Linear Dynamic Output Feedback: Invariants and Stability

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Abstract—**A full invariant under linear dynamic output feedback is derived. Some of its applications to the design of internally stable feedback control systems are considered.**

I. Introduction

Let Σ and Σ' be linear time invariant systems. We shall say that Σ and Σ' are *output feedback equivalent* if there exists a causal (dynamic) linear time invariant system ρ such that $\Sigma' = \Sigma_\rho$, where Σ_ρ is represented by Fig. 1. The system ρ serves then as output feedback. The equality $\Sigma' = \Sigma_\rho$ is to be understood in an input–output sense, namely, that Σ' and Σ_ρ give rise to equal transfer matrices.

Output feedback equivalence is, of course, an equivalence relation. Thus, the set of linear systems can be decomposed in a natural way into disjoint *output feedback equivalence classes*, where two systems belong to the same class if and only if they are output feedback equivalent. Assume now that to each system Σ there is assigned a quantity $F(\Sigma)$ in a specified parameter set. We shall say that (the function) F is a full *output feedback invariant* whenever the following condition is satisfied: Σ and Σ' are output feedback equivalent if and only if $F(\Sigma) = F(\Sigma')$.

In this paper we derive a full output feedback invariant for linear time invariant systems. The invariant is in the form of a polynomial matrix, and can be readily calculated from the system transfer matrix. In the case where Σ and Σ' are output feedback equivalent, we also obtain an explicit formula for a feedback system ρ satisfying $\Sigma' = \Sigma_\rho$. In the last section we derive complete and explicit conditions for the existence of a causal feedback system ρ such that $\Sigma' = \Sigma_\rho$ and Σ_ρ is "internally stable." Thus, using the output feedback invariant and our discussion in the last section, one can conclude whether a given system Σ can be transformed, through a stable output feedback configuration, into a specified system Σ'.

The main stimulation for this paper comes from the recent striking discovery by Kalman [15] that the algebraic theory of linear partial realization is closely related to the Euclidian algorithm. When this result is incorporated into the algebraic theory of linear dynamic output feedback [11], the full output feedback invariant follows in a natural way.

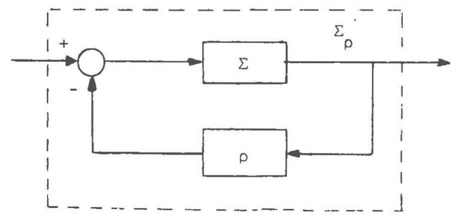

Fig. 1.

Invariants played an important role in the evolution of linear system theory. Thus, the theory of static state feedback (e.g., [2], [4], [14], [24], [29]) experienced a substantial enhancement after the incorporation of the reachability indexes ([5], [15], [19], [22]) and the reachability subspaces ([1], [29], [30]). The problem of invariants under dynamic precompensation was considered in [3], [18], and [28]. Certain invariants under dynamic output feedback and under dynamic compensation were considered in [11].

The paper is organized as follows. In Section II we derive the full output feedback invariant for single-input single-output systems, singling out this case because of its simplicity. The general multivariable case is considered in Section III, and Section IV is devoted to internal stability. We remark that much of our present discussion can be generalized to the case of nonlinear systems, and we shall do so in a future paper.

II. The Single-Input Single-Output Case

For the sake of intuitive insight, we consider first the case of single-input single-output systems. In this case, a full output feedback invariant can be derived using the realization structure in [15]. Let Σ be a nonzero rational single-input single-output system. Let $\bar{f}$ denote the transfer function of Σ and let $\bar{f} = \pi/\chi$ be a polynomial fraction representation of $\bar{f}$. We say that Σ is *causal* whenever $\deg \pi \leqslant \deg \chi$, and *strictly causal* whenever $\deg \pi < \deg \chi$. In the case where $\bar{f}\,(=\pi/\chi)$ is strictly causal, it follows by

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the theory of continued fractions (see [15], [21], and [25]) that it can be represented in the form

$$\bar{f}=1/(\zeta+\bar{h}), \tag{2.1}$$

where ζ is a nonzero polynomial, and $\bar{h}$ is strictly causal. The polynomial ζ is obtained through the Euclidian algorithm according to

$$\chi=\zeta\pi+\rho, \qquad \text{where either } \rho=0 \text{ or } \deg\rho<\deg\pi, \tag{2.2}$$

and $\bar{h} := \rho/\pi$ is strictly causal. The system represented by $1/\zeta$ is called by [15] the first atom of Σ.

Assume now that we apply to $\bar{f}$ a causal output feedback with transfer matrix $\bar{r}$. The resulting system $\bar{f}_{\bar{r}}$ is then given by

$$\bar{f}_{\bar{r}}=1/[\zeta+(\bar{h}+\bar{r})]. \tag{2.3}$$

Now, let $\zeta=a_nz^n+\cdots+a_o$, and denote

$$F(\bar{f}) := \zeta-a_o. \tag{2.4}$$

Then, by the strict causality of $\bar{f}$, $F(\bar{f})\neq 0$, and using the additivity property of output feedback, we obtain the following.

Theorem 2.5: Let $\bar{f}$ and $\bar{f}'$ be nonzero transfer functions of strictly causal and rational single-input single-output systems. Then, there exists a causal output feedback $\bar{r}$ such that $\bar{f}'=\bar{f}_{\bar{r}}$ if and only if $F(\bar{f}')=F(\bar{f})$. If $F(\bar{f}')=F(\bar{f})$, then $\bar{r}=\bar{f}'^{-1}-\bar{f}^{-1}$.

Thus, the polynomial $F(\bar{f})$, which is obtained from the transfer function $\bar{f}$ by means of the Euclidian algorithm, forms a full output feedback invariant in the single-input single-output case.

III. The Output Feedback Invariant

We now generalize our discussion in Section II to the case of multivariable systems. Our derivation will require certain notions from [11], and so we start with a brief review of the notation used there. Let K be a field, and let S be a K-linear space. We denote by ΛS the set of all Laurent series of the form

$$s=\sum_{t=t_0}^{\infty} s_t z^{-t} \tag{3.1}$$

where, for all t, $s_t\in S$. Then, under coefficientwise addition and convolution as scalar multiplication, the set ΛK is endowed with a field structure, and ΛS forms a ΛK-linear space. Moreover, whenever the K-linear space S is finite dimensional, so also is ΛS as a ΛK linear space, and $\dim_{\Lambda K}\Lambda S=\dim_K S$.

Now, let Σ be a linear system, admitting inputs from the K-linear space U and having its outputs in the K-linear space Y. (For intuitive convenience we assume that Σ is a discrete time system.) Then, every element $u=\Sigma u_t z^{-t}$ in ΛU can be regarded as an input sequence to Σ (with t being identified as the time marker). The corresponding output sequence is then an element y in ΛY. Thus, Σ induces a K-linear map $\bar{f}$: $\Lambda U\to\Lambda Y$. In particular, if the map $\bar{f}$ is also ΛK-linear, then we clearly have $z\bar{f}=\bar{f}z$, so that in this case the system Σ is *time invariant* [12], [16], [31]. Below, we shall consider ΛK-linear maps $\bar{f}$: $\Lambda U\to\Lambda Y$, where U and Y are K-linear spaces of finite dimension. We shall denote

$$m := \dim_K U, \quad p := \dim_K Y.$$

A ΛK-linear map $\bar{f}$: $\Lambda U\to\Lambda Y$ can, of course, be represented as a matrix, relative to specified bases $u_1,\cdots,u_m$ of ΛU and $y_1,\cdots,y_p$ of ΛY. When considering matrix representations, we shall *always* assume that the bases were chosen so that $u_1,\cdots,u_m$ are in U and $y_1,\cdots,y_p$ are in Y. In this case, the matrix representation of $\bar{f}$ is called a *transfer matrix*. For the sake of conciseness, we shall make no sharp distinction between a map and its transfer matrix.

Further, let $s=\sum_{t=t_o}^{\infty}s_tz^{-t}\in\Lambda S$ be an element. The *order* of s is defined as $\operatorname{ord} s := \min_t\{s_t\neq 0\}$ if $s\neq 0$ and $\operatorname{ord} s := \infty$ if $s=0$. The *leading coefficient* $\hat{s}$ of s is $\hat{s} := s_{\operatorname{ord} s}$ if $s\neq 0$, and $\hat{s} := 0$ if $s=0$. Sometimes we use the *degree* of s, which is simply $\deg s := -\operatorname{ord} s$. A ΛK-linear map $\bar{f}$: $\Lambda U\to\Lambda Y$ is called *causal* (resp. *strictly causal*) if, for every $u\in\Lambda U$, $\operatorname{ord}\bar{f}u\geqslant\operatorname{ord}u$ (resp. $\operatorname{ord}\bar{f}u>\operatorname{ord}u$). Equivalently, $\bar{f}$ is causal (resp. strictly causal) if and only if all the entries in its transfer matrix have nonnegative (resp. strictly positive) orders. Also, $\bar{f}$ is called *polynomial* if all the entries in its transfer matrix are polynomials, and $\bar{f}$ is called *rational* if there exists a nonzero polynomial ψ such that $\psi\bar{f}$ is polynomial. A ΛK-linear map $\bar{f}$: $\Lambda U\to\Lambda Y$ is called a *linear i/o (input–output) map* if it is both strictly causal and rational. A linear i/o map represents a linear time invariant system which has an internal delay of at least one step (i.e., strictly causal) and a finite dimensional realization (i.e., rational). Finally, a ΛK-linear map $\bar{l}$: $\Lambda U\to\Lambda U$ is called *bicausal* if it is causal and if it has a causal inverse.

Our discussion below heavily depends on the concept of proper bases, which we will review next. A set of elements $s_1,\cdots,s_k\in\Lambda S$ is called *properly independent* if their leading coefficients $\hat{s}_1,\cdots,\hat{s}_k\in S$ are K-linearly independent. A basis consisting of properly independent elements is called a *proper basis*. A proper basis $s_1,\cdots,s_n$ is *ordered* if, for all $i=1,\cdots,n-1$, $\operatorname{ord}s_i\geqslant\operatorname{ord}s_{i+1}$. One can prove that *every properly independent set is ΛK-linearly independent*. The concept of proper bases has found many applications in mathematics as well as in system theory (see [8], [11], [12], [26], and [27]). If Q: $\Lambda U\to\Lambda Y$ is any polynomial matrix, then there exists a polynomial unimodular matrix M: $\Lambda U\to\Lambda U$ such that the nonzero columns of QM form an ordered properly independent set [8], [27]. In the case of ΛK-linear subspaces, the following holds [11].

Theorem 3.2: Every nonzero ΛK-linear subspace $R\subset\Lambda S$ has a proper basis. Moreover, every properly independent subset of R can be extended into a proper basis of R.

The relevance of proper bases to our present discussion is related to the fact that they can be used as a finite test set for causality, as follows [11], [27].

Theorem 3.3: Let $\bar{f}$: $\Lambda U \to \Lambda Y$ be a ΛK-linear map, and let $u_1,\cdots,u_m$ be a proper basis of ΛU. Then $\bar{f}$ is causal if and only if, for every $i=1,\cdots,m$, ord $\bar{f}u_i \geqslant$ ord u_i.

We now turn to the derivation of the output feedback invariant. In Fig. 1 we represent the system Σ by the linear i/o map $\bar{f}$: $\Lambda U \to \Lambda Y$, the feedback ρ by the causal rational ΛK-linear map $\bar{r}$: $\Lambda Y \to \Lambda U$, and the resulting system Σ_ρ by the linear i/o map $\bar{f}_{\bar{r}}$: $\Lambda U \to \Lambda Y$. Then, through a routine calculation, we obtain

$$\bar{f}_{\bar{r}} = \bar{f}\bar{l}_{\bar{r}} \tag{3.4}$$

where the (equivalent) precompensator $\bar{l}_{\bar{r}}$ is given by

$$\bar{l}_{\bar{r}} = [I + \bar{r}\bar{f}]^{-1}: \Lambda U \to \Lambda U. \tag{3.5}$$

By the causality of $\bar{r}$ and the strict causality of $\bar{f}$, it readily follows that $\bar{l}_{\bar{r}}$ is a (rational) *bicausal* ΛK-linear map. In particular, $\bar{l}_{\bar{r}}$ is nonsingular, so that we obtain

$$\operatorname{Im} \bar{f}_{\bar{r}} = \operatorname{Im} \bar{f}. \tag{3.6}$$

Thus, the equality of images is a necessary condition for output feedback equivalence.

Before proceeding with our discussion, we introduce some notations. Let D: $\Lambda U \to \Lambda U$ be a transfer matrix with columns $D_1,\cdots,D_m$, and let $\xi_1,\cdots,\xi_m$ be a set of m integers. We express the columns of D as series $D_i = \Sigma_t u_t^i z^{-t}$ where, for all t, $u_t^i \in U$. Then, we denote by $D|(\xi_1,\cdots,\xi_m)$ the (square) matrix consisting of the columns $[D|(\xi_1,\cdots,\xi_m)]_i := \Sigma_{t<\xi_i} u_t^i z^{-t}$ (i.e., the truncation of the column D_i at ξ_i), $i=1,\cdots,m$.

Further, let $\bar{f}$: $\Lambda U \to \Lambda Y$ be an injective (one-to-one) linear i/o map, and let P: $\Lambda U \to \Lambda Y$ and Q: $\Lambda U \to \Lambda U$, where Q is nonsingular, be any pair of rational matrices satisfying $\bar{f} = PQ^{-1}$. (In particular, P and Q may be polynomial matrices.) By the rationality and injectivity of P, there exists a polynomial unimodular matrix M: $\Lambda U \to \Lambda U$ such that the columns of $N := PM$ are properly independent. We denote by ξ_i the degree of the ith column of N, and let $D_N := QM$, so that $\bar{f} = ND_N^{-1}$. Now, let $\bar{r}$: $\Lambda Y \to \Lambda U$ be a causal ΛK-linear map, and consider the feedback configuration $\bar{f}_{\bar{r}} = \bar{f}[I + \bar{r}\bar{f}]^{-1}$. Letting $D'_N := D_N + \bar{r}N$, we have found that $\bar{f}_{\bar{r}} = ND_N'^{-1}$ where, since the causality of $\bar{r}$ implies ord $\bar{r}N_i \geqslant$ ord $N_i = -\xi_i$, it follows that the matrix D'_N satisfies

$$D'_N|(\xi_1,\cdots,\xi_m) = D_N|(\xi_1,\cdots,\xi_m).$$

Conversely, let $\bar{f}'$: $\Lambda U \to \Lambda Y$ be an injective linear i/o map, and assume that there exists a fraction representation $\bar{f}' = ND_N'^{-1}$, where N is as above, and where D'_N satisfies the condition $D'_N|(\xi_1,\cdots,\xi_m) = D_N|(\xi_1,\cdots,\xi_m)$. We claim that in such a case there exists a causal ΛK-linear map $\bar{r}$: $\Lambda Y \to \Lambda U$ such that $\bar{f}' = \bar{f}_{\bar{r}}$. Indeed, since $N_1,\cdots,N_m$ form a properly independent set, it follows from Theorem 3.2 that there exist vectors $N_{m+1},\cdots,N_p \in \Lambda Y$ such that $N_1,\cdots,N_p$ form a proper basis of ΛY. We define now a ΛK-linear map $\bar{r}$: $\Lambda Y \to \Lambda U$ through its values on this basis as $\bar{r}N_i := (D'_N - D_N)_i$ for $i=1,\cdots,m$, and $\bar{r}N_i = 0$ for $i=m+1,\cdots,p$. Then, since by our assumption, ord$(D'_N - D_N)_i \geqslant -\xi_i =$ ord N_i for all $i=1,\cdots,m$, it follows from Theorem 3.3 that $\bar{r}$ is causal, and, by construction, we also have found that $\bar{f}' = \bar{f}_{\bar{r}}$, as claimed.

Clearly, the matrix N can be replaced by any matrix, the columns of which form a proper basis of the ΛK-linear space Im $\bar{f}$. Also, by the injectivity of $\bar{f}$, the matrices D_N and D'_N are uniquely determined by N. Finally, we note that the existence of a nonsingular matrix D: $\Lambda U \to \Lambda U$ satisfying $\bar{f}' = ND^{-1}$ is evidently equivalent to the condition Im $\bar{f}' =$ Im $\bar{f}$, which, in view of (3.6), is necessary for output feedback equivalence. We summarize our discussion in the following

Proposition 3.7: Let $\bar{f}, \bar{f}'$: $\Lambda U \to \Lambda Y$ be injective linear i/o maps, and assume that Im $\bar{f}' =$ Im $\bar{f}$. Also, let N: $\Lambda U \to \Lambda Y$ be a matrix, the columns of which form a proper basis of Im $\bar{f}$. Denote $\xi_i := \deg N_i$, $i=1,\cdots,m$, and let D_N, D'_N: $\Lambda U \to \Lambda U$ be (the unique) matrices satisfying $\bar{f}D_N = \bar{f}'D'_N = N$. Then, there exists a causal output feedback $\bar{r}$: $\Lambda Y \to \Lambda U$ such that $\bar{f}' = \bar{f}_{\bar{r}}$ if and only if $D'_N|\xi_1,\cdots,\xi_m) = D_N|(\xi_1,\cdots,\xi_m)$.

As a consequence, we see that for an injective linear i/o map $\bar{f}$, the pair of matrices N and $D_N|(\xi_1,\cdots,\xi_m)$ forms a full invariant under linear dynamic output feedback. It is worthwhile to note that the matrix $D_N|(\xi_1,\cdots,\xi_m)$ is nonsingular. Indeed, by our above discussion it follows that there exists a causal ΛK-linear map $\bar{r}_o$: $\Lambda Y \to \Lambda U$ such that $D_N|(\xi_1,\cdots,\xi_m) = [I + \bar{r}_o\bar{f}]D_N$. But then, since, by the strict causality of $\bar{f}$, the map $[I + \bar{r}_o\bar{f}]$ is nonsingular, so also is $D_N|(\xi_1,\cdots,\xi_m)$.

Lemma 3.8: The matrix $D_N|(\xi_1,\cdots,\xi_m)$ of Theorem 3.7 is nonsingular.

Next, we wish to choose N in such a way as to reduce the number of parameters in the pair N, $D_N|(\xi_1,\cdots,\xi_m)$. Basically, we do so by choosing N as a "minimal" polynomial proper basis of the ΛK-linear space Im $\bar{f}$. Intuitively speaking, such a minimal basis of Im $\bar{f}$ will consist of polynomial vectors, and it will actually be a proper basis of the "polynomial part" of Im $\bar{f}$ (compare to [8]). In order to construct such a basis and to see its system theoretic interpretation, we need to review some additional properties of ΛK-linear maps.

First we note that the set ΛS contains, as a subset, the set Ω^+S of all (polynomial) elements of the form $\Sigma_{t=t_o}^{o} s_t z^{-t}$, where $t_o \leqslant 0$ and $s_t \in S$. In particular, the set Ω^+K (sometimes denoted $K[z]$) is the set of polynomials with coefficients in K, and, as is well known, forms a principal ideal domain under the operations defined in ΛK. Also, Ω^+S forms a free Ω^+K-module of rank equal to $\dim_K S$. The ΛK-linear space ΛS is evidently an Ω^+K-module as well, and we can consider the quotient module $\Lambda S/\Omega^+S$. Qualitatively, this module represents all the (strictly) future sequences. We shall repeatedly employ the Ω^+K-module projection

$$\pi^+: \Lambda S \to \Lambda S/\Omega^+S.$$

Finally, we shall say that an Ω^+K-module $\Delta \subset \Lambda S$ is *full* if it contains a basis of the ΛK-linear space ΛS (we borrow this term from Fuhrmann [9]).

Now, let $\bar{f}$: $\Lambda U \to \Lambda Y$ be a ΛK-linear map, and consider the Ω^+K-module $\operatorname{Ker}\pi^+\bar{f}$. Intuitively speaking, this module consists of *all* input sequences that lead to zero future output. It forms an extension of the classical Kalman [15] realization module, which, in present notation, is $\operatorname{Ker}\pi^+\bar{f} \cap \Omega^+U$ (that is, all *past* input sequences that lead to zero future output). As shown in [11] $\operatorname{Ker}\pi^+\bar{f}$ determines many of the structural properties of the system represented by $\bar{f}$. We next show that this module also induces a minimal output feedback invariant for $\bar{f}$. We start with some basic properties. First, by definition, one has

$$\bar{f}\left[\operatorname{Ker}\pi^+\bar{f}\right] = \operatorname{Im}\bar{f} \cap \Omega^+Y \tag{3.9}$$

(i.e., $\operatorname{Ker}\pi^+\bar{f}$ is the set of all inputs that generate polynomial outputs). We shall denote (the set of all polynomial outputs of $\bar{f}$) by

$$o(\bar{f}) := \operatorname{Im}\bar{f} \cap \Omega^+Y$$

and shall call $o(\bar{f})$ the *output module* of $\bar{f}$. Intuitively speaking, $o(\bar{f})$ is the "polynomial part" of $\operatorname{Im}\bar{f}$. Evidently, if $\bar{f}'$: $\Lambda U \to \Lambda Y$ is any ΛK-linear map satisfying $\operatorname{Im}\bar{f}' = \operatorname{Im}\bar{f}$, then $o(\bar{f}') = o(\bar{f})$. Being a polynomial submodule of Ω^+Y, the module $o(\bar{f})$ possesses an ordered proper basis $d_1,\cdots,d_n$ (see [8], [12]). The integers $\xi_i := \deg d_i$, $i = 1,\cdots,n$, are uniquely determined by the Ω^+K-module $o(\bar{f})$ (also [11]), and we shall refer to them as the *output indexes* of $\bar{f}$. We note that the output indexes are always nonnegative.

Now, let $\bar{f}$: $\Lambda U \to \Lambda Y$ be a linear i/o map. It can then be shown (see [11]) that $\operatorname{Ker}\pi^+\bar{f}$ is a full module and, in case $\bar{f}$ is injective, $\operatorname{Ker}\pi^+\bar{f}$ is finitely generated. Assume further that $\bar{f}$ is injective, and let $d_1,\cdots,d_m$ be a basis of $\operatorname{Ker}\pi^+\bar{f}$. Denoting by $D' := [d_1,\cdots,d_m]$ the corresponding matrix, it follows that D' is nonsingular and $\operatorname{Ker}\pi^+\bar{f} = D'[\Omega^+U]$. In view of (3.9), the matrix $N' := \bar{f}D'$: $\Lambda U \to \Lambda Y$ satisfies $N'[\Omega^+U] = \bar{f}D'[\Omega^+U] = \bar{f}[\operatorname{Ker}\pi^+\bar{f}] = o(\bar{f})$, so that the columns of N' generate $o(\bar{f})$. Also, since $\bar{f}$ is injective, the columns of N' are ΛK-linearly independent, and there exists a polynomial unimodular matrix M: $\Lambda U \to \Lambda U$ such that the columns of $N := N'M$ form an ordered properly independent set. Denoting $D_N := D'M$, we still have $D_N[\Omega^+U] = \operatorname{Ker}\pi^+\bar{f}$, $o(\bar{f}) = N[\Omega^+U]$, and $N = \bar{f}D_N$. We note that, by definition, the degrees of the columns of N are the output indexes of $\bar{f}$. We call N a *proper generating matrix* of $o(\bar{f})$. Applying now Proposition 3.7, we directly obtain the following.

Theorem 3.10: Let $\bar{f},\bar{f}'$: $\Lambda U \to \Lambda Y$ be injective linear i/o maps, and assume that $\operatorname{Im}\bar{f} = \operatorname{Im}\bar{f}'$. Let $\xi_1,\cdots,\xi_m$ be the output indexes of $\bar{f}$, let N be a proper generating matrix of $o(\bar{f})$, and let D_N, D_N': $\Lambda U \to \Lambda U$ be (the unique) matrices satisfying $\bar{f}D_N = \bar{f}'D_N' = N$. Then, there exists a causal output feedback $\bar{r}$: $\Lambda Y \to \Lambda U$ such that $\bar{f}' = \bar{f}_{\bar{r}}$ if and only if $D_N'|(\xi_1,\cdots,\xi_m) = D_N|(\xi_1,\cdots,\xi_m)$.

The output feedback invariant induced by the matrices N and $D := D_N|(\xi_1,\cdots,\xi_m)$ described in Theorem 3.10 is minimal in the following sense. First, we note that both of the matrices N and D are polynomial, and that, by Lemma 3.8, D is nonsingular. Thus, we can combine N and D into the ΛK-linear map $\bar{f}_* := ND^{-1}$. By Theorem 3.10 there exists a causal ΛK-linear map $\bar{r}_o$: $\Lambda Y \to \Lambda U$ such that $\bar{f}_* = \bar{f}_{\bar{r}_o} = \bar{f}\bar{l}_{\bar{r}_o}$. Then, we have

$$\begin{aligned}\operatorname{Ker}\pi^+\bar{f}_* &= \operatorname{Ker}\pi^+\bar{f}\bar{l}_{\bar{r}_0} = \bar{l}_{\bar{r}_o}^{-1}\left[\operatorname{Ker}\pi^+\bar{f}\right]\\ &= \bar{l}_{\bar{r}_o}^{-1}D_N[\Omega^+U] = D[\Omega^+U] \subset \Omega^+U\end{aligned}$$

where the last inclusion holds since D is polynomial. Consequently (see [11]) $\bar{f}_*$ is *strictly observable*, and hence is of minimal MacMillan degree in the output feedback equivalence class $\boldsymbol{F}$ of $\bar{f}$. Thus, the pair N, D describes a coprime polynomial matrix fraction representation of a minimal system in $\boldsymbol{F}$. The minimal MacMillan degree for systems in $\boldsymbol{F}$ is then simply $\deg\det D$. Moreover, by the definition of D it also follows that, among all systems of minimal MacMillan degree in $\boldsymbol{F}$, the transfer function $\bar{f}_*$ has the largest possible number of poles at the origin (and this number is greater or equal to $\dim_K U$). Finally, we remark that an outline of the explicit calculation of the matrix D_N is given in [11].

The conditions of Theorem 3.10 take a particularly simple form in the case of a nonsingular linear i/o map $\bar{f}$: $\Lambda U \to \Lambda U$. In such a case, clearly $o(\bar{f}) = \operatorname{Im}\bar{f} \cap \Omega^+U = \Lambda U \cap \Omega^+U = \Omega^+U$, so that we have $\xi_i = 0$ for all $i = 1,\cdots,m$, and we can choose $N = I$ and $D_N = \bar{f}^{-1}$. Thus, we obtain the following.

Corollary 3.11: Let $\bar{f},\bar{f}'$: $\Lambda U \to \Lambda U$ be nonsingular linear i/o maps. Then, there exists a causal ΛK-linear map $\bar{r}$: $\Lambda U \to \Lambda U$ such that $\bar{f}' = \bar{f}_{\bar{r}}$ if and only if $\bar{f}'^{-1}|0,0,\cdots,0) = \bar{f}^{-1}|(0,0,\cdots,0)$.

In the single-input single-output case, Corollary 3.11 coincides with Theorem 2.5.

In the next section we shall calculate a feedback $\bar{r}$ which relates two specific output feedback equivalent systems. In the remainder of the present section, we extend Theorem 3.10 to the case of noninjective systems. We start with some general considerations. Let $\bar{f},\bar{f}'$: $\Lambda U \to \Lambda Y$ be output feedback equivalent linear i/o maps. As we have already noticed, we have then $\operatorname{Im}\bar{f}' = \operatorname{Im}\bar{f}$. It is also true that $\operatorname{Ker}\bar{f}' = \operatorname{Ker}\bar{f}$. Indeed, $\bar{f}' = \bar{f}[I + \bar{r}\bar{f}]^{-1}$, so that $\operatorname{Ker}\bar{f}' = (I + \bar{r}\bar{f})[\operatorname{Ker}\bar{f}] = \operatorname{Ker}\bar{f}$. Thus, both of the conditions $\operatorname{Im}\bar{f}' = \operatorname{Im}\bar{f}$ and $\operatorname{Ker}\bar{f}' = \operatorname{Ker}\bar{f}$ are necessary for output feedback equivalence. We summarize below (without proof) some elementary properties of maps satisfying the previous conditions.

Lemma 3.12: Let $\bar{f},\bar{f}'$: $\Lambda U \to \Lambda Y$ be linear i/o maps, and assume that both $\operatorname{Im}\bar{f} = \operatorname{Im}\bar{f}'$ and $\operatorname{Ker}\bar{f} = \operatorname{Ker}\bar{f}'$. Then the following are true. 1) There exists a ΛK-linear isomorphism $\bar{l}$: $\Lambda U \to \Lambda U$ such that $\bar{f}' = \bar{f}\bar{l}$. 2) Let $\bar{l}_1$: $\Lambda U \to \Lambda U$ be a ΛK-linear isomorphism. Then, $\bar{f}' = \bar{f}\bar{l}_1$ if and only if $(\bar{l}_1^{-1} - \bar{l}^{-1})[\Lambda U] \subset \operatorname{Ker}\bar{f}$.

As a consequence, we directly obtain the following.

Lemma 3.13: Let $\bar{f}$: $\Lambda U \to \Lambda Y$ be a linear i/o map, and let $\bar{r}, \bar{r}'$: $\Lambda Y \to \Lambda U$ be causal ΛK-linear maps. Then, $\bar{f}_{\bar{r}} = \bar{f}_{\bar{r}'}$ if and only if $(\bar{r} - \bar{r}')\bar{f}[\Lambda U] \subset Ker \bar{f}$.

Next, we construct a suitable canonical representation of the Ω^+K-module $\operatorname{Ker}\pi^+\bar{f}$. Let $\bar{f}$: $\Lambda U \to \Lambda Y$ be a linear i/o map, and let U_o be the set of leading coefficients of all elements in $\operatorname{Ker}\bar{f}$. It is readily seen that U_o is a K-linear subspace of U. Now, let $u_1,\cdots,u_m$ be a basis of the K-linear space U, denote $q:=\dim_{\Lambda K}\operatorname{Ker}\bar{f}(=\dim_K U_o)$, and assume $q \geqslant 1$. We say that the basis $u_1,\cdots,u_m$ is *$\bar{f}$-matched* if $u_1,\cdots,u_q$ form a basis of U_o. In case $\operatorname{Ker}\bar{f}=0$, we adopt the convention that every basis of U is $\bar{f}$-matched. Below, *we represent all matrices relative to an $\bar{f}$-matched basis $u_1,\cdots,u_m \in U$ of ΛU, and an arbitrary basis $y_1,\cdots,y_m \in Y$ of ΛY.* We say that a matrix N: $\Lambda K^n \to \Lambda Y$ is a *proper generating matrix* of $o(\bar{f})$ if N has ordered properly independent columns and $o(\bar{f}) = N[\Omega^+K^n]$.

Lemma 3.14: Let $\bar{f}$: $\Lambda U \to \Lambda Y$ be a rational ΛK-linear map, denote $n:=\dim_{\Lambda K} Im\,\bar{f}$ and $q:=\dim_{\Lambda K} Ker \bar{f}$, and assume that $n>0$. Let N: $\Lambda K^n \to \Lambda Y$ be a proper generating matrix of $o(\bar{f})$. Then, there exists a nonsingular matrix D_N: $\Lambda K^n \to \Lambda K^n$ satisfying the following. 1) $N=\bar{f}\begin{pmatrix}0\\ D_N\end{pmatrix}$, where 0 is the $q \times n$ zero matrix. 2) $Ker\pi^+\bar{f} = \begin{pmatrix}0\\ D_N\end{pmatrix}\Omega^+K^n + Ker\bar{f}$. 3) D_N is uniquely determined by 1).

Proof: By definition, the columns $N_1,\cdots,N_n$ of N are ΛK-linearly independent, and there exist (ΛK-linearly independent) elements $d_1',\cdots,d_n' \in \operatorname{Ker}\pi^+\bar{f}$ such that $N_i = \bar{f}d_i'$, $i=1,\cdots,n$. Next, let U_o be the leading space of $\operatorname{Ker}\bar{f}$, and let $u_1,\cdots,u_m$ be an $\bar{f}$-matched basis of U. We define the projection $\bar{p}$: $\Lambda U \to \Lambda U_o$ through its values as follows: $\bar{p}u_i = u_i$ for $i=1,\cdots,q$, and $\bar{p}u_i = 0$ for $i=q+1,\cdots,m$. Then, $\bar{p}[\operatorname{Ker}\bar{f}] = \Lambda U_o$ and hence, for every $i=1,\cdots,n$, there exists an element $k_i \in \operatorname{Ker}\bar{f}$ satisfying $\bar{p}k_i = \bar{p}d_i'$. Denoting $d_i:=d_i'-k_i$, we still have $N_i = \bar{f}d_i$, $i=1,\cdots,n$, and, moreover, the matrix $[d_1,\cdots,d_n]$ is of the form $\begin{pmatrix}0\\ D_N\end{pmatrix}$, where D_N: $\Lambda K^n \to \Lambda K^n$ is nonsingular. This proves 1).

Further, 2) follows by the definition of N, and we turn to 3). Assume that $N=\bar{f}\begin{pmatrix}0\\ D\end{pmatrix}$, where D: $\Lambda K^n \to \Lambda K^n$, and, for $i=1,\cdots,n$, let d_i'' denote the ith column of $\begin{pmatrix}0\\ D\end{pmatrix}$. Then, for all $i=1,\cdots,n$ and d_i as defined above, we have $d_i - d_i'' \in \operatorname{Ker}\bar{f}$ and $\bar{p}(d_i - d_i'')=0$. But then, by definition of $\bar{p}$, it follows that $d_i - d_i'' = 0$ for all $i=1,\cdots,n$, concluding our proof. □

The matrix D_N: $\Lambda K^n \to \Lambda K^n$ described in Lemma 4.9 will be called the *reduced generating matrix of* $\operatorname{Ker}\pi^+\bar{f}$ (corresponding to N).

Finally, using Lemmas 3.13 and 3.14, and a method similar to the one employed in proof of Theorem 3.10, one can also prove the following.

Theorem 3.15: Let $\bar{f}, \bar{f}'$: $\Lambda U \to \Lambda Y$ be linear i/o maps, and assume that both $Im\,\bar{f}' = Im\,\bar{f}$ and $Ker\bar{f}' = Ker\bar{f}$. Let $\xi_1,\cdots,\xi_n$ be the output indexes of $\bar{f}$, let N: $\Lambda K^n \to \Lambda Y$ be a proper generating matrix of $o(\bar{f})$, and let D_N, D_N': $\Lambda K^n \to \Lambda K^n$ be the corresponding reduced generating matrices of $Ker\pi^+\bar{f}$ and $Ker\pi^+\bar{f}'$, respectively. Then, there exists a causal output feedback $\bar{r}$: $\Lambda Y \to \Lambda U$ such that $\bar{f}' = \bar{f}_{\bar{r}}$ if and only if $D_N'|(\xi_1,\cdots,\xi_n) = D_N|(\xi_1,\cdots,\xi_n)$.

IV. Internal Stability

Let $\bar{f}, \bar{f}'$: $\Lambda U \to \Lambda Y$ be linear i/o maps. In the present section we derive explicit necessary and sufficient conditions for the existence of a causal ΛK-linear map $\bar{r}$: $\Lambda Y \to \Lambda U$ such that $\bar{f}' = \bar{f}_{\bar{r}}$ and $\bar{f}_{\bar{r}}$ is "internally stable." We shall sometimes refer to this problem as model matching by dynamic output feedback. We start with a brief survey of some definitions and notation from [10]. Let $\sigma \subset \Omega^+K$ be a multiplicative set (i.e., for every pair of elements $k_1, k_2 \in \sigma$ also $k_1k_2 \in \sigma$). We say that σ is a *stability set* if both 1) $0 \notin \sigma$, and 2) there exists an element $\alpha \in K$ such that $(z+\alpha) \in \sigma$ (see also, [18]). Further, let $\bar{f}$: $\Lambda U \to \Lambda Y$ be a ΛK-linear map, and let $\sigma \subset \Omega^+K$ be a stability set. Then, $\bar{f}$ is called *i/o (input/output) stable* (in the sense of σ) if there exists an element $\Psi \in \sigma$ such that $\Psi\bar{f}$ is a polynomial map. When the field K is the field of real numbers, this definition includes, of course, the classical notion of stability in linear control theory, where all poles are required to lie within a certain region of the complex plane (which intersects the real line). We now fix the stability set $\sigma \subset \Omega^+K$, and all our considerations below are in the sense of σ.

The definition of i/o stability leads to the introduction of a class of subrings of ΛK as follows. Let Ω_σ^+K be the set of all elements $\alpha \in \Lambda K$ which can be expressed as a polynomial fraction $\alpha = \beta/\gamma$, where $\beta \in \Omega^+K$ and $\gamma \in \sigma$ (that is, all i/o stable elements in ΛK). Then, it can be shown (e.g., [10], [13]) that Ω_σ^+K forms a *principal ideal domain* under the operations defined in ΛK. Clearly, a ΛK-linear map $\bar{f}$: $\Lambda U \to \Lambda Y$ is i/o stable if and only if all entries in its transfer matrix belong to Ω_σ^+K.

When the notion of i/o stability is combined with the notion of causality, it leads to a consideration of the following additional class of rings. First, we denote by Ω^-K the set of all causal elements in ΛK, that is, the set of all elements of the form $\Sigma_{t \geqslant 0}s_tz^{-t}$, where $s_t \in K$. Then, Ω^-K is the set of a power series in z^{-1}, and, as is well known, forms a principal ideal domain under the operations defined in ΛK. Further, we define $\Omega_\sigma^-K := \Omega_\sigma^+K \cap \Omega^-K$, that is, the set of all elements in ΛK which is both i/o stable and causal. Then, again, it can be shown [18] that Ω_σ^-K is endowed with a *principal ideal domain* structure under the operations defined in ΛK. As before, a ΛK-linear map $\bar{f}$: $\Lambda U \to \Lambda Y$ is both causal and i/o stable if and only if all entries in its transfer matrix belong to Ω_σ^-K.

Finally, a ΛK-linear map $\bar{l}$: $\Lambda U \to \Lambda U$ is called Ω^+K- (resp. Ω^-K-, Ω_σ^+K-, Ω_σ^-K-) *unimodular* if $\bar{l}$ has an inverse $\bar{l}^{-1}$ and if both $\bar{l}$ and $\bar{l}^{-1}$ are polynomial (resp. causal, i/o stable, both causal and i/o stable). Thus, an

$\Omega^+ K$-unimodular map is the (usual) polynomial unimodular map, and the $\Omega^- K$-unimodular map is the bicausal map. Also, every $\Omega_\sigma^- K$-unimodular map is necessarilly bicausal as well.

We observe that the employment of the ring theoretic framework for the study of linear system stability allows the advantage of utilizing the theory of matrices with entries in a principal ideal domain, a theory which has accumulated a remarkable wealth of results.

We now turn to internal stability. Informally, we say that a system is internally stable if all its modes, including the uncontrollable and the unobservable ones, are stable. Internal stability is, of course, stronger than i/o stability, and is of fundamental significance whenever composite systems are considered. We are interested in the case of feedback systems, and next we give conditions for internal stability in terms of i/o stability for this case.

Let $\bar{f}$: $\Lambda U \to \Lambda Y$ be a linear i/o map, and let $\bar{r}$: $\Lambda Y \to \Lambda U$ be a causal rational ΛK-linear map. We assume that both of $\bar{f}$ and $\bar{r}$ are completely described by their canonical realization. Let $\bar{f} = ND^{-1}$ (resp. $\bar{r} = Q^{-1}R$) be a right (resp. left) coprime polynomial matrix fraction representation. (For simplicity of notation, we use the same symbol for maps and their transfer matrices.) We then have $\bar{f}_{\bar{r}} = N[QD + RN]^{-1}Q$. The system represented by $\bar{f}_{\bar{r}}$ is said to be *internally stable* (in the sense of σ) if the map $[QD + RN]^{-1}$ is i/o stable. Using the fact that N, D and Q, R are pairs of coprime matrices, the following can be verified through an explicit computation.

Proposition 4.1: Let $\bar{f}$: $\Lambda U \to \Lambda Y$ be a linear i/o map, and let $\bar{r}$: $\Lambda Y \to \Lambda U$ be a causal rational ΛK-linear map. Denote $\bar{l}_{\bar{r}} := [I + \bar{r}\bar{f}]^{-1}$. Then, $\bar{f}_{\bar{r}}$ is internally stable if and only if all of the maps $\bar{f}_{\bar{r}}$, $\bar{l}_{\bar{r}}$, $\bar{f}_{\bar{r}}\bar{r}$, and $\bar{l}_{\bar{r}}\bar{r}$ and i/o stable.

Various alternative sets of conditions for internal stability appear in the linear control literature. One such alternative set was used in [6] and [7]. Our present set of conditions has the advantage of bringing into focus the role of the equivalent precompensator $\bar{l}_{\bar{r}}$, which is a quantity of physical significance, and which can be readily computed when given the systems $\bar{f}$ and $\bar{f}'$ in the model matching problem. The implications of the requirement that $\bar{l}_{\bar{r}}$ be i/o stable were studied in detail in [10]. We note that, in the case where the system represented by $\bar{f}$ does not contain hidden (i.e., unreachable or unobservable) unstable modes, there always exists an output feedback $\bar{r}$ such that $\bar{f}_{\bar{r}}$ is internally stable (see [3], [7], [23]). Reference [7] also contains a parametrization of all the feedbacks $\bar{r}$ for which $\bar{f}_{\bar{r}}$ is internally stable.

We return now to our main question, namely, given linear i/o maps $\bar{f}, \bar{f}'$: $\Lambda U \to \Lambda Y$, does there exist a causal feedback $\bar{r}$: $\Lambda Y \to \Lambda U$ such that $\bar{f}' = \bar{f}_{\bar{r}}$ *and* $\bar{f}_{\bar{r}}$ is internally stable. We note that, when the system $\bar{r}$ exists, it is, in general, not uniquely determined by the specified systems $\bar{f}$ and $\bar{f}'$ (see Lemma 3.13). Our next objective is to find simple characterizing conditions for internal stability, expressible in terms of $\bar{f}$ and $\bar{f}'$ only. In the case where $\bar{r}$ exists, we shall also find an explicit expression for it in terms of (the transfer matrices of) $\bar{f}$ and $\bar{f}'$. We first need an instrumental result (below, $m := \dim_K U$ and $p := \dim_K Y$).

Let $\bar{f}$: $\Lambda U \to \Lambda Y$ be a rational ΛK-linear map, and let $n := \dim_{\Lambda K} \operatorname{Im} \bar{f}$. It can be readily seen that there exists a nonzero element $\psi \in \Omega_\sigma^- K$ such that the transfer matrix of $\psi\bar{f}$ has all its entries in the principal ideal domain $\Omega_\sigma^- K$. Then, using classical results in the theory of matrices with entries in a principal ideal domain (e.g. [17]), it follows that there exists an $\Omega_\sigma^- K$-unimodular map $\bar{l}$: $\Lambda Y \to \Lambda Y$ such that $\bar{l}(\psi\bar{f}) = \begin{pmatrix} \bar{f}_o' \\ 0 \end{pmatrix}$, where $\bar{f}_o'$: $\Lambda U \to \Lambda K^n$ is surjective, and 0 denotes the $(p-n)\times m$ zero matrix. Dividing out ψ, we obtain the following.

Lemma 4.2: Let $\bar{f}$: $\Lambda U \to \Lambda Y$ be a rational ΛK-linear map, and denote $n := \dim_{\Lambda K} \operatorname{Im} f$. Then, there exists an $\Omega_\sigma^- K$-unimodular map $\bar{l}$: $\Lambda Y \to \Lambda Y$ such that $\bar{l}\bar{f} = \begin{pmatrix} \bar{f}_o \\ 0 \end{pmatrix}$, where $\bar{f}_0$: $\Lambda U \to \Lambda K^n$ is surjective, and 0 is the $(p-n)\times m$ zero matrix.

Let $\bar{f}, \bar{f}'$: $\Lambda U \to \Lambda Y$ be injective linear i/o maps, and assume that $\bar{f}$ and $\bar{f}'$ are output feedback equivalent. By Lemma 4.2 there exists an $\Omega_\sigma^- K$-unimodular map $\bar{l}$: $\Lambda Y \to \Lambda Y$ such that $\bar{l}\bar{f} = \begin{pmatrix} \bar{f}_o \\ 0 \end{pmatrix}$ where, by injectivity, $\bar{f}_o$: $\Lambda U \to \Lambda U$ is an isomorphism (we identified $\operatorname{Im} \bar{f}$ with ΛU). Also, since output feedback equivalence implies that $\operatorname{Im} \bar{f}' = \operatorname{Im} \bar{f}$, we obtain that $\bar{l}\bar{f}' = \begin{pmatrix} \bar{f}_o' \\ 0 \end{pmatrix}$, where $\bar{f}_o'$: $\Lambda U \to \Lambda U$ is again an isomorphism. Moreover, if $\bar{r}$: $\Lambda Y \to \Lambda U$ is a causal ΛK-linear map satisfying $\bar{f}' = \bar{f}_{\bar{r}}$, then the ΛK-linear map $\bar{r}_o := \bar{r}\bar{l}^{-1}$ is still causal, and we have $\bar{l}\bar{f}[I + \bar{r}_o\bar{l}\bar{f}]^{-1} = \bar{l}\bar{f}_{\bar{r}} = \bar{l}\bar{f}'$, which shows that $\bar{l}\bar{f}$ and $\bar{l}\bar{f}'$ are output feedback equivalent as well. A slight reflection leads then to the following observation (the converse of which is, of course, also true).

Lemma 4.3: Let $\bar{f}, \bar{f}'$: $\Lambda U \to \Lambda Y$ be injective output feedback equivalent linear i/o maps. Let $\bar{l}$: $\Lambda Y \to \Lambda Y$ be an $\Omega_\sigma^- K$-unimodular map such that $\bar{l}\bar{f} = \begin{pmatrix} \bar{f}_o \\ 0 \end{pmatrix}$, where $\bar{f}_o$: $\Lambda U \to \Lambda U$ is an isomorphism. Then, 1) $\bar{l}\bar{f}' = \begin{pmatrix} \bar{f}_o' \\ 0 \end{pmatrix}$, where $\bar{f}_o'$: $\Lambda U \to \Lambda U$ is an isomorphism, and 2) $\bar{f}_o$ and $\bar{f}_o'$ are output feedback equivalent.

Next, still adhering to the same notation, we define the ΛK-linear map

$$\bar{r}(\bar{l}) := \bar{f}_o'^{-1} - \bar{f}_0^{-1} \tag{4.4}$$

which, of course, depends on $\bar{l}$. Then, we have the following explicit representation of the feedbacks relating $\bar{f}$ and $\bar{f}'$.

Proposition 4.5: Let $\bar{f}, \bar{f}'$: $\Lambda U \to \Lambda Y$ be injective output feedback equivalent linear i/o maps, and, in case $p - m > 0$, let $u_1, \cdots, u_{p-m} \in \Omega^- K^m$ be any (causal) elements. Define the augmented matrix $\bar{r} := [\bar{r}(\bar{l}), u_1, \cdots, u_{p-m}]\bar{l}$: $\Lambda Y \to \Lambda U$, where $\bar{r}(\bar{l})$ is given by (4.4). Then, 1) $\bar{r}$ is causal, and 2) $\bar{f}' = \bar{f}_{\bar{r}}$.

Proof: 1) Clearly, since $\bar{l}$ is $\Omega_\sigma^- K$-unimodular, $\bar{r}$ is causal if and only if $\bar{r}(\bar{l})$ is causal, so we show that the latter is true. By Lemma 4.3, $\bar{f}_o$ and $\bar{f}_o'$ are output feedback equivalent, so there exists a causal ΛK-linear map $\bar{r}_o$: $\Lambda U \to \Lambda U$ such that $\bar{f}_o' = \bar{f}_{o\bar{r}_o}$. Also, since $\bar{f}_o$ is an isomorphism, it follows by Lemma 3.13 that $\bar{r}_o$ is uniquely determined by the latter condition. Now, by a direct calculation, we obtain that $\bar{f}_o' = \bar{f}_{o\bar{r}(\bar{l})}$, so that $\bar{r}(\bar{l}) = \bar{r}_o$, and $\bar{r}(\bar{l})$ is causal. 2) By direct computation,

$$\bar{f}_{\bar{r}} = \bar{f}\,[I + \bar{r}\bar{f}\,]^{-1} = \bar{l}^{-1}(\bar{l}\bar{f}\,)[I + \bar{r}\bar{l}^{-1}(\bar{l}\bar{f}\,)]^{-1}$$
$$= \bar{l}^{-1}\begin{pmatrix} \bar{f}_{o\bar{r}(\bar{l})} \\ 0 \end{pmatrix} = \bar{l}^{-1}\begin{pmatrix} \bar{f}_o' \\ 0 \end{pmatrix} = \bar{f}',$$

concluding our proof. □

Essentially, we also proved the following.

Corollary 4.6: Let $\bar{f}, \bar{f}'$: $\Lambda U \to \Lambda Y$ be injective output feedback equivalent linear i/o maps, and let r: $\Lambda Y \to \Lambda U$ be a causal ΛK-linear map satisfying $\bar{f}' = \bar{f}_r$. Then, there exists a set of elements $u_1, \cdots, u_{p-m} \in \Omega^- K^m$ (empty in case $p = m$) such that $\bar{r} = [\bar{r}(\bar{l}), u_1, \cdots, u_{p-m}]$, where $\bar{r}(\bar{l})$ is given by (4.4).

We can now state an explicit criterion for internal stability in terms of $\bar{f}$ and $\bar{f}'$.

Theorem 4.7: Let $\bar{f}, \bar{f}'$: $\Lambda U \to \Lambda Y$ be injective linear i/o maps with $Im\, \bar{f} = Im\, \bar{f}'$, and let $\bar{l}$: $\Lambda Y \to \Lambda Y$ be an $\Omega_\sigma^- K$-unimodular map such that $\bar{l}\bar{f} = \begin{pmatrix} \bar{f}_o \\ 0 \end{pmatrix}$ where $\bar{f}_o$ is square. Define the ΛK-linear map $\bar{r}_s$: $= [\bar{r}(\bar{l}), 0, \cdots, 0]\bar{l}$: $\Lambda Y \to \Lambda U$, where $\bar{r}(\bar{l})$ is given by (4.4). There exists a causal output feedback $\bar{r}$: $\Lambda Y \to \Lambda U$ such that $\bar{f}' = \bar{f}_{\bar{r}}$ and $\bar{f}_{\bar{r}}$ is internally stable if and only if $\bar{r}_s$ is causal and $\bar{f}_{\bar{r}_s}$ is internally stable.

Thus, the model matching problem can be decided through the feedback $\bar{r}_s$, and, in any concrete situation, $\bar{r}_s$ can also be implemented as the actual feedback to achieve model matching.

Proof: The "if" direction holds since, by construction, $\bar{f}' = \bar{f}_{\bar{r}_s}$, and thus we proceed to the "only if" direction. Assume that r: $\Lambda Y \to \Lambda U$ is a causal ΛK-linear map such that $\bar{f}' = \bar{f}_{\bar{r}}$ and $\bar{f}_{\bar{r}}$ is internally stable. By Corollary 4.6, there exist elements $u_1, \cdots, u_{p-m} \in \Omega^- K^m$ such that $\bar{r} = [\bar{r}(\bar{l}), u_1, \cdots, u_{p-m}]\bar{l}$. Then, since $\bar{l}$ is, in particular, bicausal, $\bar{r}(\bar{l})$ is causal, and hence $\bar{r}_s$ is causal as well. By Proposition 4.5 it follows that $\bar{f}_{\bar{r}} = \bar{f}_{\bar{r}_s}$, so that, by the injectivity of f, $\bar{l}_{\bar{r}} = \bar{l}_{\bar{r}_s}$. Now, by Proposition 4.1, $\bar{f}_{\bar{r}}$, $\bar{l}_{\bar{r}}$, $\bar{f}_{\bar{r}}r$, and $\bar{l}_{\bar{r}}\bar{r}$ are i/o stable so that, identically, $\bar{f}_{\bar{r}_s}$ and $l_{\bar{r}_s}$ are i/o stable. Further, since both $\bar{l}$ and $\bar{l}^{-1}$ are i/o stable, the fact that $\bar{f}_{\bar{r}}\bar{r}$ and $\bar{l}_{\bar{r}}\bar{r}$ and i/o stable obviously implies that $\bar{f}_{\bar{r}}\bar{r}_s$ $(= \bar{f}_{\bar{r}_s}\bar{r}_s)$ and $\bar{l}_{\bar{r}}\bar{r}_s$ $(= \bar{l}_{\bar{r}_s}\bar{r}_s)$ are i/o stable as well. Thus, $\bar{f}_{\bar{r}_s}$, $\bar{l}_{\bar{r}_s}$, $\bar{f}_{\bar{r}_s}\bar{r}_s$, and $\bar{l}_{\bar{r}_s}\bar{r}_s$ are i/o stable, and our assertion follows by Proposition 4.1. □

The problem of model matching can also be approached through alternative results in the literature. Thus, Desoer *et al.* [7] obtained a parametrization of all the feedback compensators $\bar{r}$ for which $\bar{f}_{\bar{r}}$ is internally stable. Let $\bar{r}(\alpha)$, where $\alpha \in A$, denote this parametrization. One can then study the equation $\bar{f}' = \bar{f}_{\bar{r}(\alpha)}$, where $\bar{f}'$ is the desired model to be matched. In a case where there exists a solution $\alpha_o \in A$ such that $\bar{f}' = \bar{f}_{\bar{r}(\alpha_o)}$, then $\bar{r}(\alpha_o)$ is a suitable feedback compensator. Alternatively, Pernebo [20] characterizes the family $\mathfrak{F}$ of all internally stable systems of the form $\bar{f}_{\bar{r}}$, and, when $\bar{f}' \in \mathfrak{F}$, he also describes a procedure to compute a feedback $\bar{r}$ that satisfies $\bar{f}' = \bar{f}_{\bar{r}}$. In comparison, the approach represented in Theorem 4.7 stresses the very simple relationship that exists between the feedback compensator $\bar{r}$, and the transfer matrices $\bar{f}$ and $\bar{f}'$. Also, Corollary 4.6 leads to an explicit characterization of the fixed and of the free parameters in the feedback compensator $\bar{r}$. Finally, the derivation of Theorem 4.7 basically employs properties of linear maps which are also shared by more general classes of input/output maps, and much of the present discussion can be generalized to suitable nonlinear situations as well.

We conclude with a brief outline of the extension of Theorem 4.7 to the case of noninjective systems. Let $\bar{f}, \bar{f}'$: $\Lambda U \to \Lambda Y$ be output feedback equivalent linear i/o maps, and let $\bar{r}$: $\Lambda Y \to \Lambda U$ be a causal ΛK-linear map such that $\bar{f}' = \bar{f}_r$. We denote n: $= \dim_{\Lambda K} \operatorname{Im} \bar{f}$, and assumes that $n < m$ $(= \dim_K U)$. By an argument dual to the one used in Lemma 4.2, it follows that there exists an $\Omega_\sigma^- K$-unimodular map $\bar{l}$: $\Lambda U \to \Lambda U$ such that $\bar{f}\bar{l} = (\bar{f}_o, 0)$, where $\bar{f}_o$: $\Lambda K^n \to \Lambda Y$ is injective and 0 denotes the $p \times (m-n)$ zero matrix. Now, since $\operatorname{Ker} \bar{f}' = \operatorname{Ker} \bar{f}$, we also have that $\bar{f}'\bar{l} = (\bar{f}_o', 0)$, where $\bar{f}_o'$: $\Lambda K^n \to \Lambda Y$ is again injective. Further, since $\bar{f}\bar{l}[I + \bar{l}^{-1}\bar{r}\bar{f}\bar{l}]^{-1} = \bar{f}[I + \bar{r}\bar{f}]^{-1}\bar{l} = \bar{f}'\bar{l}$, it follows that $\bar{f}'\bar{l}$ and $\bar{f}\bar{l}$ are still output feedback equivalent (through the causal feedback $\bar{l}^{-1}\bar{r}$). As a consequence, the injective linear i/o maps $\bar{f}_o'$ and $\bar{f}_o$ are output feedback equivalent as well. The following statements can be directly verified by computation. 1) There exists a causal feedback $\bar{r}$: $\Lambda Y \to \Lambda U$ such that $\bar{f}' = \bar{f}_{\bar{r}}$ and $\bar{f}_{\bar{r}}$ is internally stable if and only if there exists a causal feedback $\bar{r}_o$: $\Lambda Y \to \Lambda K^n$ such that $\bar{f}_o' = \bar{f}_{o\bar{r}_o}$ and $\bar{f}_{o\bar{r}_o}$ is internally stable. 2) In 1), the feedback $\bar{r}$ can be chosen as $\bar{r} = \bar{l}\begin{pmatrix} r_o \\ 0 \end{pmatrix}$, where 0 denotes the $(m-n) \times p$ zero matrix. Thus the general case is reduced to the injective case, and the solution is completed through Theorem 4.7.

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