Internally Stable Nonlinear Systems with Disturbances: A Parameterization

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***Abstract*— This paper deals with the control of a nonlinear system whose output is subjected to an additive disturbance. The main result is a simple parameterization of the set of all system responses that can be obtained through internally stable control of the given system. The parameterization provides a clear indication of the effects of the disturbance on the response of the stabilized closed loop system. The class of achievable responses is determined by the "numerator" of a right coprime fraction representation of the system being controlled.**

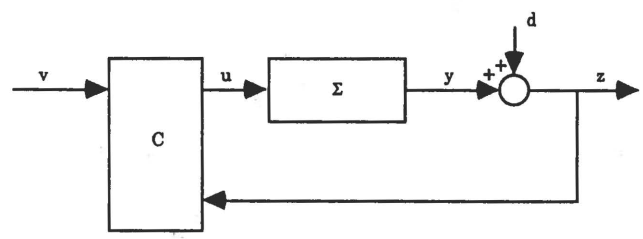

Fig. 1.1

I. Introduction

CONSIDER the problem of controlling a nonlinear system Σ whose output signal is corrupted by an additive disturbance d. Without making any particular assumptions on the nature of the control scheme used to control Σ, we can represent it in the form as shown in Fig. 1.1. In the configuration, C represents an equivalent controller that incorporates all the control elements of the loop. The external input (or reference) signal is denoted by v; the disturbance signal is denoted by d, and z is the output signal. The closed loop system is required to be internally stable, where internal stability signifies that the configuration can tolerate small disturbances on its external and internal ports (including ports within the equivalent controller C) without loosing stability.

The output signal z is determined by the signals v and d and depends on the system Σ as well as on the controller C. To make these facts explicit, we use the notation

$$z = \Sigma_c(v, d) \tag{1.2}$$

where Σ_c is the appropriate equivalent system.

The objective of this paper is to provide a characterization of the class of all equivalent systems Σ_c that can be obtained from an internally stable control configuration around the system Σ. This characterization will provide us with an understanding of the capabilities and limitations of nonlinear control systems; at the same time, it will create a foundation for the development of a theory aimed at minimizing the effects of the disturbance d on the response. The characterization derived here is in a global context and is not restricted to "small" disturbance signals d. The presentation is for discrete-time nonlinear systems, but the basic results are transferable to continuous-time nonlinear systems as well.

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The characterization of Σ_c derived in the present paper is rather simple and is reminiscent in its form of its linear analog. It is derived within the framework of the theory of fraction representations of nonlinear systems, which provides the tools for a compact and lucid statement of the results. Recall that a *right fraction representation* of a nonlinear system Σ is a factorization of Σ into a composition of the form $\Sigma = PQ^{-1}$, where P and Q are stable systems with Q being invertible. The fraction representation $\Sigma = PQ^{-1}$ is said to be *coprime* when the systems P and Q are right coprime. Qualitatively, a right coprime fraction representation $\Sigma = PQ^{-1}$ is characterized by the fact that every instability of the inverse system Q^{-1} is also an instability of the system Σ. In other words, no cancellations of instabilities are possible within the composition PQ^{-1} (see [4], [6] for details).

In general terms, our discussion depends on the assumptions that the system Σ being controlled is stabilizable and strictly causal. The assumption that Σ is stabilizable is obviously necessary, since nonstabilizable systems are not amenable to control. On the other hand, the strict causality assumption on Σ is used here only as a convenient means to guarantee that the closed loop system is well posed; it does not represent a fundamental restriction and can be replaced by plain causality combined with a well-posedness requirement.

The main result of this paper can be summarized as follows. Let $\Sigma = PQ^{-1}$ be a right coprime fraction representation (with a bicausal "denominator" Q) of the system being controlled. Then,

1) For every causal equivalent controller C for which the closed loop system of Fig. 1.1 is internally stable, there exists a stable and causal system $\phi(v, d)$ such that

$$\Sigma_c(v, d) = d + P\phi(v, d) = [I + P\phi(v, \cdot)]d \tag{1.3}$$

where P is the "numerator" of the right coprime fraction representation of Σ and I denotes the identity system.

2) Conversely, for every stable and causal system $\phi(v, d)$, there is an internally stable control configuration around

the system Σ for which the equivalent system $\Sigma_c(v,d)$ satisfies $\Sigma_c(v,d) = [I + P\phi(v,\cdot)]d$. The implementation of such a configuration is described in Section III.

Thus, (1.3) provides a complete parameterization of the class of all responses $\{\Sigma_c\}$ that can be obtained by internally stable control of the system Σ, with the stable and causal system ϕ serving as the sole parameter. Every equivalent controller C that internally stabilizes Σ induces a certain ϕ, and, conversely, for every ϕ, there is an equivalent controller C that internally stabilizes Σ and yields the response (1.3). The parameterization permits the dissolution of the design process into two basic steps: 1) the specification of the desired response, which simply amounts to the selection of ϕ, and 2) the implementation of a controller yielding this response. Once ϕ is selected, a method of deriving an internally stable implementation of the response $\Sigma_c(v,d) = [I + P\phi(v,\cdot)]d$ is outlined in Section III. The selection of ϕ depends, of course, on the objectives of the design at hand.

In many cases, an important consideration in the selection of ϕ is the desire to achieve maximal attenuation of the effects of the disturbance d on the output signal z. We shall consider a global theory of optimal nonlinear disturbance attenuation in a separate report. Some basic limitations on the achievable performance become already legible from a casual inspection of (1.3). It is fairly clear that the essential limitation on the achievable performance is imposed by the "numerator" system P of Σ, since, apart from the identity, it is the only fixed quantity in (1.3). For instance, assume hypothetically that the system $-P := (-1)P$ has a stable and causal inverse $(-P)^{-1}$ satisfying $(-P)(-P)^{-1} = I$, the identity; then, selecting $\phi(v,d) := (-P)^{-1}d$ in (1.3) yields $\Sigma_c = 0$, the zero system. This shows that if P has a stable and causal inverse, it is possible to completely eliminate the effects of the disturbance d on the output, within an internally stable control configuration around Σ. Note, however, that the existence of a causal system $(-P)^{-1}$ is precluded by the strict causality of Σ (since Q is bicausal here), whereas the existence of a stable inverse $(-P)^{-1}$ is restricted to so called "minimum phase systems."

The situation discussed in the previous paragraph is closely analogous to the well-known linear theory. In the linear case, the stable and causal system ϕ decomposes into the sum $\phi(v,d) = \phi_1 v + \phi_2 d$; this reduces the parameterization (1.3) in the linear case to the standard result $\Sigma_c(v,d) = [I + P\phi_2]d + P\phi_1 v$, which played a crucial role in the formulation of the linear theory of optimal disturbance attenuation [17]. The implications of our current results on the solution of the global nonlinear optimal disturbance attenuation problem will be considered in a separate report.

The ensuing discussion depends on the theory of fraction representations of nonlinear systems [3], [4], [6]–[8], [10]. Section II contains some refinements and a brief review of those aspects of the theory that are acutely relevant here. The main results of this paper are presented in Section III. To streamline the presentation of the main ideas, the proofs of some statements have been delegated to a section of proofs, Section IV.

Alternative recent investigations into the theory of nonlinear systems can be found in [1], [2], [12]–[16], the references cited in these works, and others.

II. Fraction Representations of Nonlinear Systems and Stabilization

We start by introducing our basic notation and setup. Let R^m be the set of all m-dimensional real vectors. Denote by $S(R^m)$ the set of all sequences $u_0, u_1, u_2, \cdots$ of m-dimensional real vectors $u_j \in R^m$, $j = 0, 1, 2, \cdots$. Adopting the input/output point of view, a system Σ is regarded as a map $\Sigma: S(R^m) \to S(R^p)$, transforming input sequences of m-dimensional real vectors into output sequences of p-dimensional real vectors. The image of a subset $S \subset S(R^m)$ through Σ is denoted by $\Sigma[S]$, and $Im\Sigma := \Sigma[S(R^m)]$ is the entire image of the system Σ.

We shall perform two kinds of binary operations on systems—composition and addition. Composition is the usual composition of maps. Regarding addition, the sum of two systems $\Sigma_1, \Sigma_2: S(R^m) \to S(R^p)$ is defined, as always, by $(\Sigma_1 + \Sigma_2)u := \Sigma_1 u + \Sigma_2 u$ for all sequences $u \in S(R^m)$; the right side of the last formula is the usual elementwise addition of sequences of real vectors.

For a sequence $u \in S(R^m)$, the ith element is denoted by u_i; the set of elements $u_i, u_{i+1}, \cdots, u_j$, where $j \geq i \geq 0$ are integers, is denoted by u_i^j. Letting $y := \Sigma u$ be the response of the system $\Sigma: S(R^m) \to S(R^p)$ to the input sequence u, we sometimes use $\Sigma u]_i^j$ to denote y_i^j, the corresponding elements of the output sequence.

In these terms, a system $\Sigma: S(R^m) \to S(R^p)$ is *causal* (respectively, *strictly causal*) if the following holds true. For every pair of input sequences $u, v \in S(R^m)$ and for every integer $j \geq 0$ for which the equality $u_0^j = v_0^j$ holds, one has $\Sigma u]_0^j = \Sigma v]_0^j$ (respectively, $\Sigma u]_0^{j+1} = \Sigma v]_0^{j+1}$). In other words, a system is causal if the outputs are equal for at least as long as the inputs are equal.

The notion of causality is of critical importance to control theory, since only causal systems can be implemented in a real-time environment. Many systems encountered in practice are, in fact, strictly causal, a stronger form of causality. For instance, the class of strictly causal systems includes every system $\Sigma: S(R^m) \to S(R^p)$ that can be represented in the form

$$x_{k+1} = f(x_k, u_k)$$
$$y_k = h(x_k), \qquad k = 0, 1, 2, \cdots. \tag{2.1}$$

Here, $u \in S(R^m)$ is the input sequence, $y \in S(R^p)$ is the output sequence, and $x \in S(R^n)$ is an intermediate sequence of "states." In case the functions $f: R^n \times R^m \to R^n$ and $h: R^n \to R^p$ are continuous, then (2.1) constitutes a *continuous realization* of the system Σ.

A system $M: S(R^m) \to S(R^m)$ is *bicausal* if it is causal and if it possesses a causal inverse. It can be readily shown that the composition of two bicausal systems is bicausal. Likewise, the composition of two causal systems is causal, and the composition of a causal system with a strictly causal one is strictly causal. Regarding sums of systems, we shall frequently use the fact that the sum of a bicausal system and a strictly causal one is always bicausal, as follows.

Lemma 2.2: For every strictly causal system $\Gamma: S(R^p) \to S(R^p)$, the sum $(I - \Gamma): S(R^p) \to S(R^p)$, where $I: S(R^p) \to S(R^p)$ is the identity, is a bicausal system.

A proof is provided in Section IV.

Remark 2.3: Note that the lemma implies the following statement. Every sum of the form $(M - \Gamma)$ is bicausal whenever $M: S(R^p) \to S(R^p)$ is bicausal and $\Gamma: S(R^p) \to S(R^p)$ is strictly causal. Indeed, one has

$$M - \Gamma = \left(I - \Gamma M^{-1}\right) M. \tag{2.4}$$

Now, the bicausality of M and the strict causality of Γ imply that ΓM^{-1} is strictly causal; whence, by the lemma, $\left[I - \Gamma M^{-1}\right]$ is bicausal. On account of (2.4), $(M - \Gamma)$ is then the composition of two bicausal systems and is, therefore, bicausal. ♦

Returning to our review of notation, let $\theta > 0$ be a real number and denote by $[-\theta, \theta]^m$ the set of vectors in R^m all of whose components belong to the interval $[-\theta, \theta]$. Let $S(\theta^m)$ be the set of all sequences $u \in S(R^m)$ having all their elements in $[-\theta, \theta]^m$, i.e., $u_i \in [-\theta, \theta]^m$ for all integers $i \geq 0$. Thus, $S(\theta^m)$ is the set of all sequences "bounded" by θ. We can say then that a system $\Sigma: S(R^m) \to S(R^p)$ is *BIBO* (Bounded-Input, Bounded-Output)-*stable* whenever there is, for every real number $\theta > 0$, a real number $M > 0$ satisfying $\Sigma[S(\theta^m)] \subset S(M^p)$. Finally, in this context, a sequence $u \in S(R^m)$ is said to be *bounded* if there is a real number $\theta > 0$ such that $u \in S(\theta^m)$.

The basic notion of stability we use is related to continuity with respect to a norm. Two norms are particularly useful in this context: the ℓ^∞-norm and the weighted ℓ^∞-norm. The ℓ^∞-norm is denoted by $|\cdot|$; for a vector $a = (a_1, \cdots, a_m) \in R^m$, it is simply $|a| := \max\{|a_1|, \cdots, |a_m|\}$, the maximal absolute value of a coordinate. For a sequence $u \in S(R^m)$, it is given by $|u| := \sup_{i \geq 0} |u_i|$. The weighted ℓ^∞-norm is denoted by ρ and is given by

$$\rho(u) := \sup_{i \geq 0} 2^{-i} |u_i| \tag{2.5}$$

for all $u \in S(R^m)$. We comment that the number "2" in (2.5) can be replaced by $1 + \varepsilon$, $\varepsilon > 0$, without affecting any of our results.

To examine the norm ρ, suppose for a moment that we are interested in the response of our systems only over a finite interval of time, say $[0, T]$, where $T > 0$ is a fixed integer. Let $C_T(R^n)$ be the set of all functions $h: [0, T] \to R^n$. Denote by $|h| := \mathrm{Sup}_{i \in [0,T]} |h(i)|$ the ℓ^∞-norm on $C_T(R^n)$ and by $\rho(h) := \mathrm{Sup}_{i \in [0,T]} 2^{-i} |h(t)|$ the norm ρ on $C_T(R^n)$. It is easy to see that on $C_T(R^n)$ the norm ρ is equivalent to the ℓ^∞-norm.

Indeed, let $h \in C_T(R^n)$ be any function. Then $\mathrm{Sup}_{i \in [0,T]} 2^{-i} |h(i)| \leq \mathrm{Sup}_{i \in [0,T]} |h(i)|$ and $2^T \mathrm{Sup}_{i \in [0,T]} 2^{-i} |h(i)| = \mathrm{Sup}_{i \in [0,T]} 2^{T-i} |h(i)| \geq \mathrm{Sup}_{i \in [0,T]} |h(i)|$, so that

$$\rho(h) \leq |h| \leq 2^T \rho(h) \quad \text{and}$$
$$2^{-T} |h| \leq \rho(h) \leq |h|$$

on $C_T(R^n)$. Using well-known properties of normed spaces, these inequalities imply the equivalence of the two norms $|\cdot|$ and ρ on $C_T(R^n)$. Since this is true for every finite $T > 0$, we arrive at the following qualitative conclusion: the two norms ρ and $|\cdot|$ on $S(R^n)$ differ "only" at the time $i = \infty$, the main difference being that a function which is bounded with respect to the norm ρ is not necessarily bounded with respect to the norm $|\cdot|$ over the infinite time axis $[0, \infty]$. For functions that are *known* to be ℓ^∞-bounded over the entire time axis $[0, \infty]$, however, the difference between the two norms ρ and $|\cdot|$ is quite minor from a practical standpoint; they are equivalent over any finite time interval, and in practical situations only the response over finite time intervals is relevant. Thus, with little, if any, compromise of practical significance, we can replace the standard definition of stability, which requires continuity with respect to the ℓ^∞-norm, with the requirement of continuity with respect to the norm ρ, combined with a separate ℓ^∞-boundedness requirement. This replacement yields a substantial simplification of mathematical arguments, mainly because the bounded set of sequences $S(\theta^m)$ is compact with respect to ρ. Formally, the notions of stability employed in the sequel are as follows.

Definition 2.6: A system $\Sigma: S(R^m) \to S(R^p)$ is *stable* (with respect to the norm ρ) if it is BIBO-stable and if the restriction $\Sigma: S(\alpha^m) \to S(R^p)$ is continuous (with respect to ρ) for every real number $\alpha > 0$. The system Σ is ℓ^∞-*stable* if the restriction $\Sigma: S(\alpha^m) \to S(R^p)$ is continuous with respect to the ℓ^∞-norm for every real number $\alpha > 0$. ♦

Note that the plain term *stable* refers to the norm ρ.

In the case of linear time-invariant finite dimensional systems, the present notion of stability coincides with the standard one. Indeed, it is well known that such a linear system is BIBO-stable if and only if all its poles are located within the open unit disc in the complex plane. When all poles are within the open unit disc, the map induced by the system is continuous with respect to the ℓ^∞-norm and, *a-fortiori*, with respect to the norm ρ.

As a general comment, we remark that the main use of the definition of stability in our present context is to restrict the class of permissible hidden modes in a composite system. As discussed in Section I, the input/output modes of the system, which determine the efficacy of the system, can be assigned (see also [7]). They can be chosen from a more restrictive class of responses when the parameterization (1.3), which encompasses all possible responses, allows such a choice.

The notion of differential boundedness is also important to our discussion [5]. In qualitative terms, it is a weak form of uniform continuity with respect to the ℓ^∞-norm. It guarantees that a small deviation of the input sequence always causes a bounded deviation of the output sequence. A more detailed discussion of the intuitive significance of differential boundedness is provided in [5].

Definition 2.7: A stable $\Sigma: S(R^m) \to S(R^p)$ is *differentially bounded* if there is a pair of real numbers $\varepsilon, \theta > 0$ such that, for every pair of sequences $u \in S(R^m)$ and $v \in S(\varepsilon^m)$, one has $|\Sigma(u + v) - \Sigma(u)| \leq \theta$. ♦

Linearity directly implies that every stable linear system is, in fact, differentially bounded.

So far, we have only mentioned stability properties of individual systems, and Definition 2.6 is usually referred to as *input/output stability*. When several individual systems are combined into a composite system, one has to use a stronger notion of stability, usually referred to as *internal stability*. The latter guarantees desirable stability properties of the composition and takes into account the effects of various disturbances and noises that may affect the component systems. Consider a composite system $\Sigma^{(s)}$ that consists of s individual systems, labeled $\Sigma^1, \cdots, \Sigma^s$, where $\Sigma^i:S(R^{m(i)}) \to S(R^{p(i)})$, $i = 1, \cdots, s$. The list $\Sigma^1, \cdots, \Sigma^s$ also includes summers, multipliers, etc., each of which is regarded as an individual system. Let $u \in S(R^m)$ be the external input sequence of the composite system, and let $y \in S(R^p)$ be its output sequence. Let $u^j \in S(R^{m(j)})$ be the input sequence of the system Σ^j within the configuration, and let $y^j \in S(R^{p(j)})$ be its output sequence. The interconnections among the subsystems are then characterized by a set of equalities $u^i = y^{j(i)}$, which determine to which output each input is connected. We now augment the external input signal u by s new input signals $\varepsilon^i \in S(R^{m(i)})$, $i = 1, \cdots, s$, and set $u^i := y^{j(i)} + \varepsilon^i$. Each one of the ε^i acts as an additive disturbance on the input port of the system Σ^i. The disturbances are all assumed to be bounded by $\delta > 0$, so that in fact $\varepsilon^i \in S(\delta^{m(i)})$, $i = 1, \cdots, s$.

Let $\Sigma^{*s*}:S(R^m) \times S(R^{m(1)}) \times \cdots \times S(R^{m(s)}) \to S(R^p) \times S(R^{p(1)}) \times \cdots \times S(R^{p(s)}):(u, \varepsilon^1, \cdots, \varepsilon^s) \mapsto \Sigma^{*s*}(u, \varepsilon^1, \cdots, \varepsilon^s)$ denote the system induced by the interconnected system $\Sigma^{(s)}$ and the disturbances, having the input signals $u, \varepsilon^1, \cdots, \varepsilon^s$ and the output signals $y, y^1, \cdots, y^s$, respectively.

Definition 2.8: The composite system $\Sigma^{(s)}$ is *internally stable* if the system Σ^{*s*} is stable in the sense of Definition 2.6. The composite system $\Sigma^{(s)}$ is *strictly internally stable* if, besides being stable, the system Σ^{*s*} is also differentially bounded. ♦

Note that this is a rather strong definition of internal stability. It requires boundedness of all internal signals under noisy conditions, along with continuity of all internal and external outputs with respect to all outside inputs, including disturbances.

In the case of linear systems, the present definition reduces to the standard definition of internal stability. Furthermore, in the linear case, every internally stable system is also strictly internally stable.

Finally, the following term is sometimes convenient to use.

Definition 2.9: A system $\Sigma:S(R^m) \to S(R^p)$ *entirely stabilizable* if there is a strictly internally stable control configuration that stabilizes Σ over the entire input space $S(R^m)$. ♦

All the systems we consider are required to be stabilizable, since otherwise they would not be amenable to control. Note that every reachable and observable linear time-invariant finite dimensional system is entirely stabilizable.

We summarize now the basic premises on which our theory rests.

Basic Assumptions 2.10: Throughout our discussion, the following assumptions will be in effect.

1) The system Σ that needs to be controlled is strictly causal and entirely stabilizable.
2) Only bounded input sequences and disturbances are applied to the closed loop system; i.e., for every pair (v, d) in Fig. 1.1, there is a real number $\alpha > 0$ such that $(v, d) \in S(\alpha^m) \times S(\alpha^p)$. The number α may, however, vary from one pair (v, d) to another, and no absolute bound is assumed here.

Further discussion of the assumptions is provided in the next section.

We now briefly review some basic issues in the theory of fraction representations of nonlinear systems. Let $\Sigma:S(R^m) \to S(R^p)$ be a system. A right fraction representation of Σ is determined by three quantities: a subset $S \subset S(R^q)$, $q > 0$, called the *factorization space*, and two stable systems $P:S \to S(R^p)$ and $Q:S \to S(R^m)$, with Q being a set isomorphism, such that $\Sigma = PQ^{-1}$. A right fraction representation $\Sigma = PQ^{-1}$ is *coprime* whenever the stable systems P and Q are right coprime according to the following definition [4], [6]. (Let $G:S_1 \to S_2$ be a map, where $S_1 \subset S(R^m)$ and $S_2 \subset S(R^n)$ are subsets; for a subset $S \subset S(R^n)$, we denote by $G^*[S]$ the inverse image of S through G, i.e., the set of all sequences $u \in S_1$ satisfying $Gu \in S$.)

Definition 2.11: Let $S \subset S(R^q)$ be a subset. Two stable systems $P:S \to S(R^p)$ and $Q:S \to S(R^m)$ are *right coprime* whenever the conditions below hold.

1) For every real number $\tau > 0$ there is a real number $\theta > 0$ such that
$$P^*[S(\tau^p)] \cap Q^*[S(\tau^m)] \subset S(\theta^q).$$
2) For every real number $\tau > 0$ the set $S \cap S(\tau^q)$ is a closed subset of $S(\tau^q)$ (with respect to the topology induced by ρ). ♦

In qualitative terms, two nonlinear systems P and Q with a common domain are right coprime if, for every unbounded input sequence u, at least one of the output sequences Pu or Qu is unbounded. In the linear case, the above definition reduces to the requirement that P and Q have no unstable zeros in common.

As mentioned earlier, for a right coprime fraction representation $\Sigma = PQ^{-1}$, every instability of the system Q^{-1} is also an instability of Σ, and no cancellations of instabilities can occur in the composition PQ^{-1}.

We shall need some basic results on the existence of right coprime fraction representations and their properties. In particular, right coprime fraction representations $\Sigma = PQ^{-1}$ in which the "denominator" system Q is bicausal are especially useful in our context. The existence of such fraction representations was shown in [10], in conjunction with the theory of static reversible state feedback for nonlinear systems. We review the basic facts here. Recall that a system $\Sigma : S(R^m) \to S(R^p)$ has a continuous realization if there are continuous functions $f : R^q \times R^m \to R^q$ and $h : R^q \to R^p$ such that the system can be represented in the form $x_{k+1} = f(x_k, u_k)$, $y_k = h(x_k)$, where $u \in S(R^m)$ is the input sequence of the system, $y = \Sigma u \in S(R^p)$ is the output sequence, and $x \in S(R^q)$ is an intermediate

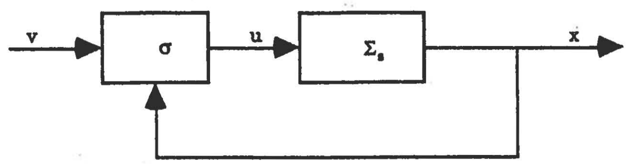

Fig. 2.12

sequence of "states." The system represented by the recursion $x_{k+1} = f(x_k, u_k)$ is called the *input/state part* of Σ and is denoted by Σ_s.

As shown in Fig. 2.12, consider a static state feedback loop around the input/state part Σ_s of Σ, where $\sigma : R^q \times R^m \to R^m : (x, v) \mapsto \sigma(x, v)$ is a continuous function, serving as state feedback. The state feedback function σ is said to be *reversible* if it is injective in v for every state x. We shall say that the system Σ_s is *stabilizable by state feedback* if there is a reversible state feedback function σ such that the closed loop of Fig. 2.12 is internally stable. An explicit and verifiable characterization of stabilizability by state feedback was derived in [9]. It basically amounts to a certain nonlinear analog of the linear reachability requirement. The following result is reproduced here from [10]. It states that, over a bounded input space, every system having a continuous realization with stabilizable input/state part has a right coprime fraction representation in which the denominator system is bicausal.

Theorem 2.13: Let $\Sigma : S(\alpha^m) \to S(R^p)$ be a system with the bounded input space $S(\alpha^m)$, $\alpha > 0$. Assume Σ has a continuous realization with an input/state part that is stabilizable by state feedback. Then, Σ has a right coprime fraction representation $\Sigma = PQ^{-1}$, with Q being a bicausal system. ♦

A basic property of right coprime fraction representations is the fact that the denominator system contains the exact information about the instabilities of the system. In formal terms, this fact can be stated as follows (the injective version was proved in [4], [6]; a proof of the present version is provided in Section IV).

Proposition 2.14: Let $\Sigma{:}S(R^m) \to S(R^p)$ be a system having a right coprime fraction representation $\Sigma = PQ^{-1}$, where $P{:}S \to S(R^p)$ and $Q : S \to S(R^m)$, and $S \subset S(R^q)$. Let $D{:}S(R^r) \to S(R^m)$ be any stable system for which the composition $\Sigma D{:}S(R^r) \to S(R^p)$ is stable. Then, there is a stable system $\phi{:}S(R^r) \to S$ such that $D = Q\phi$.

In particular, if the right coprime fraction representation $\Sigma = PQ^{-1}$ has a bicausal denominator Q, and if the stable system D of Proposition 2.14 is causal, then the system ϕ will be stable and causal. Indeed, the stability of ϕ is stated in the proposition, and, since Q^{-1} and D are both causal, the equality $\phi = Q^{-1}D$ shows that ϕ is causal as well. For future reference, we state this fact as a corollary.

Corollary 2.15: Let $\Sigma{:}S(R^m) \to S(R^p)$ be a system having a right coprime fraction representation $\Sigma = PQ^{-1}$, where $P{:}S \to S(R^p)$ and $Q{:}S \to S(R^m)$, where Q is bicausal. Let $D{:}S(R^r) \to S(R^m)$ be any stable and causal system for which the composition $\Sigma D{:}S(R^r) \to S(R^p)$ is stable. Then, there is a stable and causal system $\phi{:}S(R^r) \to S$ such that $D = Q\phi$. ♦

Finally, left fraction representations of nonlinear systems also play an important role in our discussion. As one would expect, a left fraction representation of a nonlinear system $\Sigma{:}S(R^m) \to S(R^p)$ is determined by three quantities: a subset $S_L \subset S(R^r)$, $r > 0$, called the *factorization space,* and a pair of two stable systems $T{:}S(R^m) \to S_L$ and $G{:}Im\Sigma \to S_L$, where G is a set isomorphism, and $\Sigma = G^{-1}T$.

III. Parameterizing Internally Stable Nonlinear Systems

We turn now to a detailed examination of Fig. 1.1. In that figure, $\Sigma{:}S(R^m) \to S(R^p)$ is the system that needs to be controlled, and it is strictly causal. Control is achieved by a causal nonlinear dynamic controller $C{:}S(R^m) \times S(R^p) \to S(R^m) : (v, z) \mapsto C(v, z)$. The controller C is to be interpreted as an equivalent controller, possibly consisting of several control elements or feedback loops. Later in this section we shall discuss various possible implementations of C. The output sequence z of the configuration is, of course, determined by the input sequence v, by the disturbance sequence d, and by the systems Σ and C; one can then write $z = \Sigma_c(v, d)$. We shall refer to the system Σ_c as the *input&disturbance/output* relation of Fig. 1.1. The composite system represented by the diagram is required to be internally stable.

Our discussion in this section can be divided into two main parts. The first part shows that internal stability of Fig. 1.1 implies that Σ_c can be represented in the form (1.3). The second part considers the internally stable implementation of a response of the form (1.3). The combination of both parts shows that (1.3) is a parameterization of all internally stable control configurations with disturbances.

A. An Implication of Internal Stability

In this part of the section we assume that Fig. 1.1 is internally stable, and we show that this implies that Σ_c can be represented in the form (1.3). During some parts of the analysis, it will be convenient to regard the external input sequence v as a fixed "parameter," while regarding the disturbance d as an external input, i.e., to consider the appropriate partial function. Since no restrictions will be placed on v or on d, this will have no effect on the validity of the final results, all of which are valid for all v and d. In the same spirit, we shall frequently use the notation

$$\psi(v)z := C(v, z)$$

in which v can be intuitively viewed as a parameter of the system $\psi(v)$, while z is its input. This simply amounts to concentrating on the partial function $C(v, \cdot)$. Of course, the use of this notation has no deep implications, but it is convenient for intuitive purposes. When v is regarded as a parameter, the disturbance d becomes the only external input in Fig. 1.1, and the diagram can be redrawn as shown in Fig. 3.1.1.

Note that the input v appears in the new diagram as an implicit variable, embedded within the system $\psi(v){:}S(R^p) \to S(R^m){:}z \mapsto \psi(v)z = u$. The disturbance d has been moved to the left end to emphasize that it is now the input of

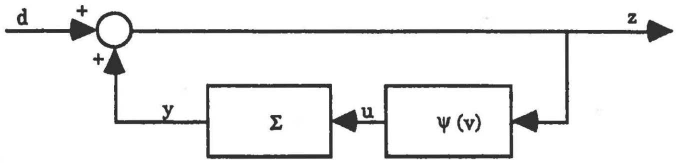

Fig. 3.1.1

interest (of course, no influence is assumed over this input). When the dependence of ψ on the input sequence v is made explicit again, we shall write $\psi(\cdot):S(R^m) \times S(R^p) \to S(R^m):(v,z) \mapsto \psi(v)z$, and $\psi(\cdot)$ is identical to the controller C.

Using this notation, the equations that describe Fig. 3.1.1 (and whence also Fig. 1.1) become

$$\begin{aligned} z &= d + y \\ y &= \Sigma\psi(v)z. \end{aligned} \tag{3.1.2}$$

This yields

$$z = d + \Sigma\psi(v)z \tag{3.1.3}$$

or

$$[I - \Sigma\psi(v)]z = d \tag{3.1.4}$$

where $I:S(R^p) \to S(R^p)$ is the identity system. Now, in view of the causality of the controller C, the system $\psi(v)$ is causal for each v (with z being the input). Combining this with the strict causality of Σ, it follows that the composition $\Sigma\psi(v):S(R^p) \to S(R^p):z \mapsto \Sigma\psi(v)z$ is a strictly causal system (as in the system $\Sigma\psi(\cdot):S(R^m) \times S(R^p) \to S(R^m):(v,z) \mapsto \Sigma\psi(v)z$). Lemma 2.2 then implies that the combination $[I - \Sigma\psi(v)]:S(R^p) \to S(R^p):z \mapsto [I - \Sigma\psi(v)]z$ is bicausal and whence has a causal inverse $[I - \Sigma\psi(v)]^{-1}:S(R^p) \to S(R^p)$. We can then solve (3.1.4) to obtain

$$z = [I - \Sigma\psi(v)]^{-1}d \tag{3.1.5}$$

which is, of course, valid for any input sequence $v \in S(R^m)$.

Equation (3.1.5) shows that $\Sigma_c = [I - \Sigma\psi(\cdot)]^{-1}$; since internal stability of Fig. 1.1 clearly implies that Σ_c is stable, it follows that the system $[I - \Sigma\psi(\cdot)]^{-1}:S(R^m) \times S(R^p) \to S(R^p):(v,d) \mapsto [I - \Sigma\psi(v)]^{-1}d$ is stable. In particular, the inverse system $[I - \Sigma\psi(v)]^{-1} : S(R^p) \to S(R^p)$ is stable whenever $v \in S(\alpha^m)$ for some real $\alpha > 0$.

Using (3.1.5) together with $u = \psi(v)z$ yields

$$u = \psi(v)[I - \Sigma\psi(v)]^{-1}d. \tag{3.1.6}$$

Define now the system

$$\begin{aligned} \psi_d(v) :=& \psi(v)[I - \Sigma\psi(v)]^{-1} : S(R^p) \to S(R^m) : \\ & d \mapsto \psi_d(v)d = u. \end{aligned} \tag{3.1.7}$$

By the causality of the systems $\psi(v)$ and $[I - \Sigma\psi(v)]^{-1}$, it follows that $\psi_d(v)$ is causal. Whence, the combination $\Sigma\psi_d(v):S(R^p) \to S(R^p):d \mapsto \Sigma\psi_d(v)d$ is strictly causal on account of the strict causality of Σ. Lemma 2.2 then implies that the system $[I + \Sigma\psi_d(v)]:S(R^p) \to S(R^p):d \mapsto [I + \Sigma\psi_d(v)]d$ is bicausal and, consequently, has a causal inverse $[I + \Sigma\psi_d(v)]^{-1}:S(R^p) \to S(R^p)$. Furthermore, we claim that

$$[I + \Sigma\psi_d(v)]^{-1} = [I - \Sigma\psi(v)]. \tag{3.1.8}$$

Indeed, by direct calculation,

$$\begin{aligned} [I + \Sigma\psi_d(v)][I - \Sigma\psi(v)] &= I - \Sigma\psi(v) \\ &\quad + \Sigma\psi_d(v)[I - \Sigma\psi(v)] \\ &= I - \Sigma\psi(v) + \Sigma\psi(v) = I \end{aligned}$$

where (3.17) was used. This shows that $[I - \Sigma\psi(v)]$ is a right inverse of $[I + \Sigma\psi_d(v)]$. But, $[I + \Sigma\psi_d(v)]:S(R^p) \to S(R^p)$ is a set isomorphism since it is bicausal and, whence, $[I - \Sigma\psi(v)]$ is also left inverse of $[I + \Sigma\psi_d(v)]$, and (3.1.8) is valid.

Several interesting consequences follow directly from (3.1.8). First, when this formula is substituted into the equation $\psi(v) = \psi_d(v)[I - \Sigma\psi(v)]$, which is just another form of (3.1.7), one obtains

$$\psi(v) = \psi_d(v)[I + \Sigma\psi_d(v)]^{-1}. \tag{3.1.9}$$

This shows that ψ is determined by ψ_d (and the system Σ, of course) and can be directly computed when ψ_d is given.

Further, we have noted earlier that the system $[I - \Sigma\psi(\cdot)]^{-1}$ is stable; in view of (3.1.8), this implies that $[I + \Sigma\psi_d(\cdot)]:S(R^m) \times S(R^p) \to S(R^p)$ is a stable system. Now, $\Sigma\psi_d(v)d = [I + \Sigma\psi_d(v)]d - Id$, so the system $\Sigma\psi_d(\cdot)$ is the difference of the two stable systems $[I + \Sigma\psi_d(\cdot)]$ and the identity system I and, whence, is itself stable. Using (3.1.5) and (3.1.8), one can write

$$z = [I + \Sigma\psi_d(v)]d \tag{3.1.10}$$

and

$$\Sigma_c(v,d) = [I + \Sigma\psi_d(v)]d. \tag{3.1.11}$$

Consider next the signal u of (3.1.1). Combining (3.1.6) with (3.1.7) yields

$$u = \psi_d(v)d. \tag{3.1.12}$$

Because of the fact that the loop is internally stable, the transmission from (v,d) to u must be stable. This, together with (3.1.12), implies that $\psi_d(\cdot):S(R^m) \times S(R^p) \to S(R^m)$ is a stable system.

For the signal y we obtain from (3.1.1) and (3.1.5) that

$$\begin{aligned} y &= \Sigma\psi(v)z = \Sigma\psi(v)[I - \Sigma\psi(v)]^{-1}d \\ &= \Sigma\psi_d(v)d. \end{aligned} \tag{3.1.13}$$

Invoking again the internal stability requirement, the transmission from (v,d) to y must be stable, which implies that $\Sigma\psi_d(\cdot):S(R^m) \times S(R^p) \to S(R^p)$ is stable, as we have already concluded earlier.

The subsequent lemma is a summary of our main conclusions so far.

Lemma 3.1.14: Let $\Sigma: S(R^m) \to S(R^p)$ be a strictly causal system controlled within Fig. 1.1 with the causal equivalent controller $C: S(R^m) \times S(R^p) \to S(R^m): (v, z) \mapsto C(v, z)$. Assume that the configuration is internally stable, and denote $\psi(v)z := C(v, z)$. Then, the following systems are stable and causal.

a) $[I - \Sigma\psi(\cdot)]^{-1}: S(R^m) \times S(R^p) \to S(R^p): (v, d) \mapsto [I - \Sigma\psi(v)]^{-1}d$.
b) $[I + \Sigma\psi_d(\cdot)]: S(R^m) \times S(R^p) \to S(R^p): (v, d) \mapsto [I + \Sigma\psi_d(v)]d$.
c) $\psi_d(\cdot): S(R^m) \times S(R^p) \to S(R^m): (v, d) \mapsto \psi_d(v)d$.
d) $\Sigma\psi_d(\cdot): S(R^m) \times S(R^p) \to S(R^p): (v, d) \mapsto \Sigma\psi_d(v)d$. ♦

An obvious, though rather important, consequence of (3.1.10) and (3.1.7) is that, apart of the system Σ which is given and fixed, the effect of the disturbance d on the output signal z is entirely determined by the system $\psi(v)$; the latter describes the transmission of the equivalent controller C from its input z to its output u, for the external input sequence v. This implies that, as far as the effect of the disturbance on the output is concerned, the particular structure of the equivalent controller C is irrelevant; only the input/output characteristics of C (as expressed by $\psi(v)$) matter. Any controller C whose input/output behavior is equal to that of $\psi(v)$ yields the same influence of the disturbance d on the output z. Though this observation seems obvious, it will be of considerable importance later on, when we shall consider the internally stable implementation of controllers that achieve a desirable input&disturbance/output response. At that point, we shall decompose the equivalent controller C into two nested feedback loops to obtain an internally stable implementation. The present observation indicates that this decomposition will not affect the input&disturbance/output response of the entire configuration, as long as it preserves the input/output map of the (then composite) controller.

As we have seen, internal stability of Fig. 1.1 entails that the composition $\Sigma\psi_d(\cdot)$ is a stable system. To further examine the implications of this fact, assume that the given system Σ has a right coprime fraction representation $\Sigma = PQ^{-1}$, where $Q: S(R^m) \to S(R^m)$ is a bicausal system. (Theorem 2.13 considers the existence of fraction representation of this nature.) Invoking Corollary 2.15, while taking into account the stability of the composition $\Sigma\psi_d(\cdot)$ and the causality of $\psi_d(\cdot)$, we conclude that there is a stable and causal system $\phi(\cdot): S(R^m) \times S(R^p) \to S(R^m): (v, d) \mapsto \phi(v)d$ satisfying

$$\psi_d(v)d = Q\phi(v)d \tag{3.1.15}$$

for all $v \in S(R^m)$ and all $d \in S(R^p)$. Inserting this into (3.1.10) and recalling the fraction representation $\Sigma = PQ^{-1}$, the expression for the output signal z takes the form

$$z = [I + P\phi(v)]d \tag{3.1.16}$$

with $\phi(\cdot): S(R^m) \times S(R^p) \to S(R^m)$ being a stable and causal system. This validates the following statement, which is the main result of the present subsection.

Proposition 3.17: Let $\Sigma: S(R^m) \to S(R^p)$ be a strictly causal system having a right coprime fraction representation $\Sigma = PQ^{-1}$, where the denominator system $Q: S(R^m) \to S(R^m)$ is bicausal. Let $C: S(R^m) \times S(R^p) \to S(R^m)$ be any causal equivalent controller for which Fig. 1.1 is internally stable. Then, there is a stable and causal system $\phi: S(R^m) \times S(R^p) \to S(R^m): (v, d) \mapsto \phi(v)d$ such that the output sequence z is given by $z = [I + P\phi(v)]d$, where $v \in S(R^m)$ is the external input and $d \in S(R^p)$ is the disturbance. ♦

Note that the equivalent controller C can be directly recovered from the stable and causal system ϕ. Indeed, using (3.1.15) and (3.1.9), we have $\psi(v)z = Q\phi(v)[I + \Sigma Q\phi(v)]^{-1}z = Q\phi(v)[I + P\phi(v)]^{-1}z$, and, since $C(v, z) = \psi(v)z$, we obtain

$$C(v, z) = Q\phi(v)[I + P\phi(v)]^{-1}z. \tag{3.1.18}$$

The fundamental significance of Proposition 3.1.17 originates from the fact that its converse is also true. Specifically, we show that, for *any* stable and causal system $\phi: S(R^p) \times S(R^m) \to S(R^m)$, there is an internally stable control configuration around the system Σ whose output sequence z is given by the expression $z = [I + P\phi(v)]d$. In general, this control configuration may contain more than one feedback loop; all the control elements are then represented by the equivalent controller C of Fig. 1.1. When the proposition is combined with its converse, one obtains a parameterization of all possible responses of internally stable control configurations around the nonlinear system Σ. The arbitrary stable and causal system ϕ serves as the sole parameter in this parameterization.

Note that the proof of the converse direction of Proposition 3.1.17 is not an entirely trivial matter. True, once the system $\phi(\cdot)$ is specified, the equivalent controller C can be obtained directly from (3.1.18). When the system Σ is unstable, however, this controller cannot be used directly as the controller through which the loop is closed, since it requires an exact model of the denominator Q of Σ. In other words, if C is regarded as the actual controller, rather than as the equivalent controller it is, the closed loop may not be internally stable. The next subsection deals with the design of an internally stable control configuration whose *equivalent* controller is given by (3.1.18). This will provide a physically meaningful implementation of the response $\Sigma_c = [I + P\phi(\cdot)]$ for any stable and causal ϕ, demonstrating the converse direction of Proposition 3.1.17.

B. Internally Stable Implementations

To derive an internally stable implementation of the equivalent controller C of (3.1.18), we decompose Fig. 1.1 into two nested loops: an inner loop and an outer loop, as depicted in Fig. 3.2.1. The inner loop internally stabilizes the system Σ, and the outer loop complements the inner loop so as to yield an overall equivalent controller equal to C. This approach can be viewed as a "separation method," whereby the system Σ is first stabilized (by the inner loop), and then an outer loop is built around the stabilized system to achieve the desired performance. It validates a general "separation principle" according to which issues of stabilization and performance can be dealt with separately.

We denote by C_i the inner loop controller; its domains are $S(R^m) \times S(R^p) \to S(R^m): (s, z) \mapsto C_i(s, z)$. The outer loop

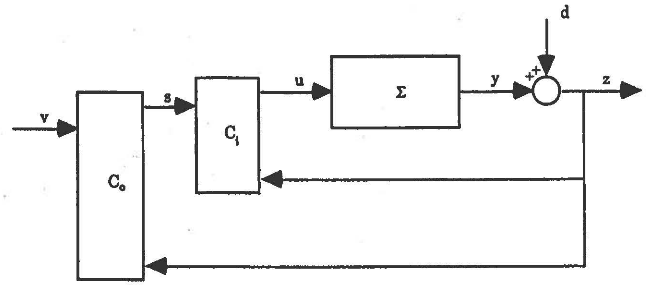

Fig. 3.2.1

is governed by the controller $C_o{:}S(R^m) \times S(R^p) \to S(R^m) : (v, z) \mapsto C_o(v, z)$. Both controllers are causal systems. It will be convenient to denote by Σ_i the system described by the inner loop and by Σ_{io} the overall system represented by Fig. 3.2.1. We shall write the response of the inner loop as $z = \Sigma_i(d)s$ and the response of the outer loop as $z = \Sigma_{io}(d)v$. The entire configuration is, of course, required to be internally stable.

As mentioned, the purpose of the controller C_i is twofold: 1) to provide internal stabilization of the given system Σ, and 2) to facilitate the computation of an outer loop controller C_o which, when combined with C_i, creates an internally stable configuration having the equivalent controller C of (3.1.18). These objectives are particularly easy to achieve when the controller C_i permits a left fraction representation of the form

$$C_i(s, z) = G^{-1}(z)[s + Tz] = u \tag{3.2.2}$$

where $T{:}S(R^p) \to S(R^m)$ and $G{:}S(R^m) \times S(R^p) \to S(R^m){:}(u, z) \mapsto G(z)u$ are causal and stable systems, with the partial system $G(z){:}S(R^m) \to S(R^m){:}(u) \mapsto G(z)u$ being invertible for every sequence $z \in S(R^p)$. To simplify our discussion, we shall assume throughout the remaining part of the paper that the inner controller C_i permits a representation of the form (3.2.2). Below are three general examples of controllers that possess representations of this form.

Example 3.2.3: Consider the classical control configuration in Fig. 3.2.4. Here, $\Sigma{:}S(R^m) \to S(R^p)$ is a strictly causal system that needs to be controlled. It is particularly convenient to choose the compensators $\pi{:}S(R^m) \to S(R^m)$ and $\varphi{:}S(R^p) \to S(R^m)$ in the special form

$$\begin{aligned} \varphi &= A, \\ \pi &= B^{-1} \end{aligned} \tag{3.2.5}$$

where $A{:}S(R^p) \to S(R^m)$ and $B{:}S(R^m) \to S(R^m)$ are stable systems with A being causal and B bicausal [4]. Letting $C_{(\pi,\varphi)}{:}S(R^m) \times S(R^p) \to S(R^m){:}(s, z) \mapsto C_{(\pi,\varphi)}(s, z) =: u$ denote the controller induced by π, φ, and the summer, we obtain

$$u = C_{(\pi,\varphi)}(s, z) = \pi[s - \varphi z] = B^{-1}[s - Az] \tag{3.2.6}$$

which is clearly of the form (3.2.2) with $G(z) := B$ and $T := -A$. Consequently, when the compensators π and φ internally stabilize the loop Fig. 3.2.4, one can take $C_i := C_{(\pi,\varphi)}$ in

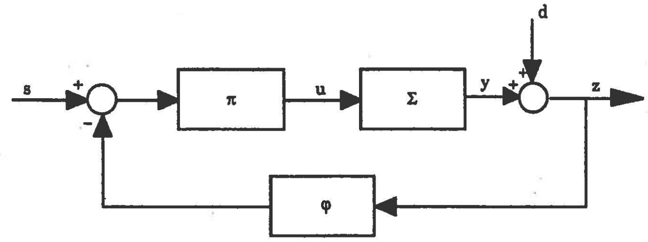

Fig. 3.2.4

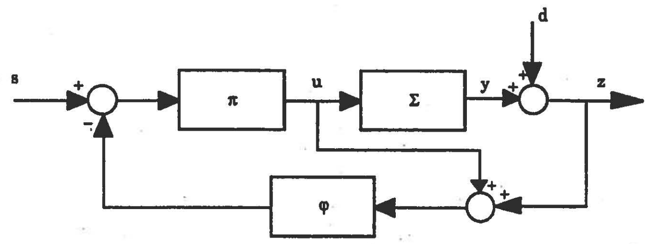

Fig. 3.2.8

Fig. 3.2.1, and (3.2.2 is satisfied. (see [6] for a more detailed discussion of this control configuration.) ♦

Example 3.2.7: As another example, consider the modification of Fig. 3.2.4 as shown in Fig. 3.2.8, which was employed in [7], [8] for the robust stabilization of nonlinear systems. In Fig. 3.2.8, it is necessary that the input space and the output space of the system Σ be of the same dimension, i.e., that $\Sigma : S(R^m) \to S(R^m)$; this, however, does not impair generality, since, as shown in [8], it can always be achieved by simple (formal) augmentation of one of those spaces. The compensators π and φ are again of the form (3.2.5), with $\varphi = A$ presently being strictly causal. Let $C_{1(\pi,\varphi)}{:}S(R^m) \times S(R^m) \to S(R^m){:}(s, z) \mapsto C_{1(\pi,\varphi)}(s, z) = u$ be the controller of this loop. Then, reading from the diagram, we have

$$u = \pi[s - \varphi(z + u)] = B^{-1}[s - A(z + u)]\,. \tag{3.2.9}$$

Note that the controller $C_{1(\pi,\varphi)}$ consists only of the compensators π and φ and of the summers and is examined now independently of the loop into which it will ultimately be inserted. It has the variables v and z as independent inputs and produces the output u. Define the partial system $A_+(z){:}S(R^m) \to S(R^m){:}u \mapsto A_+(z)u := A(z + u)$, which inherits the stability and strict causality of A. Then, (3.2.9) can be rewritten in the form

$$Bu = s - A_+(z)u \tag{3.2.10}$$

and

$$[B + A_+(z)]u = s\,. \tag{3.2.11}$$

On account of the bicausality of B and the strict causality of $A_+(z)$, it follows by Remark 2.3 that the system $[B + A_+(z)]{:}S(R^m) \to S(R^m){:}u \mapsto [B + A_+(z)]u$ is bicausal for any z, and we can invert it to obtain

$$u = [B + A_+(z)]^{-1}s\,. \tag{3.2.12}$$

Consequently, the controller in this case is given by

$$C_{1(\pi,\varphi)}(v,z) := [B + A_+(z)]^{-1}s \tag{3.2.13}$$

which is again of the form (3.2.2), with $G(z) := [B + A_+(z)]$ and $T = 0$, the zero system. Thus, $C_{1(\pi,\varphi)}$ is another possible candidate for the inner controller C_i of Fig. 3.2.1. The design of appropriate systems A and B for which the loop in Fig. 3.2.8 is internally stable was discussed in [8]. ♦

Example 3.2.14: As a third example of a controller that can be expressed in the form (3.2.2), consider the theory of reversible state feedback developed in [9] and briefly reviewed in Section II. Let $\Sigma: S(R^m) \to S(R^p): u \mapsto \Sigma u =: y$ be a system having the continuous realization $x_{k+1} = f(x_k, u_k)$, $y_k = h(x_k)$. Referring to Fig. 2.12, let $\Sigma_s: S(R^m) \to S(R^q)$ be the input/state part of Σ, which is given by the recursion

$$x_{k+1} = f(x_k, u_k) \tag{3.2.15}$$

with $f: S(R^q) \times S(R^m) \to S(R^q)$ a continuous function. The loop is closed through the continuous function $\sigma: R^q \times R^m \to R^m: (x_k, v_k) \mapsto \sigma(x_k, v_k)$. The feedback function σ is *strictly reversible* if the partial function $\sigma(x_k, \cdot): R^m \to R^m$ is a homeomorphism for every $x_k \in R^q$. (Remark: We use a somewhat stronger reversibility notion here, as compared to the one used in [9] and reviewed earlier in Section II, since we do not assume here that the input space is bounded.) In more explicit terms, strict reversibility means that for every vector $a \in R^q$, there is a continuous inverse function $\sigma^{-1}(a, \cdot): R^m \to R^m$ satisfying $\sigma^{-1}(a, \sigma(a, \cdot)) = \sigma(a, \sigma^{-1}(a, \cdot)) = I$, the identity function. In fact, the full function $\sigma^{-1}(\cdot, \cdot): R^q \times R^m \to R^m$ is also continuous, as indicated by Lemma 4.3, which is included with other technical results in Section IV.

We are interested in the use of state feedback to control systems that have continuous realizations with no direct access to the state. One method by which this can be accomplished relies on the following generalization of the standard linear notion of "reconstructibility."

Definition 3.2.16: Let $\Sigma: S(R^m) \to S(R^p): u \mapsto \Sigma u =: y$ be a system having the continuous realization $x_{k+1} = f(x_k, u_k)$, $y_k = h(x_k)$, where $f: R^q \times R^m \to R^q$ and $h: R^q \to R^p$ are continuous functions. We say that this realization is *reconstructible* if there is a strictly causal and ℓ^∞-stable system $\Xi: S(R^p) \times S(R^m) \to S(R^q): (y, u) \mapsto \Xi(y, u)$ such that $x = \Xi(y, u)$, where $x \in S(R^q)$ is the sequence of states corresponding to the input sequence u and the output sequence y of Σ. ♦

To clarify definition 3.2.16, consider for a moment the special case of a linear time-invariant discrete-time system, having the realization $x_{k+1} = Ax_k + Bu_k$, $y_k = Cx_k$ with state space dimension q. In this case, it is well known that reconstructibility is a notion somewhat weaker than observability. Specifically, the realization is reconstructible if and only if Ker $A^q \supset$ Ker $\mathcal{O}$, where $\mathcal{O}$ is the observability matrix of the realization. When this realization is reconstructible, one can express

$$\begin{aligned} x_{k+q} = {} & R_0 y_k + R_1 y_{k+1} + \cdots + R_{q-1} y_{k+q-1} \\ & + T_0 u_k + T_1 u_{k+1} + \cdots + T_{q-1} u_{k+q-1} \\ =: {} & \xi(y_k, \cdots, y_{k+q-1}, u_k, \cdots, u_{k+q-1}). \end{aligned}$$

Here, R_j, $j = 0, \cdots, q-1$, and T_i, $i = 0, \cdots, q-1$, are constant matrices, and ξ denotes the resulting function. The system Ξ of the definition is in this case given by $\Xi(y, u)]_{k+q} = \xi(y_k, \cdots, y_{k+q-1}, u_k, \cdots, u_{k+q-1})$ for all k, and it is clearly strictly causal and ℓ^∞-stable.

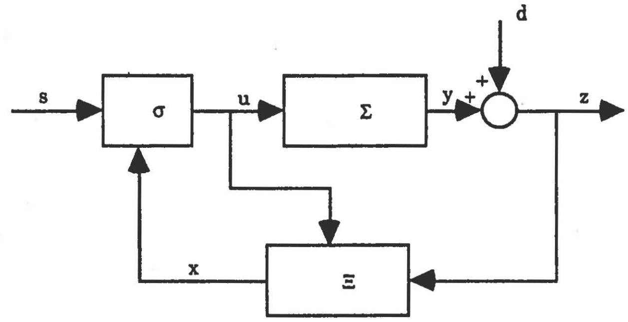

Fig. 3.2.17

Consider now a nonlinear system $\Sigma: S(R^m) \to S(R^p)$ that possesses a continuous and reconstructible realization, and let Ξ be the system of Definition 3.2.16. Combining reconstruction with state feedback yields the configuration shown in Fig. 3.2.17. By inspection,

$$u = \sigma(x, s) = \sigma(\Xi(z, u)s). \tag{3.2.18}$$

Assume now that σ is a strictly reversible state feedback function that internally stabilizes the configuration. (For the case where the input space is bounded and the disturbance d is small, σ can be derived using the theory developed in [9].) In element form, (3.2.18) takes the form

$$u_k = \sigma(\Xi(z, u)]_k, s_k) \tag{3.2.19}$$

$k = 0, 1, 2, \cdots$. The strict reversibility of σ implies then that

$$s_k = \sigma^{-1}(\Xi(z, u)]_k, u_k) \tag{3.2.20}$$

for all integers $k \geq 0$, or, in sequence form,

$$s = \sigma^{-1}(\Xi(z, u), u). \tag{3.2.21}$$

Define now, for each $z \in S(R^p)$, the system $B(z): S(R^m) \to S(R^m): u \mapsto B(z)u$ given by

$$B(z)u := \sigma^{-1}(\Xi(z, u), u) \tag{3.2.22}$$

for all $u \in S(R^m)$. The continuity of the function σ^{-1} combined with the ℓ^∞-stability of Ξ imply that $B(\cdot): S(R^p) \times S(R^m) \to S(R^m): (z, u) \mapsto B(z)u$ is ℓ^∞-stable. By (3.3.20), $B(\cdot)$ is also causal. Furthermore, by Lemma 4.4, which is included with other technical results in Section IV, the system $B(z)$ is, in fact, bicausal. But then we can write

$$u = B^{-1}(z)s \tag{3.2.23}$$

which is of the form (3.2.2) with $G(z) = B(z)$ and $T = 0$, the zero system. Thus, we have another stabilizing scheme of the required form. ♦

To summarize, the previous examples indicate that various control configurations that are commonly utilized to stabilize nonlinear systems possess controllers that permit left fraction representations of the form (3.2.2). We shall therefore assume from now on that the controller C_i that is used to internally stabilize the inner loop of Fig. 3.2.1 admits the representation (3.2.2). This will assist us in proving the converse direction of Proposition 3.1.17, i.e., that every response of the form (3.1.16) has an internally stable implementation. We proceed now to prove the latter.

Let $\Sigma{:}S(R^m) \to S(R^p)$ be the system that needs to be controlled. We assume that Σ is strictly causal and entirely stabilizable and that a right coprime fraction representation $\Sigma = PQ^{-1}$ with a bicausal denominator system $Q{:}S(R^m) \to S(R^m)$ is available. The numerator system $P{:}S(R^m) \to S(R^p)$ is then strictly causal, since $P = \Sigma Q$ and Σ is strictly causal. Given a stable and causal system $\phi(\cdot) : S(R^m) \times S(R^p) \to S(R^m)$, our objective is to find an internally stable control configuration whose output sequence z is given by $z = [I + P\phi(v)]d$, where v is the external input signal and d is the disturbance. As mentioned earlier, we achieve this objective with the control configuration as shown in Fig. 3.2.1. The construction of appropriate controllers C_i and C_o proceeds as follows.

Step 1. Design any strictly internally stable closed loop around the system Σ (see Definition 2.8), with a controller C_i that permits a left fraction representation of the form (3.2.2). Indications of some suitable configurations are provided in Examples 3.2.3, 3.2.7, and 3.2.14, and in the references mentioned therein (we do not discuss in detail the stabilization problem per se in this paper). This controller is then used as the inner controller in Fig. 3.2.1. Note that this step of the design process is concerned only with stabilization of the system Σ and is independent of the system ϕ. The same controller C_i can be used for all ϕ. ♦

Step 2. Once the inner loop controller C_i has been computed, the outer loop controller of C_o of Fig. 3.2.1 can be derived in the following rather straightforward way. Let $\phi(\cdot) : S(R^m) \times S(R^p) \to S(R^m) : (v, d) \mapsto \phi(v)d$ be any stable and causal system. Recalling from (3.2.2) that $C_i = G^{-1}(z)[s + Tz]$, simply set

$$C_o(v, z) = G(z)C(v, z) - Tz \tag{3.2.24}$$

where C is given by

$$C(v, z) := Q\phi(v)[I + P\phi(v)]^{-1}z\,. \tag{2.2.25}$$

♦

We assert that the controllers C_i and C_o render Fig. 3.2.1 internally stable and assign to it the response $z = [I + P\phi(v)]d$. Consequently these controllers fulfill our objective.

We prove now the validity of Step 2. First, note that C_o is causal, due to the causality of G, C, and T. Reading from Fig. 3.2.1, we have

$$\begin{aligned} s &= C_o(v, z) \\ u &= C_i(s, z) \end{aligned} \tag{3.2.26}$$

so that

$$u = C_i(C_o(v, z), z)\,. \tag{3.2.27}$$

Using (3.2.2) for C_i we obtain

$$u = G^{-1}(z)[C_o(v, z) + Tz]\,. \tag{3.2.28}$$

Let $C' : S(R^m) \times S(R^p) \to S(R^m) : (v, z) \mapsto C'(v, z)$ be the equivalent controller induced by C_i and C_o in Fig. 3.2.1, so that $u = C'(v, z)$. Then, by (3.2.28),

$$C'(v, z) = G^{-1}(z)[C_o(v, z) + Tz]\,. \tag{3.2.29}$$

For the controller C_o of (3.2.24), this yields $C'(v, z) = C(v, z)$, with C being given by (3.2.25). Consequently, with the present controllers C_i and C_o, we obtain

$$u = Q\phi(v)[I + P\phi(v)]^{-1}z\,. \tag{3.2.30}$$

But then

$$\begin{aligned} z &= y + d = \Sigma u + d \\ &= PQ^{-1}u + d \\ &= PQ^{-1}\Big\{Q\phi(v)[I + P\phi(v)]^{-1}z\Big\} + d \\ &= P\phi(v)[I + P\phi(v)]^{-1}z + d \\ &= \{I + P\phi(v)\}[I + P\phi(v)]^{-1}z \\ &\quad - [I + P\phi(v)]^{-1}z + Id \\ &= Iz - [I + P\phi(v)]^{-1}z + d \\ &= z - [I + P\phi(v)]^{-1}z + d\,. \end{aligned}$$

Canceling the z term on both sides and rearranging, we obtain $[I + P\phi(v)]^{-1}z = d$, or $z = [I + P\phi(v)]d$. Thus, the proposed configuration achieves the desired response, and it only remains to show that internal stability holds. We proceed now to prove the latter.

Consider then Fig. 3.2.1 with the controllers C_i and C_o constructed in Steps 1 and 2 above. Recall that $\Sigma_i : S(R^p) \times S(R^m) \to S(R^p) : (d, s) \mapsto \Sigma_i(d)s$ denotes the system induced by the inner loop of Fig. 3.2.1. Since, according to Step 1, the controller C_i renders the inner loop strictly internally stable, the system Σ_i is stable and differentially bounded. In particular, the partial system $\Sigma_i(d) : S(R^m) \times S(R^p) \to S(R^p) : s \mapsto \Sigma_i(d)s$ is stable for all bounded d and is differentially bounded. By the strict causality of Σ, the partial system $\Sigma_i(d)$ is also strictly causal for all $d \in S(R^p)$ (this follows directly from (3.1.11) when specialized to the present case by replacing C by C_i; v by s; and Σ_c by Σ_i; while regarding d as fixed). We can now state the following result, whose proof is listed in Section IV below.

Proposition 3.2.31: Let $\Sigma : S(R^m) \to S(R^p)$ be a strictly causal system having a right coprime fraction representation $\Sigma = PQ^{-1}$, with $Q : S(R^m) \to S(R^m)$ bicausal. Assume that controllers C_i and C_o were derived for Σ using Steps 1 and 2 above. Finally, let $\phi : S(R^m) \times S(R^p) \to S(R^m) : (v, d) \mapsto \phi(v)d$ be any stable and causal system. Then, the controllers C_i and C_o render Fig. 3.2.1 internally stable and assign to it the response $z = [I + P\phi(v)]d$.

We provide now a simple example to illustrate the computation of the controller C_o of Step 2.

Example 3.2.32: Consider the strictly stable and strictly causal system $\Sigma : S(R) \to S(R)$ given by the recursion

$$x_{k+1} = 0.5\sin(x_k) + u_k, \qquad x_0 = 0,$$
$$y_k = x_k$$

where u is the input sequence of Σ and y is its output sequence. We take the stable and causal parameter system $\phi : S(R) \times S(R) \to S(R) : (v, d) \mapsto \phi(v)d$ to be

$$\xi_{k+1} = 0.5\xi_k + v_k d_k, \qquad \xi_0 = 0,$$
$$\eta_k = \xi_k$$

where η is the output sequence of ϕ. Due to the stability of Σ, we can choose a right coprime fraction $\Sigma = PQ^{-1}$ with $Q = I$, the identity system, and $P = \Sigma$. The stability of Σ also renders the inner loop controller C_i unnecessary, so we can choose $C_i = I$. Referring to (3.2.2), we then have $G(z) = I$ and $T = 0$. By (3.2.24), this yields $C_o(v, z) = C(v, z)$, and, utilizing (3.2.25), we obtain

$$C_o(v, z) = \phi v[I + \Sigma\phi(v)]^{-1} z \tag{3.2.33}$$

in this case. We now derive a realization for the controller C_o, which is the only controller in this case.

First, we write the equations for the composition $\Sigma\phi(v)$. Here, the sequence v is regarded as a "parameter," d is the input, and y is the ouput. This simply yields

$$\Sigma\phi(v)d : \begin{cases} \xi_{k+1} = 0.5\xi_k + v_k d_k \\ x_{k+1} = 0.5\sin(x_k) + \xi_k \\ y_k = x_k\,. \end{cases}$$

Next, let z be the output sequence of the system $[I + \Sigma\phi(v)]$. Then, $z_k = d_k + y_k = d_k + x_k$, or $d_k = z_k - x_k$. The system $[I + \Sigma\phi(v)]^{-1}$ has the input sequence z and the output sequence d and is given by

$$[I + \Sigma\phi(v)]^{-1} : \begin{cases} \xi_{k+1} = 0.5\xi_k + v_k(z_k - x_k) \\ x_{k+1} = 0.5\sin(x_k) + \xi_k \\ d_k = z_k - x_k\,. \end{cases}$$

Finally, the controller C_o of (3.2.33) is a system of order 3, with the following representation, where z and v are the input sequences and u is the output sequence.

$$C_o : \begin{cases} \xi_{k+1} = 0.5\xi_k + v_k(z_k - x_k) \\ x_{k+1} = 0.5\sin(x_k) + \xi_k \\ \zeta_{k+1} = 0.5\zeta_k + v_k(z_k - x_k) \\ u_k = \zeta_k\,. \end{cases}$$

♦

To conclude, Proposition 3.2.31 shows that any response of the form $z = [I + P\phi(v, \cdot)]d$ can be assigned to the system Σ by an internally stable control configuration. When this is combined with Proposition 3.1.17, we obtain the following parameterization of all responses that can be achieved through internally stable control of Σ. This is the main result of this paper.

Theorem 3.2.34: Let $\Sigma : S(R^m) \to S(R^p)$ be a strictly causal system having a right coprime fraction representation $\Sigma = PQ^{-1}$ with a bicausal denominator $Q : S(R^m) \to S(R^m)$. Assume that Σ can be strictly internally stabilized by a controller that admits the representation (3.2.2). Then, referring to (1.2), the following is true. The class of all input&disturbance/output responses Σ_c that can be achieved through internally stable control of Σ is given by

$$\Big\{\Sigma_c(v, d) = [I + P\phi(v)]d, \phi(\cdot) : S(R^m) \times S(R^p) \to S(R^m) : (v, d) \mapsto \phi(v)d \text{ is a stable and causal system.}\Big\}.$$ ♦

We have shown in Examples 3.2.3, 3.2.7, and 3.2.14 that many of the controllers used to stabilize nonlinear systems admit a representation of the form (3.2.2). Right coprime fraction representations with bicausal denominators are discussed in Theorem 2.13.

Theorem 3.2.34 provides a rather simple and transparent parameterization of the class of all systems that can be obtained from a given system Σ by internally stable control. The most intriguing application of this result would be, of course, a solution of the nonlinear optimal disturbance attenuation problem. In that context, ϕ will be determined by an optimization process aimed at minimizing the effect of the disturbance d on the output signal $z = \Sigma_c(v, d)$. Once ϕ is determined, the present subsection outlines an internally stable implementation of the optimal system. These topics form the subject of a separate report.

It is also interesting to note that Step 2 above provides an explicit formula for the outer loop controller C_o, so that no equations need to be solved to obtain C_o once the inner loop controller C_i is known. The computation of C_i involves the solution of the internal stabilization equations for Σ, but is independent of the parameter ϕ that characterizes the performance of the final system, as discussed earlier. This shows the validity of a general principle of separation, whereby stabilization and performance can be treated as separate issues.

We conclude our present discussion by specializing Theorem 3.2.34 to the case where the system Σ is stable. In that case, we can simply take $P = \Sigma$ and $Q = I$, and we obtain the following.

Corollary 3.2.35: Let $\Sigma : S(R^m) \to S(R^p)$ be a strictly causal, stable, and differentially bounded system. Then, referring to (1.2), the following is true. The class of all input&disturbance/output responses Σ_c that can be achieved through internally stable control of Σ is given by

$$\Big\{\Sigma_c(v, d) = [I + \Sigma\phi(v)]d, \phi(\cdot) : S(R^m) \times S(R^p) \to S(R^m) : (v, d) \mapsto \phi(v)d \text{ is a stable and causal system.}\Big\}.$$ ♦

IV. Proofs and Technicalities

Proof of Lemma 2.2: We start by showing that the system $(I - \Gamma) : S(R^p) \to S(R^p)$ is injective (one to one). Indeed, let $z, \zeta \in S(R^p)$ be two sequences satisfying

$$(I - \Gamma)z = (I - \Gamma)\zeta\,. \tag{4.1}$$

We show that $z = \zeta$, which proves that $(I - \Gamma)$ is injective. Note that the strict causality of Γ implies that, for every input sequence $u \in S(R^p)$, the output value $y_0 := \Gamma u]_0$ at the time zero is entirely determined by the initial conditions of Γ and does not depend on u. Consequently, at the time zero, we have $(I - \Gamma)z]_0 = z_0 - y_0$ and $(I - \Gamma)\zeta]_0 = \zeta_0 - y_0$. Together with (4.1), this yields $z_0 = \zeta_0$.

In preparation for an induction, assume that $z_0^i = \zeta_0^i$ for some integer $i \geq 0$. By the strict causality of the system Γ, this implies that $\Gamma z]_0^{i+1} = \Gamma\zeta]_0^{i+1}$, so that, in particular, $\Gamma z]_{i+1} = \Gamma\zeta]_{i+1} =: y_{i+1}$. Substituting the latter into (4.1), we obtain $(I - \Gamma)z]_{i+1} = z_{i+1} - y_{i+1} = (I - \Gamma)\zeta]_{i+1} = \zeta_{i+1} - y_{i+1}$, which directly yields $z_{i+1} = \zeta_{i+1}$. By induction, this proves that $z = \zeta$, and the system $(I - \Gamma) : S(R^p) \to S(R^p)$ is injective.

Next we show that the system $(I - \Gamma) : S(R^p) \to S(R^p)$ is surjective (onto). To this end, let $d \in S(R^p)$ be an arbitrary sequence. The following argument proves the existence of a sequence $z \in S(R^p)$ satisfying $d = (I - \Gamma)z$, which implies surjectivity. First, at the time zero, set $z_0 := d_0 + y_0$, where y_0 is the initial output of the system Γ. The remaining elements of the sequence z are determined via recursion in the following way. Let $i \geq 0$ be some integer for which the elements $z_0, z_1, \cdots, z_i$ of the sequence z have been determined. Let $z^* \in S(R^p)$ be any sequence having $z_0, z_1, \cdots, z_i$ as its first elements, i.e., $z^*]_o^i = \{z_0, z_1, \cdots, z_i\}$. By the strict causality of Γ, the output value $y_{i+1} := \Gamma z^*]_{i+1}$ is uniquely determined by the elements $z_0, z_1, \cdots, z_i$. Setting $z_{i+1} := d_{i+1} + y_{i+1}$ provides the element number $i+1$ of the sequence z. By induction, this process defines a unique sequence $z \in S(R^p)$, which, by construction, clearly satisfies $d = (I - \Gamma)z$; since $d \in S(R^p)$ was arbitrary, it follows that $(I - \Gamma)$ is surjective. Furthermore, note that the determination of the sequence z from the sequence d is causal, since the computation of the element z_i involves only the elements $d_0, \cdots .d_i$ of the sequence d.

To conclude, we have shown that the system $(I - \Gamma) : S(R^p) \to S(R^p)$ is both injective and surjective and, whence, is a set isomorphism. Furthermore, the system $[I - \Gamma]$, being the sum of two causal systems, is causal. The last sentence of the previous paragraph shows that the inverse $(I - \Gamma)^{-1} : S(R^p) \to S(R^p)$ is likewise causal. Thus, $(I - \Gamma)$ is bicausal, as asserted. ♦

Proof of Proposition 2.14: We show that the premise of the proposition implies that the system $Q^{-1}D : S(R^r) \to S$ is stable. The proposition follows then simply by setting $\phi := Q^{-1}D$.

Let $\theta > 0$ be a real number. By the stability of the system ΣD, there is a real number $\tau > 0$ such that $\Sigma D[S(\theta^r)] \subset S(\tau^p)$. Furthermore, by the stability of the system D, there is a real number $\beta > 0$ such that $D[S(\theta^r)] \subset S(\beta^m)$. Let $\gamma := \max[\tau, \beta]$, and denote $S_\theta := Q^{-1}D[S(\theta^r)]$. (The set S_θ is closed: compactness of $S(\theta^r)$ and stability of D imply that $D[S(\theta^r)]$ is compact, but then S_θ is the inverse image of a closed set through a continuous map and must be closed.) The definition of γ yields $S_\theta \subset P^*[S(\gamma^p)] \cap Q^*[S(\gamma^m)]$. By the Definition 2.11 of right coprimeness, there is then a real number $\zeta > 0$ such that $S_\theta \subset S(\zeta^q)$. Since θ was arbitrary, this shows directly that the system $Q^{-1}D$ is BIBO-stable.

To show that the restriction $Q^{-1}D : S(\theta^r) \to S$ is continuous as well, note that condition 2) of the right coprimeness Definition 2.11 directly implies that the set S_θ is compact, since $S_\theta \subset S \cap S(\zeta^q)$, and a closed subset of a compact set is compact. Now, let $u_i \in S(\theta^r)$, $i = 0, 1, 2, \cdots$, be any convergent sequence, and denote $u := \lim_{i\to\infty} u_i$. Let $s_i := Q^{-1}Du_i$, $i = 0, 1, 2, \cdots$. Since $s_i \in S_\theta$ for all $i = 0, 1, 2, \cdots$ and S_θ is compact, it follows that the sequence $\{s_i\}$ has a convergent subsequence $s_{i(j)}$, $j = 0, 1, 2, \cdots$; let the limit of this subsequence be $s := \lim_{j\to\infty} s_{i(j)}$. According to our notation, we have $s_{i(j)} = Q^{-1}Du_{i(j)}$, so that

$$Qs_{i(j)} = Du_{i(j)}, \qquad j = 0, 1, 2, \cdots \tag{4.2}$$

Now, by continuity of the systems Q and D, it follows directly that $\lim_{j\to\infty} Qs_{i(j)} = Qs$ and $\lim_{j\to\infty} Du_{i(j)} = Du$. In view of (4.2), this yields $Qs = Du$ or $s = Q^{-1}Du$. Since the latter is valid for any convergent subsequence of $\{s_i\}$, it follows that the sequence $\{s_i\}$ has the single accumulation point $s = Q^{-1}Du$ to which it converges. Finally, since this holds for any convergent sequence $\{u_i\}$ in $S(\theta^r)$, we conclude that the restriction of $Q^{-1}D$ to $S(\theta^r)$ is continuous for any real $\theta > 0$. Combining this with the conclusion of the second paragraph of the proof, we obtain that $Q^{-1}D$ is a stable system. ♦

The following two technical results were employed in Example 3.2.14, where the partial inverse function σ^{-1} is defined.

Lemma 4.3: If $\sigma : R^q \times R^m \to R^m : (x, v) \mapsto \sigma(x, v)$ is a continuous and strictly reversible function, then the partial inverse function $\sigma^{-1}(\cdot, \cdot) : R^q \times R^m \to R^m$ is continuous.

Proof: By definition of strict reversibility, the partial function $\sigma^{-1}(x, \cdot)$ is continuous for all $x \in R^q$. Consider now the continuity of the partial function $\sigma^{-1}(\cdot, v)$ for a fixed $v \in R^m$. Let $\Delta \in R^q$, and write $\alpha := \sigma^{-1}(x + \Delta, v)$, $\beta := \sigma^{-1}(x, v)$, and $\zeta := \sigma(x + \Delta, \alpha) - \sigma(x, \alpha)$. Noting that $v = \sigma(x + \Delta, \alpha) = \sigma(x, \beta)$, it follows that $\sigma(x, \alpha) = v - \zeta$. Consequently, $\alpha = \sigma^{-1}(x, v - \zeta)$ and $\beta = \sigma^{-1}(x, v)$.

Now, let $\delta > 0$ be a real number. By continuity of the function $\sigma^{-1}(x, \cdot)$, there is a real number $\eta > 0$ such that $|\alpha - \beta| = |\sigma^{-1}(x, v - \zeta) - \sigma^{-1}(x, v)| < \delta$ whenever $|\zeta| < \eta$. Furthermore, by continuity of the function σ, there is a real number $\varepsilon > 0$ such that $|\zeta| = |\sigma(x + \Delta, \alpha) - \sigma(x, \alpha)| < \eta$ whenever $|\Delta| < \varepsilon$. This yields that $|\alpha - \beta| = |\sigma^{-1}(x + \Delta, v) - \sigma^{-1}(x, v)| < \delta$ whenever $|\Delta| < \varepsilon$, and $\sigma^{-1}(\cdot, v)$ is a continuous function.

Finally, the inequality

$$\begin{aligned}
|\sigma^{-1}(x + \Delta, v + \xi) - \sigma^{-1}(x, v)| &= \big|[\sigma^{-1}(x + \Delta, v + \xi) \\
&\quad - \sigma^{-1}(x + \Delta, v)] \\
&\quad + [\sigma^{-1}(x + \Delta, v) \\
&\quad - \sigma^{-1}(x, v)]\big| \\
&\leq \big|[\sigma^{-1}(x + \Delta, v + \xi) \\
&\quad - \sigma^{-1}(x + \Delta, v)]\big| \\
&\quad + \big|[\sigma^{-1}(x + \Delta, v) \\
&\quad - \sigma^{-1}(x, v)]\big|
\end{aligned}$$

combined with the fact that we have uniform continuity over bounded domains for the partial functions, implies that $\sigma^{-1}(\cdot,\cdot)$ is continuous. ♦

Lemma 4.4: The system $B(z) : S(R^m) \to S(R^m) : u \mapsto B(z)u$ of (3.2.22) is bicausal for every $z \in S(R^p)$.

Proof: We have already noticed that $B(z)$ is causal. To prove that it is bicausal, it is enough to show that u is uniquely determined by $s := B(z)u$ in a causal manner, for every fixed $z \in S(R^p)$. To this end, fix a sequence $z \in S(R^p)$ and consider (3.2.19). First, since $\Xi(z,\cdot)$ is a strictly causal system by Definition 3.2.16, the first element $\Xi(z,u)]_0$ is determined by the initial conditions and does not depend on the sequence u (that starts at the time 0). Since $u_0 = \sigma(\Xi(z)u]_0, s_0)$ by (3.2.19), it follows that u_0 can be determined from s_0 (and the initial conditions). Next, preparing for recursion, assume that, for some integer $i \geq 0$, the elements u_0^i have been determined from the elements s_0^i. To determine the element u_{i+1}, let $w = (u_0, \cdots, u_i, w_{i+1}, \cdots) \in S(R^m)$ be any sequence starting with the elements $u_0, \cdots, u_i$. By the strict causality of the system $\Xi(z,\cdot)$, we have $\Xi(z,u)]_{i+1} = \Xi(z,w)]_{i+1}$, since $\Xi(z,u)]_{i+1}$ is determined by elements $0, \cdots, i$ of the sequence. Hence, $\Xi(z,u)]_{i+1}$ is determined by the elements $s_0, \cdots, s_i$. But $u_{i+1} = \sigma(\Xi(z)w]_{i+1}, s_{i+1})$ according to (3.2.19), and it follows that u_{i+1} is determined by the elements $s_0, \cdots, s_{i+1}$. This shows that u depends causally on s. Having shown earlier that s also depends causally on u, we conclude that $B(z) : S(R^m) \to S(R^m) : u \mapsto B(z)u$ is a bicausal system for any $z \in S(R^p)$. ♦

The following statement simplifies the proof of Proposition 3.2.31, since it implies that in the course of proving internal stability, each disturbance signal within the configuration can be examined separately.

Lemma 4.5: Let $F : E_1 \times E_2 \times \cdots \times E_n \to E : (e_1, e_2, \cdots, e_n) \mapsto F(e_1, e_2, \cdots, e_n)$ be a function, where $E_1, \cdots, E_n$, E are compact subsets of normed spaces. Denote by $\|\cdot\|$ the norm on each one of those spaces, and define a norm on the product space by setting $\|(e_1, \cdots, e_n)\| := \max_{k=1,\cdots n}\{\|e_k\|\}$. Assume that, for each $i = 1, \cdots, n$, the partial function $F(e_1, \cdots, e_{i-1}, \cdot, e_{i+1}, \cdots, e_n) : E_i \to E : e_i \mapsto F(e_1, e_2, \cdots, e_n)$ is continuous for all $e_1, \cdots, e_{i-1}, e_{i+1}, \cdots, e_n$. Then, F is a continuous function.

Proof: We use $n = 2$ to simplify notation; the same basic argument applies to the general case. Fix two elements $e_1 \in E_1$ and $e_2 \in E_2$, and let S_i be the set of all elements $\varepsilon_i \in E_i$ for which $e_i + \varepsilon_i \in E_i$, $i = 1, 2$. Notice that the sets S_i, $i = 1, 2$ are also compact. Clearly,

$$\begin{aligned}
&\|F((e_1+\varepsilon_1),(e_2+\varepsilon_2)) - F(e_1,e_2)\| \\
&\quad = \|[F((e_1+\varepsilon_1),(e_2+\varepsilon_2)) - F((e_1+\varepsilon_1),e_2)] \\
&\qquad + [F((e_1+\varepsilon_1),e_2) - F(e_1,e_2)]\| \\
&\quad \leq \|F((e_1+\varepsilon_1),(e_2+\varepsilon_2)) - F((e_1+\varepsilon_1),e_2)\| \\
&\qquad + \|F((e_1+\varepsilon_1),e_2) - F(e_1,e_2)\| \qquad (4.6)
\end{aligned}$$

Now, fix some real number $\delta > 0$. By continuity of the partial functions, there are then real numbers $\xi_1, \xi_2(\varepsilon_1) > 0$ such that $\|F((e_1+\varepsilon_1),(e_2+\xi_2)) - F((e_1+\xi_1),e_2)\| < \delta/2$ whenever $\|\varepsilon_2\| < \xi_2(\varepsilon_1)$ and $\|F((e_1+\varepsilon_1),e_2) - F(e_1,e_2)\| < \delta/2$ whenever $\|\varepsilon_1\| < \xi_1$. Now, compactness implies that there is a real number $\xi' > 0$ such that the function $\xi_2(\varepsilon_1)$ can be chosen to satisfy $\xi_2(\varepsilon_1) > \xi'$ for all $\varepsilon_1 \in S_1$. Otherwise, by a standard convergent sequence argument, there would be an element $\varepsilon_1' \in S_1$ for which the partial function $F((e_1+\varepsilon_1'),\cdot)$ is not continuous at the point e_2. Defining $\xi := \min[\xi_1, \xi']$, we clearly have that $\xi > 0$ and, using (4.6), yields $\|F((e_1+\varepsilon_1),(e_2+\varepsilon_2)) - F(e_1,e_2)\| \leq \|F((e_1+\varepsilon_1),(e_2+\varepsilon_2)) - F((e_1+\varepsilon_1),e_2)\| + \|F((e_1+\varepsilon_1),e_2) - F(e_1,e_2)\| < \delta/2 + \delta/2 = \delta$ for all $\varepsilon_1 \in S_1$ and all $\varepsilon_2 \in S_2$ satisfying $\|\varepsilon_1\| < \xi$ and $\|\varepsilon_2\| < \xi$. This proves the lemma. ♦

Proof of Proposition 3.2.31: We have shown earlier in Section III that the controllers listed in the proposition achieve the response $z = [I + P\phi(v)]d$. Thus, it only remains to show that the configuration is internally stable. According to (3.2.30), the equivalent controller C induced by the controllers C_i and C_o of the proposition is $C(v,z) = Q\phi(v)[I + P\phi(v)]^{-1}z$. Adhering to our usual practice of denoting $\psi(v)z := C(v,z)$, we have

$$\psi(v)z = C(v,z) = Q\phi(v)[I + P\phi(v)]^{-1}z. \qquad (4.7)$$

The proof is divided into several parts enumerated below.

1) Inserting (4.7) into (3.1.12) yields

$$\begin{aligned}
u = \psi_d(v)d &= \psi(v)[I - \Sigma\psi(v)]^{-1}d \\
&= Q\phi(v)[I + P\phi(v)]^{-1} \\
&\quad \cdot \left\{I - PQ^{-1}Q\phi(v)[I + P\phi(v)]^{-1}\right\}^{-1} d \\
&= Q\phi(v)[(I + P\phi(v)) - P\phi(v)]^{-1}d \\
&= Q\phi(v)d
\end{aligned}$$

so that

$$u = \psi_d(v)d = Q\phi(v)d. \qquad (4.8)$$

Since Q and ϕ are both stable and causal, we conclude that $\psi_d(\cdot) : S(R^m) \times S(R^p) \to S(R^m)$ is a stable and causal system, and the transmission from (v,d) to u in Fig. 3.2.1 is stable.

2) The relation $z = [I + P\phi(v)]d$ shows directly that the transmission from (v,d) to the output z in Fig. 3.2.1 is stable, since P and ϕ are stable.

3) Consider now the signal s in Fig. 3.2.1. By (3.2.26), we have $u = C_i(s,z)$, and, together with (3.2.2), this yields

$$s = G(z)u - Tz. \qquad (4.9)$$

Substituting for u and z the formulas from parts 1) and 2) of the present proof, we obtain

$$s = G([I + P\phi(v)]d)Q\phi(v)d - T[I + P\phi(v)]d. \qquad (4.10)$$

This indicates that the transmission from (v,d) to s is stable, since G, P, ϕ, Q, and T are all stable systems.

4) Note that an additive disturbance on the signal y can be regarded as part of the disturbance d and does not require separate consideration.

5) We examine now the effect of a disturbance $n \in S(R^m)$ being added to the signal s of Fig. 3.2.1. For this purpose it is convenient to redraw Fig. 3.2.1 in the form as depicted in

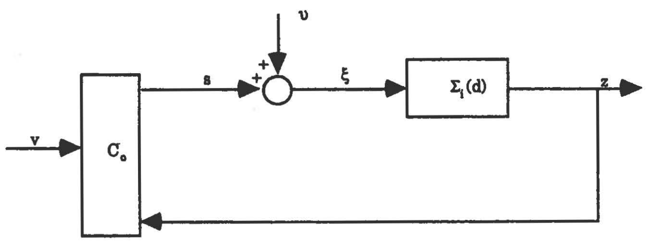

Fig. 4.11

Fig. 4.11. Recall that $\Sigma_i(d)$ is a strictly causal system for every fixed $d \in S(R^m)$, and C_o is causal. The diagram indicates that

$$\begin{aligned} z &= \Sigma_i(d)\xi \\ \xi &= n + s = [n + C_o(v, z)] \end{aligned} \tag{4.12}$$

Using the notation $\psi_o(v)z := C_o(v, z)$ we get

$$\xi = n + \psi_o(v)z = n + \psi_o(v)\Sigma_i(d)\xi \tag{4.13}$$

or

$$n = \xi - \psi_o(v)\Sigma_i(d)\xi = [I - \psi_o(v)\Sigma_i(d)]\xi. \tag{4.14}$$

Now, the combination $\psi_o(v)\Sigma_i(d) : S(R^m) \to S(R^m) : \xi \mapsto \psi_o(v)\Sigma_i(d)\xi$ is strictly causal for fixed d and v, since $\psi_o(v)$ is causal and $\Sigma_i(d)$ is strictly causal. By Lemma 2.2, this implies that the system $[I - \psi_o(v)\Sigma_i(d)] : S(R^m) \to S(R^m)$ is bicausal (for each d and v) and has a causal inverse $[I - \psi_o(v)\Sigma_i(d)]^{-1} : S(R^m) \to S(R^m)$. We can then write

$$\xi = [I - \psi_o(v)\Sigma_i(d)]^{-1}n \tag{4.15}$$

and substituting this into (4.12) yields

$$z = \Sigma_i(d)\xi = \Sigma_i(d)[I - \psi_o(v)\Sigma_i(d)]^{-1}n. \tag{4.16}$$

This shows that the output signal z is well defined as a function of the disturbance signal n. Since $s = C_o(v, z)$, it follows that s is also uniquely determined by n, for any v and d. Similar arguments show that all other signals within the configuration remain well defined when n is incorporated, and the control configuration remains well posed.

6) We fix now the signals n and d and consider the stability with respect to the disturbance signal v. By our basic premise that only bounded signals and disturbances are applied, there is a real number $\alpha > 0$ such that $v \in S(\alpha^m)$ and $d \in S(\alpha^p)$. Denote by $s(n)$ and $z(n)$ the respective signals generated by the disturbance n under those circumstances. On account of the strict internal stability of the inner loop of Fig. 3.2.1, there are two real numbers $\varepsilon, \theta > 0$ such that $|\Sigma_i(d)(s(n) + n) - \Sigma_i(d)s(n)| \le \theta$ for all $n \in S(\varepsilon^m)$ and all $s(n) \in S(R^m)$. For a sequence $n \in S(\varepsilon^m)$, let $\delta z := \Sigma_i(d)[s(n) + n] - \Sigma_i(d)s(n)$, so that $|\delta z(n)| \le \theta$. Notice that

$$\begin{aligned} s(n) &= C_o(v, \Sigma_i(d)(s(n) + n)) \\ &= C_o(v, [\Sigma_i(d)s(n) + \delta z(n)]). \end{aligned} \tag{4.17}$$

Remove now the disturbance n (so that $\xi = s$ in Fig. 4.11), and add the disturbance $\delta z(n)$ to the output signal z. Denote by s^*, z^* the values of the signals s and z, respectively, for this new configuration. We claim that $s^* = s(n)$. To prove the claim note that, by definition, $z^* = \Sigma_i(d)s^*$, and

$$s^* = C_o(v, (z^* + \delta z(n)) = C_o(v, [\Sigma_i(d)s^* + \delta z(n)]). \tag{4.18}$$

Combine now the disturbances $\delta z(n)$ and d, both of which are additive disturbances on z^*. Clearly, this has no effect on any of the signals and yields

$$C_o(v, [\Sigma_i(d)s^* + \delta z(n)]) = C_o(v, \Sigma_i(d + \delta z(n))s^*). \tag{4.19}$$

Thus, we have

$$s^* = C_o(v, \Sigma_i(d + \delta z(n))s^*). \tag{4.20}$$

We claim that (4.20) determines s^* uniquely. Indeed, using the notation $\psi_o(v)z = C_o(v, z)$, we obtain $s^* = \psi_o(v)\Sigma_i(d + \delta z(n))s^*$ or

$$[I - \psi_o(v)\Sigma_i(d + \delta z(n))]s^* = 0. \tag{4.21}$$

Invoking now Lemma 2.2, while taking into account the strict causality of the system $\Sigma_i(d + \delta z(n)) : S(R^m) \to S(R^p)$ and the causality of the system $\psi_o(v) : S(R^p) \to S(R^m)$ (with v, d, and n fixed), we obtain that $[I - \psi_o(v)\Sigma_i(d + \delta z(n))]$ is bicausal. Thus, the unique solution of (4.21) is

$$s^* = [I - \psi_o(v)\Sigma_i(d + \delta z(n))]^{-1}0 \tag{4.22}$$

which is also the unique solution of (4.19) and (4.18). Finally, a comparison of (4.18) with (4.17) shows that $s^* = s(n)$ is also a solution of (4.18); since the latter has a unique solution, this must be it. Thus,

$$s^* = s(n). \tag{4.23}$$

Also, $\Sigma_i(d+\delta z(n))s(n) = \Sigma_i(d)s(n)+\delta z(n) = \Sigma_i(d)[s(n)+n] = z(n)$, so that

$$\Sigma_i(d + \delta z(n))s(n) = z(n). \tag{4.24}$$

7) Still regarding $\delta z(n)$ as part of the disturbance d and using (4.10) in combination with (4.23), we obtain

$$\begin{aligned} s(n) = G(&[I + P\phi(v)](d + \delta z(n)))Q\phi(v)(d + \delta z(n)) \\ &- T[I + P\phi(v)](d + \delta z(n)). \end{aligned} \tag{4.25}$$

Recall that $v \in S(\alpha^m)$, $d \in S(\alpha^p)$, and $\delta z(n) \in S(\theta^p)$. Consequently, $d + \delta z(n) \in S((\alpha + \theta)^p)$, and the stability of the systems G, P, ϕ, and Q in (4.25) entails that there is a real number $\beta > 0$ such that $s(n) \in S(\beta^m)$, i.e., $s(n)$ is bounded. Since $\delta z(n) \in S(\theta^p)$ whenever $n \in S(\varepsilon^m)$, it follows that $s(n) \in S(\beta^m)$ for all $n \in S(\varepsilon^m)$, $v \in S(\alpha^m)$, and $d \in S(\alpha^p)$.

When these facts are used in conjunction with (4.24), we obtain that $z(n) \in \Sigma_i[S((\alpha + \theta)^p) \times S(\beta^m)]$. The stability of Σ_i then implies that $z(n)$ is bounded, and there is a real number $\eta > 0$ such that $z(n) \in S(\eta^p)$ for all $n \in S(\varepsilon^m)$, $v \in S(\alpha^m)$, and $d \in S(\alpha^p)$.

Now, the set $S((\beta + \varepsilon)^m)$ is compact with respect to the topology induced by the norm ρ, and thus the restriction of $\Sigma_i(d)$ to $S((\beta+\varepsilon)^m)$ is uniformly continuous. This means that for every real number $\gamma > 0$ there is a real number $\omega > 0$ such that, for all $s \in S(\beta^m)$, one has $\rho[\Sigma_i(d)(s+n) - \Sigma_i(d)s] < \gamma$ for all $n \in S(\varepsilon^m)$ satisfying $\rho(n) < \omega$. Since we have $s(n) \in S(\beta^m)$ for all $n \in S(\varepsilon^m)$, the latter directly implies that $\delta z(n) = \Sigma_i(d)[s(n)+n] - \Sigma_i(d)s(n)$ is a continuous function of n, as long as $n \in S(\varepsilon^m)$. When this fact is used in conjunction with (4.25), taking into account the stability of the systems G, P, ϕ, and Q, it implies that $s(n)$ is a continuous function of n over $S(\varepsilon^m)$. Having shown earlier that $s(n)$ is bounded, it follows that the transmission from n to s is stable over $S(\varepsilon^m)$.

Finally, when these facts are combined with the stability of the system $\Sigma_i(\cdot)$ and (4.24), they lead us to the conclusion that $z(n)$ is a continuous function of n as long as $n \in S(\varepsilon^m)$. Since we have already shown earlier that $z(n)$ is bounded for all $n \in S(\varepsilon^m)$, it follows that the transmission from n to z is stable over $S(\varepsilon^m)$.

8) We consider next the effect of a disturbance signal ζ that is added to the signal u in (3.2.1), i.e., inside the inner loop. The objective is to show that the transmission from ζ to z is stable. We use an argument similar to the one used for the disturbance v.

We regard the disturbance ζ as a new input of the inner loop, so that the inner loop now has the two inputs s and ζ. To express this fact, we represent the response of the inner loop in the form $\Sigma_i(d)(s, \zeta)$. Clearly, within the closed loops shown in Fig. 3.2.1, the signals s and z depend on ζ, so we denote by $s(\zeta)$, $z(\zeta)$ the values of those signals with the disturbance ζ present. The signals v and d are temporarily regarded as fixed, and, as before, we take $v \in S(\alpha^m)$ and $d \in S(\alpha^p)$.

Now, by the strict internal stability of the inner loop, there is a real number $\varepsilon > 0$ such that $|\Sigma_i(d)(s(\zeta), \zeta) - \Sigma_i(d)(s(\zeta), 0)| \le \theta$ for all $\zeta \in S(\varepsilon^m)$ and all $s(\zeta) \in S(R^m)$. Denote $\delta z(\zeta) := \Sigma_i(d)(s(\zeta), \zeta) - \Sigma_i(d)(s(\zeta), 0)$, and note that $|\delta z(\zeta)| \le \theta$ for all $\zeta \in S(\varepsilon^m)$. Notice that $s(\zeta) = \psi_o(v)z(\zeta) = \psi_o(v)\Sigma_i(d)(s(\zeta), \zeta) = \psi_o(v)[\Sigma_i(d)(s(\zeta), 0) + \delta z(\zeta)] = \psi_o(v)\Sigma_i[d + \delta z(\zeta)](s(\zeta), 0) = \psi_o(v)\Sigma_i[d + \delta z(\zeta)]$ $s(\zeta)$, so that

$$s(\zeta) = \psi_o(v)\Sigma_i[d + \delta z(\zeta)]s(\zeta). \tag{4.26}$$

Eliminate now the disturbance ζ, and consider $\delta z(\zeta)$ as an additive disturbance on the signal z. Let s' and z' be the respective values of the signals s and z under these circumstances. We claim that $s' = s(\zeta)$. To prove the claim, notice that $\delta z(\zeta)$ can be regarded as part of the disturbance d, so that $z' = \Sigma_i(d + \delta z(\zeta))s'$, and

$$s' = \psi_o(v)z' = \psi_o(v)\Sigma_i(d + \delta z(\zeta))s'. \tag{4.27}$$

Notice that $d + \delta z(\zeta) \in S((\alpha + \theta)^m)$. As shown in part 7) of the present proof, (4.27) has a unique solution, which, in view of (4.26) is given by

$$s' = s(\zeta). \tag{4.28}$$

Applying (4.10) to our present situation, we obtain

$$\begin{aligned} s(\zeta) = G([I + P\phi(v)](d + \delta z(\zeta)))Q\phi(v)(d + \delta z(\zeta)) \\ - T[I + P\phi(v)]((d + \delta z(\zeta))) \end{aligned} \tag{4.29}$$

which, on account of the stability of the systems G, P, Q, ϕ, and T, shows directly that $s(\zeta)$ is bounded and is a continuous function of $\delta z(\zeta)$. An argument identical to the one used in 7) leads us then to the conclusion that $s(\zeta)$ and $z(\zeta)$ are bounded and continuous functions of the disturbance ζ, whenever $\zeta \in S(\varepsilon^m)$, with $v \in S(\alpha^m)$ and $d \in S(\alpha^p)$ for any real $\alpha > 0$. This shows that the transmissions from ζ to s and z are stable for all $\zeta \in S(\varepsilon^m)$, $v \in S(\alpha^m)$ and $d \in S(\alpha^p)$.

The stability of all other transmissions within Fig. 3.2.1 follows from the above, or along similar lines.

Within the proof, the effect of each signal and disturbance was considered separately, whereas the definition of internal stability requires simultaneous consideration of the effect of all signals and disturbances. Nevertheless, since bounded signals and disturbances make all relevant domains compact, internal stability follows by Lemma 4.5. ♦

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