in "Mathematical System Theory: the influence of R.E. Kalman",
Edited by A. Antoulas, Springer Verlag, Berlin, 1990.

Linear Realization Theory, Integer Invariants and Feedback Control[1]

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The algebraic module theoretic stability framework for linear time-invariant systems is reviewed. The main theme is that Kalman's algebraic realization theory has evolved much beyond its initial objective of providing an abstract framework for the derivation of mathematical models of systems. It has become a powerful tool for the extraction of structural invariants, permitting the exact characterization of all options for dynamics assignment through internally stable linear dynamic compensation. This characterization is provided by a set of integers—the stability indices of the system.

1 Introduction

In a sense, realization theory is the basic mechanism of science through which the conceptualization of observation is achieved. It formulates the mathematical guiding principles that lead from measurement of behavior to laws of nature. Duly stated, realization theory is the abstract theory of mathematical modeling. It forms the bridge from experiment to theory, and, in a way, is a grand mathematical scheme for data compression, facilitating compact mathematical description of vast realms of experimental data. All this is, of course, well known. The main point of the present note is to show that realization theory has matured beyond its innate mission of providing guiding principles for modeling, and has become a refined tool for scientific analysis, capable of singling out the important aspects of experimental data and filtering away the clutter of secondary details. In mathematical terms, realization theory has become a sophisticated tool for the extraction of structural invariants of systems.

Historically and philosophically, realization theory may be conceived as the driving force behind the scientific revolution that started in the eighteenth century; Nevertheless, it seems that elicit mathematical treatments of basic aspects of realization theory had not appeared in the scientific literature until around the middle of the present century. At that time, realization theory formed

[1] This research was supported in part by the National Science Foundation, USA, Grant number 8896182 and 8913424. Partial support was also provided by the Office of Naval Research, USA, Grant number N00014-86-K0538, by US Air Force Grant AFOSR-87-0249, and by the US Army Research Office, Grant number DAAL03-89-0018, through the Center for Mathematical System Theory, University of Florida, Gainesville, FL 32611, USA.

a basic component of the research in automata theory, which culminated in the well known Nerode principle (Nerode [1958]). The basic principles of realization theory the way it is known today were set out in the pioneering contributions of R.E. Kalman (Kalman [1965, 1968], and Kalman, Falb, and Arbib [1969]), which formed the mathematical foundation of modern system theory and control. The broad implications of realization theory to a variety of other scientific disciplines were also pointed out by R.E. Kalman (e.g. Kalman [1980]).

Perhaps, one of the most important contributions of R.E. Kalman in the context of realization theory was the development of an explicit mathematical framework within which the realization issue can be explored. In this way, the realization problem was transformed from a vague philosophical entity into a concrete set of mathematical problems, complete with tools and techniques for exploration. Kalman's mathematization of realization theory formed the cradle for the evolution of mathematical system theory, giving birth to an entirely new branch of Mathematics, and forming a lead toward the mathematization of Engineering.

The present note is concerned with some implications of a particular contribution of R.E. Kalman to mathematical realization theory—the introduction of algebraic module theory as a fundamental tool for the solution of the realization problem for linear time-invariant systems (Kalman [1965]). The main objective here is to show that basically the same abstract formalism can be used to derive structural invariants of linear time-invariant systems, invariants that determine and characterize the limitations on the performance of a linear system within a control engineering environment. The material covered in this note is a review and re-interpretation of results presented in Hammer [1983a, b, and c].

Over the years, much thought has been given to the issue of what should the true nature of Engineering theory be, and how should it relate to Engineering practice. Ideally, one might say, Engineering theory should provide the formulas for solving practical engineering problems. However, further reflection shows this approach to be quite simplistic; while being scientifically founded, Engineering practice comprises a substantial component of what one might call 'Engineering Art'. The wide spectrum of design constraints and refined performance criteria facing the Engineer make an individualistic approach to design imperative. And the unavoidable interaction between engineering systems and human operators adds to the performance evaluation an aspect of aesthetics and values. Thus, Engineering practice consists of much more than the application of pre-derived formulas.

It seems that the most important role of Engineering theory is to provide the designer with a clear indication of the limitations on design performance. These limitations should be extracted by the theory from the description of the system, and presented in concise and clear form, so as to provide the designer with a clear characterization of the entire spectrum of achievable design specifications. Furthermore, in case critical design specifications cannot be met for the given system, the theory should provide a clear indication of the ways in which the system has to be modified to facilitate the desired performance. From a mathematical point of view, the limitations on design performance for a given class of systems are usually presented in terms of *system invariants*. In qualitative terms, the invariants characterize the fundamental underlying structure of the system which cannot be altered by external intervention; they provide the skeleton upon which the designer may build.

The module theoretic approach to the linear realization problem initiated in Kalman [1965] has matured into a powerful tool for the derivation of structural invariants for linear time-invariant systems. Specifically, it yields a set of integer invariants that entirely characterize all possible dynamical behaviors that can be assigned to a system through the use of external dynamic compensation, subject to the requirement that the final closed loop configuration be internally stable. This result is derived by developing an algebraic module theoretic realization theory over certain rings of stable rational functions, which replace the ring of polynomials used in the original Kalman realization theory. In this way, algebraic realization theory becomes more than just a tool for obtaining dynamical models of linear systems; it provides the means to extract the inherent structure of the system from the input/output data on its behavior.

It is quite fascinating that the fundamental invariant structure of a linear time-invariant system can be entirely characterized by a finite set of integers. When investigating the possible dynamical properties that can be endowed to a given system through an internally stable closed loop configuration, all one needs to know are these integer invariants, despite the fact that the complete description of the dynamical model of the system requires a much larger number of *real* parameters. The question of whether certain dynamical properties can or cannot be assigned to a system by internally stable compensation simply reduces to the comparison of some integers. This fact provides a deep indication of the fundamental simplicity of linear time-invariant control systems. It is, of course, in line with the classical results on pole assignment (Wonham [1967]) and on the assignment of invariant factors (Rosenbrock [1970]).

2 Time Invariance, Linearity, and Stability Rings

Consider a causal linear time-invariant system Σ having an input space U of finite dimension m and an output space Y of finite dimension p. For the sake of intuitive clarity, assume that Σ is a discrete-time system (the framework discussed herein applies to continuous-time systems as well; one only needs to use the Laplace transform instead of the Z-transform employed here). Adopting the input/output point of view, the system Σ is regarded as a map that transforms input sequences of vectors of U into output sequences of vectors of Y. An input sequence $u = \{u_{t_0}, u_{t_0+1}, u_{t_0+2}, \ldots\}$ is commonly represented as a Laurent series of the form

$$u = \sum_{t=t_0}^{\infty} u_t z^{-t}, \tag{2.1}$$

which is interpreted as the Z-transform of u. In this notation, the Laurent series zu simply represents the input sequence obtained by shifting u by one step to the left, and z represents the one-step shift operator. The set of all Laurent sequences of the form (2.1), where the (finite) initial time t_0 may vary from sequence to sequence, is denoted by ΛU. Clearly, the output sequence generated by Σ from the input sequence u is then an element of the space ΛY. The time invariance of the system Σ implies that it commutes with the shift operator z, so that

$$\Sigma z = z\Sigma \tag{2.2}$$

In his [1972] lecture notes, Wyman noted that, through (2.2), time invariance is related to the linearity of the system Σ over the field of scalar Laurent series, and this idea was then further expounded in Hautus and Heymann [1978]. We review this point next.

Let K be a field, let S be a K-linear space, and let ΛS denote the set of all Laurent series of the form

$$s = \sum_{t=t_0}^{\infty} s_t z^{-t}, \tag{2.3}$$

where the initial integer t_0 may vary from one sequence to another, and where $s_t \in S$ for all integers $t \geqq t_0$. In particular, taking $S = K$, it is easy to see that the set ΛK forms a field under the operations of coefficientwise addition as addition and series convolution as multiplication. The set ΛS becomes then a ΛK-linear space. Furthermore, it can readily be shown that if the dimension of S as a K-linear space is n, then ΛS is a finite dimensional ΛK-linear space of dimension n as well. The importance of the notion of ΛK-linearity is that it permits the fusion of two seemingly disparate notions—the notion of linearity and the notion of time invariance. In fact, every ΛK-linear map $f: \Lambda U \to \Lambda Y$ clearly satisfies (2.2), and whence *represents a K-linear time-invariant system*, which has the K-linear space U as its input space, and the K-linear space Y as its output space (Wyman [1972]). Moreover, it can be shown that *every K-linear time-invariant system Σ which is causal and has a finite dimensional state-space represents a ΛK-linear map $f: \Lambda U \to \Lambda Y$*, where U is the input space of Σ and Y is its output space. Thus, for the systems we intend to consider in this note, the notion of a ΛK-linear map is equivalent to the input/output description of a system.

As any linear map between finite dimensional linear spaces, a ΛK-linear map $f: \Lambda U \to \Lambda Y$ may be represented by a matrix, relative to specified bases of its domain ΛU and its codomain ΛY. Among the bases of the space ΛU, a particularly significant role is played by bases of the original K-linear space U.

It is easy to verify that every basis $u_1, u_2, \ldots, u_m$ of the K-linear space U is also a basis of the ΛK-linear space ΛU. A matrix representation of the ΛK-linear map f with respect to bases $u_1, u_2, \ldots, u_m$ of U and $y_1, y_2, \ldots, y_m$ of Y is called a *transfer matrix* of f, and it coincides with the standard notion of a transfer matrix used in linear control.

We consider next some objects in the infrastructure of the space ΛS. First, let ΩS denote the set of all polynomial elements within ΛS, namely, the set of all elements of the form $\sum_{t=t_0}^{0} s_t z^{-t}$, where $t_0 \leq 0$ and $s_t \in S$. Then, for $S = K$, the set ΩK is simply the principal ideal domain of polynomials with coefficients in the field K. More generally, the space ΩS is an ΩK-module of rank equal to $\dim_K S$. Also, let $\Omega^- S$ denote the set of all elements in ΛS of the form $\sum_{t=0}^{\infty} s_t z^{-t}$. The set $\Omega^- K$ is then the principal ideal of all power series with coefficients in K, and $\Omega^- S$ is a module over $\Omega^- K$ with rank given by $\dim_K S$.

The space ΛS is itself an ΩK-module as well, and one may consider the quotient module $\Gamma S := \Lambda S/\Omega S$. We denote by

$$j: \Omega S \to \Lambda S$$

the identity injection, which assigns to each polynomial vector the formal Laurent series equal to it. We let

$$\pi: \Lambda S \to \Gamma S$$

be the canonical projection onto the quotient module.

Several classes of ΛK-linear maps are important in this context. A *polynomial map* is a ΛK-linear map $f: \Lambda U \to \Lambda Y$ which satisfies $f[\Omega U] \subset \Omega Y$, and is simply a map that has a polynomial transfer matrix. A ΛK-linear map $f: \Lambda U \to \Lambda Y$ is *rational* if there is a nonzero polynomial $\psi \in \Omega K$ for which ψf is a polynomial map. A *causal map* is a ΛK-linear map $f: \Lambda U \to \Lambda Y$ that satisfies $f[\Omega^- U] \subset \Omega^- Y$, and it describes a causal linear time-invariant system; The map f is *strictly causal* if $f[\Omega^- U] \subset z^{-1}[\Omega^- Y]$. A ΛK-linear map $M: \Lambda U \to \Lambda U$ is *bicausal* if it is invertible, and if it and its inverse M^{-1} are both causal maps. Finally, a rational and strictly causal ΛK-linear map is called a *linear input/output map*.

Perhaps, one of the most intriguing properties of linear systems is the fact that they permit treatment of the notion of stability in an algebraic setup, without requiring direct reference to topological properties. This fact, which is crucial to the algebraic theory of linear systems, has been observed by several investigators during the seventies. One of the earliest references to this point is Morse [1975]. The basic observation in this context is that stability theory for linear time-invariant systems can be developed within the algebraic framework of localizations of the ring of polynomials. To be specific, let $\theta \subset \Omega K$ be a subset of polynomials satisfying the following three properties: (i) the product of every two elements of θ is still in θ; (ii) the zero polynomial is *not* in θ; and (iii) θ contains a polynomial of degree one. A subset θ satisfying these properties is called a *stability set*. The algebraic definition of stability is then given by the following

2.4. Definition. Let θ be a stability set. A ΛK-linear map $f:\Lambda U \to \Lambda Y$ is *input/output stable* (in the sense of θ) if there is an element $\psi \in \theta$ such that ψf is a polynomial map.

Note that this notion of stability conforms with the classical notion of stability used in linear control. Indeed, let K be the field of real numbers. We can take the stability set θ as the set of all polynomials having all their roots within the unit disc in the complex plane. Then, Definition (2.4) becomes identical with the classical definition of stability for discrete-time systems. Alternatively, we can take θ to be the set of all polynomials having all their roots in the open left half of the complex plane. Then, (2.4) reduces to the classical definition of stability for continuous time systems. Thus, the present definition of stability generalizes the classical ones.

In order to permit the development of a complete algebraic theory of internally stable control, it is necessary to refine somewhat the notion of a stability set in the following way (Hammer [1983a]). A *strict stability set* θ is a stability set for which there exists a polynomial of degree one *not* in θ. Thus, for a strict stability set θ, there is a polynomial of degree one in θ, and there is (another) polynomial of degree one which is not in θ. The examples provided above are, in fact, all strict stability sets. Throughout the present note, all stability sets are assumed to be strict stability sets.

Let $\Omega_\theta K$ denote the set of all rational elements $\alpha \in \Lambda K$ which can be expressed as a fraction of the form $\alpha = \beta/\gamma$, where β and γ are polynomials and $\gamma \in \theta$. The set $\Omega_\theta K$ simply describes the set of all stable scalar transfer functions (in the sense of θ). The following is then a direct consequence of the theory of localized rings (e.g. Zariski and Samuel [1958]).

(2.5) Proposition. *$\Omega_\theta K$ is a principal ideal domain.*

The space ΛS is, of course, an $\Omega_\theta K$-module as well. Let $s_1, s_2, \ldots, s_n$ be a basis of the K-linear space S, and let $\Omega_\theta S$ be the $\Omega_\theta K$-submodule of ΛS generated by this basis, namely,

$$\Omega_\theta S = \left\{ s = \sum_{i=1}^{m} \alpha_i s_i, \alpha_1, \ldots, \alpha_m \in \Omega_\theta K \right\}$$

Then, it is easy to see that $\Omega_\theta S$ is the same for any basis $s_1, s_2, \ldots, s_m$ of S, and its rank as an $\Omega_\theta K$-module is equal to the dimension of S as a K-linear space. We denote by

$$j_\theta : \Omega_\theta S \to \Lambda S \tag{2.6}$$

the identity injection which maps each element of $\Omega_\theta S$ into the same element in ΛS. By

$$\pi_\theta : \Lambda S \to \Lambda S / \Omega_\theta S \tag{2.7}$$

we denote the canonical projection which maps each element $s \in \Lambda S$ into the equivalence class $s + \Omega_\theta S$ in $\Lambda S/\Omega_\theta S$. It follows then that *a ΛK-linear map $f:\Lambda U \to \Lambda Y$ is input/output stable if and only if $f[\Omega_\theta U] \subset \Omega_\theta Y$.*

The final algebraic structure that we need to review deals with the combination of the notions of causality and stability. Let $\Omega_\theta^- K := \Omega_\theta K \cap \Omega^- K$, which consists of all the stable and causal elements in ΛK. Then, the following is true (Morse [1975])

(2.8) Proposition. *$\Omega_\theta^- K$ is a principal ideal domain.*

Using this notation, *a ΛK-linear map $f:\Lambda U \to \Lambda Y$ is causal and input/output stable if and only if $f[\Omega_\theta^- U] \subset \Omega_\theta^- Y$.* It is convenient to employ the following terminology. A ΛK-linear map $M:\Lambda U \to \Lambda U$ is ΩK- (respectively, $\Omega^- K$-, $\Omega_\theta K$-, $\Omega_\theta^- K$-) *unimodular* if it has an inverse M^{-1}, and if both M and M^{-1} are polynomial (respectively, causal, stable, causal and stable) maps. Clearly, the ΩK-unimodular maps are the usual polynomial unimodular maps, and the $\Omega^- K$-unimodular maps are the bicausal maps. Every $\Omega_\theta^- K$-unimodular map must also be bicausal.

3 Realization, Strict Observability, and Stability Modules

The basic structure of Kalman's algebraic realization theory (Kalman [1965]) can be briefly described as follows. First, in order to provide an intuitive background, note that every element $u = \sum_{t=t_0}^{\infty} u_t z^{-t}$ with $t_0 \leqq 0$ in the space ΛU can be decomposed into two (non-disjoint) parts: the *past* part $u_p := \sum_{t=t_0}^{0} u_t z^{-t}$, which is the polynomial part of u; and the *future* part $u_F := \sum_{t=0}^{\infty} u_t z^{-t}$, which can be identified with the projection πu. The 'present' component u_0 is contained in both parts. Now, with each ΛK-linear map $f:\Lambda U \to \Lambda Y$, one associates a *restricted* map $\tilde{f}$ given by

$$\tilde{f} := \pi f j : \Omega U \to \Gamma Y \tag{3.1}$$

In intuitive terms, the map $\tilde{f}$ associates with each input sequence terminating at the present, the future part of the output sequence generated by it. Of particular importance is the set of all past input sequences that generate zero future outputs, namley the set

$$\Delta_K := \ker \tilde{f}, \tag{3.2}$$

which, by definition, is a subset of ΩU. In fact, since $\tilde{f}$ is clearly an ΩK-homomorphism, the set Δ_K is an ΩK-module, and it is usually referred to as the *Kalman realization module.* In order to point out the significance of the module Δ_K, we review the basic definition of an abstract realization, as conceived in Kalman [1965]. Let $f:\Lambda U \to \Lambda Y$ be a linear input/output map. An *abstract realization* of f is a triple (X, g, h), where X is an ΩK-module and $g:\Omega U \to X$

and $h: X \to \Gamma Y$ are ΩK-homomorphisms, such that the following diagram is commutative.

$$\begin{tikzcd} \Omega U \arrow[rr, "\tilde{f}"] \arrow[dr, "g"'] & & \Gamma Y \\ & X \arrow[ur, "h"'] & \end{tikzcd}$$

This commutative diagram gives rise to a representation of the system represented by the input/output map f in the standard form

$$x_{k+1} = Fx_k + Gu_k,$$
$$y_k = Hx_k,$$

in a way that we do not detail here (see Kalman, Falb, and Arbib [1969] or Hautus and Heymann [1978]). The realization (X, g, h) is called *reachable* whenever g is surjective; *observable* whenever h is injective; and *canonical* whenever it is both reachable and observable. The pair (X, g) is called a *semirealization* of f. It can be readily seen from the diagram that every realization (X, g, h) of f must satisfy $\ker g \subset \ker \tilde{f}$; and a reachable realization (X, g, h) is canonical if and only if $\ker g = \ker \tilde{f}$ $(= \Delta_K)$. A canonical semirealization can be simply constructed by taking g as the projection $\Omega U \to \Omega U/\Delta_K$, and the canonical state-space is simply given by the quotient module $X = \Omega U/\Delta_K$. These facts indicate that the module Δ_K is the basic quantity in realization theory for linear time-invariant systems.

In stronger terms, the polynomial module Δ_K exactly contains the critical information that is necessary in order to construct a dynamical mathematical model of a given linear time-invariant system from its input/output behavior f. However, as we shall show in the sequel, this information does not directly provide the designer with a clear indication of the design options at his disposition, when trying to control the given system using an internally stable control configuration. The information provided by Δ_K, although complete, contains much too many details whithin which the critical information characterizing the design options is buried. Nevertheless, the module theoretic stability framework which we have briefly reviewed earlier will extract the sought after information from the deep underlying algebraic structure of the input/output map f. It will provide an accurate characterization of all design options in the most obvious and condensed form, in terms of a set of m integers, where m is the dimension of the input space U.

Perhaps, the most fundamental quantity in the analysis of the invariant structure of linear time-invariant systems is the *strict observability* module associated with an input/output map $f: \Lambda U \to \Lambda Y$, and given by $\ker \pi f$ (Hammer and Heymann [1983]). The strict observability module $\ker \pi f$ consists of all input sequences (not necessarily restricted to the past) which produce zero future outputs from the system described by the input/output map f. Since f and π are both ΩK-homomorphisms, it is an ΩK-module, namely, a module over the polynomials. In contrast to the realization module Δ_K, which contains only polynomial vectors as elements, the module $\ker \pi f$ may also contain non-polynomial elements. As a direct consequence of the definition, we have that

$$\Delta_K = \ker \pi f \cap \Omega U,$$

so that $\Delta_K \subset \ker \pi f$. In view of the well known fact that the realization module Δ_K is of full rank whenever f is a rational function, the last containment implies that the module $\ker \pi f$ contains a basis of the ΛK-linear space ΛU.

The basic algebraic quantity of our present discussion is the *stability module* Δ^θ of Hammer [1983a], given by

$$\Delta^\theta := \ker \pi f \cap \Omega_\theta U \tag{3.3}$$

It consists of all *stable* input sequences that produce zero future outputs for the system described by the input/output map f. Since $\ker \pi f$ and $\Omega_\theta U$ are both ΩK-modules (the latter being implied by the fact that $\Omega K \subset \Omega_\theta K$), it follows that Δ^θ is an ΩK-module. The module Δ^θ will allow us to derive a complete set of invariants that characterize the set of all dynamical properties that can be assigned to the system described by the input/output map f by internally stable control. These invariants are derived through a standard procedure for the extraction of integer invariants from ΩK-modules, which is described in the next section.

Another module which is critical to our discussion is the *pole module* Δ_θ, also introduced in Hammer [1983a], and given by

$$\Delta_\theta := \ker \pi_\theta f \cap \Omega U, \tag{3.4}$$

where π_θ is defined in (2.7). This module consists of all polynomial (i.e. past) input sequences which produce stable output sequences from the system described by $f: \Lambda U \to \Lambda Y$. It is an ΩK-module, and is obviously contained in ΩU.

4 Module Indices and Stability Indices

The derivation of structural invariants for linear time-invariant systems seems to be intimately linked to the notion of *proper bases* (or 'minimal bases'), which have found various applications in algebra (e.g. Wedderburn [1934]), and whose significance to the theory of linear control systems has been pointed out by numerous authors (Rosenbrock [1970], Wolovich [1974], Forney [1975], Hautus and Heymann [1978], Hammer and Heymann [1981]). We start this section with a brief review of the notion of proper bases.

Let $s = \sum_{t=t_0}^{\infty} s_t z^{-t}$ be an element of the ΛK-linear space ΛS. The *order* of s is defined by $\operatorname{ord} s := \min_t \{s_t \neq 0\}$ if $s \neq 0$ and $\operatorname{ord} s := \infty$ if $s = 0$; sometimes, it is more common to use the notion of *degree*, given by $\deg s := -\operatorname{ord} s$. The *leading coefficient* $\hat{s}$ of s is an element of the k-linear space S given by $\hat{s} := s_{\operatorname{ord} s}$ if $s \neq 0$ and $\hat{s} := 0$ if $s = 0$. A set of elements $u_1, u_2, \ldots, u_n$ in the space ΛS is said to be *properly independent* if the leading coefficients $\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_n$ are K-linearly independent. A *proper basis* of the ΛK-linear space ΛS is a basis of ΛS that consists of properly independent elements. An *ordered* proper basis of an ΩK-module $\Delta \subset \Lambda S$ is a basis $d_1, \ldots, d_n$ of Δ that consists of properly independent elements, and $\deg d_i \leqq \deg d_{i+1}$ for all $i = 1, \ldots, n-1$. As it turns out, the degrees of the elemens of ordered proper bases are of deep significance in linear time-invariant system theory. The origin of this fact stems from the notion of causality, but we will not explore this connection here in great detail (see Hammer and Heymann [1981, 1983] and Hammer [1983a]).

It is quite interesting that the degrees of the elements of an ordered proper basis of an ΩK-submodule Δ of the ΛK-linear space ΛS are uniquely determined by Δ, and can be derived without the explicit construction of any proper bases. This fact has probably first been noticed (somewhat implicitly) in Rosenbrock [1970], but the specific procedure used here to derive these degrees was developed in Hautus and Heymann [1978], Hammer and Heymann [1981, 1983] and Hammer [1983a]. To describe the procedure, let $\Delta \subset \Lambda S$ be an ΩK-module. For every integer k, let S_k be the K-linear subspace of S spanned by the leading coefficients of all elements $s \in \Delta$ satisfying $\operatorname{ord} s \geqq k$. Since Δ is an ΩK-module (thus permitting shifts to the left in the discrete-time interpretation), it follows that the sequence of subspaces $\{S_k\}$ creates a chain $\cdots \supset S_{-1} \supset S_0 \supset S_1 \supset \cdots$, which is called the *order chain* of Δ. The sequence of the dimensions of the elements of this chain, namely, the sequence of integers $\eta_k := \dim_K S_k, k = \ldots, -1, 0, 1, \ldots$, is called the *order list* of Δ. An ΩK-module $\Delta \subset \Lambda S$ is said to be *rational* if the intersection $\Delta \cap \Omega S$ is of rank equal to the dimension of the K-linear space S. Also, the ΩK-module Δ is said to be *bounded* if there is an integer α such that $\operatorname{ord} s \leqq \alpha$ for all nonzero elements $s \in \Delta$.

Now, let Δ be a rational ΩK-submodule of the ΛK-linear space ΛS, let $\{\eta_k\}$ be the order list of Δ, and let n be the dimension of the K-linear space S. The *degree indices* $\mu_1 \leqq \mu_2 \leqq \cdots \leqq \mu_n$ of Δ are then defined as follows. For every integer j satisfying $\eta_i \leqq j < \eta_{i-1}$, the degree index $\mu_j := -i$; if $\lim_{k \to \infty} \eta_k \neq 0$, set $\mu_j := 0$ for $j = 1, \ldots, \lim_{k \to \infty} \eta_k$. It can then be shown that, for a rational and bounded ΩK-module Δ, the degree indices describe the degrees of every ordered proper basis of Δ, as follows (Hammer and Heymann [1983]).

(4.1) Theorem. *Let Δ be a rational and bounded ΩK-submodule of the ΛK-linear space ΛS, and let $\mu_1 \leqq \mu_2 \leqq \cdots \leqq \mu_n$ be its degree indices. Then, Δ is of* rank $n = \dim_K S$, *and*

(*i*) *Δ has an ordered proper basis;*
(*ii*) *Every ordered proper basis $d_1, \ldots, d_n$ of Δ satisfies* $\deg d_i = \mu_i, i = 1, \ldots, n$.

In order to provide an indication of the profound significance of the degree indices in the context of the theory of linear time-invariant systems, consider the following fundamental result, which is due to Rosenbrock [1970], Wolovich

[1974], Forney [1975], and Hautus and Heymann [1978] (the present version of the result is taken from the last reference).

(4.2) Theorem. *Let $f: \Lambda U \to \Lambda Y$ be a linear input/output map, and let Δ_K be the Kalman realization module of f. Then, the degree indices of Δ_K are the reachability indices of a canonical realization of the system represented by f.*

As well known, the reachability indices play an important role in the theory of linear control, as evidenced by the Rosenbrock Theorem (Rosenbrock [1970]). They are also referred to as the 'Kronecker invariants' of the system (Kalman [1971]).

Of fundamental importance to the theory of internally stable linear control are the degree indices of the stability module Δ^θ which were introduced and studied in Hammer [1983a, b, and c], and which form the main motto of this note. We review from these references the following basic definition. (Recall that a linear input/output map is simply a strictly causal and rational ΛK-linear map.)

(4.3) Definition. Let $f: \Lambda U \to \Lambda Y$ be a linear input/output map, let $\Delta^\theta = \ker \pi f \cap \Omega_\theta U$ be its stability module, and let $m = \dim_K U$. The *stability indices* $\sigma_1 \leqq \sigma_2 \leqq \cdots \leqq \sigma_m$ of f (in the sense of the stability set θ) are the degree indices of Δ^θ.

As it turns out, the stability indices exactly characterize the set of all possible dynamical behaviors that can be assigned to the system represented by f through internally stable closed loop control. Thus, the m integers $\sigma_1, \ldots, \sigma_m$ represent the *entire* information that a designer needs to know about the system represented by f, when weighing the options available for designing the dynamical behavior of a closed loop control configuration that internally stabilizes the system. In this way, the formalism of the algebraic theory of linear realization provides a mechanism for the extraction of the underlying invariant structure of a linear system in the context of dynamical design and stabilization. It is also quite surprising to note how little data is needed about the system for this purpose—only a set of m nonnegative integers. The verification of these facts is the subject of the next section.

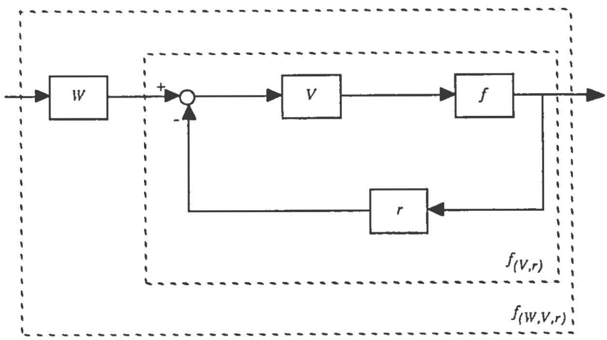
Before concluding the present section, we provide one more definition.

(4.4) Definition. Let $f: \Lambda U \to \Lambda Y$ be a linear input/output map, let $\Delta_\theta = \ker \pi_\theta f \cap \Omega U$ be its pole module, and let $m = \dim_K U$. The *pole indices* $\rho_1 \leqq \rho_2 \leqq \cdots \leqq \rho_m$ of f (in the sense of the stability set θ) are the degree indices of Δ_θ. The *pole degree* ρ of the system represented by f is defined as $\rho(f) := \rho_1 + \rho_2 + \cdots + \rho_m$.

It can be shown the *the pole degree is equal to the number of unstable poles of the system* (see Hammer [1983a] for a detailed discussion of this and other topics mentioned in this section).

5 Invariants, Dynamics Assignment, and Internal Stability

In order to discuss the implications of the stability indices on the theory of internally stable linear control, consider the following classical control configuration.



Here, f is the transfer matrix of the system that needs to be controlled; V is an in-loop dynamic precompensator; r is a dynamic output feedback compensator; and W is an external dynamic precompensator. We denote by $f_{(V,r)}$ the input/output relation (transfer matrix) of the loop alone, and by $f_{(W,V,r)}$ the input/output relation of the entire composite system. To set up the notation, we take $f: \Lambda U \to \Lambda Y$, in which case $V: \Lambda U \to \Lambda U; r: \Lambda Y \to \Lambda U$; and $W: \Lambda U \to \Lambda U$. In order to preserve the degrees of freedom available for the input of the system, we shall require the precompensators V and W to represent invertible systems (i.e. systems with nonsingular transfer matrices). As is common practice, we shall also assume that the given system f is strictly causal. Of course, the compensators W, V, and r are all required to be causal.

When discussing composite systems, the notion of internal stability is of uttermost importance. A composite system is *internally stable* if all its modes, including the unobservable and the unreachable ones, are stable, where stability is in the sense of the stability set θ. Our basic objective here is to characterize all possible dynamical behaviors that can be assigned to the transfer matrix $f_{(W,V,r)}$ of the composite system by appropriately choosing the compensators W, V, and r, under the requirement that the system be internally stable. We shall see that the set of all such possible dynamical behaviors is completely characterized by the stability indices of the given system f (Hammer [1983a, b, and c]).

We discuss next explicit conditions for the internal stability of the closed loop configuration (5.1). First, we list some input/output relations which can be directly derived through simple standard computations. Denoting

$$l_r := [I + rfV]^{-1}, \tag{5.2}$$

it can be seen that

$$f_{(V,r)} = fVl_r, \tag{5.3}$$

and

$$f_{(W,V,r)} = f_{(V,r)}W \tag{5.4}$$

Further, some terminology. The series combination fV is said to be *θ-detectable* if the pole degrees satisfy

$$\rho(fV) = \rho(f) + \rho(V) \tag{5.5}$$

In intuitive terms, fV is θ-detectable if and only if there occur no cancellations of unstable poles and unstable zeros when the transfer matrices of f and V are multiplied (Hammer [1983b]). We can now state a characterization of internal stability for the configuration (5.1) (Hammer [1983b]).

(5.6) Proposition. *The composite system $f_{(W,V,r)}$ is internally stable (in the sense of the stability set θ) if and only if the following conditions hold.*

(i) *fV is θ-detectable;*
(ii) *All of the maps W, $f_{(V,r)}$, l_r, $f_{(V,r)}r$, and $l_r r$ are input/output stable (in the sense of θ).*

Consider now a state representation

$$x_{k+1} = Fx_k + Gu_k,$$
$$y_k = Hx_k,$$

of the input/output relation $f_{(W,V,r)}$. As is well known (Rosenbrock [1970]), every such canonical state representation corresponds to a left coprime polynomial fraction representation $f_{(W,V,r)} = G^{-1}H$ of the transfer matrix of $f_{(W,V,r)}$ (which we denote by the same symbol as the map for simplicity of notation). Here, G and H are left coprime polynomial matrices with G being invertible. Also, the (nontrivial) invariant factors of the polynomial matrix G are the same as the invariant factors of the matrix F (where the invariant factors of the matrix F over the field K are defined as the invariant factors of the polynomial matrix $(zI - F)$). Furthermore, the dynamical properties of the input/output map $f_{(W,V,r)}$ are entirely determined by the invariant factors of the matrix F. In fact, the invariant factors of F, which characterize the invariant structure of F under similarly transformations (e.g. Maclane and Birkhoff [1979]), provide the only significant data about F, since F can be replaced by any matrix similar to it by inducing an isomorphic transformation of the state-space X. Consequently, *the canonical dynamical behavior of the closed loop system (5.1) is entirely described by the invariant factors of the denominator matrix G in a left coprime polynomial fraction representation $f_{(W,V,r)} = G^{-1}H$.*

Thus, in order to determine the entire set of canonical dynamical behaviors that can be assigned to (5.1) by appropriately choosing the compensators, all we need to know is the set of all possible invariant factors that may appear as the invariant factors of the denominator matrix G in a left coprime polynomial fraction representation $f_{(W,V,r)} = G^{-1}H$ of the composite system. This set is entirely characterized by the next result, which is reproduced here from Hammer [1983c]. (We say that a polynomial ϕ is *stable* if $1/\phi \in \Omega_\theta K$. Also, note the reverse ordering of the stability indices here.)

(5.7) Theorem. *Let $f: \Lambda U \to \Lambda Y$ be a linear input/output map with stability indices $\theta_1 \geqq \theta_2 \geqq \cdots \geqq \theta_m$, and let $k := \operatorname{rank}_{\Lambda K} f$. Let $\phi_1, \ldots, \phi_k$ be a set of monic stable polynomials, where ϕ_{i+1} divides ϕ_i for all $i = 1, \ldots, k-1$. Then, the following statements are equivalent.*

(i) $\sum_{i=1}^{j} \deg \phi_i \geqq \sum_{i=1}^{j} \theta_i$ *for all* $j = 1, \ldots, k$.

(ii) *There exist causal dynamic compensators $W: \Lambda U \to \Lambda U$, $V: \Lambda U \to \Lambda U$, and $r: \Lambda Y \to \Lambda U$, where W and V are nonsingular, such that the closed loop system $f_{(W,V,r)}$ is internally stable and has a left coprime polynomial fraction representation $f_{(W,V,r)} = G^{-1}H$, with G having $\phi_1, \ldots, \phi_k$ as its (nontrivial) invariant factors.*

Whence, we have a complete characterization of all the possible dynamical properties that can be assigned to the given system f by dynamic compensation, within an internally stable closed loop control configuration. This characterization is entirely determined by m integers—the stability indices of the given system f. It follows then that the stability indices provide all the information a designer needs to know in order to be able to evaluate all the available options for the assignment of input/output dynamical properties through internally stable control.

A detailed proof of Theorem (5.7), as well as explicit descriptions of the construction of dynamic compensators that achieve desired invariant factors for the closed loop system, are given in Hammer [1983b, c]. These references also contain a variety of other results on dynamics assignment, including dynamics assignment by pure dynamic output feedback and by unity output feedback.

To conclude, we have seen that algebraic realization theory for linear time-invariant systems has matured into more than just an abstract framework for the derivation of dynamical models of systems. It has become a refined tool for the extraction of structural invariants from the input/output behavior of the system, and has elicited the simplicity of the fundamental structure of linear time-invariant control systems.

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(For a more complete list of references refer to Hammer [1983a, b, and c].)

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