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INVARIANT FACTORS AND OUTPUT FEEDBACK

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Abstract

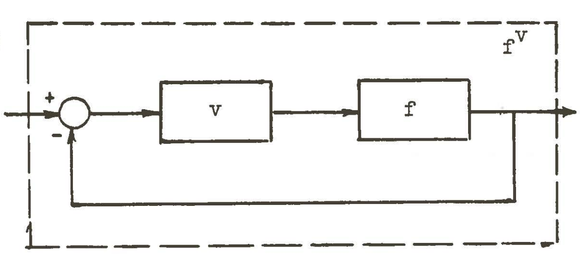
The problem of assigning invariant factors by internally stable output feedback configurations is considered. The emphasis is placed on the input-output invariant factors of the final feedback configuration, whereas the internal hidden modes of the final system are disregarded after their stability is ensured. Two types of output feedback configurations are considered: (i) a combination of unity feedback and dynamic pre-compensation, and (ii) pure dynamic output feedback. It is shown that in both cases the possibilities of assigning invariant factors depend on certain integer invariants which are determined, roughly speaking, by the unstable poles, by the unstable zeros, and by the zeros at infinity of the transfer matrix of the given system .

1. INTRODUCTION

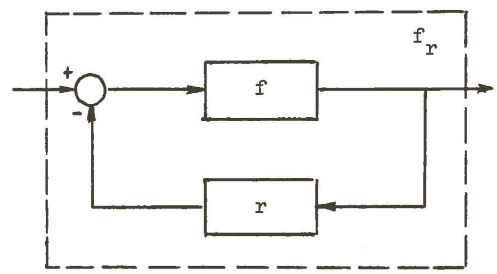
The purpose of the present note is to report some results on the assignment of invariant factors through application of output feedback. We mention here only the simplest cases and a brief outline of proof . A detailed presentation is given in HAMMER [1982], which also contains a discussion of the assignment of characteristic polynomials, omitted in our present survey.

Let f be the transfer matrix of a strictly causal linear time-invariant system, and consider the following two feedback configurations around f ,

(1.1)



(1.2)



where v and r are transfer matrices of appropriate causal linear time-invariant systems, and where f^v and f_r represent the respective composite systems. We assume that the precompensator v is nonsingular, so that no degrees of freedom of the control variables are being destroyed. The configuration (1.1) is the classical unity feedback configuration, which has found many applications in tracking control systems, and (1.2) is a pure dynamic output feedback configuration. We require throughout our discussion that the systems represented by f^v and by f_r are <u>internally stable</u>, that is, that all their modes, including the unobservable and the unreachable ones, are stable. By <u>stable</u> we mean that the respective poles are located within a prescribed region of the complex plane, which we call the <u>stability region</u>. Our objective is to study the <u>input-output</u> dynamic properties that can be assigned to f^v and f_r by appropriate choices of v and r . We show that these dynamic properties depend on certain integer invariants which are determined, roughly speaking, by the unstable poles, by the unstable zeros, and by the zeros at infinity of the given transfer matrix f. In exact terms, the problems that we are interested in are stated as follows.

Let $f^v = D_v^{-1}N_v$ and $f_r = D_r^{-1}N_r$ be polynomial matrix fraction representations. The invariant factors of

D_v and of D_r provide a detailed description of the (observable) dynamical properties of f^v and of f_r, respectively. We are interested in the following problems related to these invariant factors. Let $\phi_1, \ldots, \phi_m$, where ϕ_{i+1} divides ϕ_i for all $i = 1, \ldots, m-1$, be a set of polynomials having all their roots in the stability region of the complex plane.

(1.3) Under what conditions (on $\phi_1, \ldots, \phi_m$) does there exist a nonsingular causal precompensator v such that $\phi_1, \ldots, \phi_m$ are the invariant factors of D_v (and f^v is internally stable).

(1.4) Under what conditions (on $\phi_1, \ldots, \phi_m$) does there exist a causal feedback r such that $\phi_1, \ldots, \phi_m$ are the invariant factors of D_r (and f_r is internally stable).

The study of the effect of feedback on the invariant factors was initiated by ROSENBROCK [1970] (see also DICKINSON [1974], and MUNZNER and PRATZEL-WOLTERS [1978]) with a study of state feedback. Later the problem (1.4) was considered in ROSENBROCK and HAYTON [1978], where the following result was obtained. Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m$ and $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_m$ be the reachability indices and the observability indices of f, respectively. If

(1.5) $\sum_{j=1}^{i} \deg \phi_j \geq \sum_{j=1}^{i} (\lambda_j + \mu_1 - 1)$ for all $i=1, \ldots m$,

then there exists a feedback r such that $\phi_1, \ldots, \phi_m$ are the invariant factors of D_r.

Actually, ROSENBROCK and HAYTON [1978] consider the invariant factors of a full realization of the composite system (1.2), and therefore their condition also applies to (1.3). Presently, however, we are interested in the input-output properties of f^v and f_r, and, as we shall see, in this case (1.3) and (1.4) lead to separate conditions, both of which are stronger than (1.5). Our present conditions are stated in terms of certain structural system invariants, which we next discuss.

2. SYSTEM INVARIANTS

In the present section we review certain integer invariants from HAMMER [1981 and 1982] which play a central role in our present discussion. These system invariants are determined by rigid structural features of the given system, like its unstable poles, its unstable zeros, its internal delay, and its singularity. Their derivation is algebraically analogous to the derivation of the classical reachability indices (see ROSENBROCK [1970], BRUNOVSKY [1970], KALMAN [1971], and, in particular, WOLOVICH [1974] and FORNEY [1975]). In order to emphasize this analogy we start with a review of some concepts from realization theory.

Let R denote the real numbers, let $R(z)$ denote the set of all rational functions in z with real coefficients, and let $R^m(z)$ denote the set of all m-dimensional vectors with entries in $R(z)$. Every element $d \in R^m(z)$ can be expressed as a formal Laurent series $d = \sum_{t=t_0}^{\infty} d_t z^{-t}$, where, for all t, the coefficient d_t is in R^m. The order of d is defined as $\text{ord}\, d := \min_t \{d_t \neq 0\}$ if $d \neq 0$, and $\text{ord}\, d := \infty$ if $d = 0$. The leading coefficient $\hat{d}$ of d is defined as $\hat{d} := d_{\text{ord}\, d}$ if $d \neq 0$, and $\hat{d} := 0$ if $d = 0$. A set of elements $d_1, \ldots, d_m \in R^m(z)$ is called properly independent if their leading coefficients $\hat{d}_1, \ldots, \hat{d}_m$ are linearly independent over the real numbers R (see WEDDERBURN [1936], WOLOVICH [1974], FORNEY [1975], HAUTUS and HEYMANN [1978], and HAMMER and HEYMANN [1981]). The set $d_1, \ldots, d_m \in R^m(z)$ is ordered if $\text{ord}\, d_{i+1} \leq \text{ord}\, d_i$ for all $i = 1, \ldots, m-1$.

Now, let f be a $p \times m$ rational transfer matrix, and let $f = PQ^{-1}$ be a right coprime polynomial matrix fraction representation. There exists an $m \times m$ polynomial unimodular matrix M such that the columns $d_1, \ldots, d_m$ of the matrix QM are properly independent and ordered. Then, the integers $\lambda_i := -\text{ord}\, d_i$, $i = 1, \ldots, m$, are the reachability indices (or Kronecker invariants) of (a canonical realization of) f (see ROSENBROCK [1970], KALMAN [1972], WOLOVICH [1974], FORNEY [1975]). Using analogous procedures, we construct below several additional kinds of system integer-invariants.

First, some terminology. A rational transfer matrix h is called i/o (input/output) stable if all its canonical poles are located within the stability region of the complex plane. A polynomial matrix P is called completely unstable if all the roots of its invariant factors are located outside the stability region. Now, let f be a $p \times m$ rational transfer matrix, and let $f = ND^{-1}$ be a right coprime polynomial matrix fraction representation. Employing the Smith canonical form of N, one can factor $N = N_0 N_1$, where N_0 is a completely unstable $p \times m$ polynomial matrix, and where N_1 is an $m \times m$ nonsingular polynomial matrix having a stable inverse. The matrix N_0 exactly characterizes the unstable zeros of f, and we call it a zero matrix of f. The factorization $N = N_0 N_1$ is a particular case of the classical left standard factorizations of N as employed by GOKHBERG and KREIN [1960] and, in a different sense, by YOULA [1961], and the matrix N_0 was also employed in PERNEBO [1981]. We now define the i/o stable matrix $D_0 := DN_1^{-1}$, so that $f = N_0 D_0^{-1}$, and we call this matrix fraction representation a zero representation of f (HAMMER [1981]).

Further, still letting $f = ND^{-1}$ be a right coprime polynomial matrix fraction representation, we factor $D = D_pD_1$, where D_p is an $m\times m$ completely unstable polynomial matrix, and D_1 is an $m\times m$ polynomial matrix having a stable inverse. The matrix D_p exactly characterizes the unstable poles of f, and we call it a pole matrix of f. Defining the i/o stable matrix $N_p := ND_1^{-1}$, we obtain a matrix fraction representation $f = N_pD_p^{-1}$, which we call a right pole representation of f (HAMMER [1981]). A left pole representation is defined dually.

Now, let f be a $p\times m$ injective transfer matrix (i.e., with linearly independent columns), and let $f = N_oD_o^{-1}$ be a zero representation of f. There exists an $m\times m$ polynomial unimodular matrix M such that the columns $d_1,\dots,d_m$ of the matrix D_oM are properly independent and ordered. We define the stability indices $\sigma_1\geq\sigma_2\geq\dots\geq\sigma_m$ of f by $\sigma_i := -\text{ord}\, d_i$, $i = 1,\dots,m$ (HAMMER [1981]). (In the noninjective case, the stability indices are defined similarly, except that the matrix D_o, which is nonunique then, has to be suitably chosen; see HAMMER [1981].) It can be shown that, if $\lambda_1\geq\lambda_2\geq\dots\geq\lambda_m$ are the reachability indices of f, then $\sigma_i \leq \lambda_i$ for all $i = 1,\dots,m$ (HAMMER [1981]).

Next, let f be a rational $p\times m$ transfer matrix, and let $f = D_p^{-1}N_p$ be a left pole representation of f. There exists a $p\times p$ polynomial unimodular matrix M such that the rows $d_1',\dots,d_p'$ of the matrix MD_p are properly independent and ordered. We define then the left pole indices $\rho_1\geq\rho_2\geq\dots\geq\rho_p$ of f by $\rho_i := -\text{ord}\, d_i'$, $i = 1,\dots,p$ (HAMMER [1981]). It can then be shown that, if $\mu_1\geq\mu_2\geq\dots\geq\mu_p$ are the observability indices of f, then $\rho_i \leq \mu_i$ for all $i = 1,\dots,p$ (HAMMER [1981]). It is easy to see that the sum $\rho := \sum_{i=1}^{p} \rho_i$ is equal to the number of unstable poles of f.

We nextdefine an additional set of integer invariants. Let f be an injective rational transfer matrix, and let $f = P^{-1}Q$ be a left coprime polynomial matrix fraction representation of f. We again factor $Q = Q_1Q_o$, where, this time, Q_o is an $m\times m$ nonsingular and completely unstable polynomial matrix, and Q_1 is a polynomial matrix which has an i/o stable left inverse. Then, the rational matrix $g := fQ_o^{-1}$ has no unstable zeros, and we let M be an $m\times m$ polynomial unimodular matrix such that the columns $g_1,\dots,g_m$ of gM are properly independent and ordered. We define now the σ-latency indices $\upsilon_1\geq\upsilon_2\geq\dots\geq\upsilon_m$ of f as $\upsilon_i := \text{ord}\, g_i$, $i = 1,\dots,m$ (HAMMER [1982]). The sum $\upsilon := \sum_{i=1}^{m} \upsilon_i$ is called the σ-latency degree of f, and it is equal to the number of those zeros of f which are either unstable or at infinity.

For a detailed discussion of the invariants mentioned in this section see HAMMER [1981 and 1982].

3. THE MAIN RESULTS

We describe now the main results regarding problems (1.3) and (1.4) obtained in HAMMER [1982]. The conditions that we obtain are of the same general form as the condition (1.5) obtained by ROSENBROCK and HAYTON [1978], exept that the reachability and the observability indices therein are replaced by the invariants discussed in section 2. In order to avoid mentioning some more delicate definitions, we consider in our present short note only the simplest situation, where we assume that the given transfer matrix f in diagrams (1.1) and (1.2) is square and nonsingular. The conditions for a general transfer matrix f are similar to the ones in this particular case (see HAMMER [1982]). We first recall a few terms. A transfer matrix f is a i/o (input/output) map if it is both rational and strictly causal ("strictly proper"). We note that for a linear i/o map f, the σ-latency indices $\upsilon_1,\dots,\upsilon_m$ always satisfy $\upsilon_i \geq 1$ for all $i = 1,\dots,m$. Given an integer a, we denote by $[a]^+ := \max\{a,0\}$. The following is from HAMMER [1982].

(3.1) THEOREM. Let f be an $m\times m$ nonsingular linear i/o map with stability indices $\sigma_1\geq\sigma_2\geq\dots\geq\sigma_m$. Also, let $\phi_1,\dots,\phi_m$ be a set of monic polynomials having all their roots in the stability region of the complex plane, and for which ϕ_{i+1} divides ϕ_i for all $i = 1,\dots,m-1$.

(i) Let $\rho_1\geq\rho_2\geq\dots\geq\rho_m$ be the left pole indices of f. If

$$\sum_{j=1}^{i} \deg \phi_j \geq \sum_{j=1}^{i} (\sigma_j + [\rho_1 - 1]^+) \quad \text{for all } i = 1,\dots,m,$$

then there exists a nonsingular and causal precompensator v such that the unity feeback configuration f^v has a polynomial fraction representation $f^v = G^{-1}H$, where G has $\phi_1,\dots,\phi_m$ as its invariant factors.

(ii) Let $\upsilon_1\geq\upsilon_2\geq\dots\geq\upsilon_m$ be the σ-latency indices of f. If

$$\sum_{j=1}^{i} \deg \phi_j \geq \sum_{j=1}^{i} (\sigma_j + \upsilon_1 - 1) \quad \text{for all } i = 1,\dots,m,$$

then there exists a causal feedback compensator r such that the pure output feedback configuration f_r has a polynomial fraction representation $f_r = G^{-1}H$, where G has $\phi_1,\dots,\phi_m$ as its invariant factors.

Comparing the conditions in parts (i) and (ii) of Theorem 3.1, we see that in (i) there is a strong dependence on the unstable poles of f (through ρ_1), whereas in (ii) there is a strong dependence on the unstable and on the infinite zeros of f (through υ_1). We illustrate the numerical difference between the present conditions and the ones of ROSENBROCK and HAYTON [1978] by the following

EXAMPLE. Let $f = [(z-1)(z+1)^5]/[(z-2)(z+2)^6]$, and let the stability region be the left hand side of the complex plane. Then, using the above notation, we have $\sigma_1 = 2$; $\upsilon_1 = 2$; and $\rho_1 = 1$; whereas the reachability and the observability indices are $\lambda_1 = \mu_1 = 7$. Now, let ϕ be any monic polynomial with stable roots, and suppose that one is required to assign ϕ as a characteristic polynomial. For the given f, we obtain the following sufficient conditions on ϕ :

ROSENBROCK and HAYTON (Condition (1.5)): $\deg \phi \geq 13$.

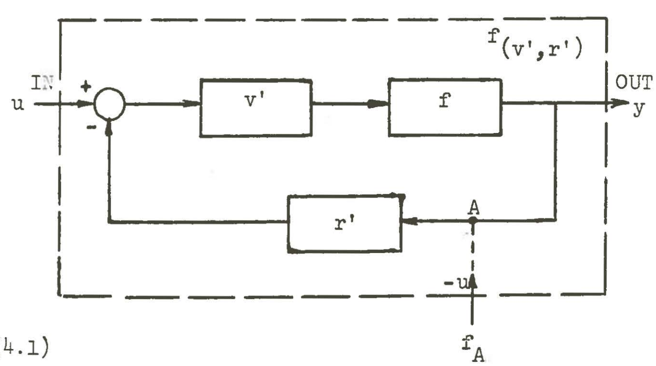
Theorem 3.1, Condition (i) : $\deg \phi \geq 2$.

Theorem 3.1, Condition (ii): $\deg \phi \geq 3$.

As we can see, certain stable components of the system have no effect on the conditions of Theorem 3.1.

4. OUTLINE OF PROOF

We summerize now the main ingredients of the proof of Theorem 3.1 (a detailed and general proof is given in HAMMER [1982]). We again assume that f is a square, say $m\times m$, nonsingular, rational, and stricly causal transfer matrix. We use as our starting point the following configuration,



(4.1)

where v' is an $m\times m$ nonsingular and causal precompensator, r' is an $m\times m$ causal feedback, and where $f_{(v',r')}$ represents the resulting transfer matrix from IN to OUT. Explicitly, we have

(4.2) $$f_{(v',r')} = f\iota_{(v',r')} ,$$

where $\iota_{(v',r')} = v'[I + r'fv']^{-1}$.

From the configuration (4.1) we can obtain the configuration (1.1) simply by transfering the input u, and substracting it at point A instead of adding it at point IN. The transfer matrix f_A obtained in this way is then clearly

(4.3) $$f_A = f^v,$$

where $v := v'r'$, and it can be readily verified that

(4.4) $$f^v = f_{(v',r')}r' .$$

It is also clear that if $f_{(v',r')}$ is internally stable, then so is also f^v.

In order to obtain from (4.1) a configuration of the form (1.2), we use the nonsingularity of v', and, defining $r := v'r'$, we have that

(4.5) $$f_r = f_{(v',r')}v'^{-1}.$$

In the following steps (1) to (5) we construct compensators v' and r' such that f^v satisfies condition (i) of Theorem 3.1.

(1) Let $f = N_oD_o^{-1}$ be a zero representation of f, where D_o has ordered and properly independent columns. Then, by definition, the stability induces $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_m$ of f are the degrees of the columns of D_o.

(2) Assume that $\phi_1, \ldots, \phi_m$ satisfy the condition

$$\sum_{j=1}^{i} \deg \phi_j \geq \sum_{j=1}^{i} (\sigma_j + [\rho_1 - 1]^+) \quad \text{for all } i = 1, \ldots, m.$$

This condition implies (see ROSENBROCK [1970], and also MUNZNER and PRATZEL-WOLTERS [1978]) that there exists a polynomial matrix Q, having ordered and properly indipendent columns $d_1, \ldots, d_m$, such that (a) $\phi_1, \ldots, \phi_m$ are the invariant factors of Q, and (b) the degrees $\lambda_i := -\text{ord}\, d_i$ satisfy $\lambda_i \geq (\sigma_i + [\rho_1 - 1]^+)$ for all $i = 1, \ldots, m$.

(3) Let α be a real number in the stability region of the complex plane, and denote $\beta := [\rho_1 - 1]^+$. Now, since $\lambda_i \geq \sigma_i + \beta$ for all $i = 1, \ldots, m$, the matrix $\iota := D_oQ^{-1}(z - \alpha)^\beta$ is causal, and $f\iota = N_oQ^{-1}(z - \alpha)^\beta$.

(4) By HAMMER [1982, section 4], there exists a pair of $m\times m$ causal matrices v' and r', where v' is nonsingular, such that (a) $f\iota = f_{(v',r')}$ and $f_{(v',r')}$ is internally stable ; and (b) the matrix $P := (z - \alpha)^\beta r'$ is a <u>polynomial</u> matrix.

(5) Combining now steps (1) to (4), defining $v := v'r'$, and using (4.4), we obtain that $f^v = f_{(v',r')}r' = f\iota r' = N_oQ^{-1}(z - \alpha)^\beta r' = N_oQ^{-1}P$, where all of N_o , Q, and P are polynomial matrices. Whence, since N_o and Q are coprime, it follows that f^v has a left polynomial fraction representation $f^v = G^{-1}H$, where G has the invariant factors of Q, that is, $\phi_1, \ldots, \phi_m$. Thus, v satisfies part (i) of Theorem 3.1 .

The proof of part (ii) of Theorem 3.1 is analogous. We use (4.5) and the fact that, by HAMMER [1982, section 6], the pair v', r' in step (4) can be chosen so that the matrix $P := (z - \alpha)^{\upsilon_1 - 1} v'^{-1}$ is a polynomial matrix.

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For an extended list of references see HAMMER [1982].

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