots,s_m \in Q_{\Omega K}S$ is called ΩK-adapted if for every set of scalars $\alpha_1,\ldots,\alpha_m \in Q_{\Omega K}$ the condition

$$(4.8) \qquad \bigcap_{i=1}^{m} \operatorname{ord} \alpha_i s_i = \operatorname{ord} \sum_{i=1}^{m} \alpha_i s_i$$

holds. A basis of ΩK-adapted elements $s_1,\ldots,s_n$ of $Q_{\Omega K}S$ is called an ΩK-adapted basis

It is easily verified that in Definition 4.7 we could replace $Q_{\Omega K}$ by ΩK, i.e., $s_1,\ldots,s_m$ is ΩK-adapted if and only if (4.8) holds for every set $\alpha_1,\ldots,\alpha_m \in \Omega K$.

In the case when $\Omega K=\Omega^- K$, it can be seen that $\Omega^- K$-adapted sets coincide with properly independent sets (see HAMMER and HEYMANN [1981]) and minimal bases (see FORNEY [1975]) which have found many applications in system theory (see also WOLOVICH [1974], HAUTUS and HEYMANN [1978] and KAILATH [1980]).

Next we have the following theorem

(4.9) THEOREM. An ΩK-adapted set of nonzero elements $s_1,\ldots,s_m \in Q_{\Omega K}S$ is ΛK-linearly independent.

Let $s_1,\ldots,s_m \in \Lambda S$ be a set of elements and let $\Lambda[s_1,\ldots,s_m]$ denote the ΛK-linear space spanned by $s_1,\ldots,s_m$. We then have the following characterization of ΩK-adapted sets.

(4.10) THEOREM. Consider a set of nonzero elements $s_1,\ldots,s_m \in Q_{\Omega K}S$ with $\operatorname{ord} s_i = (\gamma_i)_{\Omega K}$, $i=1,\ldots,m$. Then $\{s_1,\ldots,s_m\}$ is an ΩK-adapted set if and only if $\{\gamma_1 s_1,\ldots,\gamma_m s_m\}$ forms a basis for the ΩK-module $\Lambda[s_1,\ldots,s_m] \cap \Omega S$.

As an immediate consequence of the above theorem we have the following characterizatic of ΩK-adapted bases.

(4.11) COROLLARY. Assume the set $s_1,\ldots,s_n \in Q_{\Omega K}S$ is a basis for ΛS with $\operatorname{ord} s_i = (\gamma_i)_{\Omega K}$, $i=1,\ldots,n$. Then the set $\{s_1,\ldots,s_n\}$ is ΩK-adapted if and only if $\{\gamma_1 s_1,\ldots,\gamma_n s_n\}$ generates ΩS.

(4.12) EXAMPLE. Corollary 4.11 provides a particularly simple way for determining whether a basis $s_1,\ldots,s_n$ of a ΛK-linear space ΛS is ΩK-adapted. Indeed, the main clause of the Corollary can be restated to read: The basis $s_1,\ldots,s_n$ of ΛS is ΩK-adapted if and only if $\det[s_1,\ldots,s_n] = \gamma_1^{-1}\cdot\gamma_2^{-1}\cdot\ldots\cdot\gamma_n^{-1}$. Using this simple criterion, we show that the columns

$$s_1 = \begin{bmatrix} z \\ z^3 \\ z^4 \end{bmatrix}, \quad s_2 = \begin{bmatrix} z^2+1 \\ (z^2+1)^2 \\ z^4(z^2+1) \end{bmatrix}, \quad s_3 = \begin{bmatrix} 0 \\ 0 \\ z^3+1 \end{bmatrix}$$

form an (unordered) Ω^+K-adapted basis of ΛK^3. Indeed, we have $\operatorname{ord}_{\Omega^+K} s_1 = (z^{-1})_{\Omega^+K}$, $\operatorname{ord}_{\Omega^+K} s_2 = ((z^2+1)^{-1})_{\Omega^+K}$ and $\operatorname{ord}_{\Omega^+K} s_3 = ((z^3+1)^{-1})_{\Omega^+K}$, whence $\gamma_1^{-1}\cdot\gamma_2^{-1}\cdot\gamma_3^{-1}=z(z^2+1)(z^3+1)$

which is equal to $\det[s_1,s_2,s_3]$. If however, s_1, say, is replaced by $s_1'=(2z,z^3,z^4)^T$, the resulting set will no longer be Ω^+K-adapted since
$\det[s_1',s_2,s_3] = (z^3+1)(z^2+1)(z^3+2z)$.

□

We turn now to the characterization of ΩK-maps with the aid of ΩK-adapted bases. As a further consequence of Theorem 4.10 we have the following

(4.13) PROPOSITION. Let $\bar{f}: \Lambda U \to \Lambda Y$ be a ΛK-linear map and assume that $u_1,\dots,u_n$ is an ΩK-adapted basis for ΛU. Then $\bar{f}$ is an ΩK-map if and only if $\text{ord } u_i \subset \text{ord } \bar{f}(u_i)$ for all $i=1,\dots,n$.

(4.14) DEFINITION. A ΛK-linear map $\bar{f}: \Lambda U \to \Lambda Y$ is called ΩK-order preserving (or, simply, order preserving) if for each $u \in Q_{\Omega K}U$, $\text{ord } u = \text{ord } \bar{f}(u)$.

(4.15) THEOREM. Let $\bar{f}: \Lambda U \to \Lambda Y$ be a ΛK-linear map and let $u_1,\dots,u_n \in Q_{\Omega K}U$ be an ΩK-adapted basis for ΛU. Then $\bar{f}$ is ΩK-order preserving if and only if (i) $\bar{f}(u_1),\dots,\bar{f}(u_n)$ is ΩK-adapted and (ii) for all $i=1,\dots,n$, $\text{ord } u_i = \text{ord } \bar{f}(u_i)$.

(4.16) THEOREM. Let $\bar{f}: \Lambda U \to \Lambda U$ be a surjective ΛK-linear map. Then $\bar{f}$ is ΩK-unimodular if and only if it is ΩK-order preserving.

5. Bounded K-Modules

Let $\Delta \subset \Lambda S$ be an ΩK-module. We say that Δ is ΩK-bounded (or simply bounded) if there exists a nonzero element $\gamma \in Q_{\Omega K}$ such that $\gamma \in \text{ord } s$ for all $s \in \Delta$ (i.e., $\gamma s \in \Omega S$ for all $s \in \Delta$). It is clear that if Δ is a bounded ΩK-submodule of ΛS, it consists only of ΩK-rational elements. An ΩK-module consisting of ΩK-rational elements is called rational. If $\Delta \subset \Lambda S$ is bounded ΩK-submodule, we define the order of Δ, denoted ord Δ, as the class of all elements $\gamma \in Q_{\Omega K}$ such that $\gamma \in \text{ord } s$ for all $s \in \Delta$. It is easily seen that $\text{ord } \Delta = \bigcap_{s \in \Delta} \text{ord } s$ whence if $\Delta \neq 0$, ord Δ is a cyclic ΩK-module and is generated by an element $\psi \in Q_{\Omega K}$. Explicitly, ψ is a least common ΩK-multiple of all order generators $\gamma = \gamma(s)$ of elements $s \in \Delta$.

Next, we have the following :

(5.1) LEMMA. Let $\Delta \subset \Lambda S$ be a rational ΩK-submodule. Then Δ is bounded if and only if Δ has finite rank (i.e., is finitely generated) in which case $\text{rank } \Delta \leq \dim S$.

Below we make use of the Smith canonical form theorem for matrices over a principal ideal domain (see e.g. MACDUFFEE [1934] and NEWMAN [1972]). We shall identify ΛK-linea maps with their transfer function matrices. In particular, we shall speak of an ΩK-matrix if its entries are in ΩK and of an ΩK-unimodular matrix if both it and its inverse are ΩK-matrices. Smith's theorem is stated as follows:

(5.2) THEOREM. Let T be an mxn ΩK-matrix. Then there are ΩK-unimodular matrices M_L and M_R of dimensions mxm and nxn, respectively, and elements $\delta_1,\dots,\delta_r \in \Omega K$, uniquely defined up to multiples of units of ΩK, with $r \leq \min(m,n)$ and $\delta_{i+1} | \delta_i$, $i=1,\dots,r-1$, such that

$$(5.3) \qquad T = M_L \, D \, M_R$$

where D is the mxn matrix given by $D = \text{diag}(\delta_1,\ldots,\delta_r, 0,\ldots,0)$.

The elements $\delta_1,\ldots,\delta_r$ in Theorem 5.2 are called the invariant factors of T and the theorem itself is sometimes called the invariant factor theorem.

Assume now that $\Delta\subset\Lambda S$ is a nonzero and bounded ΩK-module with ord $\Delta = (\psi)_{\Omega K}$ and (in view of Lemma 5.1) let $d_1,\ldots,d_r \in \Delta$ be a basis for Δ. Then $\psi d_1,\ldots,\psi d_r \in \Omega S$ and the mxr matrix $\psi T := [\psi d_1,\ldots,\psi d_r]$ (where ψd_i is viewed as a column vector) has Smith representation

$$(5.4) \qquad \psi T = M_L \, D \, M_R$$

where

$$D = \begin{bmatrix} \delta_1 & & 0 \\ & \ddots & \\ 0 & & \delta_r \\ \hline & 0 & \end{bmatrix},$$

and the $\delta_i \in \Omega K$ (with $\delta_{i+1}|\delta_i$) are the invariant factors of ψT. We note that, by assumption, $\delta_1,\ldots,\delta_r$ are nonzero. Dividing both sides of (5.4) by ψ yields

$$(5.5) \qquad T = M_L \, D_o \, M_R$$

where D_o is the Mcmillan form of D, and is given by

$$D_o = \begin{bmatrix} \delta_1/\psi & & 0 \\ & \ddots & \\ 0 & & \delta_r/\psi \\ \hline & 0 & \end{bmatrix}.$$

Let d_{oi} denote the ith column of D_o. It is easily observed that the columns $d_{o1},\ldots,d_{or} \in Q_{\Omega K}S$ constitute an ΩK-adapted set. Indeed, for every set $\alpha_1,\ldots,\alpha_r \in Q_{\Omega K}$ we have that

$$d := \sum_{i=1}^{r} \alpha_i d_{oi} = \begin{bmatrix} \alpha_1 \frac{\delta_1}{\psi} \\ \vdots \\ \alpha_r \frac{\delta_r}{\psi} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and clearly ord $d = (\frac{\alpha_1\delta_1}{\psi})_{\Omega K} \cap\ldots\cap (\frac{\alpha_r\delta_r}{\psi})_{\Omega K}$.

Furthermore, we have

$$\Delta = T[\Omega S] = M_L D_o M_R[\Omega S] = M_L D_o[\Omega S] ,$$

the last equality following since M_R is ΩK-unimodular (see Corollary 3.9). Now, M_L is ΩK-unimodular, so that by Theorem 4.15 the columns of $M_L D_o$, given by $\frac{\delta_1}{\psi}M_{L1},\ldots,\frac{\delta_r}{\psi}M_{Lr}$ (where M_{Li} is the ith column of M_L) are also ΩK-adapted. Further, since M_L is ΩK-unimodular, it also follows that ord $M_{Li} = \Omega K$, whence, ord $\frac{\delta_i}{\psi}M_{Li} = \frac{\psi}{\delta_i}\Omega K = \left(\frac{\psi}{\delta_i}\right)_{\Omega K}$. We see immediately that the set $\frac{\delta_1}{\psi}M_{L1},\ldots,\frac{\delta_r}{\psi}M_{Lr}$ constitutes an ordered ΩK-adapted basis for Δ. We make the following further observation. Since $\delta_{i+1}|\delta_i$, it follows that

$$\left(\frac{\psi}{\delta_r}\right)_{\Omega K} \subset \left(\frac{\psi}{\delta_{r-1}}\right)_{\Omega K} \subset\ldots\subset\left(\frac{\psi}{\delta_1}\right)_{\Omega K}$$

so that

$$\text{ord } \Delta = (\psi)_{\Omega K} = \left(\frac{\psi}{\delta_r}\right)_{\Omega K}$$

and we conclude that δ_r is a unit in ΩK which, in particular, can always be chosen as $\delta_r = 1$.

We summarize the foregoing discussion with the following important theorem

(5.6) THEOREM. Let $\Delta\subset\Lambda S$ be a nonzero bounded ΩK-module. Then

(i) Δ has an ordered ΩK-adapted basis $d_1,\ldots,d_r$.

(ii) If $d_1',\ldots,d_r'$ is any other ordered ΩK-adapted basis of Δ then ord d_i' = ord d_i, $i = 1,\ldots,r$.

If $\Delta\subset\Lambda S$ is a bounded ΩK-module with ordered ΩK-adapted basis $d_1,\ldots,d_r$, then the set of ΩK-modules ord $d_i = \left(\frac{\delta_i}{\psi}\right)_{\Omega K}$, $i = 1,\ldots,r$ constitutes an important invariant of Δ and we call it the order trace of Δ.

Let $\Delta\subset\Lambda S$ be a bounded ΩK-module of rank r and let $d_1,\ldots,d_r$ be a basis of Δ. We can form the matrix $D: = [d_1,\ldots,d_r]$ and view Δ as the image of an ΩK-homomorphism $\Omega K^r\to\Lambda S$ defined by $e_i \mapsto De_i = d_i$. With this convention we then write Δ as $\Delta = D\Omega K^r$. We say that Δ is full (in ΛS) if rank Δ = dim S, i.e., if $\Delta = D\Omega S$ and D is nonsingula

(5.7) THEOREM. Let $\Delta_1, \Delta_2 \subset \Lambda S$ be bounded ΩK-submodules given by $\Delta_1 = D_1\Omega S$ and $\Delta_2 = D_2\Omega S$, respectively. Then $\Delta_2 \subset \Delta_1$ if and only if there exists an ΩK-matrix R (i.e., with entries in ΩK) such that $D_2 = D_1 R$.

(5.8) COROLLARY. Let $\Delta_1, \Delta_2 \subset \Lambda S$ be bounded ΩK-submodules given by $\Delta_1 = D_1\Omega S$ and $\Delta_2 = D_2\Omega S$. Assume Δ_1 is full and define $R: = D_1^{-1}D_2$. Then $\Delta_2 \subseteq \Delta_1$ if and only if R is an ΩK-matrix, with equality holding if and only if R is ΩK-unimodular.

We turn now to the existence of ΩK-adapted bases for ΛK-linear spaces. A ΛK-linear subspace $\mathcal{R}\subset\Lambda S$ is called ΩK-rational if it has a basis $s_1,\ldots,s_k$ consisting of ΩK-rational vectors.

(5.9) THEOREM. Let $\dim S = n$ and let $R \subset \Lambda S$ be a nonzero ΩK-rational ΛK-linear subspace. Then (i) R has an ΩK-adapted basis, and (ii) every ΩK-adapted subset $s_1,\ldots,s_\ell \in R$ can be extended to an ΩK-adapted basis for R.

Next, we give the following characterization of the order trace.

(5.10) PROPOSITION. Let $\Delta,\Delta' \subset \Lambda S$ be nonzero and bounded ΩK-modules of equal rank m. Then there exists an ΩK-unimodular map $M: \Lambda S \to \Lambda S$ such that $M[\Delta] = \Delta'$ if and only if Δ and Δ' have the same order traces.

Related to the notion of ΩK-adapted bases is also the following

(5.11) DEFINITION. Let $R_1,\ldots,R_k \subset \Lambda S$ be ΩK-rational ΛK-linear subspaces. Then $R_1,\ldots,R_k$ are called ΩK-adapted if for every set of elements $s_1,\ldots,s_k$ where $s_i \in R_i$, $i = 1,\ldots,k$,

$$\operatorname{ord}(s_1 + \ldots + s_k) = \bigcap_{i=1}^{k} \operatorname{ord} s_i \quad .$$

It follows readily from the above definition that the concept of ΩK-adapted subspaces is equivalent to the following : Let $R_1,\ldots,R_k \in \Lambda S$ be ΩK-rational ΛK-linear subspaces and let $d_{i1},\ldots,d_{i\ell_i}$ be a basis for R_i, $i = 1,\ldots,k$. Then the subspaces $R_1,\ldots,R_k$ are ΩK-adapted if and only if $d_{11},\ldots,d_{1\ell_1},\ldots,d_{k1},\ldots,d_{k\ell_k}$ is an ΩK-adapted basis for $R_1 + \ldots + R_k$. Naturally, ΩK-adapted spaces are ΛK-linearly independent so that the above sum of subspaces is, in fact, a direct sum. Accordingly, we speak of ΩK-adapted direct sums of ΛK-linear spaces.

The concept of ΩK-adapted subspaces is of course a generalization to arbitrary P.I.D.' of the concept of properly independent and stably independent spaces as defined in HAMMER and HEYMANN [1981], and in HAUTUS and HEYMANN [1980a], [1980b].

Theorem 5.9 leads to the following useful result.

(5.12) COROLLARY. Let $R_1 \subset R_2$ $(\subset \Lambda S)$ be ΩK-rational ΛK-linear subspaces. Then R_1 has an ΩK-adapted direct summand in R_2.

6. ΩK-Factorization and Invertibility

Consider two ΛK-linear maps $\bar{f}_1: \Lambda U \to \Lambda Y$ and $\bar{f}_2: \Lambda U \to \Lambda W$ and assume there exists an ΩK-map $h: \Lambda Y \to \Lambda W$ such that $\bar{f}_2 = \bar{h}\cdot\bar{f}_1$. Let $u \in \Lambda U$ satisfy the condition that $\bar{f}_1(u) \in \Omega Y$, or, in the notation of (3.7), that $u \in \ker\pi_{\Omega K}\bar{f}$. Then, obviously, $\bar{f}_2(u) = \bar{h}\cdot\bar{f}_1(u) \in \Omega W$ so that $u \in \ker\pi_{\Omega K}\bar{f}_2$, and the existence of the ΩK-map $\bar{h}$ such that $\bar{f}_2 = \bar{h}\cdot\bar{f}_1$, implies that $\ker\pi_{\Omega K}\bar{f}_1 \subset \ker\pi_{\Omega K}\bar{f}_2$. In case the maps $\bar{f}_1$ and $\bar{f}_2$ are ΩK-rational, the converse of the above statement is also true and we have the following central

(6.1) THEOREM. Let $\bar{f}_1: \Lambda U \to \Lambda Y$ and $\bar{f}_2: \Lambda U \to \Lambda W$ be ΩK-rational ΛK-linear maps. There exists an ΩK-map $\bar{h}: \Lambda Y \to \Lambda W$ such that $\bar{f}_2 = \bar{h}\cdot\bar{f}_1$ if and only if $\ker\pi_{\Omega K}\bar{f}_1 \subset \ker\pi_{\Omega K}\bar{f}_2$.

Theorem 6.1 depends on the following lemmas.

(6.2) LEMMA. Let $\bar{f}$: $\Lambda U \to \Lambda Y$ be an ΩK-rational ΛK-linear map. Let $r := \dim_{\Lambda K} \mathrm{Im}\bar{f}$ and let $Y_o \subset Y$ be any r-dimensional subspace. Then there exists an ΩK-unimodular map M: $\Lambda Y \to \Lambda Y$ such that $\mathrm{Im} M\cdot\bar{f} = \Lambda Y_o$.

(6.3) LEMMA. Let $\bar{f}$: $\Lambda U \to \Lambda Y$ be a ΛK-linear map. If $R \subset \ker\pi_{\Omega K}\bar{f}$ is a ΛK-linear subspace, then $R \subset \ker\bar{f}$.

Theorem 6.1 admits the following

(6.4) COROLLARY. Let $\bar{f}_1, \bar{f}_2$: $\Lambda U \to \Lambda Y$ be ΩK-rational ΛK-linear maps. There exists an ΩK-unimodular map M: $\Lambda Y \to \Lambda Y$ such that $\bar{f}_2 = M\cdot\bar{f}_1$, if and only if $\ker\pi_{\Omega K}\bar{f}_1 = \ker\pi_{\Omega K}\bar{f}_2$.

We call a ΛK-linear map $\bar{f}$: $\Lambda U \to \Lambda Y$ ΩK-left invertible if it has an ΩK-map as a left inverse. The following further corollary to Theorem 6.1 is also useful.

(6.5) COROLLARY. An ΩK-rational ΛK-linear map $\bar{f}$: $\Lambda U \to \Lambda Y$ is ΩK-left invertible if and only if $\ker\pi_{\Omega K}\bar{f} \subset \Omega U$.

Before concluding the section, we wish to express in an explicit form the main quantities that appeared in our discussion. Let $\bar{f}$: $\Lambda U \to \Lambda Y$ be an ΩK-rational ΛK-linear map. We start with an explicit representation of the ΩK-module $\ker\pi_{\Omega K}\bar{f}$. We shall identify the map $\bar{f}$ with its transfer matrix, and shall denote $r := \dim_{\Lambda K}\mathrm{Im}\bar{f}$. Let M_L: $\Lambda Y \to \Lambda Y$ and M_R: $\Lambda U \to \Lambda U$ be ΩK-unimodular maps such that $\bar{f} = M_L\cdot D\cdot M_R$, where the matrix D: $\Lambda U \to \Lambda Y$ is of the form $D = \begin{pmatrix} D_o & 0 \\ 0 & 0 \end{pmatrix}$, with D_o: $\Lambda K^r \to \Lambda K^r$ (square) nonsingular. One possible choice of D is, of course, the McMillan canonical form of $\bar{f}$. Also, we let $U_o \oplus U_1 = U$ be a direct sum decomposition, where $\Lambda U_o = \ker D$ and ΛU_1 is the domain of D_o.

Now, $\ker\pi_{\Omega K}\bar{f} = \ker\pi_{\Omega K} M_L D M_R = M_R^{-1}[\ker\pi_{\Omega K} M_L D]$, and, applying corollary 6.4, we obtain that $\ker\pi_{\Omega K}\bar{f} = M_R^{-1}[\ker\pi_{\Omega K} D]$. Further, it is readily seen that $\ker\pi_{\Omega K} D = D_o^{-1}[\Omega U_1] \oplus \Lambda U_o$ and, consequently, we have

(6.6) $$\ker\pi_{\Omega K}\bar{f} = M_R^{-1}[D_o^{-1}[\Omega U_1] \oplus \Lambda U_o], \quad \text{and}$$

(6.7) $$\ker\bar{f} = M_R^{-1}[\Lambda U_o].$$

Defining now the map

$$\bar{f}_* := M_R^{-1}\begin{pmatrix} D_o^{-1} \\ 0 \end{pmatrix} : \Lambda U_1 \to \Lambda U ,$$

we have that

(6.8) $$\ker\pi_{\Omega K}\bar{f} = \bar{f}_*[\Omega U_1] + \ker\bar{f} ,$$

so that $\bar{f}_*$ generates the "bounded part" of $\ker\pi_{\Omega K}\bar{f}$.

Next, let $\bar{f}'$: $\Lambda U \to \Lambda Y'$ be a linear i/o map. We express now the condition of theorem 6.1 in more explicit form. The condition $\ker\pi_{\Omega K}\bar{f} \subset \ker\pi_{\Omega K}\bar{f}'$ is clearly equivalent to $\bar{f}'[\ker\pi_{\Omega K}\bar{f}] \subset \Omega Y'$. Substituting now (6.8), and noting that $\ker\bar{f}$ is a ΛK-linear subspace, the latter condition can be split into the two conditions: (i) $\bar{f}'\bar{f}_*[\Omega U_1] \subset \Omega Y'$, and (ii) $\bar{f}'[\ker\bar{f}] = 0$. These conditions are then equivalent to simply

(i_a) $\bar{f}\bar{f}_*$ is an ΩK-map, and (ii_a) $\ker\bar{f}\subset\ker\bar{f}'$, respectively.

Returning now to theorem 6.1, we can summarize as follows: There exists an ΩK-map $\bar{h}$: $\Lambda Y\to\Lambda Y$ such that $\bar{f}' = \bar{h}\cdot\bar{f}$ if and only if $\bar{f}'\bar{f}_*$ is an ΩK-map, and $\ker\bar{f}\subset\ker\bar{f}'$. Moreover, through a direct computation, one can show that, if $\bar{h}$ exists, then it is necessarily of the form

$$\bar{h} = (\bar{f}'\bar{f}_*,\ y_1,\ldots,y_{p-r})M_L^{-1}\ ,$$

where p: $= \dim_K Y$, and $y_1,\ldots,y_{p-r}$ are (arbitrary) elements in $\Omega Y'$. Thus, the map $\bar{f}_*$, which generates the "bounded part" of $\ker\pi_{\Omega K}\bar{f}$, plays a central role in factorization theory.

7. Precompensation and Stable Output Feedback

We turn now to a brief discussion of some applications of the above factorization theory to stable (and causal) output feedback. We assume throughout the section that ΩK is either the ring $\Omega_{\mathcal{D}}K$ or the ring $\Omega_{\mathcal{D}}^{-}K$.

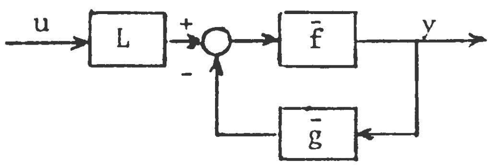
Let $\bar{f}$: $\Lambda U\to\Lambda Y$ be a linear i/o map and let $\bar{\ell}$: $\Lambda U\to\Lambda U$ be a bicausal ΛK-linear map (i.e., Ω^{-}K-unimodular) which we regard as a precompensator for $\bar{f}$. We can express $\bar{\ell}^{-1}$ as

(7.1) $$\bar{\ell}^{-1} = L^{-1}(I + \bar{h})$$

where L is static (see HAUTUS and HEYMANN [1978]) and where $\bar{h}$ is strictly causal. If, additionally, we can express $\bar{h}$ as $\bar{h} = \bar{g}\bar{f}$ for some causal map $\bar{g}$: $\Lambda Y\to\Lambda U$ then we can give $\bar{\ell}$ an output feedback interpretation through the formula

$$\bar{f}\bar{\ell} = \bar{f}(I + \bar{g}\bar{f})^{-1}L\ ,$$

which is the i/o map of the composite system



The map $\bar{g}$ is then clearly a causal (dynamic) output feedback compensator and L is a coordinate transformation map in the input value space. We may require additionally that the feedback compensator $\bar{g}$ be stable, i.e., an $\Omega_{\mathcal{D}}$K-map. We are then faced with the question of when can $\bar{h}$ of (7.1) be factored over $\bar{f}$ through an $\Omega_{\mathcal{D}}^{-}$K-map $\bar{g}$. The answer is provided by Theorem 6.1 and we have the following

(7.2) THEOREM. Let $\bar{f}$: $\Lambda U\to\Lambda Y$ be an $\Omega_{\mathcal{D}}$K-rational linear i/o map, let $\bar{\ell}$: $\Lambda U\to\Lambda U$ be an $\Omega_{\mathcal{D}}$K-rational bicausal precompensator for $\bar{f}$ and express $\bar{\ell}$ as in (7.1). There exists a causal and stable output feedback representation for $\bar{\ell}$ if and only if $\ker\pi_{\Omega_{\mathcal{D}}^{-}K}\bar{f}\subset\ker\pi_{\Omega_{\mathcal{D}}^{-}K}\ \bar{h}$.

We say that a linear i/o map $\bar{f}$: $\Lambda U\to\Lambda Y$ is $\Omega_{\mathcal{D}}$K-minimum phase (or, simply, minimum phase if it is an $\Omega_{\mathcal{D}}K$-map (i.e., stable) and is $\Omega_{\mathcal{D}}K$-left invertible. Thus $\bar{f}$ is $\Omega_{\mathcal{D}}^{-}K$-minimum phase precisely whenever

$$(7.3) \qquad \ker\pi_{\Omega_{\mathcal{D}}K}\bar{f} = \Omega_{\mathcal{D}}U .$$

We recall further (see HAMMER and HEYMANN [1981]) that a linear i/o map $\bar{f}$ is called nonlatent if and only if

$$\ker\pi_{\Omega^{-}K}\bar{f} = z\Omega^{-}U ,$$

i.e., if and only if $z\bar{f}$ has a causal left inverse. Now, if $\bar{f}$ is $\Omega_{\mathcal{D}}K$-left invertible, so is also $(z+\alpha)\bar{f}$ where $(z+\alpha)\in\mathcal{D}$. In case $\bar{f}$ is nonlatent as well, then $(z+\alpha)\bar{f}$ also has a causal left inverse. Thus, one can readilly see that an i/o map $\bar{f}$ is nonlatent and minimum phase if and only if

$$(7.4) \qquad \ker\pi_{\Omega_{\mathcal{D}}^{-}K}\bar{f} = (z+\alpha)\Omega_{\mathcal{D}}^{-}U .$$

We now have the following Theorem which is an analog to Corollary 5.4 in HAMMER and HEYMANN [1981]

(7.5) THEOREM. Assume that for some $\alpha,\beta\in K$, both $(z+\alpha)$ and $(z+\beta)$ are in $\mathcal{D}$, and let $\bar{f}$: $\Lambda U\to\Lambda Y$ be an $\Omega_{\mathcal{D}}K$-rational and stable linear i/o map. Then $\bar{f}$ is nonlatent and minimum phase if and only if every $\Omega_{\mathcal{D}}^{-}K$-unimodular ΛK-linear precompensator $\bar{\ell}$: $\Lambda U\to\Lambda U$ has a causal and stable feedback representation $(L,\bar{g})$, i.e., there exists a pair $(L,\bar{g})$ with L static and $\bar{g}$ causal and $\mathcal{D}$-stable such that $\bar{\ell} = (I+\bar{g}\bar{f})^{-1}L$.

The interest in Theorem 7.5 derives from the fact that stable injective linear i/s maps are always nonlatent and minimum phase. This fact is seen as follows. It was shown in HAMMER and HEYMANN [1980] that if $\bar{f}$: $\Lambda U\to\Lambda Y$ is an injective linear i/s map, it is strictly observable, i.e., $\ker\pi_{\Omega^{+}K}\bar{f}\subset\Omega^{+}U$. Let D be an $\Omega^{+}K$-adapted basis matrix for $\ker\pi_{\Omega^{+}K}\bar{f}$, that is, $D\Omega^{+}U=\ker\pi_{\Omega^{+}K}\bar{f}$. It is easily verified that we then also have that $D\Omega_{\mathcal{D}}U = \ker\pi_{\Omega_{\mathcal{D}}K}\bar{f}$. Now, the strict observability of $\bar{f}$ implies that D is a polynomial matrix and thus $D\Omega_{\mathcal{D}}U\subset\Omega_{\mathcal{D}}U$ (since $\Omega^{+}K\subset\Omega_{\mathcal{D}}K$). We conclude that $\ker\pi_{\Omega_{\mathcal{D}}K}\bar{f}\subset\Omega_{\mathcal{D}}U$, and if the i/s map $\bar{f}$ is also stable, the minimum phase property (see (7.3)) follows. That injective linear i/s maps are nonlatent was proved in HAMMER and HEYMANN [1981] (Theorem 5.5). We summarize the above in the following.

(7.6) PROPOSITION. If $\bar{f}$: $\Lambda U\to\Lambda Y$ is a stable injective linear i/s map, then it is nonlatent and minimum phase.

We can now combine Theorem 7.5 with Proposition 7.6 to obtain the following result.

(7.7) COROLLARY. Let $\bar{f}$: $\Lambda U\to\Lambda Y$ be a stable, injective linear i/o map and let $\bar{\ell}$: $\Lambda U\to\Lambda U$ be an $\Omega_{\mathcal{D}}^{-}K$-unimodular precompensator for $\bar{f}$. Then $\bar{\ell}$ has a stable causal (dynamic) state feedback representation in every stable realization of $\bar{f}$.

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